Equivalence of Normal States on von Neumann Algebras and the Flow of Weights

UFFE HAAGERUP

Institut for Matematik og Datalogi, Odense Universitet, DK-5230 Odense M, Denmark

AND

ERLING STØRMER

Matematisk Institutt, Oslo Universitet, Blindern, Oslo 3, Norway

Communicated by Richard V. Kadison

Let $M$ be a von Neumann algebra. Two positive normal functionals $\varphi, \psi$ on $M$ are called equivalent, $\varphi \sim \psi$, if $\psi$ is in the norm-closure of the orbit of $\varphi$ under the action of inner automorphisms. Our main result is an isometric characterization of the quotient space $M^+_*/\sim$: We construct a natural isometry $[\varphi] \rightarrow \hat{\varphi}$ of $M^+_*/\sim$ into the set of positive normal functionals on "the smooth flow of weights" of $M$, where the smooth flow of weights is realized as the center $Z(N)$ of the crossed product $N = M \times_{\omega} \mathbb{R}$ for some faithful normal semifinite weight $\omega$ on $M$. As an application we obtain that an automorphism $\alpha$ on a factor $M$ with separable predual acts trivially on $M^+_*/\sim$ if and only if $\alpha$ acts trivially on the smooth flow of weights, i.e., the Connes-Takesaki modulus $\text{mod}(\alpha)$ of $\alpha$ vanishes. We also obtain a new proof of the diameter formula

$$\text{diam}(S_\lambda(M)/\sim) = 2 \frac{1 - \sqrt{\lambda}}{1 + \sqrt{\lambda}}$$

for the quotient of the state space of a factor of type $\text{III}_\lambda$, $0 \leq \lambda \leq 1$. © 1990 Academic Press, Inc.

CONTENTS

1. Introduction and formulation of the main theorem.
2. The equivalence relation on $M^+_*$.
3. The crossed product $M \times_{\omega} \mathbb{R}$.
4. Proof of the main theorem for semifinite factors.
5. Proof of the main theorem for factors of type $\text{III}_\lambda$, $0 < \lambda < 1$.
6. One-parameter automorphism groups on abelian von Neumann algebras.
7. A martingale theorem.
10. The quotient space $S_\lambda(M)/\sim$.
11. States with centralizers of type $\text{II}_1$. 

180
12. Extensions of automorphisms to $M \times M$. 

1. INTRODUCTION AND FORMULATION OF THE MAIN THEOREM

Let $M$ be a von Neumann algebra with predual $M^*$, with positive cone $M^*$. For each $\varphi \in M^*$ let $[\varphi]$ be the norm closure of its orbit under the action of the inner automorphisms by $\varphi \mapsto u \varphi u^* = \varphi \circ Ad(u)$. From the norm identity
\[ \| \varphi - u \varphi u^* \| = \| u^* \varphi u - \psi \|, \]
it is clear that we have an equivalence relation on $M^*$ by $\varphi \sim \psi$ if $\psi \in [\varphi]$. The quotient space $M^*/\sim$ is a metric space with metric
\[ d([\varphi], [\psi]) = \inf \{ \| \varphi' - \psi' \| : \varphi' \sim \varphi, \psi' \sim \psi \}. \]

The first result on $M^*/\sim$ is due to Powers [15], who showed that for factors of type $I_n$, $n \in \mathbb{N}$, $M^*/\sim$ is isometric to the set of decreasing nonnegative functions on the set $\{1, 2, \ldots, n\}$ with $L_1$-norm due to the counting measure. The map was as follows: Let $Tr$ be the usual trace and let $\varphi \in M^*$ be defined by $\varphi(x) = Tr(hx)$. Then $h$ has spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, and the function corresponding to $\varphi$ is $f_\varphi(j) = \lambda_j$.

Later on, Connes and the second named author [4] showed that for factors of type $III$, with separable predual any two states are equivalent, from which it follows that $[\varphi] \to \varphi(1)$ defines an isometry of $M^*/\sim$ onto $\mathbb{R}_+ \cup \{0\}$. In her thesis [1] Bion-Nadal characterized $M^*/\sim$ for factors of type $III$ with separable predual by exhibiting a bijection of $M^*/\sim$ onto a certain class of positive Borel measures on a point-set realization of the flow of weights. Particularly for factors of type $III_A$, $0 < \lambda < 1$. $M^*/\sim$ was mapped onto the set of positive bounded measures on the unit circle. The next step towards our understanding of the quotient space was taken by Connes and the present authors in [3], where the diameter of the quotient state space $S_n(M)/\sim$ of the normal states was shown to be 2 for factors of types $II$ and $I_\infty$ and $2((1 - \sqrt{\lambda})/(1 + \sqrt{\lambda}))$ for factors of type $III_A$ with separable predual, $\lambda \in [0, 1]$.

In the present paper we shall complete the program by giving a rather concrete isometric characterization of $M^*/\sim$. The result is easily described for semifinite factors and type $III_A$-factors, $0 < \lambda < 1$, with separable preduals. Assume first $M$ is semifinite with trace $\tau$, and let $J = \{\tau(p) : p$ finite projection in $M\}$. Then we show that $M^*/\sim$ is isometric to the set of nonnegative decreasing $L^1$-functions on $\mathbb{R}_+ = (0, \infty)$ with values in $J$ which are continuous from the right (Thm. 4.4). The connection to Powers' characterization of $M^*/\sim$ in the type $I_n$ case is given by (generalized)
inverse functions: The list of eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_n \) of \( h = d\varphi/d\text{Tr} \), \( \varphi \in M^+_\tau \), can be considered as a function from \( J \setminus \{0\} \) to \([0, \infty)\) and the corresponding function \( f^\varphi \) from \([0, \infty)\) to \( J \) in Theorem 4.4 is
\[
f^\varphi(s) = \# \{ i | \lambda_i > s \},
\]
where \( \# \) denotes the number of elements in the set.

If \( M \) is of type III\(_2\), then \( M^+_{\tau}/\sim \) is isometric to the set of nonnegative decreasing functions on \( \mathbb{R}_+ \), satisfying \( \lambda f(\lambda a) = f(a) \), \( a \in \mathbb{R}_+ \), considered as a subset of \( L^1(\lambda, 1) \).

If \( M \) is of type III\(_0\) there is no such simple characterization of \( M^+_{\tau}/\sim \). As pointed out by Bion-Nadal [1] it is necessary to consider the “flow of weights” of \( M \) and to reformulate the theorem. We shall do this as follows; more details will be given in Section 3.

Let \( M \) be a von Neumann algebra and \( \omega \) a faithful normal semifinite weight on \( M \). Let \( \sigma^\omega \) be the associated modular automorphism group and \( N = M \times_{\sigma^\omega} \mathbb{R} \). It was shown by Takesaki [21] that \( N \) is independent of the choice of \( \omega \), and that \( N \) is semifinite with a faithful normal trace \( \tau \), such that \( \tau \circ \theta_s = e^{-s}\tau \), where \( \{\theta_s\}_{s \in \mathbb{R}_+} \) is the dual automorphism group of \( \sigma^\omega \) on \( N \). For \( \varphi \in M^+_\tau \) let \( \tilde{\varphi} \) denote its dual weight on \( N \), and let \( d\tilde{\varphi}/dt \) denote the Radon–Nikodym derivative of \( \tilde{\varphi} \) with respect to \( \tau \). Then \( d\tilde{\varphi}/dt \) is a positive self-adjoint operator affiliated with \( N \); hence the spectral projection \( e_{\varphi} = \chi_{(1, \infty)}(d\tilde{\varphi}/dt) \) belongs to \( N \). Now \( \tau(e_{\varphi}) = \varphi(1) \), so we can obtain a positive normal functional \( \tilde{\varphi} \) on the center \( Z(N) \) of \( N \) (also called the flow of weights) defined by
\[
\tilde{\varphi}(z) = \tau(e_{\varphi} z), \quad z \in Z(N).
\]

Notice that from the unitary invariance of \( \tau \) we have \( (u\varphi u^*) = \tilde{\varphi} \) for every unitary operator \( u \in M \), so we obtain by continuity a map \( M^+_{\tau}/\sim \to Z(N)^+ \) by \( [\varphi] \to \tilde{\varphi} \). Our main result can now be formulated.

**Main Theorem.** Let \( M \) be a von Neumann algebra. Then with notation as above we have

(i) The map \([\varphi] \to \tilde{\varphi}\) is an isometry of \( M^+_{\tau}/\sim \) into \( Z(N)^+ \).

(ii) If \( M \) is properly infinite with no direct summand of type I then the range of the map \([\varphi] \to \tilde{\varphi}\) is the cone
\[
P(M) = \{ \chi \in Z(N)^+: \chi \circ \theta_s \geq e^{-s}\chi, s \in \mathbb{R}_+ \}.
\]

(iii) If \( M \) is of type II\(_1\) then the range of the map \([\varphi] \to \tilde{\varphi}\) is the set \( \{ \chi \in P(M): \chi \leq \tau |_{Z(N)} \} \).

The proof of this theorem constitutes the main bulk of the present paper and forms the contents of Sections 2–9. The different sections are quickly
described as follows. In Section 2 we prove the basic facts needed on the equivalence relation ~ on $M_+^\star$. In Section 3 we present the crossed product construction $N = M \times_{\varphi^\omega} \mathbb{R}$ in more detail and prove the basic results, in particular that the map $[\varphi] \mapsto \phi$ is a contraction and that $\phi \in P(M)$ for all $\varphi \in M_+^\star$. In Sections 4 and 5 the theorem is proved for, respectively, semifinite factors and III$_0$-factors, $0 < \lambda < 1$, with separable preduals. To show the theorem for factors of type III$_0$ with separable preduals, it is necessary to use the fact that such factors are limits of increasing sequences of (global) II$_\infty$-algebras. Thus we need a martingale theorem saying roughly that the theorem holds for a limit of an increasing net of von Neumann algebras for which it holds. This is done in Section 7 after we prove some preliminary facts on flows on abelian algebras in Section 6. In Section 8 we use direct integral theory to show the Main Theorem first for II$_\infty$-algebras with separable preduals, then via the martingale theorem for III$_0$-factors, and then finally for all von Neumann algebras with separable preduals. The proof is completed in Section 9. It is first done for $\sigma$-finite algebras by use of the martingale theorem to an increasing net of separable algebras. In order to do this we show that if $M$ is $\sigma$-finite of type III then there exists a countably generated subalgebra $N$ of $M$ such that whenever $P$ is a von Neumann algebra with $N \subset P \subset M$, then $P$ is of type III. The general case follows easily from the $\sigma$-finite case, since any von Neumann algebra is a direct sum of algebras of the form $P \otimes B(H)$, where $P$ is $\sigma$-finite and $B(H)$ all bounded operators on a Hilbert space $H$.

The remaining four sections of the paper include applications of the Main Theorem. It is shown in Section 10, partly as a consequence of the flow results in Section 6, that if $M$ is properly infinite with no direct summand of type I then the quotient state space $S_n(M)/\sim$ is isometric to a (in general noncompact) norm complete Choquet simplex. Furthermore, we obtain a new proof of the diameter formula in [3]. In Section 11 it is shown that if $M$ has no direct summand of type I and $\varphi \in M_+^\star$ is faithful, then there exists a faithful functional $\psi \in [\varphi]$ with centralizer of type II$_1$. This extends a result in [4].

Finally, in the last two sections we study the automorphism group $\text{Aut}(M)$ of $M$. If $\alpha \in \text{Aut}(M)$ then $\alpha$ has a canonical extension to $\tilde{\alpha} \in \text{Aut}(N)$ ($N = M \times_{\varphi^\omega} \mathbb{R}$ as before). $\alpha$ also induces an isometry $\tilde{\alpha}_*^\star$ of $M_+^\star / \sim$ arising from its adjoint action $\alpha_\star^\star : M_+^\star \to M_+^\star$. The main result in Section 12 states that the homomorphisms $\alpha \to \tilde{\alpha}_\star^\star |_{Z(N)}$ and $\alpha \to \tilde{\alpha}_*$ have the same kernels, namely the closed normal subgroup of $\text{Aut}(M)$ consisting of those $\alpha$ for which there is for each $\varphi \in M_+^\star$ and $\varepsilon > 0$ a unitary operator $u = u(\varphi, \varepsilon) \in M$, such that

$$\|\varphi \circ \alpha - u\varphi u^*\| < \varepsilon.$$
If $M$ is properly infinite with separable predual it is deduced in Section 13 from [5] and [21] that there is an isomorphism $\beta$ of the flow of weights on $M$ onto $Z(N)$, such that $\tilde{\beta}': Z(N) = \beta \circ \text{mod}(x) \circ \beta^{-1}$ for $x \in \text{Aut}(M)$, where $\text{mod}$ is the fundamental homomorphism of $\text{Aut}(M)$ into the automorphisms of the flow of weights found in [5]. Thus the results in Section 12 in particular describe the kernel of $\text{mod}$ for all von Neumann algebras.

2. THE EQUIVALENCE RELATION ON $M_+^*$

As described in the Introduction, if $M$ is a von Neumann algebra and $\varphi, \psi \in M_+^*$ then $\varphi$ and $\psi$ are called equivalent, written $\varphi \sim \psi$, if $\psi$ belongs to the norm closure $[\varphi]$ of the set of functionals $u\varphi u^*$, $u \in U(M)$—the unitary group in $M$. Here we used the notation $u\varphi u^*(x) = \varphi(u^*xu)$, $u \in U(M)$. The quotient space $M_+^*/\sim$ is a metric space with metric

$$d([\varphi], [\psi]) = \inf\{\|\varphi' - \psi'\|, \varphi' \in [\varphi], \psi' \in [\psi]\}.$$

**Lemma 2.1.** With the metric $d$, $M_+^*/\sim$ is a complete metric space.

**Proof.** Let $[\varphi_n]$ be a Cauchy sequence in $M_+^*/\sim$. It suffices to show that a subsequence converges to an element in $M_+^*/\sim$, so we may assume

$$d([\varphi_n], [\varphi_{n-1}]) < 2^{-n}.$$

Thus there is $v_n \in U(M)$ such that $\|v_n\varphi_nv_n^* - \varphi_{n-1}\| < 2^{-n}$. Choose $\varphi'_n \in [\varphi_n]$ recursively as follows. Let $\varphi'_1 = \varphi_1$. Suppose $\varphi'_1, \ldots, \varphi'_{n-1}$ are chosen in $[\varphi_1], \ldots, [\varphi_{n-1}]$, respectively, so that $\|\varphi_k - \varphi_{k-1}\| < 2^{-k}$, $k = 1, \ldots, n-1$, and $\varphi'_k = u_k\varphi_k u_k^*$, $u_k \in U(M)$. Let $u_n = u_{n-1}v_n$ and $\varphi'_n = u_n\varphi_n u_n^*$. Then clearly $\|\varphi'_n - \varphi_n\| < 2^{-n}$, whence $(\varphi'_k)$ is Cauchy and therefore converges to $\varphi \in M_+^*$. Since $d([\varphi_n], [\varphi]) \leq \|\varphi'_n - \varphi\|$, $[\varphi_n] \rightarrow [\varphi]$, and $M_+^*/\sim$ is complete. Q.E.D.

In the definition of the equivalence relation $\sim$ it is often preferable to work with partial isometries instead of unitaries. The following theorem will therefore be useful. We use the notation that $\text{supp}(\varphi)$ denotes the support projection of $\varphi$ in $M$ when $\varphi \in M_+^*$.

**Theorem 2.2.** Let $M$ be a von Neumann algebra and $\varphi, \psi \in M$. Then the following are equivalent.

(i) $\varphi \sim \psi$.

(ii) For every $\varepsilon > 0$ there exists a partial isometry $u \in M$ such that $u^*u = \text{supp}(\varphi)$, $uu^* = \text{supp}(\psi)$, and

$$\|u\varphi u^* - \psi\| < \varepsilon.$$
For the proof we shall need two lemmas.

**Lemma 2.3.** With \( \phi \sim \psi \) in \( M^+ \) then \( \text{supp}(\phi) \sim \text{supp}(\psi) \) (as projections in \( M \)).

**Proof.** By the GNS-construction we can assume that \( \phi \) is a vector state, \( \phi = \omega_\xi \), for a cyclic vector \( \xi \) in the underlying Hilbert space \( H \). Choose a sequence \((u_n)_{n \in \mathbb{N}}\) in \( U(M) \) such that
\[
\| \psi - u_n \phi u_n^* \| \to 0 \quad \text{for} \quad n \to \infty.
\]
Since \( u_n \phi u_n^* = \omega_{u_n \xi} \) it follows from [12, Thm. 7.3.11] that \( \psi \) is also a vector state, \( \psi = \omega_\eta \), with \( \eta \in H \). Since \( \xi \) is cyclic \([M\eta] \leq [M\xi]\), so by the Coupling Theorem [12, Thm. 7.2.12]
\[
\text{supp}(\psi) = [M\eta] \subseteq [M\xi] = \text{supp}(\phi).
\]
By symmetry of \( \phi \sim \psi \) we get \( \text{supp}(\phi) \sim \text{supp}(\psi) \). Q.E.D.

**Lemma 2.4.** Let \( \phi, \psi \in M^+ \). Suppose \( \| \phi \| = \| \psi \| \) and that for every \( \varepsilon > 0 \) there is \( a \in M, \| a \| \leq 1 \), such that \( \| a \phi a^* - \psi \| < \varepsilon \). Then \( \phi \sim \psi \).

**Proof.** It is sufficient to consider the case \( \| \phi \| = \| \psi \| = 1 \). By multiplying \( a \) by a positive scalar slightly less than 1 we can obtain that \( \| a \| < 1 \) and still have \( \| a \phi a^* - \psi \| < \varepsilon \). By the Russo–Dye–Palmer Theorem (see [11]) \( a \) is a convex combination of unitaries in \( M \), say
\[
a = \sum_{i=1}^n \lambda_i u_i, \quad u_i \in U(M), \quad \lambda_i > 0, \quad \sum \lambda_i = 1.
\]
We may assume \( \phi \) is a vector state, \( \phi = \omega_\xi \). Since \( \| \phi \| = \| \psi \| = 1 \) we have \( \| \xi \|^2 - 1 \), and
\[
\| a \xi \|^2 = \| a \phi a^* \| > 1 - \varepsilon.
\]
Hence
\[
\sum_{i=1}^n \lambda_i \| u_i \xi - a \xi \|^2 = \sum_{i=1}^n \lambda_i (\| \xi \|^2 + \| a \xi \|^2 - 2 \text{Re}(u_i \xi, a \xi))
\]
\[
= \| \xi \|^2 + \| a \xi \|^2 - 2 \text{Re}(a \xi, a \xi)
\]
\[
= \| \xi \|^2 - \| a \xi \|^2
\]
\[
< \varepsilon.
\]
Therefore, \( \|u_i \xi - a \xi\| < \epsilon \) for at least one \( i \in \{1, \ldots, n\} \). For two vector states \( \omega_{\xi_1}, \omega_{\xi_2} \), we always have

\[
\|\omega_{\xi_1} - \omega_{\xi_2}\| \leq \|\xi_1 - \xi_2\| \|\xi_1 + \xi_2\|.
\]

Therefore,

\[
\|u_i \varphi u_i^* - a \varphi a^*\| \leq \|u_i \xi - a \xi\| \|u_i \xi + a \xi\| < 2 \epsilon^{1/2}
\]

for some \( i \in \{1, \ldots, n\} \). It follows that

\[
\|u \varphi u^* - \psi\| < \epsilon + 2 \epsilon^{1/2}
\]

for some \( u \in U(M) \); hence \( \varphi \sim \psi \). Q.E.D.

Proof of Theorem 2.2. The implication (ii) \( \Rightarrow \) (i) is a special case of Lemma 2.4. To show the converse we can by Lemma 2.3 choose a partial isometry \( v \in M \) such that

\[
v^*v = \text{supp}(\varphi), \quad vv^* = \text{supp}(\psi).
\]

Put \( \psi' = v^* \psi v \). Then \( \text{supp}(\psi') = v^*v = \text{supp}(\varphi) \), and by (ii) \( \psi' \sim \psi \); hence also \( \psi \sim \varphi \). Let \( \epsilon > 0 \) and choose \( u \in U(M) \) such that

\[
\|u \varphi u^* - \psi\| < \epsilon.
\]

Put \( p = \text{supp}(\varphi) = \text{supp}(\psi') \), and \( a = pu u^* \). Then \( \|a\| \leq 1 \), \( a \in pMp \), and

\[
\|a \varphi a^* - \psi\| = \|p(u \varphi u^* - \psi') p\| < \epsilon.
\]

It is clear that \( \varphi \sim \psi' \) implies \( \|\varphi\| = \|\psi'\| \), so by Lemma 2.4 we can find a unitary \( w \) in the reduced algebra \( pMp \) such that

\[
\|w \varphi w^* - \psi'\| < \epsilon.
\]

The inequality still holds when we consider \( w \) as a partial isometry in \( M \) with support and range projection equal to \( p \). Put now \( u_1 = vw \). Then

\[
u_1^* u_1 = \text{supp}(\varphi), \quad u_1 u_1^* = \text{supp}(\psi),
\]

and

\[
\|u_1 \varphi u_1^* - \psi\| = \|v(w \varphi w^* - \psi') v^*\| < \epsilon.
\]

This proves (ii). Q.E.D.

If \( p \) is a projection in \( M \) and \( pMp \) is the reduced von Neumann algebra, then each \( \varphi \in (pMp)_* \) has a natural extension in \( M_+ \), still denoted by \( \varphi \),
such that \( \varphi(1 - p) = 0 \). This extension identifies \((pMp)^+\) with the set of \( \varphi \in M^+_\ast \) with \( \text{supp}(\varphi) \leq p \). We shall write \([\varphi]_p\) for the equivalence class of \( \varphi \) in \((pMp)^+\). Recall from [12, Thm. 6.3.4] that if \( p \) is a properly infinite projection in \( M \) with central support 1, and \( e \) is a \( \sigma \)-finite projection in \( M \), then \( e \leq p \).

**Lemma 2.5.** Let \( M \) be a von Neumann algebra and \( p \) a properly infinite projection in \( M \) with central support 1. Let \( \psi \in M^+_\ast \) and let \( e \) be a \( \sigma \)-finite projection in \( M \) such that \( e \geq \text{supp}(\psi) \). Let \( v \) be a partial isometry in \( M \) such that \( vv^* \leq p \), \( v^*v = e \). Then we have

(i) \( v\psi v^* \in (pMp)^+_\ast \cap [\psi] \).

(ii) If \( \varphi \in M^+_\ast \), \( \varphi \sim \psi \), and \( w \) is a partial isometry in \( M \) such that \( ww^* \leq p \), \( w^*w \geq \text{supp}(\varphi) \), then

\[ [v\psi v^*]_p = [w\varphi w^*]_p. \]

**Proof.** Note that since \( \text{supp}(\psi) \) is \( \sigma \)-finite the choice of \( v \) is possible. By Theorem 2.2 applied to the partial isometry \( v_1 = v \text{supp}(\psi) \) and using that \( v_1 v_1^* = v_1 \), \( \psi v_1^* \), it follows that \( v\psi v^* \in [\psi] \), proving (i). Let now \( \varphi \in M^+_\ast \) and \( w \) be as described in (ii). By (i) applied to \( \varphi \), \( w\varphi w^* \sim \varphi \), so by (i) and the assumption \( \varphi \sim \psi \), \( v\psi v^* \sim w\varphi w^* \). Let \( \varepsilon > 0 \). Then by Theorem 2.2 there is a partial isometry \( u \in M \) such that

\[ u^*u = \text{supp}(v\psi v^*), \quad uu^* = \text{supp}(w\varphi w^*), \]

and

\[ \|uu\psi v^* - w\varphi w^*\| < \varepsilon. \]

Since \( uu^* \leq p \) and \( u^*u \leq p \), \( v\psi v^* \) and \( w\varphi w^* \) are equivalent in \((pMp)^+_\ast \) by Theorem 2.2. Thus \([v\psi v^*]_p = [w\varphi w^*]_p\). Q.E.D.

By Lemma 2.5, if \( p \) is a properly infinite projection in \( M \) with central support 1, we obtain a natural map

\[ \Pi: M^+_\ast / \sim \to (pMp)^+_\ast / \sim \]

defined by

\[ \Pi([\psi]) = [v\psi v^*]_p, \]

with \( v \) any partial isometry in \( M \) such that \( vv^* \leq p \), \( v^*v \geq \text{supp}(\psi) \).

**Proposition 2.6.** Let \( M \) be a von Neumann algebra and \( p \) a properly infinite projection in \( M \) with central support 1. Then the map \( \Pi \) defined above is an isometry of \( M^+_\ast / \sim \) onto \((pMp)^+_\ast / \sim \).
Proof. Since each element in $\mathcal{(pM_p)}^+$ has a natural extension in $M_+^*$, $\Pi$ is clearly surjective. To show it is isometric, let $\phi, \psi \in M_+^*$ and let $v$ and $w$ be partial isometries in $M$, such that

$$vv^* \leq p, \quad v^*v \geq \text{supp}(\psi), \quad ww^* \leq p, \quad w^*w \geq \text{supp}(\phi).$$

By definition of $d([\phi], [\psi])$ it is immediate from Lemma 2.5 that

$$d([\phi], [\psi]) = d([w\phi w^*], [v\psi v^*]) = d(\Pi([\phi]), \Pi([\psi])).$$

To show the opposite inequality, let $e > 0$ and choose $\phi_0 \sim \phi$, $\psi_0 \sim \psi$, such that

$$d([\phi], [\psi]) > \|\phi_0 - \psi_0\| - e.$$  

Let $e = \text{supp}(\phi_0) \cup \text{supp}(\psi_0)$. Then $e$ is $\sigma$-finite, so there is a partial isometry $u \in M$ such that $uu^* \leq p$, $u^*u = e$. By Lemma 2.5 $u\psi_0 u^* \in \Pi([\psi])$, $u\phi_0 u^* \in \Pi([\phi])$. Thus

$$d(\Pi([\phi]), \Pi([\psi])) \leq \|u\phi_0 u^* - u\psi_0 u^*\|$$

$$\leq \|\phi_0 - \psi_0\|$$

$$< d([\phi], [\psi]) + e,$$

completing the proof that $\Pi$ is isometric. Q.E.D.

We conclude this section by giving a formula for the norm difference of two functionals. The result has previously been obtained for type I factors by Powers [15, Lem. 5.5]. Recall that if $\phi \in M_*$ is self-adjoint, then $\phi$ has a positive and negative part such that $\phi = \phi^+ - \phi^-$, $\phi^\pm \in M_+^*$, and $\|\phi\| = \|\phi^+\| + \|\phi^-\| [12, Thm. 7.4.7]$. If $\phi, \psi \in M_+^*$, we denote by

$$\phi \vee \psi = \phi + (\phi - \psi)^- = \psi + (\phi - \psi)^+.$$  

**Lemma 2.7.** If $\phi, \psi \in M_+^*$ then

$$\|\phi - \psi\| = 2\phi \vee \psi(1) - \phi(1) - \psi(1)$$

$$= \inf\{2\omega(1) - \phi(1) - \psi(1) : \omega \in M_+^*, \omega \geq \phi, \omega \geq \psi\}.$$

**Proof.** If $\omega \geq \phi, \psi$ then, since $\|\rho\| = \rho(1)$ for $\rho \in M_+^*$,

$$\|\phi - \psi\| \leq \|\omega - \phi\| + \|\omega - \psi\| = 2\omega(1) - \phi(1) - \psi(1).$$

Furthermore,

$$\|\phi - \psi\| = \|(\phi - \psi)^+\| + \|(\phi - \psi)^-\| = \|\phi \vee \psi - \psi\| + \|\phi \vee \psi - \phi\|$$

$$= 2\phi \vee \psi(1) - \phi(1) - \psi(1)$$

proving the lemma. Q.E.D.
3. The Crossed Product $M \times_{\sigma^\omega} \mathbb{R}$

Let $M$ be a von Neumann algebra acting on a Hilbert space $H$. Let $\omega$ be a faithful normal semifinite weight on $M$ with modular group $\sigma^\omega$. Then the crossed product $N_\omega = M \times_{\sigma^\omega} \mathbb{R}$ is defined as follows. Let $\pi_\omega$ and $\lambda$ be the representations of $M$ and $\mathbb{R}$, respectively, on the Hilbert space $L^2(\mathbb{R}, H)$:

$$(\pi_\omega(x)\xi)(s) = \sigma_{-s}^\omega(x)\xi(s), \quad \xi \in L^2(\mathbb{R}, H).$$

$$(\lambda(t)\xi)(s) = \xi(s-t), \quad s \in \mathbb{R}.$$ 

Then $\lambda$ is a unitary representation of $\mathbb{R}$ on $L^2(\mathbb{R}, H)$ such that

$$\lambda(t)\pi_\omega(x)\lambda(t)^* = \pi_\omega(\sigma_t^\omega(x)), \quad x \in M, \quad t \in \mathbb{R}.$$ 

$N_\omega$ is the von Neumann algebra generated by $q_\omega(x), x \in M$ and $\lambda(t), t \in \mathbb{R}$. The dual automorphism group $\Theta^\omega$ of $\sigma^\omega$ on $N_\omega$ is the automorphism group determined by

$$\Theta^\omega_\omega(\pi_\omega(x)) = \pi_\omega(x), \quad x \in M, \quad s \in \mathbb{R}$$

$$\Theta^\omega_\omega(\lambda(t)) = e^{-it\lambda(t)}, \quad s, t \in \mathbb{R}.$$ 

Then $\pi_\omega(M)$ is the fixed point algebra of $\Theta^\omega$ in $N_\omega$. By [9] there is a faithful normal semifinite operator valued weight $T_\omega$ from $N_\omega$ on $\pi_\omega(M)$ given by

$$T_\omega(y) = \int_{-\infty}^{\infty} \Theta^\omega_\omega(y) ds, \quad s \in N_\omega^+,$$

where $ds$ denotes the Lebesgue measure on $\mathbb{R}$. Then for any normal semifinite weight $\phi$ on $M$ its dual weight $\tilde{\phi}_\omega$ on $N_\omega$ is given by

$$\tilde{\phi}_\omega = \phi \circ \pi_\omega^{-1} \circ T_\omega.$$ 

By [21, Lem. 8.2] there is a positive self-adjoint operator $h$ affiliated with $N_\omega$ such that $\lambda(t) = h^t$, and the weight $\tau_\omega$ defined by

$$\tau_\omega(y) = \tilde{\phi}_\omega(h^{-1}y)$$

is a faithful normal semifinite trace on $N_\omega$ such that

$$\tau_\omega \circ \Theta^\omega_\omega = e^{-s}\tau_\omega, \quad S \in \mathbb{R}.$$ 

We call $\tau_\omega$ the canonical trace on $N_\omega$.

Suppose $\psi$ is another faithful normal semifinite weight on $M$. By [21, Props. 3.5 and 4.2] there is a natural isomorphism $\chi$ of $N_\omega$ onto $N_\psi$ such
that $\chi \circ \pi_\omega = \pi_\psi$ and $\chi \circ \theta_\psi^\omega = \theta_\psi^\omega \circ \chi$. By the above discussion it follows that $\bar{\phi}_\omega = \bar{\phi}_\psi \circ \chi$ for any normal semifinite weight on $M$. Since operator valued weights preserve cocycle Radon–Nikodym derivates (cf. [7]), one has

$$(D\bar{\omega}_\omega : D\bar{\psi}_\psi \circ \chi)_t = (D\bar{\omega}_\omega : D\bar{\psi}_\omega)_t = \pi_\omega((D\omega : D\psi)_t).$$

Therefore, by the chain rule for cocycle Radon–Nikodym derivates

$$(D\tau_\omega : D\tau_\psi \circ \chi)_t = h^{-it}\pi_\omega((D\omega : D\psi)_t) \chi^{-1}(h^t)$$

$$= \lambda(-t) \pi_\omega((D\omega : D\psi)_t) \chi^{-1}(\lambda(t)).$$

By [21, Prop. 3.5] $\chi(\lambda(t)) = \pi_\psi((D\omega : D\psi)_t) \lambda(t)$. Thus $\chi^{-1}(\lambda(t)) = \pi_\omega((D\omega : D\psi)_t^{-1}) \lambda(t)$, which implies that $(D\tau_\omega : D(\tau_\psi \circ \chi))_t = 1$ for all $t$, i.e.,

$$\tau_\omega = \tau_\psi \circ \chi.$$ 

Hence $\chi$ carries the quadruple $(N_\omega, \pi_\omega, \theta_\omega, \tau_\omega)$ onto $(N_\psi, \pi_\psi, \theta_\psi, \pi_\psi)$ together with all dual weights $\bar{\phi}_\omega$ onto the corresponding weights $\bar{\phi}_\psi$. We shall therefore refer to the quadruple $(N_\omega, \pi_\omega, \theta_\omega, \tau_\omega)$ as $(N, \pi, \theta, \tau)$ even when we use a specific choice of $\omega$. Also, we write $\bar{\phi}$ instead of $\bar{\phi}_\omega$.

If $\varphi$ is a normal semifinite weight on $M$ we put

$$h_\varphi = \frac{d\bar{\varphi}}{dt}.$$ 

Then $h_\varphi$ is a positive self-adjoint operator affiliated with $N$ such that $\bar{\varphi}(y) = \tau(h_\varphi y)$, $y \in N^+$. 

**Lemma 3.1.** With the above notation let $e_\varphi = \chi_{(1, \infty)}(h_\varphi)$ for $\varphi \in M^+$. Then $\tau(e_\varphi) = \varphi(1)$.

**Proof** (See also [10, 22]). Since $h_\varphi = d\bar{\varphi}/dt$ and $e_\varphi \leq \text{supp}(h_\varphi)$ we have

$$\tau(e_\varphi) = \bar{\varphi}(h_\varphi^{-1} e_\varphi),$$

where $h_\varphi^{-1}$ is the inverse of $h_\varphi$ on $\text{supp}(h_\varphi)(L^2(\mathbb{R}, H))$. From its definition $\bar{\varphi} = \bar{\varphi} \circ \theta_\varphi$. Thus the identity $\tau \circ \theta_\varphi = e^{-t} \tau$ yields the formula

$$\theta_\varphi(h_\varphi) = e^{-t} h_\varphi.$$ 

It follows that $\theta_\varphi(f(h_\varphi)) = f(e^{-t} h_\varphi)$ for every bounded Borel function $f$. Put $f(t) = 0$ for $t \in [0, 1]$ and $f(t) = 1/t$ for $t > 1$. Then $f(e^{-t}) = 0$ for $t \in [0, 1]$ and $f(e^{-t}) = e^t$ for $t > e^t$. Hence we have

$$\int_{-\infty}^{0} f(e^{-s}) \, ds = \int_{-\infty}^{\log t} \frac{e^s}{t} \, ds = \begin{cases} 1, & t > 0 \\ 0, & t = 0. \end{cases}$$
Since \( h_{\varphi} f(h_{\varphi}) = \chi_{(1, \infty)}(h_{\varphi}) = e_{\varphi} \) we thus have

\[
\tau(e_{\varphi}) = \tilde{\varphi}(h_{\varphi}^{-1} e_{\varphi}) = \varphi \circ \pi^{-1} \circ T(f(h_{\varphi})) = \varphi \circ \pi^{-1} \left( \int_{-\infty}^{\infty} \theta_s(f(h_{\varphi})) \, ds \right) = \varphi \circ \pi^{-1} \left( \int_{-\infty}^{\infty} f(e^{-s} h_{\varphi}) \, ds \right) = \varphi \circ \pi^{-1}(\text{supp}(h_{\varphi})) = \varphi(1),
\]

where we have used that \( \text{supp}(\varphi) = \pi^{-1}(\text{supp}(h_{\varphi})) \). Q.E.D.

**Definition 3.2.** If \( \varphi \in M_+^* \) we denote by \( \hat{\varphi} \in Z(N)_+^* \) the functional defined by

\[
\hat{\varphi}(z) = \tau(e_{\varphi} z), \quad z \in Z(N).
\]

By the previous lemma \( \hat{\varphi}(1) = \varphi(1) \), in particular \( \|\hat{\varphi}\| = \|\varphi\| \). Note that it is immediate from the definition of \( \hat{\varphi} \) that

\[
h_{u \varphi u^*} = \pi(u) h_{\varphi} \pi(u)^*, \quad u \in U(M)
\]

\((U(M)\) is the unitary group of \( M \)); hence \( e_{u \varphi u^*} = \pi(u) e_{\varphi} \pi(u)^* \), and therefore \((u \varphi u^*)^\perp = \hat{\varphi}, \ u \in U(M)\).

It follows from our next result that \( \hat{\psi} = \hat{\varphi} \) for all \( \psi \in [\varphi] \).

**Lemma 3.3.** The map \( \varphi \rightarrow \hat{\varphi} \) of \( M_+^* \) into \( Z(N)_+^* \) preserves order, and for \( \varphi, \psi \in M_+^* \) we have

\[
\|\hat{\varphi} - \hat{\psi}\| \leq d([\varphi], [\psi]).
\]

**Proof.** It is well known that if \( h, k \) are positive self-adjoint operators affiliated with a von Neumann algebra \( N \) and \( h \leq k \) (in the sense that \( D(h^{1/2}) \supset D(k^{1/2}) \) and \( \|h^{1/2} \xi\| \leq \|k^{1/2} \xi\| \) for all \( \xi \in D(k^{1/2}) \)) then

\[
\chi_{(a, \infty)}(h) \preceq \chi_{(a, \infty)}(k), \quad a > 0
\]

in the Murray–von Neumann sense as projections in \( N \). Hence if \( \varphi, \psi \in M_+^* \), \( \varphi \leq \psi \), then \( e_{\varphi} \leq e_{\psi} \), and therefore

\[
\tau(e_{\varphi} z) \leq \tau(e_{\psi} z), \quad z \in Z(N)^+,
\]

i.e., \( \hat{\varphi} \leq \hat{\psi} \).
Now let \( \phi, \psi \in M_*^+ \) be given. From the previous paragraph \( (\phi \vee \psi)^\wedge \geq \phi \) and \( (\phi \vee \psi)^\wedge \geq \psi \); hence by Lemma 2.7
\[
\| \hat{\phi} - \hat{\psi} \| \leq 2(\phi \vee \psi)^\wedge (1) - \phi(1) - \psi(1)
= 2(\phi \vee \psi)(1) - \phi(1) - \psi(1)
= \| \phi - \psi \|.
\]
In particular, if \( u \in U(M) \) then
\[
\| \hat{\phi} - \hat{\psi} \| = \| (u\phi u^*)^\wedge - \hat{\psi} \| \leq \| u\phi u^* - \psi \|;
\]

hence
\[
\| \hat{\phi} - \hat{\psi} \| \leq d([\phi], [\psi]). \quad \text{Q.E.D.}
\]

It follows from the last lemma that the map \( \varphi \rightarrow \hat{\phi} \) induces a norm decreasing map \([\varphi] \rightarrow \hat{\phi} \) by \( M_*^+/\sim \) into \( Z(N)^+ \). Our next result points out the key property of the functionals in the range of this map and proves the easy part of the Main Theorem.

**Lemma 3.4.** For all \( \varphi \in M_*^+ \), \( \phi \circ \theta_s \geq e^{-s} \phi \) for \( s > 0 \).

*Proof.* Since \( \theta_s(h_{\varphi}) = e^{-s} h_{\varphi} \), it follows that
\[
\theta_{-s}(e_{\varphi}) = \chi_{(e^{-s}, \infty)}(h_{\varphi}) \geq e_{\varphi} \quad \text{for } s \geq 0.
\]
Hence for \( z \in Z(N)^+ \) and \( s > 0 \),
\[
\phi \circ \theta_s(z) = \tau(e_{\varphi} \theta_s(z)) = e^{-s} \tau(\theta_s(e_{\varphi})z)
\geq e^{-s} \tau(e_{\varphi}z) = e^{-s} \phi(z). \quad \text{Q.E.D.}
\]

**Lemma 3.5.** The map \( \varphi \rightarrow e_{\varphi} \) sends \( M_*^+ \) onto the set of projections in \( L^1(N, \tau) \) such that \( \theta_s(e) \leq e \) for \( s > 0 \).

*Proof.* From the proof of the above lemma \( \theta_s(e_{\varphi}) \leq e_{\varphi} \) whenever \( s > 0 \). Suppose \( e \) is a projection in \( N \) such that \( \theta_s(e) \leq e \), \( s > 0 \), and \( \tau(e) < \infty \). Since the map \( s \rightarrow \theta_s(e) \) is continuous and decreasing, there exists a positive self-adjoint operator \( h \) affiliated with \( N \) such that
\[
\theta_s(e) = \chi_{(e^{-s}, \infty)}(h), \quad s \in \mathbb{R}.
\]
From the identity
\[
t = \int_0^\infty \chi_{(s, \infty)}(t) \, ds, \quad t > 0
\]
it follows that
\[ h = \int_0^\infty \theta_{\log s}(e) \, ds. \]

If \( t > 0 \) we have
\[ \theta_{\log t}(h) = \int_0^\infty \theta_{\log t \log s}(e) \, ds = \int_0^\infty \theta_{\log s}(e) \, ds = \frac{1}{t} \int_0^\infty \theta_{\log s}(e) \, ds = \frac{1}{t} h. \]

Thus \( \theta_t h = e^{-\theta h} \) for \( t \in \mathbb{R} \).

Define a normal semifinite weight \( \hat{\varphi} \) on \( N \) by
\[ \hat{\varphi}(y) = \tau(hy), \quad y \in N^+. \]

Then
\[ \hat{\varphi}(\theta_s(y)) = \tau(h \theta_s(y)) = e^{-\tau(\theta_s(h) y)} = \hat{\varphi}(y). \]

If \( \hat{\varphi} \) is faithful there is by [8, Thm. 3.7] a faithful normal semifinite weight \( \varphi \) on \( M \) such that \( \varphi = \hat{\varphi} \). If \( \hat{\varphi} \) is not faithful, its support \( f \) is \( \theta \)-invariant, since \( \hat{\varphi} \) is; hence \( f \in \pi(M) \). We can find a normal semifinite weight \( \tilde{\omega} \) on \( M \) such that \( \tilde{\omega} \) has support \( 1 - f \) and apply [8, Thm. 3.7] to \( \varphi + \tilde{\omega} \). It is then immediate that \( \tilde{\omega} = \hat{\varphi} \) for a normal semifinite weight \( \varphi \) on \( M \) with support \( \pi^{-1}(f) \). If we apply Lemma 3.1 we find
\[ \varphi(1) = \tau(\chi_{(1, \infty)}(h_{\varphi})) = \tau(\chi_{(1, \infty)}(h)) = \tau(e) < \infty. \]

Thus \( \varphi \in M^+_\tau \), and \( e_{\varphi} = e \). Q.E.D.

4. Proof of the Main Theorem for Semifinite Factors

Throughout this section \( M \) is a semifinite factor with separable predual. \( \tau_0 \) is a faithful normal semifinite trace on \( M \), normalized so that \( \tau_0(e) = 1 \) if \( M \) is of type I and \( e \) a minimal projection in \( M \), and \( \tau_0(1) = 1 \) if \( M \) is of type II\(_1\). Let
\[ J = \{ \tau_0(p) : p \text{ finite projection in } M \}. \]
Then $J = \{0, 1, \ldots, n\}$ (resp. $\{0, 1, 2, \ldots\}$, $[0, 1]$, $[0, \infty)$) if $M$ is of type $I_n$ (resp. $I_\infty$, $II_1$, $II_\infty$). Let $P_f(M)$ denote the set of finite projections in $M$. Thus $J = \{\tau_0(p) : p \in P_f(M)\}$.

**Lemma 4.1.** With the above notation there is a map $p: J \to P_f(M)$ such that

(i) $s, t \in J$, $s \leq t \Rightarrow p(s) \leq p(t)$.

(ii) $\tau_0(p(t)) = t$, $t \in J$.

**Proof.** If $M$ is of type $I_n$, $n \in \mathbb{N} \cup \{\infty\}$, let $\{e_k : k \in J \setminus \{0\}\}$ be an orthogonal sequence of minimal projections in $M$. Define

$$p(t) = \begin{cases} \sum_{k=1}^{\infty} e_k, & t \in J \setminus \{0\} \\ 0, & t = 0. \end{cases}$$

Then $p$ satisfies (i) and (ii).

Suppose next $M$ is of type $II_1$. Let $D$ denote the set of dyadic numbers in $[0, 1]$. By successive use of the Halving Lemma there is a map $p: D \to P_f(M)$ which satisfies (i) and (ii). If $s \leq t$, $s, t \in D$, then

$$\|p(t) - p(s)\|_1 = \tau_0(p(t) - p(s)) = |t - s|.$$ 

By symmetry this also holds for $s \geq t$. Thus $p$ is uniformly continuous on $D$ and therefore has a continuous extension $p: J \to P_f(M)$ satisfying (i) and (ii).

Finally, suppose $M$ is of type $II_\infty$ and let $\{e_k : k \in \mathbb{N}\}$ be an orthogonal sequence in $M$ with $\tau_0(e_k) = 1$. Then the reduced algebras $M_k = e_k M e_k$, $k \in \mathbb{N}$, are factors of type $II_1$, and $\tau_k = \tau_0 | M_k$ is a faithful normal tracial state. For each $k$ let by the previous paragraph $p_k: [0, 1] \to P_f(M_k)$ satisfy (i) and (ii) with respect to $\tau_k$. Define $p: [0, \infty] \to P_f(M)$ as follows. If $s \in \mathbb{R}$ then $s = m + r$, $m \in \mathbb{N}$, $r \in [0, 1)$. Then we put

$$p(s) = \sum_{k=1}^{m} e_k + p_{m+1}(r).$$

Then $p$ satisfies (i) and (ii).

**Lemma 4.2.** For $\varphi \in M_\mathbb{R}^+$ define

$$f_\varphi(a) = \tau_0 \left( \chi_{(a, \infty)} \left( \frac{d\varphi}{d\tau_0} \right) \right), \quad a > 0.$$ 

Q.E.D.
Then we have for \( \varphi, \psi \in M_+^* \):

(i) \( \varphi(1) = \int_0^\infty f_\varphi(a) \, da \).

(ii) If \( \varphi \leq \psi \) then \( f_\varphi \leq f_\psi \).

(iii) \( d([\varphi], [\psi]) \geq \int_0^\infty |f_\varphi(a) - f_\psi(a)| \, da \).

**Proof.** From the identity

\[
 t = \int_0^\infty \chi(s, \infty)(t) \, ds, \quad t > 0,
\]

we have

\[
 \frac{d\varphi}{dt_0} = \int_0^\infty \chi(a, \infty) \left( \frac{d\varphi}{dt_0} \right) \, da;
\]

hence

\[
 \varphi(1) = \tau_0 \left( \frac{d\varphi}{dt_0} \right) = \int_0^\infty f_\varphi(a) \, da,
\]

proving (i).

To show (ii) we notice as in the proof of Lemma 3.3 that if \( \varphi \leq \psi \) then \( d\varphi/dt_0 \leq d\psi/dt_0 \), whence \( \chi(a, \infty)(d\varphi/dt_0) \leq \chi(a, \infty)(d\psi/dt_0) \) for \( a > 0 \).

To show (iii) we apply Lemma 2.7. Thus if \( \varphi, \psi \in M_+^* \) then by (i) and (ii)

\[
 \| \varphi - \psi \| = 2\varphi \vee \psi(1) - \varphi(1) - \psi(1)
\]

\[
 = \int_0^\infty (2f_\varphi \vee \psi(a) - f_\varphi(a) - f_\psi(a)) \, da
\]

\[
 \geq \int_0^\infty (2f_\varphi(a) \vee f_\psi(a) - f_\varphi(a) - f_\psi(a)) \, da
\]

\[
 = \int_0^\infty |f_\varphi(a) - f_\psi(a)| \, da.
\]

Since clearly \( f_{u\psi} = f_\varphi \) for \( u \) unitary in \( M \), (iii) follows. Q.E.D.

**Lemma 4.3.** With the notation of Lemma 4.2, if \( \varphi, \psi \in M_+^* \) then

\[ [\varphi] = [\psi] \text{ if and only if } f_\varphi(a) = f_\psi(a) \text{ for all } a > 0. \]

**Proof.** By Lemma 4.2(iii) if \( [\varphi] = [\psi] \) then \( f_\varphi(a) = f_\psi(a) \) a.e. Since \( f_\psi \) and \( f_\varphi \) are decreasing and continuous from the right they are equal.

Conversely assume \( f_\varphi = f_\psi \). Let \( h = d\varphi/dt_0, k = d\psi/dt_0 \), and let \( \mu \in (0, 1) \).
Put 
\[ e_n = \chi_{(\mu^{n+1}, \mu^n)}(h), \quad f_n = \chi_{(\mu^{n+1}, \mu^n)}(k). \]
Since \( f_\phi = f_\psi \), \( \tau_0(e_n) = \tau_0(f_n) \) for \( n \in \mathbb{N} \), whence \( e_n \sim f_n \). Since \( \sum e_n = \text{supp}(h) = \text{supp}(\phi) \) and \( \sum f_n = \text{supp}(k) = \text{supp}(\psi) \) we can find a partial isometry \( u \in M \) such that 
\[ u^*u = \text{supp}(\phi), \quad uu^* = \text{supp}(\psi), \]
and \( ue_nu^* = f_n, \) \( n \in \mathbb{N} \). Then 
\[ \mu k \leq uhu^* \leq \mu^{-1}k, \]
so that \( \mu \psi \leq u\phi u^* \leq \mu^{-1} \psi \). Since \( \mu \) is arbitrary, \( [\phi] = [\psi] \). Q.E.D.

**Theorem 4.4.** The map \( \phi \to f_\phi, \) \( \phi \in M_+^* \), determines a bijective map of \( M_+^*/\sim \) onto the set of decreasing \( L^1 \)-functions \( f: \mathbb{R}^+ \to J \), which are continuous from the right. Furthermore, if \( \phi, \psi \in M_+^* \) then 
\[ d([\phi], [\psi]) = \int_0^\infty |f_{\phi}(a) - f_{\psi}(a)| \, da. \]

**Proof.** By Lemma 4.3 the map is well defined, and by Lemma 4.2 a contraction. Suppose \( f: \mathbb{R}^+ \to J \) belongs to \( L^1(\mathbb{R}^+) \) and is decreasing and continuous from the right. By spectral theory we can find a positive self-adjoint operator \( h \) affiliated with \( M \) such that 
\[ \chi_{(a, \infty)}(h) = p(f(a)). \]
Let \( \phi(x) = \tau_0(hx), \) \( x \in M \). By Lemma 4.1 
\[ f_\phi(a) = \tau_0(\chi_{(a, \infty)}(h)) = \tau_0(p(f(a))) = f(a), \]
so the map \( [\phi] \to f_\phi \) is surjective.

Let \( \phi, \psi \in M_+^* \) and choose as above positive self-adjoint operators \( h \) and \( k \) affiliated with \( M \) such that 
\[ \chi_{(a, \infty)}(h) = p(f_\phi(a)), \quad \chi_{(a, \infty)}(k) = p(f_\psi(a)), \quad a > 0. \]
Put \( \phi' = \tau_0(h\cdot), \) \( \psi' = \tau_0(k\cdot) \). By the previous paragraph \( f_\phi = f_{\phi'} \) and \( f_\psi = f_{\psi'} \), so by Lemma 4.3 \( [\phi] = [\phi'], [\psi] = [\psi'] \). From the proof of Lemma 4.1 we have 
\[ |\tau_0(p(t) - p(s))| = |t - s|, \quad s, t \in \mathbb{R}. \]
Thus we have
\[ \| \varphi' - \psi' \| = \| h - k \|_1 \]
\[ = \tau_0 \left( \left| \int_0^\infty \chi_{(a, \infty)}(h) \, da - \int_0^\infty \chi_{(a, \infty)}(k) \, da \right| \right) \]
\[ \leq \int_0^\infty \left| \tau_0(\chi_{(a, \infty)}(h)) - \tau_0(\chi_{(a, \infty)}(k)) \right| \, da \]
\[ = \int_0^\infty |\tau_0(p(f_\varphi(a)) - p(f_\psi(a)))| \, da \]
\[ = \int_0^\infty |f_\varphi(a) - f_\psi(a)| \, da. \]

In particular
\[ \int \leq \int_0^\infty |f_\varphi(a) - f_\psi(a)| \, da, \]
completing the proof of the theorem.

Q.E.D.

**Lemma 4.5.** Let \( N = M \times_{\sigma_0} \mathbb{R} \). Then \( Z(N) \simeq L^\infty(\mathbb{R}) \) so if we identify them, then for \( \varphi \in M^+_\chi \) we have
\[ \hat{\varphi}(z) = \int_{-\infty}^\infty z(y) f_\varphi(e^{-y}) e^{-iy} \, dy, \quad z \in Z(N). \]

**Proof.** Since \( \sigma_0^{-1} = 1 \), the identity map, for all \( t \in \mathbb{R} \), \( N \) is generated by \( \Pi(x) = x \otimes 1 \in M \otimes L^\infty(\mathbb{R}) \), and \( \lambda(t) = 1 \otimes m(e^{it}) \), where \( m(f) \) denotes the multiplication operator on \( L^2(\mathbb{R}) \) for \( f \in L^\infty(\mathbb{R}) \). Thus \( N = M \otimes L^\infty(\mathbb{R}) \). The dual automorphism \( \theta \) is given by
\[ \theta_\varphi(x \otimes \lambda(t)) = x \otimes e^{-ist} \lambda(t), \]
so that
\[ \theta_\varphi(x \otimes m(f)) = x \otimes m(\beta_x f), \]
where \( (\beta_x f)(y) = f(y - s), f \in L^\infty(\mathbb{R}) \). The canonical trace \( \tau \) is given by
\[ \tau = \tau_0 \otimes e^{-y} \, dy, \]
\[ \tau(x \otimes m(f)) = \tau_0(x) \int_{-\infty}^\infty f(y) e^{-y} \, dy. \]
If we identify $M \otimes L^\infty(\mathbb{R})$ with $L^\infty(\mathbb{R}, M)$ then for $y \in N$

$$\tau(y) = \int_{-\infty}^{\infty} \tau_0(y(\gamma)) e^{-\gamma} d\gamma.$$ 

Let $\varphi \in M_+^*$, and put $k_\varphi = d\varphi/d\tau_0$. Then $\tilde{\varphi} = \varphi \otimes d\gamma$, i.e.,

$$\tilde{\varphi}(x \otimes m(f)) = \varphi(x) \int_{-\infty}^{\infty} f(\gamma) d\gamma.$$ 

Thus

$$h_\varphi = \frac{d\tilde{\varphi}}{d\tau} = \frac{d(\varphi \otimes d\gamma)}{d(\tau_0 \otimes e^{-\gamma} d\gamma)} = k_\varphi \otimes m(e^\gamma),$$

or written as a direct integral

$$h_\varphi = \int_{\mathbb{R}} e^\gamma k_\varphi(\gamma) d\gamma,$$

where $k_\varphi(\gamma) = k_\varphi$ for all $\gamma \in \mathbb{R}$. It follows that

$$e_\varphi = \chi_{(1, \infty)}(h_\varphi) = \int_{\mathbb{R}} \chi_{(e^{-\gamma}, \infty)}(k_\varphi) d\gamma.$$ 

Thus, if $z \in Z(N) = L^\infty(\mathbb{R})$,

$$\tilde{\varphi}(z) = \tau(e_\varphi z)$$

$$= \int_{-\infty}^{\infty} \tau_0(z(\gamma) \chi_{(e^{-\gamma}, \infty)}(k_\varphi)) e^{-\gamma} d\gamma$$

$$= \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma}) e^{-\gamma} d\gamma.$$ \hspace{1cm} Q.E.D.

**Lemma 4.6.** Let $\chi \in Z(N)^*_+$ satisfy $\chi \cdot \theta_s \geq e^{-s} \chi$ for $s > 0$. Let $\chi_0$ be the nonnegative measurable real function such that for all $z \in Z(N) \simeq L^\infty(\mathbb{R})$, such that

$$\chi(z) = \int_{-\infty}^{\infty} z(\gamma) \chi_0(\gamma) e^{-\gamma} d\gamma.$$ 

Then there exists $\varphi \in M_+^*$ with $\tilde{\varphi} = \chi$ if and only if $\chi_0(\gamma) \in J$ a.e.
Proof. If \( \varphi \in M_+ \) then by Lemma 4.5

\[
\hat{\varphi}(z) = \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma}) e^{-\gamma} \, d\gamma,
\]

where \( f_\varphi(e^{-\gamma}) \in J \).

To prove the converse assume \( \chi_0(\gamma) \in J \) a.e. If \( z \in Z(N)^+ \) and \( s > 0 \) we have

\[
e^{-s} \int_{-\infty}^{\infty} z(\gamma) \chi_0(\gamma) e^{-\gamma} \, d\gamma = e^{-s} \chi(z) \leq \chi \circ \theta_s(z)
\]

\[
= \int_{-\infty}^{\infty} z(\gamma - s) \chi_0(\gamma) e^{-\gamma} \, d\gamma
\]

\[
= \int_{-\infty}^{\infty} z(\gamma) \chi_0(\gamma + s) e^{-\gamma} e^{-s} \, d\gamma.
\]

Thus \( \chi_0(\gamma + s) \geq \chi_0(\gamma) \) for almost all \( \gamma \). Put

\[
f'(e^{-\gamma}) = \chi_0(\gamma).
\]

Then \( f'(e^{-\gamma - s}) \geq f'(e^{-\gamma}) \), \( s > 0 \) for a.a. \( \gamma \). Hence, if we put

\[
f(a) = \text{ess sup}_{b > a} f'(b), \quad a > 0,
\]

then \( f \) is decreasing, nonnegative, continuous from the right, and \( f = f' \) a.e. Thus for \( z \in Z(N) \),

\[
\chi(z) = \int_{-\infty}^{\infty} z(\gamma) f(e^{-\gamma}) e^{-\gamma} \, d\gamma.
\]

By Theorem 4.4 there is \( \varphi \in M_+ \) such that \( f = f_\varphi \). By Lemma 4.5

\[
\hat{\varphi}(z) = \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma}) e^{-\gamma} \, d\gamma
\]

\[
= \int_{-\infty}^{\infty} z(\gamma) f(e^{-\gamma}) e^{-\gamma} \, d\gamma
\]

\[
= \chi(z). \quad \text{Q.E.D.}
\]

**Theorem 4.7.** Let \( M \) be a semifinite factor with separable predual. Then

(i) \( d([\varphi], [\psi]) = \| \hat{\varphi} - \hat{\psi} \|, \varphi, \psi \in M_+ \).
The range of the map \([\varphi] \to \hat{\varphi}\) of \(M_+^* / \sim\) into \(Z(\mathbb{N})_+^\star\) is the set of \(\chi \in Z(\mathbb{N})_+^\star\) such that

(a) \(\chi \circ \theta_s \geq e^{-s}\chi, s > 0;\)

(b) if \(\chi(z) = \int_{-\infty}^{\infty} z(\gamma) \chi_0(\gamma) e^{-\gamma} d\gamma, z \in Z(\mathbb{N}) \cong L^\infty(\mathbb{R}),\) then \(\chi_0(\gamma) \in J = \{ \tau_0(p); p \text{ finite projection in M}\} \text{ a.e.}\)

**Proof.** (i) Let \(\varphi, \psi \in M_+^*\). By Lemma 4.5

\[
\|\hat{\varphi} - \hat{\psi}\| = \sup \{ |\varphi(z) - \psi(z)|, z \in Z(\mathbb{N}), \|z\| = 1 \}
\]

\[
= \sup \int_{-\infty}^{\infty} z(\gamma) (\varphi_\psi(e^{-\gamma}) - \varphi_\psi(e^{-\gamma})) e^{-\gamma} d\gamma
\]

\[
= \sup \int_{0}^{\infty} z(-\log a)(\varphi_\psi(a) - \varphi_\psi(a)) da
\]

\[
= \int_{-\infty}^{\infty} |\varphi_\psi(a) - \varphi_\psi(a)| da
\]

\[
= d([\varphi], [\psi]),
\]

by Theorem 4.4.

(ii) follows, since each \(\hat{\varphi}\) is by Lemma 4.5 of the form described in (b), and by Lemma 4.6 each such \(\chi\) is of the form \(\tilde{\varphi}\) with \(\varphi \in M_+^*\). Q.E.D.

**Corollary 4.8.** The Main Theorem holds for semifinite factors with separable preduals.

**Proof.** By Theorem 4.7 it remains to show that the ranges are as described in the Main Theorem. If \(M\) is of type \(\text{II}_\infty\) then \(J = [0, \infty)\), so condition (b) is always satisfied. If \(M\) is of type \(\text{II}_1\), then (b) means that \(\chi_0(\gamma) \in [0, 1]\) a.e. Since \(\tau = \tau_0 \otimes e^{-\gamma} d\gamma\), where \(\tau_0(1) = 1\), it is immediate that \(\chi_0(\gamma) \in [0, 1]\) a.e. if and only if for all \(z \in Z(\mathbb{N})\)

\[
\chi(z) = \int_{-\infty}^{\infty} z(\gamma) \chi_0(\gamma) e^{-\gamma} d\gamma \leq \tau(z);
\]

hence the range of the map \([\varphi] \to \hat{\varphi}\) is the set of \(\chi \in Z(\mathbb{N})_+^\star\) such that (a) holds and \(\chi \leq \tau|_{Z(\mathbb{N})}\). Q.E.D.

**Remark 4.9.** For \(\varphi \in M_+^*\) the function \(f_\varphi: \mathbb{R} \to J\) is decreasing and continuous from the right. Its distribution function (see [16, p. 57]) is defined by

\[
f_\varphi^\star(a) = \int_{0}^{a} \chi_{(a, \infty)}(f_\varphi(s)) ds, \quad a \in J.
\]
Then $f^*_\phi$ is a nonnegative decreasing function on $J$ which is continuous from the right. The map $f_\phi \to f^*_\phi$ is an $L^1$-isometry, so from Theorem 4.4 we obtain the known result that $M^+_\phi/\sim$ is isometric to the nonnegative decreasing functions in $L^1(J)$ which are continuous from the right.

**Remark 4.10.** If $M$ is a von Neumann algebra of type I there is no nice description of the range of the map $[\phi] \to \phi$ in $\mathcal{Z}(N)^+$. We shall not worry about this problem in the present paper. By different methods one can show that if $M$ is of type $I_\infty$, $n \in \mathbb{N} \cup \{\infty\}$, then $M^+_\phi/\sim$ is isometric to the set of $n$-tuples (resp. sequence if $n = \infty$), $\phi = (\phi_j)$ with $\phi_j \in \mathcal{Z}(M)^+_\phi$, where $\phi_1 \geq \phi_2 \geq \cdots$, and

$$\|\phi\| = \sum_{i=1}^n \|\phi_i\|.$$

5. **Proof of the Main Theorem for Factors of Type $\text{III}_\lambda$, $0 < \lambda < 1$**

Let $M$ be a factor of type $\text{III}_\lambda$ with separable predual. Then by [2, Thm. 4.4.1] or [17, Prop. 2.9.1]

$$M = P \times_\alpha \mathbb{Z},$$

where $P$ is a factor of type $\text{II}_\infty$ and $\alpha$ is an automorphism of $P$ for which

$$\text{tr} \circ \alpha = \lambda \text{tr},$$

where $\text{tr}$ is a faithful normal semifinite trace on $P$. Let $\omega$ be the dual weight on $M$. Then $\omega$ is a "Trace generalise" in the sense of Connes (see [17, Prop. 2.9.1]). In particular

$$\sigma^\omega_{t_0} = 1, \quad \text{where} \quad t_0 = -\frac{2\pi}{\log \lambda}.$$

Moreover, the dual action $\hat{\alpha}$ on $M$ coincides with $\sigma^\omega$ if we identify $\hat{\mathbb{Z}}$ with $\mathbb{R}/t_0 \mathbb{Z}$. Put

$$N_0 = M \times_{\sigma^\omega} (\mathbb{R}/t_0 \mathbb{Z}).$$

By Takesaki's double crossed product theorem [21, Thm. 4.5] $N_0 \simeq P \otimes B(l^2(\mathbb{Z}))$, so in particular $N_0$ is a factor of type $\text{II}_\infty$. The crossed product $N_0$ is generated by operators $\pi_0(x)$, $x \in M$, and $\lambda_0(t)$, $t \in \mathbb{R}$, where

$$(\pi_0(x)\xi)(s) = \sigma^\omega_{t_0}(x) \xi(s), \quad (\lambda_0(t)\xi)(s) = \xi(s-t) \quad \xi \in L^2(\mathbb{R}/t_0 \mathbb{Z}, H).$$
Note that $\lambda_0(t + t_0) = \lambda_0(t)$, $t \in \mathbb{R}$. By Stone's Theorem there exists a positive self-adjoint operator $h$ affiliated with $N_0$ such that 

$$h''(t) = \lambda_0(t), \quad t \in \mathbb{R}.$$ 

Let $(\theta_0^n)_{n \in \mathbb{Z}}$ be the dual action of $\sigma^\omega$ on $N_0$. Then 

$$\theta_0(\pi_0(x)) = \pi_0(x)$$

$$\theta_0(\lambda_0(t)) = e^{-\gamma_0 t} \lambda_0(t),$$

where $\gamma_0 = -\log \lambda$. The last equality implies that $\theta_0(h) = \lambda h$. Let $T_0$ be the operator valued weight from $N_0$ to $M$ given by 

$$T_0(u) = \sum_{n = -\infty}^{\infty} \theta_0^n(y), \quad y \in N_0^+. $$

Then the dual weight $\tilde{\omega}$ of $\omega$ is given by $\tilde{\omega} = \omega \cdot \pi_0^{-1} \cdot T_0$, and by [9] 

$$\sigma_s^\omega(\pi_0(x)) = \pi_0(\sigma_s^\omega(x)), \quad x \in M,$$

$$\sigma_s^\omega(\lambda_0(t)) = \lambda_0(t), \quad s \in \mathbb{R}.$$ 

Thus if we put $\tau_0 = \tilde{\omega}(h^{-1})$, then the modular automorphism group of $\tau_0$ is trivial, i.e., $\tau_0$ is a faithful normal trace on $N_0$. Since $\tilde{\omega}$ is $\theta_0$-invariant and $\theta_0(h) = \lambda h$, we obtain 

$$\tau_0 \circ \theta_0 = \lambda \tau_0. $$

**Lemma 5.1.** For $\varphi \in M_+^*$ we put 

$$f_\varphi(a) = \tau_0 \left( \chi(a, \infty) \left( \frac{d\tilde{\varphi}}{d\tau_0} \right) \right), \quad a > 0,$$

where $\tilde{\varphi} = \varphi \cdot \pi_0^{-1} \cdot T_0$ is the dual weight of $\varphi$ on $N_0$. Then $a \mapsto f_\varphi(a)$ is a nonnegative decreasing function on $\mathbb{R}^+$ which is continuous from the right and satisfies 

(i) $f_\varphi(\lambda a) = \lambda^{-1} f_\varphi(a), \quad a \geq 0.$

(ii) $\varphi(1) = \int_1^{\infty} f_\varphi(a) \, da.$

**Proof.** The first statement is obvious, so it remains to show (i) and (ii). To show (i) note that since $\varphi \circ \theta_0 = \varphi$ and $\tau_0 \circ \theta_0 = \lambda \tau_0$, we have 

$$\theta_0 \left( \frac{d\tilde{\varphi}}{d\tau_0} \right) = \lambda \frac{d\tilde{\varphi}}{d\tau_0}.$$
Hence \( \theta^{-1}_0(\chi_{(a,\infty)}(d\phi/d\tau_0)) = \chi_{(a,\infty)}(d\phi/d\tau_0) \), and therefore

\[
f_\phi(\lambda x) = \tau_0 \left( \theta^{-1}_0 \left( \chi_{(a,\infty)} \left( \frac{d\phi}{d\tau_0} \right) \right) \right) = \frac{1}{\lambda} f_\phi(x).
\]

(ii) Put \( g_a(t) = 0 \) if \( 0 \leq t \leq a \) and \( g_a(t) = 1/t \), \( t > a \). Put

\[
G_a(t) = \sum_{n \in \mathbb{Z}} g_a(\lambda^n t).
\]

Then \( G_a(0) = 0 \), and for \( \lambda a < t < a \) we have

\[
G_a(t) = \sum_{n=-\infty}^{-1} \frac{1}{\lambda^n t} = \frac{1}{\lambda n} \sum_{n=1}^{\infty} \lambda^n = \frac{\lambda}{1-\lambda} t.
\]

Since \( G_a(\lambda t) = G_a(t) \), \( t \in \mathbb{R} \), it follows that \( G_a \) is a bounded function with \( \| G_a \|_\infty \leq 1/(1-\lambda) a \). Since

\[
\theta_0 \left( \frac{d\phi}{d\tau_0} \right) = \lambda \frac{d\phi}{d\tau_0},
\]

\[
\theta_0^n \left( \frac{d\phi}{d\tau_0} \right) = \lambda^n \frac{d\phi}{d\tau_0}, \quad n \in \mathbb{Z},
\]

whence,

\[
\theta_0^n \left( g_a \left( \frac{d\phi}{d\tau_0} \right) \right) = g_a \left( \lambda^n \frac{d\phi}{d\tau_0} \right),
\]

and therefore

\[
T_0 \left( g_a \left( \frac{d\phi}{d\tau_0} \right) \right) = G_a \left( \frac{d\phi}{d\tau_0} \right).
\]

Thus

\[
f_\phi(a) = \tau_0 \left( \chi_{(a,\infty)} \left( \frac{d\phi}{d\tau_0} \right) \right) = \tau_0 \left( \frac{d\phi}{d\tau_0} \right) \phi \left( g_a \left( \frac{d\phi}{d\tau_0} \right) \right) = \phi \left( g_a \left( \frac{d\phi}{d\tau_0} \right) \right) = \phi \circ \pi_0^{-1} \left( G_a \left( \frac{d\phi}{d\tau_0} \right) \right).
\]
Hence \( f_{\varphi}(a) \leq \|\varphi\| \|G_a\|_{\infty} \) for all \( a \in \mathbb{R}^+ \). Now

\[
\int_{\lambda}^{1} G_a(t) \, da = \sum_{n \in \mathbb{Z}} \int_{\lambda}^{1} g_a(\lambda^n t) \, da = \sum_{n \in \mathbb{Z}} \int_{\lambda}^{1} \lambda^{-n} g_{\lambda^{-n}}(t) \, da = \sum_{n \in \mathbb{Z}} \int_{\lambda^{-n+1}}^{\lambda^{-n}} g_a(t) \, da = \int_{0}^{\infty} g_a(t) \, da = \begin{cases} 1, & t > 0 \\ 0, & t = 0. \end{cases}
\]

Hence

\[
\int_{\lambda}^{1} f_{\varphi}(a) \, da = \varphi \circ \pi_0^{-1} \left( \text{supp} \left( \frac{d\varphi}{dt_0} \right) \right) = \varphi \circ \pi_0^{-1}(\text{supp}(\varphi)) = \varphi(\text{supp}(\varphi)) = \varphi(1). \quad \text{Q.E.D.}
\]

**Lemma 5.2.** For \( x \in M \) we have

\[
\varphi(x) = \int_{\lambda}^{1} \tau_0 \left( \pi_0(x) \chi_{(a, \infty)} \left( \frac{d\varphi}{dt_0} \right) \right) \, da.
\]

**Proof.** Note that by Lemma 5.1 the integral on the right side is well defined, because

\[
\int_{\lambda}^{1} \left| \tau_0 \left( \pi_0(x) \chi_{(a, \infty)} \left( \frac{d\varphi}{dt_0} \right) \right) \right| \, da \leq \|x\| \int_{\lambda}^{1} f_{\varphi}(a) \, da = \|x\| \varphi(1).
\]

Let \( g_a \) and \( G_a \) be as in the proof of Lemma 5.1. For \( y \in M \) we have

\[
\tilde{\varphi} \left( \pi_0(y) g_a \left( \frac{d\varphi}{dt_0} \right) \pi_0(y)^* \right) = \varphi \circ \pi_0^{-1} \circ T \left( \pi_0(y) g_a \left( \frac{d\varphi}{dt_0} \right) \pi_0(y)^* \right) = \varphi \left( y \pi_0^{-1} \left( G_a \left( \frac{d\varphi}{dt_0} \right) \right) y^* \right) < \infty.
\]
By polarization, \( \pi_0(y_1) g_a(d\bar{\phi}/dt_0) \pi_0(y_2) \in m_\phi \)—the ideal of definition of \( \bar{\phi} \)—for all \( y_1, y_2 \in M \), and

\[
\varphi \left( \pi_0(y_1) g_a \left( \frac{d\bar{\phi}}{dt_0} \right) \pi_0(y_2) \right) = \varphi \left( y_1 \pi_0^{-1} \left( G_a \left( \frac{d\bar{\phi}}{dt_0} \right) \right) y_2 \right).
\]

Now put \( y_1 = x \), \( y_2 = 1 \). Then we have

\[
\varphi \left( x \pi_0^{-1} \left( G_a \left( \frac{d\bar{\phi}}{dt_0} \right) \right) \right) = \bar{\varphi} \left( \pi_0(x) g_a \left( \frac{d\bar{\phi}}{dt_0} \right) \right)
\]

\[
= \tau_0 \left( \pi_0(x) g_a \left( \frac{d\bar{\phi}}{dt_0} \right) \right)
\]

\[
= \tau_0 \left( \pi_0(x) \chi_{(a, \infty)} \left( \frac{d\bar{\phi}}{dt_0} \right) \right).
\]

From the proof of Lemma 5.1

\[
\int_{-\lambda}^{1} G_a \left( \frac{d\bar{\phi}}{dt_0} \right) da = \text{supp}(\varphi) = \pi_0(\text{supp } \varphi).
\]

Hence

\[
\varphi(x) = \varphi(x \text{ supp}(\varphi)) = \int_{-\lambda}^{1} \tau_0 \left( \pi_0(x) \chi_{(a, \infty)} \left( \frac{d\bar{\phi}}{dt_0} \right) \right) da.
\]

Q.E.D.

**Lemma 5.3.** Let \( \varphi, \psi \in M_+^* \). Then \( [\varphi] = [\psi] \) if and only if \( f_{\varphi}(a) = f_{\psi}(a) \) for all \( a > 0 \).

**Proof.** By Lemma 2.7 there is \( \omega \in M_+^* \) such that \( \omega \geq \varphi, \omega \geq \psi \), and \( \| \varphi - \psi \| = 2\omega(1) - \varphi(1) - \psi(1) \). Since \( d\bar{\phi}/dt_0 \leq d\tilde{\phi}/dt_0 \) the spectral projections satisfy

\[
\chi_{(a, \infty)} \left( \frac{d\bar{\phi}}{dt_0} \right) \leq \chi_{(a, \infty)} \left( \frac{d\tilde{\phi}}{dt_0} \right)
\]

in the Murray–von Neumann sense; hence \( f_{\varphi}(a) \leq f_\omega(a), a > 0 \), and similarly \( f_{\psi}(a) \leq f_\omega(a) \). Therefore, by Lemma 5.1

\[
\int_{-\lambda}^{1} | f_{\varphi}(a) - f_{\psi}(a) | da \leq \int_{-\lambda}^{1} (2f_\omega(a) - f_{\varphi}(a) - f_{\psi}(a)) da
\]

\[
= 2\omega(1) - \varphi(1) - \psi(1)
\]

\[
= \| \varphi - \psi \|.
\]
Since $f_\omega = f_{u\omega u^*}$ for every $u \in U(M)$ we have

$$d([\varphi], [\psi]) \geq \int_{\lambda}^{1} |f_\omega(\alpha) - f_\psi(\alpha)| \, d\alpha.$$ 

In particular, if $[\varphi] = [\psi]$ then $f_\varphi = f_\psi$ almost everywhere in the interval $[\lambda, 1)$. By Lemma 5.1(i) it follows that $f_\varphi = f_\psi$ almost everywhere on $\mathbb{R}^+$. The continuity of $f_\varphi$ and $f_\psi$ from the right implies $f_\varphi = f_\psi$.

Conversely assume $f_\varphi = f_\psi$. Let $m \in \mathbb{N}$ and put

$$\mu = \lambda^{1/m}.$$ 

Define projections

$$e_n = \chi_{(\mu^{n+1}, \mu^n)} \left( \frac{d\varphi}{dt_0} \right), \quad f_n = \chi_{(\mu^{n+1}, \mu^n)} \left( \frac{d\psi}{dt_0} \right), \quad n \in \mathbb{Z}.$$ 

Then $\tau_0(e_n) = \tau_0(f_n)$, and so $e_n \sim f_n$, $n \in \mathbb{Z}$. Choose partial isometries $v_0, ..., v_{m-1} \in N_0$ such that

$$e_n = v_n^* v_n, \quad f_n = v_n v_n^*, \quad 0 \leq n \leq m - 1.$$ 

Since $\varphi(e_n) = e_{n-m}$, $\varphi(f_n) = f_{n-m}$, $n \in \mathbb{Z}$, we can extend the set \{v_0, ..., v_{m-1}\} to a set $(v_n)_{n \in \mathbb{Z}}$ of partial isometries in $N_0$ for which

$$e_n = v_n^* v_n, \quad f_n = v_n v_n^*, \quad n \in \mathbb{Z},$$ 

and

$$\theta_0(v_n) = v_{n-m}, \quad n \in \mathbb{Z},$$ 

by putting $v_n = \theta_0^p v_{n_0}$, when $n = mp + n_0$, $p \in \mathbb{Z}$, $n_0 \in \{0, ..., m - 1\}$. Then $v = \sum_{n \in \mathbb{Z}} v_n$ is a partial isometry in $N_0$ such that

$$v^* v = \text{supp} \left( \frac{d\varphi}{dt_0} \right), \quad vv^* = \text{supp} \left( \frac{d\psi}{dt_0} \right),$$

$$ve_n v^* = f_n,$$

$$\theta_0(v) = v.$$ 

The last equality implies that $v = \pi_0(u)$ for some partial isometry $u \in M$ (see, e.g., [8, Lem. 3.6]). As in the semifinite case we get

$$\mu \frac{d\psi}{dt_0} \leq v \frac{d\varphi}{dt_0} v^* \leq \mu^{-1} \frac{d\psi}{dt_0},$$
i.e., $\mu \psi \leq v \phi v^* \leq \mu^{-1} \psi$. Since $\text{supp}(\chi) = \pi_0(\text{supp}(\chi))$ for any normal semifinite weight $\chi$ on $M$, $u^* u = \text{supp}(\phi)$, $uu^* = \text{supp}(\psi)$. From [7, Prop. 2.5(1)] it follows that the dual weight operation $\chi \to \overline{\chi}$ is an order isomorphism of $M_+^*$ onto its range. Hence

$$\mu \psi \leq u \phi u^* \leq \mu^{-1} \psi.$$ 

Using $\mu = \lambda^{1/m}$, $m \in \mathbb{Z}$ arbitrary, we get $\phi \sim \psi$. Q.E.D.

**Lemma 5.4.** Let $P_f(N_0)$ denote the set of finite projections in $N_0$. Then there is a map $p: [0, \infty) \to P_f(N_0)$ such that

(i) $s \leq t \Rightarrow p(s) \leq p(t)$,
(ii) $\tau_0(p(s)) = s$, $s \geq 0$,
(iii) $\theta_0(p(s)) = p(\lambda s)$, $s \geq 0$,
(iv) $\|p(s) - p(t)\|_1 = |s - t|$, $s, t \in \mathbb{R}$.

**Proof.** Let $e' \in P_f(N_0)$ be nonzero. Put

$$e = \bigvee_{n=0}^{\infty} \theta^n_0(e').$$

Then $e \neq 0$ and

$$\tau_0(e) \leq \sum_{n=0}^{\infty} \tau_0 \circ \theta_0^n(e') = \frac{1}{1 - \lambda} \tau_0(e') < \infty.$$ 

Clearly $\theta_0(e) \leq e$. Put $a = \tau_0(e)$ and $f = e - \theta_0(e)$. Then $f \neq 0$, since $\tau_0(f) = (1 - \lambda)a$. Since $fN_0f$ is a factor there exists as in the proof of Lemma 4.1 an increasing map $q$ from $[0, 1]$ into the projections in $fN_0f$ for which

$$\tau_0(q(t)) = (1 - \lambda)at, \quad 0 \leq t \leq 1.$$ 

Define $p: (\lambda a, a] \to P_f(N_0)$ by

$$p(\lambda a + u) = \theta_0(e) + q\left(\frac{u}{(1 - \lambda)a}\right), \quad u \in (0, (1 - \lambda)a].$$

Then $p$ is an increasing function on $(\lambda a, a]$ such that

$$\theta_0(e) \leq p(t) \leq e, \quad t \in (\lambda a, a]$$

$$p(a) = e,$$

$$\tau_0(p(t)) = t, \quad t \in (\lambda a, a].$$
Define now \( p \) on \( (\lambda^{n+1}a, \lambda^n a] \) by
\[
p(t) = \theta_0^n(\lambda^{-nt}), \quad n \in \mathbb{Z},
\]
and put
\[
p(0) = 0.
\]
Then it is clear that \( p: [0, \infty) \to P_f(N_0) \) satisfies conditions (i), (ii), and (iii). Moreover, (iv) follows from (i) and (ii) as in the proof of Lemma 4.1. Q.E.D.

**Theorem 5.5.** (i) The map \([\phi] \mapsto f_\phi\) is a bijection of \( M_+ / \sim \) onto the set of decreasing functions from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \) which are continuous from the right and satisfy
\[
f(\lambda a) = \frac{1}{\lambda} f(a), \quad a > 0.
\]

(ii) For \( \phi, \psi \in M_+ \) we have
\[
d([\phi], [\psi]) = \int_a^1 |f_\phi(a) - f_\psi(a)| \, da.
\]

*Proof.* (i) By Lemmas 5.1 and 5.3 the map \([\phi] \mapsto f_\phi\) is well defined and injective. To show that the map is surjective let \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) be decreasing, continuous from the right, and satisfying \( f(\lambda a) = (1/\lambda) f(a) \), \( a > 0 \). Let \( p: [0, \infty) \to P_f(N_0) \) be as in Lemma 5.4 and put
\[
e(a) = p(f(a)).
\]
Then \( (e(a))_{a > 0} \) is a decreasing family of projections which is continuous from the right, and \( e(a) \to 0 \) for \( a \to \infty \) because \( \tau(e(a)) = f(a) \to 0 \) for \( a \to \infty \). Hence there exists a positive self-adjoint operator \( h \) affiliated with \( N \), such that
\[
\chi_{(a, \infty)}(h) = e(a).
\]
Put
\[
\omega = \tau_0(h).
\]

Since \( p(\lambda a) = \theta_0(p(a)) \) it follows that \( \theta_0(h) = \lambda h \). Therefore \( \omega \circ \theta_0 = \omega \), so as in the proof of Lemma 3.5 there exists by [8, Thm. 3.7] a normal semifinite weight \( \phi \) on \( M \) such that \( \tilde{\phi} = \omega \). By the proof of Lemma 5.1
\[
\tilde{\phi}(1) = \int_a^1 \tau_0(\chi_{(a, \infty)}(h)) \, da = \int_a^1 f(a) \, da,
\]
which is finite, because $f(\lambda) < \infty$ and $f$ is decreasing. Hence $\varphi \in M_+^*$. By construction $f = f_\varphi$, proving surjectivity.

(ii) Let $\varphi, \psi \in M_+^*$. By the proof of (i) we can choose $\varphi', \psi' \in M_*^*$ such that $f_\varphi = f_{\varphi'}$, $f_\psi = f_{\psi'}$, and

$$\chi(a, \infty) \left( \frac{d\varphi'}{dt_0} \right) = p(f_\varphi(a)), \quad \chi(a, \infty) \left( \frac{d\psi'}{dt_0} \right) = p(f_\psi(a)).$$

Furthermore, by Lemma 5.3 $[\varphi] = [\varphi']$, $[\psi] = [\psi']$. By Lemma 5.2 and 5.4(iv) we have

$$\|\varphi - \psi\| \leq \int_{\lambda}^1 \left\| \chi(a, \infty) \left( \frac{d\varphi'}{dt_0} \right) - \chi(a, \infty) \left( \frac{d\psi'}{dt_0} \right) \right\|_1 da$$

$$= \int_{\lambda}^1 \| p(f_\varphi(a)) - p(f_\psi(a)) \|_1 da$$

$$= \int_{\lambda}^1 | f_\varphi(a) - f_\psi(a) | da;$$

hence

$$d([\varphi], [\psi]) \leq \int_{\lambda}^1 | f_\varphi(a) - f_\psi(a) | da.$$

This completes the proof, since the converse inequality was established in the proof of Lemma 5.3. Q.E.D.

The above theorem is the analogue for $\text{III}_\lambda$-factors of Theorem 4.4 for semifinite factors. Just as in that case we have to translate into the setting of the Main Theorem. For this we shall need a result which is more or less folklore. A similar result was obtained by Takesaki in [21, Sec. 10], but a complete proof of the result we need is to the best of our knowledge not in the literature.

**Proposition 5.6.** Let $M$ be a factor of type $\text{III}_\lambda$ with separable predual. As before let $\omega$ be a generalized trace, $N_0 = M \times_{\varphi_0^\omega} (\mathbb{R}/t_0 \mathbb{Z})$, where $t_0 = -2\pi/\log \lambda$, with generators $\pi_0(x)$, $x \in M$, $\lambda_0(t)$, $t \in \mathbb{R}$. Let $N - M \times_{\varphi_0^\omega} \mathbb{R}$ with generators $\pi(x)$, $\lambda(t)$. Put $\gamma_0 = -\log \lambda$ and let $m(e^{it})$ denote the multiplication operator

$$(m(e^{it})\xi)(\gamma) = e^{it\gamma}\xi(\gamma), \quad \xi \in L^2(0, \gamma_0), \quad t \in \mathbb{R}.$$
Then there is a spatial isomorphism $\Phi$ of $N$ onto $N_0 \otimes L^\infty(0, \gamma_0)$ such that

$$
\Phi(\pi(x)) = \pi_0(x) \otimes 1
$$

$$
\Phi(\lambda(t)) = \lambda_0(t) \otimes m(e^{it}).
$$

Proof. Let $H$ be the underlying Hilbert space for $M$. Then we can identify the Hilbert spaces $L^2(\mathbb{R}/t_0\mathbb{Z}, H) \otimes L^2(0, \gamma_0)$ and

$$
K = L^2((\mathbb{R}/t_0\mathbb{Z}) \times [0, \gamma_0), H).
$$

Note that the interval $[0, \gamma_0]$ can be identified with the dual group of $t_0\mathbb{Z}$, since $e^{iynt_0} = e^{2\pi i y_0 t_0}$. Let $\xi \in K$ and $t \in \mathbb{R}$, and write $t = i + mt_0$ with $i \in [0, t_0)$. Let $\xi'(i, \gamma) = \xi(i, \gamma) e^{-it}$. Define $T$ by

$$
(T\xi)(t) = \gamma_0^{-1/2} \int_0^{\gamma_0} \xi'(i, \gamma) e^{-i\gamma t} d\gamma = \gamma_0^{-1/2} \int_0^{\gamma_0} \xi'(i, \gamma) e^{-im\gamma} d\gamma, \ t \in \mathbb{R}.
$$

By Parseval's formula for Fourier series

$$
\sum_{n = -\infty}^{\infty} \|(T\xi)(t + nt_0)\|^2 = \int_0^{\gamma_0} \|\xi'(i, \gamma)\|^2 d\gamma
$$

$$
= \int_0^{\gamma_0} \|\xi(i, \gamma)\|^2 d\gamma,
$$

for every $t \in \mathbb{R}$. Hence

$$
\int_{-\infty}^{\infty} \|(T\xi)(t)\|^2 dt = \int_0^{\gamma_0} \sum_{n = -\infty}^{\infty} \|(T\xi)(t + nt_0)\|^2 dt
$$

$$
= \int_0^{\gamma_0} \left( \int_0^{\gamma_0} \|\xi(i, \gamma)\|^2 d\gamma \right) dt,
$$

which shows that $T$ is an isometry of $K$ into $L^2(\mathbb{R}, H)$. For $\eta \in C_0(\mathbb{R}, H)$ put

$$
(S\eta)(i, \gamma) = \gamma_0^{-1/2} \sum_{n = -\infty}^{\infty} \eta(t + nt_0) e^{i\gamma(t + nt_0)}
$$

for $i \in \mathbb{R}/t_0\mathbb{Z}$, where $t \in \mathbb{R}$ is an element with image $i$ under the quotient map $\mathbb{R} \to \mathbb{R}/t_0\mathbb{Z}$, and $\gamma \in [0, \gamma_0)$. The map is well defined, because the sum
on the right side is a periodic function in $t$ with period $t_0$. By Parseval's formula

$$
\int_0^{t_0} \| (S\eta)(t, \gamma) \|^2 \, d\gamma = \sum_{n = -\infty}^{\infty} \| \eta(t + nt_0) \|^2.
$$

Thus

$$
\int_0^{t_0} \int_0^{t_0} \| (S\eta)(i, \gamma) \|^2 \, d\gamma \, dt = \int_{-\infty}^{\infty} \| \eta(t) \|^2 \, dt.
$$

Hence $S$ can be extended by continuity to an isometry of $L^2(\mathbb{R}, H)$ into $K$. Finally, for $\eta \in C_0(\mathbb{R}, H)$

$$
(TS\eta)(t) = \gamma_0^{-1} \int_0^{t_0} \left( \sum_{n = -\infty}^{\infty} \eta(t + nt_0) e^{int\gamma} \right) \, d\gamma
$$

$$
= \gamma_0^{-1} \sum_{n = -\infty}^{\infty} \eta(t + nt_0) \int_0^{t_0} e^{int\gamma} \, d\gamma
$$

$$
= \eta(t).
$$

Thus $TS = 1$; hence in particular $T$ is surjective, so a unitary operator. Furthermore $S = T^{-1}$. We conclude the proof by showing the formulas in the proposition. Let $x \in M$ and $\xi \in K$. Then

$$
(T(\pi_o(x) \otimes 1)\xi)(t) = (T(\sigma_o, (x) \otimes 1)\xi)(t)
$$

$$
= \gamma_0^{-1/2} \int_0^{t_0} ((\sigma_o, (x) \otimes 1)\xi)(i, \gamma) e^{-int} \, d\gamma
$$

$$
= \sigma_o, (x) \left( \gamma_0^{-1/2} \int_0^{t_0} \xi(i, \gamma) e^{-int} \, d\gamma \right)
$$

$$
= \sigma_o, (x)(T\xi)(t)
$$

$$
= (\pi(x) \, T\xi)(t);
$$

hence $\pi_o(x) \otimes 1 = T^{-1}\pi(x) \, T$,

$$
(T(\lambda_o(s) \otimes m(e^{is})\xi))(t) = \gamma_0^{-1/2} \int_0^{t_0} e^{isy}(i - s, \gamma) e^{-int} \, d\gamma
$$

$$
= (T\xi)(t - s)
$$

$$
= (\lambda(s) \, T\xi)(t).
$$
Hence $\lambda_0(s) \otimes m(e^{i\theta}) = T^{-1} \lambda(s) T$. This completes the proof, since the isomorphism $\Phi(y) = T^{-1}yT$ maps generators onto generators, and hence $N$ onto $N_0 \otimes L^\infty(0, \gamma_0)$. Q.E.D.

**Proposition 5.7.** The Main Theorem holds for factors of type $III\lambda$, $0 < \lambda < 1$, with separable preduals.

**Proof.** Let $M$ be such a factor, $\omega$ a generalized trace, $N_0$ and $N$ as before with dual automorphisms $\theta_0$ and $\theta$, and canonical traces $\tau_0$ and $\tau$. By Proposition 5.6 $N \cong N_0 \otimes L^\infty(0, \gamma_0)$ and acts on $K = L^2((\mathbb{R}/t_0 \mathbb{Z}) \times [0, \gamma_0), H)$. Let $s \in \mathbb{R}$. Then $s = r + n\gamma_0$, $r \in [0, \gamma_0)$, $n \in \mathbb{Z}$. By Proposition 5.6 we can identify $N$ and $N_0 \otimes L^\infty(0, \gamma_0)$ such that $\pi(x) = \pi_0(x) \otimes 1$, $x \in M$, and $\lambda(t) = \lambda_0(t) \otimes m(e^{i\theta})$, $t \in \mathbb{R}$. For $r \in [0, \gamma_0)$ we let $\beta_r : L^\infty(0, \gamma_0 - r) \to L^\infty(r, \gamma_0)$ be the shift $$(\beta_r f)(\gamma) = f(\gamma - r), \quad \gamma \in [r, \gamma_0)$$ and similarly we let $\beta_{r-\gamma_0} : L^\infty(\gamma_0 - r, \gamma_0) \to L^\infty(0, r)$ be the shift $$(\beta_{r-\gamma_0} f)(\gamma) = f(\gamma - r + \gamma_0), \quad \gamma \in [0, r).$$

We claim that the dual action $\theta_s$ on $N = N_0 \otimes L^\infty(0, \gamma_0)$ can be described as $$\theta_s = (\theta_0^n \otimes \beta_r) \oplus (\theta_0^{n+1} \otimes \beta_{r-\gamma_0}), \quad (*)$$ where $s = nr_0 + r$, $n \in \mathbb{Z}$, $r \in [0, \gamma_0)$. (This is exactly the situation described by Takesaki in [21, Eq. 10.18] for the case $M_\omega$ separable.) To prove (*) it suffices to show that the right side (*) acts as $\theta$, on the generators $\pi(x)$ and $\lambda(t)$ of $N$, i.e., that

$$(\theta_0^n \otimes \beta_r) \oplus (\theta_0^{n+1} \otimes \beta_{r-\gamma_0})(\pi_0(x) \otimes 1) = \pi_0(x) \otimes 1, \quad x \in N$$

$$(\theta_0^n \otimes \beta_r) \oplus (\theta_0^{n+1} \otimes \beta_{r-\gamma_0})(\lambda_0(t) \otimes m(e^{i\theta})) = e^{-ixt} \lambda_0(t) \otimes m(e^{i\theta}), \quad t \in \mathbb{R},$$

and these two identities follow easily by an explicit computation. Writing $N$ as a direct integral

$$N = \int_{[0, \gamma_0)} N(\gamma) \, d\gamma, \quad N(\gamma) = N_0,$$

and applying the arguments of Takesaki on [21, pp. 303–304] to $[0, T_0)$ replaced by $[0, \gamma_0)$ and $\rho$ by $\theta_0$, we obtain that

$$\tau = \int_{[0, \gamma_0)} e^{-\gamma \tau_0} \, d\gamma,$$

where $\tau_0 = \tau_0$ a.e. We can formally write this as

$$\tau = \tau_0 \otimes e^{-\gamma} \, d\gamma.$$
Following the notation of Section 3 we let $\phi$ denote the dual weight on $N$ of $\varphi \in M_\ast^+$. To compute $\phi$ we have to find the operator valued weight $T = \int_{-\infty}^{\infty} \theta_s \, ds$. Recall that the operator valued weight $T_0: N_0 \to \pi_0(M)$ was defined as $T_0(y) = \sum_{-\infty}^{\infty} \theta_0^n(y)$, $y \in N_0^+$. Let $y \in N_0^+$, $f \in L^\infty(0, \gamma_0)^+$. Then we have

$$T(y \otimes m(f)) = \int_0^{\gamma_0} \sum_{-\infty}^{\infty} \theta_{\gamma_0 + r}(y \otimes m(f)) \, dr$$

$$= \int_0^{\gamma} \sum_{-\infty}^{\infty} \theta_0^n(y) \otimes m(\beta_r f) \, dr + \int_0^{\gamma_0} \sum_{-\infty}^{\infty} \theta_0^{n+1}(y) \otimes m(\beta_r, -\gamma_0 f) \, dr$$

$$= T_0(y) \otimes \left[ \int_0^{\gamma} m(\beta_r f) \, dr + \int_0^{\gamma_0} m(\beta_r, -\gamma_0 f) \, dr \right]$$

$$= T_0(y) \otimes \left[ \int_0^{\gamma} \theta_r m(f) \, dr + \int_{\gamma}^{\gamma_0} \theta_r m(f) \, dr \right]$$

$$= T_0(y) \otimes \int_0^{\gamma_0} \theta_r m(f) \, dr$$

$$= T_0(y) \otimes \left( \int_0^{\gamma_0} f(\gamma) \, d\gamma \right) 1,$$

using that $\theta$ is ergodic on $L^\infty(0, \gamma_0)$—the center of $N$. Therefore, if $\varphi \in M_\ast^+$ and $\tilde{\varphi}$ is its dual weight on $N_0$ we have in the notation introduced above for $\tau$,

$$\varphi = \tilde{\varphi} \otimes d\gamma.$$

Let $h_\varphi = d\tilde{\varphi}/d\tau_0$ and $k_\varphi = d\tilde{\varphi}/d\tau$. Then

$$k_\varphi = \frac{d(\tilde{\varphi} \otimes d\gamma)}{d(\tau_0 \otimes e^{-\gamma} \, d\gamma)} = h_\varphi \otimes m(e^\gamma),$$

or in direct integral notation

$$k_\varphi = \int_{(0, \gamma_0)} h_\varphi e^\gamma \, d\gamma.$$

Thus

$$e_\varphi = \chi_{(1, \infty)}(k_\varphi) = \int_{(0, \gamma_0)} \chi_{(1, \infty)}(h_\varphi e^\gamma) \, d\gamma$$

$$= \int_{(0, \gamma_0)} \chi_{(e^{-1}, \infty)}(h_\varphi) \, d\gamma.$$
Therefore, if \( z \in Z(N) \), \( z = \int_{(0, y_0)} z(\gamma) \, d\gamma \),

\[
\hat{\phi}(z) = \tau(e \varphi z) = \int_0^{y_0} \tau(\chi_{(-\infty, \infty)}(h_\varphi) z(\gamma)) \, e^{-\gamma} \, d\gamma
\]

\[
= \int_0^1 \tau(\chi_{(a, \infty)}(h_\varphi)) z(-\log a) \, da.
\]

Let

\[
f_\varphi(a) = \tau(\chi_{(a, \infty)}(h_\varphi)).
\]

Then

\[
\hat{\phi}(z) = \int_\lambda^1 f_\varphi(a) \, z(-\log a) \, da,
\]

whence, if \( \psi \in M_\varphi^+ \),

\[
\| \hat{\phi} - \hat{\psi} \| = \int_\lambda^1 | f_\varphi(a) - f_\psi(a) | \, da.
\]

By Theorem 5.5 we then have

\[
\| \hat{\phi} - \hat{\psi} \| = d([\varphi], [\psi]),
\]

proving the first part of the Main Theorem for \( M \).

To show surjectivity we shall use Theorem 5.5 and follow the pattern of
the proof of Lemma 4.6. If we identify \( L^\infty(0, y_0) \) with the set of periodic
functions in \( L^\infty(\mathbb{R}) \) with period \( y_0 \), the formula for \( \theta_s(y \otimes m(f)) \) gives in
the special case \( y = 1 \) that

\[
\theta_s(1 \otimes m(f)) = m_{(s, f)},
\]

where \( (s, f)(\gamma) = f(\gamma - s), \gamma \in \mathbb{R} \).

Let \( \chi \in Z(N)_\varphi^+ \) satisfy \( \chi \cdot \theta_s \geq e^{-s} \chi, s > 0 \), and choose a nonnegative
measurable function \( \rho \) on \([0, y_0] \), such that

\[
\chi(z) = \int_0^{y_0} z(\gamma) \, \rho(\gamma) \, d\gamma, \quad z \in Z(N) \cong L^\infty(0, y_0).
\]

If we extend \( z(\cdot) \) and \( \rho \) to periodic functions on \( \mathbb{R} \) with period \( y_0 \), then

\[
(\chi \cdot \theta_s)(z) = \int_0^{y_0} z(\gamma - s) \, \rho(\gamma) \, d\gamma
\]

\[
= \int_0^{y_0} z(\gamma) \, \rho(\gamma + s) \, d\gamma, \quad s \in \mathbb{R}.
\]
Hence for every \( s > 0 \),
\[
\rho(\gamma + s) \geq e^{-s} \rho(\gamma) \quad \text{for a.e. } \gamma \in \mathbb{R}.
\]
Put \( \chi_0(\gamma) = \rho(\gamma) e^\gamma, \gamma \in \mathbb{R} \). Then
\[
\chi(z) = \int_0^{\chi_0(z)} z(\gamma) \chi_0(\gamma) e^{-\gamma} \, d\gamma, \quad z \in Z(N),
\]
and for every \( s > 0 \),
\[
\chi_0(\gamma + s) \geq \chi_0(\gamma) \quad \text{for a.e. } \gamma \in \mathbb{R}.
\]
Hence if we put
\[
\chi'(\gamma) = \text{ess sup} \chi_0(\gamma_1), \quad \gamma \in \mathbb{R},
\]
then \( \chi' \) is increasing and continuous from the left. Moreover, \( \chi'_0 = \chi_0 \) a.e. The periodicity of \( \rho \) implies that
\[
\chi'_0(\gamma + \gamma_0) = e^{\gamma_0} \chi'_0(\gamma) = \frac{1}{\lambda} \chi'_0(\gamma), \quad \gamma \in \mathbb{R}.
\]
Hence, if we put \( f(a) = \chi_0(-\log a), a > 0 \), then \( f \) is decreasing, continuous from the right, and satisfies \( \lambda f(\lambda a) = f(a), a > 0 \). By Theorem 5.5, \( f \) is of the form \( f = f_\phi, \phi \in M_*^+ \). But then with \( z \in Z(N) \),
\[
\chi(z) = \int_0^{\chi_0(z)} z(\gamma) \chi'_0(\gamma) e^{-\gamma} \, d\gamma
\]
\[
= \int_1^{\lambda} z(-\log a) f_\phi(a) \, da
\]
\[
= \phi(z),
\]
proving surjectivity. Q.E.D.

6. **ONE-PARAMETER AUTOMORPHISM GROUPS ON ABELIAN VON NEUMANN ALGEBRAS**

In the Main Theorem the range of the map \([\phi] \to \hat{\phi}\) is the set of \( \chi \in Z(N)_+^* \) such that \( \chi \circ \theta_s \geq e^{-s} \chi \) for \( s > 0 \). In the present section we shall make a detailed study of this set of positive functionals. The results do not depend on \( M \) and \( N \) and will be obtained for general abelian von Neumann algebras. Only the first lemma will be used in the proof of the Main Theorem. The remaining results of this section are used in the
applications. Throughout the section $Z$ will be an abelian von Neumann algebra, and $(\theta_s)_{s \in \mathbb{R}}$ will be a continuous one-parameter group of automorphisms of $Z$.

**Lemma 6.1.** Put

$$P = \{ \chi \in Z^+_*: \chi \circ \theta_s \geq e^{-s}\chi \text{ for } s > 0 \},$$

$$P_0 = \{ \chi \in Z^+_*: \chi = \int_{-\infty}^{0} e^s \omega \circ \theta_s \, ds, \omega \in Z^+_* \}.$$

Then $P$ is the norm closure of $P_0$.

**Proof.** Let $\omega \in Z^+_*$ and

$$\chi = \int_{-\infty}^{0} e^s \omega \circ \theta_s \, ds \in P_0.$$

Then for $t > 0$

$$\chi \circ \theta_t = \int_{-\infty}^{0} e^s \omega \circ \theta_{t+s} \, ds = \int_{-\infty}^{t'} e^{u-t} \omega \circ \theta_u \, du \geq e^{-t} \int_{-\infty}^{0} e^u \omega \circ \theta_u \, du = e^{-t} \chi,$$

proving that $P_0 \subset P$, and hence the norm closure $\overline{P}_0 \subset P$.

Let $\chi \in P$ and choose an approximate unit $(f_n)_{n \in \mathbb{N}}$ for $L^1(\mathbb{R})$ consisting of nonnegative $C^1$-functions with compact support. Since $P$ is closed and $\theta$-invariant

$$\chi_n = \int_{-\infty}^{0} f_n(s) \chi \circ \theta_s \, ds \in P,$$

and $\|\chi_n - \chi\| \to 0$ as $n \to \infty$. Moreover, the function $s \mapsto \chi_n \circ \theta_s$ is a $C^1$-function from $\mathbb{R}$ to $Z^+_*$. Put

$$\omega_n = \frac{d}{ds} (e^s \chi_n \circ \theta_s) \bigg|_{s = 0} = \lim_{s \to 0^+} \frac{1}{s} (e^s \chi_n \circ \theta_s - \chi_n).$$

Since $\chi_n \in P$, $\omega_n \geq 0$. Furthermore, for $s_0 \in \mathbb{R}$ we have

$$\frac{d}{ds} (e^s \chi_n \circ \theta_s) \bigg|_{s = s_0} = \frac{d}{du} (e^{s_0 + u} \chi_n \circ \theta_{s_0 + u}) \bigg|_{u = 0}$$

$$= e^{s_0} \frac{d}{du} (e^u \chi_n \circ \theta_{u}) \bigg|_{u = 0} \circ \theta_{s_0}$$

$$= e^{s_0} \omega_n \circ \theta_{s_0}.$$
Therefore
\[ \int_{-\infty}^{0} e^{t} \omega_{n} \circ \theta_{s} \, ds = \int_{-\infty}^{0} \frac{d}{ds} (e^{s} \chi_{n} \circ \theta_{s}) \, ds = \chi_{n}. \]

Since \( \omega_{n} \in Z_{\ast}^{+} \), \( \chi_{n} \in P_{0} \) for all \( n \in \mathbb{N} \). Thus \( \chi \in P_{0} \), and \( P_{0} = P \). Q.E.D.

**Lemma 6.2.** Let \( h \) be a bounded continuous real function on \( \mathbb{R} \) such that
\[ \int_{-\infty}^{0} e^{t} h(s + t) \, ds = 0 \]
for all \( t \in \mathbb{R} \). Then \( h = 0 \).

**Proof.** Since
\[ 0 = \int_{-\infty}^{0} e^{t} h(s + t) \, ds = e^{-t} \int_{-\infty}^{t} e^{s} h(s) \, ds \]
for all \( t \in \mathbb{R} \), it follows that if \( a < b \) then
\[ \int_{a}^{b} e^{s} h(s) \, ds = 0. \]
By continuity of \( h \), \( h = 0 \). Q.E.D.

**Proposition 6.3.** \( Z_{\ast} \) is the norm closed linear span of \( P \).

**Proof.** Let \( z \in Z \) belong to the annihilator of \( P_{0} \). Since \( P_{0} \) is \( \theta \)-invariant, \( \theta_{t}(z) \) belongs to the annihilator for all \( t \in \mathbb{R} \); hence for all \( \omega \in Z_{\ast}^{+} \),
\[ \int_{-\infty}^{0} e^{t} \omega \circ \theta_{t}(z) \, ds = 0. \]
Let \( h(s) = \omega \circ \theta_{t}(z) \). Then by Lemma 6.2 \( h = 0 \); hence in particular \( \omega(z) = 0 \). Since \( \omega \) was arbitrary in \( Z_{\ast}^{+} \), \( z = 0 \), and \( P_{0} \) is separating for \( Z \).

Now let \( S \) denote the closed linear span of \( P \). By Lemma 6.1, \( S \) is the closed linear span of \( P_{0} \). Since \( Z = (Z_{\ast})^{*} \) and \( S \) is a closed subspace, which by the previous paragraph is separating for \( Z \), \( S = Z_{\ast} \) by the Hahn–Banach Theorem. Q.E.D.

We next prove the \( C^{*} \)-algebra version of Lemma 6.1.

**Lemma 6.4.** Let \( A \) be an abelian \( C^{*} \)-algebra and \( (\theta_{t})_{t \in \mathbb{R}} \) a one-parameter group of automorphisms of \( A \) such that \( s \to \theta_{s}(f) \) is continuous in
norm for all \( f \in \mathcal{A} \). Let \( \chi \in \mathcal{A}_\mathcal{W}^* \). Then \( \chi \circ \theta_s \geq e^{-s} \chi \) for all \( s > 0 \), if and only if there is \( \omega \in \mathcal{A}_\mathcal{W}^* \) such that

\[
\chi = \int_{-\infty}^0 e^s \omega \circ \theta_s \, ds.
\]

Moreover, \( \omega \) is uniquely determined by \( \chi \).

Proof. Just as in the proof of Lemma 6.1 it follows that if \( \chi = \int_{-\infty}^0 e^s \omega \circ \theta_s \, ds \), then \( \chi \circ \theta_s \geq e^{-s} \chi \) for \( s > 0 \). Conversely, if this inequality holds then we can, as in the proof of Lemma 6.1, find a sequence \((\omega_n)\) in \( \mathcal{A}_\mathcal{W}^* \) such that \( \|\chi - \chi_n\| \to 0 \), \( n \to \infty \), where

\[
\chi_n = \int_{-\infty}^0 e^s \omega_n \circ \theta_s \, ds,
\]

and \( \|\chi_n\| = \chi_n(1) = \omega_n(1) = \|\omega_n\| \). Let \( \omega \) be a \( \mathcal{W}^* \)-cluster point for \((\omega_n)\) and let \( f \in \mathcal{A} \). Since \( s \to \theta_s(f) \) is norm continuous, \( \int_{-\infty}^0 e^s \theta_s(f) \, ds \) is an element in \( \mathcal{A} \). Thus

\[
\chi(f) = \lim_n \chi_n(f)
= \lim_n \omega_n \left( \int_{-\infty}^0 e^s \theta_s(f) \, ds \right)
= \omega \left( \int_{-\infty}^0 e^s \theta_s(f) \, ds \right)
= \int_{-\infty}^0 e^s \omega \circ \theta_s(f) \, ds,
\]

proving the converse.

To show the uniqueness it suffices by linearity to show that if \( \omega \in \mathcal{A}_\mathcal{W}^* \) and

\[
\int_{-\infty}^0 e^s \omega \circ \theta_s \, ds = 0,
\]

then \( \omega = 0 \). But if \( f \in \mathcal{A} \) then for all \( t \in \mathbb{R} \)

\[
\int_{-\infty}^0 e^s \omega \circ \theta_s(\theta_s(f)) \, ds = 0.
\]

Thus by Lemma 6.2 \( \omega \circ \theta_s(f) = 0 \) for all \( s \), and in particular \( \omega(f) = 0 \). Since \( f \) is arbitrary in \( \mathcal{A} \), \( \omega = 0 \). Q.E.D.

Let \( Z \) and \((\theta_s)_{s \in \mathbb{R}}\) be as before. Let \( \mathcal{A} \) denote the set of \( z \in Z \) such that
the function \( s \to \theta_s(z) \) is norm continuous. Then \( A \) is a \( \theta \)-invariant \( \sigma \)-weakly dense \( C^* \)-subalgebra of \( Z \) (see, e.g., the proof of [21, Lem. 9.5]). Put

\[ Q = \left\{ \omega \in A^*_+ : \text{There is } x_0 \in Z^+ \text{ with } x_0 \| A = 1 \| \omega, s > 0 \right\}. \]

Recall from Lemma 6.1 that \( P = \{ \chi \in Z^+_* : \chi \circ \theta_s \geq e^{-s} \chi, s > 0 \} \).

**Lemma 6.5.** With the above notation the map \( \omega \to \chi_\omega \) is an affine bijection of \( Q \) onto \( P \).

**Proof.** By Lemma 6.4 and the density of \( A \) in \( Z \), the map \( \omega \to \chi_\omega \) is an injection of \( Q \) into \( P \). Furthermore, the map is clearly affine.

To show surjectivity let \( \chi \in P \). By Lemma 6.1 there is a sequence \( (\chi_n) \) in \( P_0 \) such that \( \| \chi_n - \chi \| \to 0, n \to \infty \). Let

\[ \chi_n = \int_{-\infty}^{0} e^s \omega_n \circ \theta_s \, ds, \]

where \( \omega_n \in Z^+_* \). Let \( \bar{\omega}_n = \omega_n \| A \). Then \( \bar{\omega}_n \in Q \) and \( \chi_n = \chi_{\bar{\omega}_n} \). As in the proof of Lemma 6.4, if \( \omega \) is a \( w^* \)-cluster point of the sequence \( (\bar{\omega}_n) \), then

\[ \chi \| A = \int_{-\infty}^{0} e^s \omega \circ \theta_s \, ds; \]

hence \( \chi = \chi_{\omega_0} \).

**Proposition 6.6.** (a) With the ordering inherited from \( Q \) via the affine isomorphism \( \omega \to \chi_\omega \), \( P \) is a cone in \( Z^+_* \) satisfying the Riesz decomposition property.

(b) The convex set \( P_1 = \{ \varphi \in P \mid \varphi(1) = 1 \} \) is a norm-closed (in general noncompact) Choquet simplex.

**Proof.** (a) By Lemma 6.5 it suffices to show that \( Q \) satisfies the Riesz decomposition property. But this is clear, since \( A^*_+ \) docs, and if \( \varphi \in A^*_+ \), \( \omega \in Q \), and \( 0 \leq \varphi \leq \omega \), then \( \varphi \in Q \), because

\[ 0 \leq \int_{-\infty}^{0} e^s \varphi \circ \theta_s \, ds \leq \int_{-\infty}^{0} e^s \omega \circ \theta_s \, ds = \chi_\omega \| A, \]

so that \( \int_{-\infty}^{0} e^s \varphi \circ \theta_s \, ds \) is majorized by the restriction of a normal functional, and hence is itself the restriction of a normal functional. Thus \( \varphi \in Q \).

(b) Clearly \( P_1 \) is norm-closed. Since \( Q \) has the Riesz decomposition property in the usual ordering \( Q_1 = \{ \chi \in Q \mid \chi(1) = 1 \} \) is a simplex. But \( \omega \to \chi_\omega \) is an affine isomorphism of \( Q_1 \) onto \( P_1 \). This proves (b).
7. A Martingale Theorem

Theorem 7.1. Suppose $M$ is a von Neumann algebra such that $M = (\bigcup_{x \in A} M_x)^-$ is the $\sigma$-weak closure of an increasing family of von Neumann subalgebras for which there is for each $x \in A$ a faithful normal conditional expectation $E_x : M \to M_x$ of $M$ onto $M_x$ satisfying $E_x E_y = E_x$ whenever $M_x \subset M_y$, and
\[
\lim_{x} \|\varphi \circ E_x - \varphi\| = 0 \quad \text{for all } \varphi \in M_*. \tag{7.1}
\]
Suppose the conclusions of the Main Theorem hold for each $M_x$. Then

(i) Part (i) of the Main Theorem holds for $M$.

(ii) If each $M_x$ is properly infinite with no type I direct summand, part (ii) of the Main Theorem holds for $M$.

(iii) If $M$ and each $M_x$ are of type $II_1$, part (iii) of the Main Theorem holds for $M$.

Proof. There is no loss of generality to assume that the index set $A$ has a minimal element $x_0$ (otherwise choose $x_0 \in A$ and consider $A' = \{ x \in A : M_x \supset M_{x_0} \}$). Let $\omega_0$ be a faithful normal semifinite weight on $M_{x_0}$ and put $\omega = \omega_0 \circ E_{x_0}$. Let $\omega_x$ denote the restriction of $\omega$ to $M_x$. Note that all $\omega_x$'s are semifinite, since the restriction to $M_{x_0}$ is semifinite. Moreover, for all $x \in A$
\[
\omega = \omega_0 \circ E_{x_0} \circ E_x = \omega \circ E_x.
\]
Hence by [20] or [17, Thm. 10.1] $M_x$ is $\sigma^\omega$-invariant for every $x$, and
\[
\sigma^\omega_t(x) = \sigma^\omega_t(x), \quad x \in M_x.
\]
It follows that

\[
M_x \times_{\sigma^\omega} \mathbb{R} \subset M \times_{\sigma^\omega} \mathbb{R}.
\]
Put $N_x = M_x \times_{\sigma^\omega} \mathbb{R}$ and $N = M \times_{\sigma^\omega} \mathbb{R}$. Then $N = (\bigcup_{x \in A} N_x)^-$. Furthermore the dual action $\theta^\omega$ of $\sigma^\omega$ on $N_x$ is simply the restriction of the dual action $\theta$ of $\sigma^\omega$ on $N$ to $N_x$. Hence the operator valued weight
\[
T_x(x) = \int_{-\infty}^{\infty} \theta^\omega_s(x) \, ds, \quad x \in N_x^+,
\]
is just the restriction of the operator valued weight $T$ on $N$. In particular the dual weight $\omega^\ast$ of $\omega_x$ on $N_x$ is simply the restriction of $\omega$ to $N_x$. 


Let $\tau_a$ and $\tau$ be the canonical traces on $N_a$ and $N$, respectively. From the discussion in Section 3,

$$\left(\frac{d\tilde{\omega}}{dt}\right)^{it} = \lambda(t) = \left(\frac{d\tilde{\omega}}{dt}\right)^{it}.$$ 

Thus $\tau_a(x) = \tau(x)$ for $x \in N_a^+$. Put

$$\tilde{E}_\alpha = E_\alpha \otimes 1_{B(L^2(\mathbb{R}))},$$

where $1_{B(L^2(\mathbb{R}))}$ denotes the identity map on $B(L^2(\mathbb{R}))$. Then $\tilde{E}_\alpha$ is a faithful normal conditional expectation of $M \otimes B(L^2(\mathbb{R}))$ onto $M_\alpha \otimes B(L^2(\mathbb{R}))$, and

$$\|\varphi \circ \tilde{E}_\alpha - \varphi\| \to 0, \quad \varphi \in (M \otimes B(L^2(\mathbb{R})))_*.$$ 

Now $N \subset M \otimes B(L^2(\mathbb{R}))$, and the definition of $\pi: M \to N$, $(\pi(x) \xi)(t) = \sigma^\omega_{-\omega}(x) \xi(t)$, $x \in M$, $\xi \in L^2(\mathbb{R}, H)$, together with the formula $E_\alpha \sigma^\omega = \sigma^\omega_{\omega}$ [17, Cor. 10.5] yield

$$\tilde{E}_\alpha(\pi(x)) = \pi(E_\alpha(x)), \quad x \in M.$$ 

Moreover, since $\lambda(t) \in 1 \otimes B(L^2(\mathbb{R}))$

$$\tilde{E}_\alpha(\lambda(t)) = \lambda(t), \quad t \in \mathbb{R}.$$ 

Hence, if we let $F_\alpha$ denote the restriction of $\tilde{E}_\alpha$ to $N$ then $F_\alpha$ is a faithful normal conditional expectation of $N$ onto $N_\alpha$ such that

(i) $F_\alpha(\pi(x)) = \pi(E_\alpha(x))$, $x \in M$,

(ii) $F_\alpha(\lambda(t)) = \lambda(t)$, $t \in \mathbb{R}$,

(iii) $\lim_x \|\varphi \circ F_\alpha - \varphi\| = 0$, $\varphi \in N_\alpha^*$. 

Since $\theta_x \pi(x) = \pi(x)$, $x \in M$, and $\theta_x \lambda(t) = e^{-ixt} \lambda(t)$, $t \in \mathbb{R}$, it follows from (i) and (ii) that $F_\alpha$ commutes with $\theta$, and so with the operator valued weight $T$. Hence for $x \in N^+$ we get, using (i),

$$\tilde{\omega}(x) = \omega \circ \pi^{-1} \circ T(x) = \omega \circ E_\alpha \circ \pi^{-1} \circ T(x) = \omega \circ \pi^{-1} \circ F_\alpha \circ T(x) = \omega \circ \pi^{-1} \circ T \circ F_\alpha(x) = (\tilde{\omega} \circ F_\alpha)(x),$$

i.e., $\tilde{\omega} = \tilde{\omega} \circ F_\alpha = \tilde{\omega} \circ F_\alpha$ for all $\alpha$. Since $F_\alpha$ preserves cocycle Radon–Nikodym derivatives [17, Cor. 10.5], we also have

$$\tau = \tau_a \circ F_\alpha = \tau \circ F_\alpha.$$ 

At this point we interrupt the discussion to prove a lemma.
LEMMA 7.2. Let $\chi \in N_\ast$. Then

$$\|\chi\|_{Z(N)} = \lim_{x} \|\chi\|_{Z(N_x)}.$$ 

Proof. Let $Z \in Z(N)$. For any $x \in N_x$

$$xF_a(z) = F_a(xz) = F_a(zx) = F_a(z)x;$$

hence $F_a(z) \in Z(N_x)$. Moreover, by (iii)

$$\lim_{x} \chi(F_a(z)) = \chi(z).$$

Therefore, for $z \in Z(N)$,

$$|\chi(z)| = \lim_{x} |\chi \circ F_a(z)|$$

$$\leq \lim_{x} \inf \|\chi\|_{Z(N_x)} \|F_a(z)\|$$

$$\leq (\lim_{x} \inf \|\chi\|_{Z(N_x)}) \|z\|.$$ 

Hence

$$\|\chi\|_{Z(N)} \leq \lim_{x} \inf \|\chi\|_{Z(N_x)}.$$ 

Choose for each $\alpha \in A$, $z_\alpha \in Z(N_\alpha)$ such that $\|z_\alpha\| \leq 1$ and $\chi(z_\alpha) = \|\chi\|_{Z(N_\alpha)}$. We can choose a subnet $(z_{\alpha_\beta})$ such that

$$\lim_{\beta} \chi(z_{\alpha_\beta}) = \lim_{\alpha} \sup \|\chi\|_{Z(N_\alpha)}.$$ 

Let $z \in N$ be a $\sigma$-weak cluster point for $(z_{\alpha_\beta})$. Then for each $\alpha' \in A$, $z$ commutes with all operators in $N_{\alpha'}$ because $\alpha_\beta \supseteq \alpha'$ eventually ($A$ is ordered by inclusion). Since $N = (\bigcup_{\alpha' \in A} N_{\alpha'})^-$ we have $z \in Z(N)$. Clearly $\|z\| \leq 1$ and therefore

$$\|\chi\|_{Z(N)} \geq |\chi(z)| = \lim_{\beta} \chi(z_{\alpha_\beta}) = \lim_{\alpha} \sup \|\chi\|_{Z(N_\alpha)}$$

proving the lemma. Q.E.D.

Proof of Part (i) of Theorem 7.1. By Lemma 3.3 the map $[\varphi] \rightarrow \hat{\phi}$ is a contraction, so it remains to prove

(i) $\|\phi - \hat{\psi}\| \geq d([\varphi], [\psi])$
for \( \varphi \) and \( \psi \) in a dense subset of \( M_+^\ast \). Since \( \| \omega \circ E_\alpha - \omega \| \to 0 \) for all \( \omega \in M_+^\ast \), it suffices to prove (i) if

\[
\varphi = \varphi \circ E_{\alpha_1}, \quad \psi = \psi \circ E_{\alpha_1},
\]

for some \( \alpha_1 \in A \). So let us assume this. Then clearly \( \varphi = \varphi \circ E_{\alpha}, \psi = \psi \circ E_{\alpha} \) for all \( \alpha \geq \alpha_1 \). Put \( \varphi_{\alpha} = \varphi \mid M_\alpha, \psi_{\alpha} = \psi \mid M_\alpha \). If \( u \in U(M_\alpha) \), \( \alpha \geq \alpha_1 \), then we get by the bimodule property of \( E_\alpha \) that

\[
\varphi u^* - \psi = (\varphi_{\alpha} u^* - \psi_{\alpha}) \circ E_{\alpha},
\]

and therefore

\[
d(\varphi, \psi) \leq d_\alpha(\varphi_{\alpha}, \psi_{\alpha}), \quad \alpha \geq \alpha_1,
\]

where \( d_\alpha \) is the metric on \( (M_\alpha)^\ast / \sim \). Hence if (i) holds for all \( M_\alpha \) then

\[
d(\varphi, \psi) \leq \| \hat{\varphi} - \hat{\psi} \|, \quad \alpha \geq \alpha_1.
\]

Let \( \hat{\varphi} \) and \( \hat{\psi} \) be the dual weights of \( \varphi \) and \( \psi \) on \( N \), and let \( \hat{\varphi}_{\alpha} \) and \( \hat{\psi}_{\alpha} \) be those of \( \varphi_{\alpha} \) and \( \psi_{\alpha} \) on \( N_\alpha \).

Since \( T_\alpha = T \mid N_\alpha \) we have

\[
\hat{\varphi}_{\alpha} = \hat{\varphi} \mid N_\alpha, \quad \hat{\psi}_{\alpha} = \hat{\psi} \mid N_\alpha, \quad \alpha \geq \alpha_1.
\]

Moreover, repeating the proof of \( \tilde{\omega} = \tilde{\omega} \circ F_\alpha \) we find \( \tilde{\varphi} = \tilde{\varphi} \circ F_\alpha, \tilde{\psi} = \tilde{\psi} \circ F_\alpha \). Let

\[
ed_{\varphi} = \chi_{(1, \infty)} \left( \frac{d\tilde{\varphi}}{d\tau} \right), \quad ed_{\varphi_{\alpha}} = \chi_{(1, \infty)} \left( \frac{d\tilde{\varphi}_{\alpha}}{d\tau_{\alpha}} \right),
\]

and similarly for \( \psi \). Then, since \( F_\alpha \) preserves Radon–Nikodym derivatives [17, Cor. 10.5],

\[
ed_{\varphi} = ed_{\varphi_{\alpha}}, \quad ed_{\psi} = ed_{\psi_{\alpha}}, \quad \alpha \geq \alpha_1.
\]

Thus by applying Lemma 7.2 to the functional \( \chi \) defined by

\[
\chi(z) = \tau((e_{\varphi} - e_{\psi}) z), \quad z \in N,
\]

we obtain

\[
\lim_{\alpha} \| \hat{\varphi}_{\alpha} - \hat{\psi}_{\alpha} \| = \| \hat{\varphi} - \hat{\psi} \|,
\]

and hence that \( \| \hat{\varphi} - \hat{\psi} \| \geq d(\{ \varphi \}, \{ \psi \}) \). \( \Box \)

Proof of Part (ii) of Theorem 7.1. Assume each \( M_\alpha \) satisfies both parts (i) and (ii) of the Main Theorem. We know that in this case \( M \) satisfies
part (i). Since \( M_*/\sim \) is a complete metric space by Lemma 2.1, it follows that the set

\[
P(M) = \{ \phi \in Z(N)_*^+ : \phi \in M_*/\sim \}
\]

is closed in \( Z(N)_*^+ \). Let \( P \) and \( P_0 \) be as in Lemma 6.1. We know that \( P(M) \subset P \) by Lemma 3.4, so to prove that \( P(M) = P \) it suffices by Lemma 6.1 to show \( P_0 \subset P(M) \). For this let \( \chi \in P_0 \); hence

\[
\chi = \int_{-\infty}^{0} e^{s \omega} \circ \theta_s \, ds
\]

for some \( \omega \in Z(N)_*^+ \). We extend \( \omega \) to a functional \( \tilde{\omega} \in N_*^+ \). Then

\[
\tilde{\chi} = \int_{-\infty}^{0} e^{s \tilde{\omega}} \circ \theta_s \, ds
\]

is a positive normal extension of \( \chi \) to \( N \) for which

\[
\tilde{\chi} \circ \theta_s \geq e^{-s \tilde{\omega}}, \quad s > 0.
\]

Put \( \chi_\omega = \tilde{\chi} |_{Z(N)_*} \). We have previously observed that the dual action \( \theta^* \) on \( N_\omega \) is the restriction of \( \theta \) to \( N_\omega \). Therefore

\[
\chi_\omega \circ \theta^*_s \geq e^{-s \chi_\omega}, \quad s > 0,
\]

and so by the assumptions on \( M_\omega \), since \( M_\omega \) is properly infinite with no type I portion, \( \chi_\omega = \phi_\omega \) for some \( \phi_\omega \in (M_\omega)_*^+ \). Put

\[
\psi_\omega = \phi_\omega \circ F_\omega \in M_\omega^+,
\]

and put

\[
e^{z}_\phi = \chi_{(1, \infty)} \left( \frac{d\tilde{\psi}_\omega}{d\tau_\omega} \right).
\]

Since \( \tilde{\psi}_\omega = \phi_\omega \circ F_\omega \) and \( \tau = \tau_\omega \circ F_\omega \), and \( F_\omega \) preserves cocycle Radon–Nikodym derivatives,

\[
e^{z}_\phi = \chi_{(1, \infty)} \left( \frac{d\tilde{\psi}_\omega}{d\tau} \right) = e^{z}_\psi.
\]

Let \( z \in Z(N) \). Then \( F_\omega(z) \in Z(N_\omega) \), whence

\[
\tilde{\psi}_\omega(z) = \tau(e^{z}_\phi z) = \tau(e^{z}_\phi z) = \tau_\omega(F_\omega(e^{z}_\phi z))
\]

\[
= \tau_\omega(e^{z}_\phi F_\omega(z)) = \phi_\omega(F_\omega(z))
\]

\[
= \chi_\omega(F_\omega(z)) = \tilde{\chi}(F_\omega(z)).
\]
Since \( \|\tilde{\chi} \circ F_s - \tilde{\chi}\| \to 0 \) we have
\[
\chi = \tilde{\chi} |_{Z(N)} = \lim_{\alpha} \hat{\psi}_e \quad \text{in} \ Z(N)_*,
\]
Thus \( \chi \in P(M) \), completing the proof in the case \( M \) and all \( M_\alpha \) are properly infinite.

**Proof of Part (iii) of Theorem 7.1.** We now assume \( M \) and all \( M_\alpha \) are of type \( \text{II}_1 \). Since \( M \) is a direct sum of \( \sigma \)-finite von Neumann algebras we may assume \( M \) has a faithful normal tracial state \( \tau_0 \). Exactly as in the factor case (see proof of Lemma 4.5) we find \( N = M \times_{\sigma} R \cong M \otimes L^\infty(R) \), \( \tilde{\lambda}(t) \) is the multiplication operator \( m(e^t) \) on \( L^\infty(R) \), and \( \Pi(x) = x \otimes 1 \).

Moreover, the canonical trace is \( \tau = \tau_0 \otimes e^{-\gamma} \, dt \). It follows that \( N \) is finite, and there is an increasing sequence \( (e_n)_{n \in \mathbb{N}} \) of projections in \( Z(N) \) such that \( e_n \to 1 \) and \( \tau(e_n) < \infty \). Let \( \Phi \) be the unique center valued trace on \( N \) such that \( \Phi |_{Z(N)} = \tau \). Let \( \chi \in Z(N)_+, \chi \circ \theta_s \geq e^{-t} \chi, \ s > 0, \) and \( \chi \leq \tau |_{Z(N)} \). Let
\[
\tilde{\chi} = \chi \circ \Phi \in N_+^*.
\]
Since by uniqueness \( \theta_s \Phi \theta_s^{-1} = \Phi, \ s \in \mathbb{R}, \)
\[
\tilde{\chi} \circ \theta_s = \chi \circ \Phi \circ \theta_s = \chi \circ \theta_s \circ \Phi \geq e^{-s} \chi \circ \Phi = e^{-s} \tilde{\chi}, \quad s > 0.
\]
Note that since \( \tau - \tau |_{Z(N)} \circ \Phi \) we have \( \tilde{\chi} \leq \tau \). Since \( \tau_a = \tau |_{N_a} \), and if we put \( \chi_a = \tilde{\chi} |_{Z(N_a)} \), then \( \chi_a \leq \tau_a |_{Z(N_a)} \). We can now complete the proof exactly as in the properly infinite case.

**Q.E.D.**

8. **Proof of the Main Theorem for von Neumann Algebras with Separable Predual**

Let \( M \) be a von Neumann algebra with separable predual, which we assume acts on a separable Hilbert space. By [12, 14.2.1] there are a locally compact complete separable metric space \( Z \), a positive measure \( \nu \) on \( Z \) with support \( Z \), a \( \nu \)-measurable field \( \zeta \to H(\zeta) \) of nonzero Hilbert spaces on \( Z \), a \( \nu \)-measurable field \( \zeta \to M(\zeta) \) of factors on \( H(\zeta) \), and an isomorphism of \( H \) on \( \int^\oplus H(\zeta) \, d\nu(\zeta) \), which carries \( M \) onto \( \int^\oplus M(\zeta) \, d\nu(\zeta) \). In the present section we shall first show that if the Main Theorem holds for the \( M(\zeta) \) a.e. then it holds for \( M \). As a consequence it will follow that it holds for all von Neumann algebras with separable preduals. We use the notation introduced above, except that we presently do not assume the \( M(\zeta) \) are factors.
Proposition 8.1. Let $M$ be a von Neumann algebra with separable predual. Let $\omega$ be a faithful normal semifinite weight on $M$. Let $N$ be the crossed product $N = M \times_{\sigma^\omega} \mathbb{R}$, and let $\theta$ be the dual automorphism group on $N$, and $\tau$ the canonical trace on $N$. Then we have

(i) $N = \int N(\xi) \times_{\sigma_{N(\xi)}} \mathbb{R} \, d\omega(\xi) = \int N(\xi) \, d\omega(\xi)$, where $N(\xi) = M(\xi) \times_{\sigma_{N(\xi)}} \mathbb{R}$, and $\omega = \int \omega(\xi) \, d\omega(\xi)$.

(ii) $Z(N) = \int Z(N(\xi)) \, d\omega(\xi)$, $Z(N(\xi)) = Z(N(\xi))$ a.e.

(iii) $\theta = \int \theta(\xi) \, d\omega(\xi)$, where $\theta(\xi)$ is the dual automorphism group on $N(\xi)$.

(iv) $\tau = \int \tau(\xi) \, d\omega(\xi)$, where $\tau(\xi)$ is the canonical trace on $N(\xi)$ a.e.

Proof. This result is more or less implicit in work of Lance [13] and Sutherland [18, 19]. Indeed, by [19] $\omega$ decomposes as a direct integral $\omega = \int \omega(\xi) \, d\omega(\xi)$, where $\omega(\xi)$ is a faithful normal weight on $M(\xi)$ a.e. By [13, Thm. 2.5] $\sigma^\omega = \int \sigma^\omega \, d\omega(\xi)$. Thus by [18, Prop. 1.4] (i) follows, and by [18, Prop. 1.5], (iii) follows. By [12, 14.4.4] (ii) holds. Finally, by [6, Chap. II, Sect. 5, Thm. 5], or by [12, 14.1.19] coupled with [19] $\tau = \int \tau(\xi) \, d\omega(\xi)$, where $\xi \rightarrow \tau(\xi)$ is a measurable field of faithful normal semifinite traces on $N(\xi)$. Since $\tau \circ \theta = e^{-\tau}$ we have by (iii)

$$\int e^{-\tau(\xi)} \, d\omega(\xi) = e^{-\tau} = \tau \circ \theta = \int \tau(\xi) \, d\omega(\xi),$$

so by uniqueness [19, Cor. A.12], $e^{-\tau(\xi)} = \tau(\xi) \circ \theta(\xi)$ a.e., proving (iv).

Q.E.D.

A few further remarks are necessary. If $\phi \in M_*$ then there is $\phi(\xi) \in M(\xi)_*$ such that

$$\phi = \int \phi(\xi) \, d\omega(\xi).$$

Moreover,

$$\|\phi\| = \int \|\phi(\xi)\| \, d\omega(\xi).$$

If $\phi \geq 0$, then $\phi(\xi) \geq 0$ a.e. so without loss of generality we can assume that $\phi(\xi) \geq 0$ for all $\xi \in Z$ when $\phi \geq 0$.

With the notation as in Proposition 8.1 let $T: N \rightarrow \pi(M)$ be the operator valued weight given by $T(x) = \int Z(\xi) \, d\omega(\xi)$, $x \in N^+$, and define similarly $T(\xi): N(\xi) \rightarrow \pi(\xi)(M(\xi))$ (cf. Sect. 3). If $\phi \in M_*$ its dual weight $\check{\phi}$ on $N^+$ is defined by

$$\check{\phi} = \phi \circ \pi^{-1} \circ T = \tau(h_{\phi}),$$
where $h_\varphi = d\bar{\varphi}/dt$ is a positive self-adjoint operator affiliated with $N$. If $\varphi = \int^\infty \varphi_\zeta \, dv(\zeta)$ we similarly have

$$\bar{\varphi_\zeta} = \varphi_\zeta \circ \pi^{-1}_\zeta \circ T_\zeta = \tau_\zeta(h_{\varphi_\zeta}).$$

If $x \in N^+$ and $T(x) \in \pi(M)^+$ then by the Fubini–Tonelli theorem we have

$$\varphi(x) = \int_{-\infty}^\infty \varphi \circ \pi^{-1} \circ \theta_s(x) \, ds$$

$$= \int_{-\infty}^\infty \left( \int_{\zeta} \varphi_\zeta \circ \pi^{-1}_\zeta \circ \theta_s(\zeta)(x(\zeta)) \, dv(\zeta) \right) \, ds$$

$$= \int_{\zeta} \left( \int_{-\infty}^\infty \varphi_\zeta \circ \pi^{-1}_\zeta \circ \theta_s(\zeta)(x(\zeta)) \, ds \right) \, dv(\zeta)$$

$$= \int_{\zeta} \varphi_\zeta \circ \pi^{-1}_\zeta \circ T_\zeta(x(\zeta)) \, dv(\zeta),$$

so from the uniqueness of the decomposition [19, Cor. A.12]

$$\varphi = \int^\oplus \bar{\varphi_\zeta} \, dv(\zeta).$$

The operator $h_\varphi$ being positive and self-adjoint affiliated with $N$ is decomposable [13, Thm. 1.8], and therefore of the form

$$h_\varphi = \int^\oplus h(\zeta) \, dv(\zeta)$$

with $h(\zeta)$ positive and self-adjoint affiliated with $N(\zeta)$. Suppose $x \in N^+$, $\bar{\varphi}(x) < \infty$. Then

$$\int \tau_\zeta(h_{\varphi_\zeta}, x(\zeta)) \, dv(\zeta) = \int \bar{\varphi_\zeta}(x(\zeta)) \, dv(\zeta) = \varphi(x)$$

$$= \tau(h_{\varphi}, x) = \int \tau_\zeta(h(\zeta), x(\zeta)) \, dv(\zeta).$$

Thus by uniqueness $h_{\varphi_\zeta} = h(\zeta)$ a.e. As before let $e_\varphi = \chi_{(1, \infty)}(h_{\varphi})$, and define $e_\varphi(\zeta)$ similarly. By [13, Thm. 1.10]

$$e_\varphi = \int^\oplus e_\varphi(\zeta) \, dv(\zeta).$$
Now $\phi \in Z(N)^*$ is defined by $\phi(z) = \tau(e_\omega z)$. Since by Proposition 8.1 $Z(N) = \int Z(N(\zeta)) \, dv(\zeta)$, we thus have

$$\phi(z) = \int \tau(e_\omega(z(\zeta))) \, dv(\zeta) = \int \phi_\zeta(z(\zeta)) \, dv(\zeta).$$

Thus in the direct integral decomposition

$$\phi = \int \phi_\zeta \, dv(\zeta)$$

we have by uniqueness

$$(2) \quad (\phi)_\zeta = \phi_\zeta \text{ a.e.}$$

**Proposition 8.2.** Let $M$ be a von Neumann algebra with separable predual and direct integral decomposition

$$M = \int Z M(\zeta) \, dv(\zeta).$$

Suppose the Main Theorem holds for each $M(\zeta)$. Then it holds for $M$.

**Proof.** Let

$$(*) \quad M = \int Z M(\zeta) \, dv(\zeta)$$

be the direct integral decomposition of $M$, and let

$$H = \int Z H(\zeta) \, dv(\zeta)$$

be the corresponding decomposition of $H$. If we exchange $M(\zeta)$ with its natural representation on $\oplus_{\zeta} H(\zeta)$ for each $\zeta \in Z$, the integral $(*)$ gives the natural representation of $M$ on $\oplus_{\zeta} H$. Hence, we may assume that the commutant $M'$ and all the commutants $M(\zeta)'$ are properly infinite. In particular

$$\dim H(\zeta) = +\infty \quad \text{for all } \zeta \in Z,$$

so by using [12, Lemma 14.1.23] we may assume that $(H(\zeta))_{\zeta \in Z}$ is a constant field of Hilbert spaces, i.e.,

$$H(\zeta) = H_0,$$

where $H_0$ is a fixed infinite dimensional separable Hilbert space.
Let \((x_i)_{i \in \mathbb{N}}\) and \((x'_i)_{i \in \mathbb{N}}\) be strongly dense sequences in the unit balls of \(M\) and \(M'\), respectively,

\[
x_i = \int_{Z} x_i(\zeta) \, dv(\zeta), \quad x'_i = \int_{Z} x'_i(\zeta) \, dv(\zeta).
\]

By the proof of \([12, \text{Prop. 14.1.24}]\), there is a \(v\)-nullset \(X_0\), such that for all \(\zeta \in Z \setminus X_0\), \((x_i(\zeta))_{i=1}^{\infty}\) and \((x'_i(\zeta))_{i=1}^{\infty}\) are strongly dense in the unit balls of \(M(\zeta)\) and \(M'(\zeta)\), respectively. Note that any \(v\)-nullset is contained in a Borel \(v\)-nullset. This combined with \([12, \text{Lemma 14.3.1}]\) shows that we can find a Borel \(v\)-nullset \(X_1 \supseteq X_0\), such that \((x_i(\zeta))_{i=1}^{\infty}\) and \((x'_i(\zeta))_{i=1}^{\infty}\) are Borel functions on \(Z \setminus X_1\) in the Borel structure of \(B(H_0)\) generated by the strong operator topology.

To show the first part of the Main Theorem for \(M\) let \(\varphi, \psi \in M_+^*\). Since \(M'\) is properly infinite, \(\varphi, \psi\) are the vector states given by the two vectors \(\xi, \eta \in H\),

\[
\xi = \int_{Z} \xi(\zeta) \, dv(\zeta), \quad \eta = \int_{Z} \eta(\zeta) \, dv(\zeta).
\]

This implies that

\[
\varphi = \int_{Z} \varphi_\zeta \, dv(\zeta),
\]

where \(\varphi_\zeta\) (resp. \(\psi_\zeta\)) is the vector state on \(M(\zeta)\) given by \(\xi(\zeta)\) (resp. \(\eta(\zeta)\)). Since \(\xi(\zeta), \eta(\zeta)\) are \(v\)-measurable \(H_0\)-valued functions, and since

\[
\zeta \rightarrow \| (\varphi_\zeta)^\wedge - (\psi_\zeta)^\wedge \|
\]

is a \(v\)-measurable \(\mathbb{R}\)-valued function on \(Z\), we can by \([12, \text{Lemma 14.3.1}]\) find a Borel \(v\)-nullset \(X_2 \supseteq X_1\), such that \(\xi(\zeta), \eta(\zeta)\), and \(\| (\varphi_\zeta)^\wedge - (\psi_\zeta)^\wedge \|\) are Borel functions on \(Z\).

Let \(\mathcal{U}\) be the unitary group in \(B(H_0)\). Note that \(\mathcal{U}\) is a Polish group in strong operator topology (cf. \([12, 14.4.9]\)). Let \(\varepsilon > 0\) and let \(S\) denote the set of points

\[
(\zeta, u) \in (Z \setminus X_2) \times \mathcal{U}
\]

for which

\[
(1) \quad u \in M(\zeta),
(2) \quad \| \varphi_\zeta - u\psi_\zeta u^* \| \leq (1 + \varepsilon) \| (\varphi_\zeta)^\wedge - (\psi_\zeta)^\wedge \|.
\]
We will show that $S$ is a Borel set. The set $S_1$ of pairs $(\zeta, u) \in (Z \setminus X_2) \times \mathcal{U}$ satisfying (1) is a Borel set, because for $\zeta \in Z \setminus X_1$, $u \in M(\zeta)$ if and only if $u$ commutes with $x_1^i(\zeta)$ for all $i \in \mathbb{N}$. For $u \in M(\zeta)$:

$$
\|\varphi - u\psi_u u^*\| = \sup_{i \in \mathbb{N}} \| (x_i \xi(\zeta), \xi(\zeta)) - (u^* x_i \eta(\zeta), \eta(\zeta)) \|
$$

for all $\zeta \in Z \setminus X_1$. Hence, by the choice of $X_2 \supseteq X_1 \supseteq X_0$ both left and right side of (2) are Borel functions on $S_1 \subseteq (Z \setminus X_2) \times \mathcal{U}$. By the assumptions each $M(\zeta)$ satisfies (i) in the Main Theorem. Hence the projection map $(\zeta, u) \mapsto \zeta$ maps $S$ onto $Z \setminus X_2$. By the measurable selection principle [12, Thm. 14.3.6], there is a $\nu$-measurable map $\zeta \mapsto v_\zeta$ of $Z \setminus X_2$ into $\mathcal{U}$, such that $(\zeta, v_\zeta) \in S$ for all $\zeta \in Z \setminus X_2$. Define $v_\zeta = 1$ for $\zeta \in X_1$. Then

$$
v = \int_Z v_\zeta d\nu(\zeta)
$$

is a unitary operator in $M$, and by integrating (2) one gets

$$
\|\varphi - v\psi_v v^*\| \leq (1 + \varepsilon) \|\hat{\varphi} - \hat{\psi}\|.
$$

Since $\varepsilon$ is arbitrary, $d([\varphi], [\psi]) \leq \|\hat{\varphi} - \hat{\psi}\|$, which together with Lemma 3.3 proves that $M$ satisfies (i) in the Main Theorem.

Assume now that $M$ is properly infinite without type I direct summand. Then $M(\zeta)$ is properly infinite without type I direct summand for almost all $\zeta \in Z$. As usual put

$$
P(M) = \{ \chi \in Z(N)^+_v | \chi \cdot \theta_s \geq e^{-s}\chi, s > 0 \}
$$

and let $\chi \in P(M)$. With the notation of Proposition 8.1 we have

$$
\chi = \int_Z \chi_\zeta d\nu(\zeta),
$$

where $\chi_\zeta \in Z(N(\zeta))_v^+$. Since

$$
\theta_s = \int_Z \theta_s(\zeta) d\nu(\zeta)
$$

it follows that

$$
\chi_\zeta \cdot \theta_s(\zeta) \geq e^{-s}\chi_\zeta, \quad s \in \mathbb{Q}_+, \quad \forall \zeta \in Z,
$$

for almost all $\zeta \in Z$, so by the continuity of $s \rightarrow \theta_s(\zeta)$, $\chi_\zeta \in P(M(\zeta))$ for almost all $\zeta$. Let $Y_0$ be a $\nu$-nullset, such that $M(\zeta)$ is properly infinite without type I direct summand and $\chi_\zeta \in P(M(\zeta))$ for all $\zeta \in Z \setminus Y_0$. 


By the first part of the proof we can assume that all the factors $M(\zeta)$ act on the same Hilbert space $H_0$. Hence all the von Neumann algebras $N(\zeta)$ in the direct integral decomposition

$$N = \int_Z N(\zeta) \ d\zeta$$

act on the Hilbert space $K_0 = L^2(\mathbb{R}, H_0)$. Using the arguments of the first part of the proof to the direct integral decompositions of $Z(N)$ and $N$, we can find a Borel $v$-nullset $Y_1 \supseteq Y_0$ and Borel functions $(z_i(\zeta))_{i=1}^{\infty}$ and $(y_i(\zeta))_{i=1}^{\infty}$ from $Z \setminus Y_1$ to $B(K_0)$, such that for fixed $\zeta \in Z \setminus Y_1$ the sequences $(z_i(\zeta))_{i=1}^{\infty}$ and $(y_i(\zeta))_{i=1}^{\infty}$ are strongly dense in the unit balls $Z(N(\zeta))$ and $N(\zeta)'$, respectively. Let

$$\tau = \int_Z \tau_\zeta \ dv(\zeta)$$

be the canonical trace on $N$ (cf. Prop. 8.1(iv)). Since $N$ has separable predual, there exists a sequence $(p_i)_{i=1}^{\infty}$ of orthogonal projections in $M$ with sum $1$, such that $\tau(p_i) < \infty$ for all $i$, so by applying [12, Thm. 7.1.8] to each of the positive functional $\tau(p_i)$, one gets that there is a sequence $(\xi_k)_{k=1}^{\infty}$ of vectors in $K_0$, such that

$$\tau(x) = \sum_{k=1}^{\infty} (x \xi_k, \xi_k), \quad x \in N_+.$$

Moreover, since $Z(N)$ is abelian, every normal state on $Z(N)$ is a vector state. Particularly,

$$\chi(z) = (z\eta, \eta), \quad z \in Z(N)_+,$$

for some $\eta \in K_0$. Let

$$\xi_k = \int_Z \xi_k(\zeta) \ d\zeta, \quad k \in \mathbb{N},$$

and

$$\eta = \int_Z \eta(\zeta) \ d\zeta.$$

Then for almost all $\chi \in Z$,

$$(*) \quad \tau_\zeta(x) = \sum_{k=1}^{\infty} (x \xi_k(\zeta), \xi_k(\zeta)), \quad x \in N(\zeta)_+$$

and

$$(**) \quad \chi(z) = (z\eta(\zeta), \eta(\zeta)), \quad z \in Z(N).$$
Again applying [12, Lemma 14.3.1] we can find a Borel $\nu$-nullset $Y_2 \supsetneq Y_1$, such that $(\xi_j(\zeta))_{j=1}^\infty$ and $\eta(\zeta)$ are $K_0$-valued Borel functions on $Z \setminus Y_2$ and such that $(*)$ holds for all $\zeta \in Z \setminus Y_2$. Let $\mathcal{P}$ denote the set of projections in $B(K_0)$. Then $\mathcal{P}$ is a Polish space in strong operator topology, because the map $e \mapsto 2e - 1$ maps $\mathcal{P}$ onto a closed subset of the unitary group of $K_0$.

Let $T$ be the set of $(\zeta, e) \in (Z \setminus Y_2) \times \mathcal{P}$ for which

1. $e \in N(\zeta)$,
2. $\theta_s(\zeta)e \leq e$, for $s > 0$,
3. $\tau(\zeta)e < \infty$,
4. $\chi(\zeta) = \tau(\zeta(e))|_{Z \setminus N(\zeta)}$.

The set $T_1$ of pairs $(\zeta, e)$ in $(Z \setminus Y_2) \times \mathcal{P}$ satisfying (1) is a Borel set, because $e \in N(\zeta)$ if and only if it commutes with $y_i(\zeta)$, $i \in \mathbb{N}$. Let $T_2$ be the subset of $T_1$ for which (2) holds. By [21], the dual action $\theta_s(\zeta)$ is implemented by the unitary operators $(u_s)_{s \in \mathbb{R}}$ on $K_0 = L^2(\mathbb{R}, H_0)$ given by

$$(v, \xi)(t) = e^{-ist}\xi(t).$$

Note that $(v, s)_{s \in \mathbb{R}}$ do not depend on $\zeta$. Put

$$\mathcal{P}_1 = \{ e \in \mathcal{P} \mid v_s e v_s^* \leq e \text{ for } s > 0 \}.$$

Then $\mathcal{P}_1$ is a closed subset of $\mathcal{P}$. Since

$$T_2 = \{ (\zeta, e) \in T_1 \mid e \in \mathcal{P}_1 \}$$

the set $T_2$ is also Borel. Since $\tau(\zeta)e < \infty$ if and only if

$$(*) \quad \sum_{k=1}^\infty \| e \xi_k(\zeta) \|^2 < \infty$$

the set $T_3 = \{ (\zeta, e) \in T_2 \mid \tau(\zeta)e < \infty \}$ is Borel. But for $(\zeta, e) \in T_3$ condition (4) is equivalent to

$$\tau(\zeta e_i(\zeta)) = \sum_{k=1}^\infty (z_i(\zeta), e \xi_k(\zeta), e \xi_k(\zeta)),$$  

where the right side of the equality is a convergent sum, because $(*)$ holds. This shows that $T$ is a Borel set. By the assumptions (ii) in the Main Theorem holds for each $M(\zeta)$, $\zeta \in Z \setminus X_2$. Hence for each $\zeta \in Z \setminus X_2$ there is a $\varphi(\zeta) \in M(\zeta)^*$, such that $(\varphi(\zeta)) - \chi(\zeta)$. By Lemma 3.5 and Definition 3.2 $(\zeta, e_\alpha(\zeta)) \in T$. Thus the projection $(\zeta, e) \mapsto \zeta$ maps $T$ onto $Z \setminus X_2$. By the measurable selection principle, there is a $\nu$-measurable function $e(\zeta)$ from
EQUIVALENCE OF NORMAL STATES

$Z \setminus X_2$ to $\mathcal{P}$, such that $(\zeta, e(\zeta)) \in T$ for all $\zeta \in Z \setminus X_2$. Put $e(\zeta) = 0$ for $\zeta \in X_2$. Then

$$f = \int_Z e(\zeta) \, dv(\zeta)$$

is a projection in $L^1(N, \tau)$ for which $\theta_s(f) \leq f$ when $s > 0$, and $\chi = \tau(f)_{|Z(N)}$. By Lemma 3.5 $f = e_\phi$ for some $\phi \in M_*$. Clearly $\chi = \phi$, which proves that $M$ satisfies (ii) in the Main Theorem.

The fact that $M$ satisfies (iii) in the Main Theorem provided that each $M(\zeta)$ satisfies (iii) follows by trivial modifications of the above arguments.

Q.E.D.

In order to prove the Main Theorem for factors of type $\text{III}_0$ with separable preduals we shall apply the martingale theorem in Section 7. Our next result makes this possible.

Proposition 8.3. Suppose $M$ is a factor of type $\text{III}_0$ with separable predual. Then there is an increasing sequence $(M_n)_{n \in \mathbb{N}}$ of subalgebras such that

(i) $M = (\bigcup_{n=1}^{\infty} M_n)^\sigma$, $\sigma$-weak closure.

(ii) Each $M_n$ is a von Neumann algebra of type $\text{II}_\infty$.

(iii) There is a faithful normal conditional expectation $E_n: M \to M_n$ such that $E_mE_n = E_m, m \leq n$, and $\lim_{n \to \infty} \| \phi \circ E_n - \phi \| = 0, \phi \in M_*$. 

Proof. By [2, Cor. 5.3.6] if $G$ is the restricted infinite product $\mathbb{Z}_2^{(\infty)}$ there are a von Neumann algebra $P$ of type $\text{II}_\infty$ and a representation $\alpha: G \to \text{Aut} P$ such that

$$M = P \times_{\alpha} G.$$ 

$G$ is the union of an increasing sequence of finite groups $(G_n)_{n \in \mathbb{N}}$ with $G_n \cong \mathbb{Z}_2^{(n)}$.

Let $\pi(x), x \in P$, and $\lambda(g), g \in G$, be the generators of $M$. Then

$$M_n = \{ \pi(P) \cup \lambda(G_n) \}, \quad n \in \mathbb{N},$$

form an increasing sequence of subalgebras of $M$ such that $M = (\bigcup_n M_n)$. Since $G_n$ is finite, $M_n$ is of type $\text{II}_\infty$. Moreover, $M_n \cong P \times_{\alpha} G_n, n \in \mathbb{N}$. Let $G_n^+$ denote the orthogonal group of $G_n$ in $\hat{G}$. Now $\hat{G} \cong \mathbb{Z}_2^{N}$, and $G_n^+$ is of finite index in $\hat{G}$. Let $\theta_n$ be the dual action of $\hat{G}$ on $M$, and let $\mu$ (resp. $\mu_n$) be the normalized Haar measure on $\hat{G}$ (resp. $G_n^+$). Define

$$E_n(x) = \int_{G_n^+} \theta_n(x) \, d\mu_n, \quad x \in M.$$
Then $E_n$ is the conditional expectation of $M$ onto the fixed point algebra of the action $\theta |_{G_n^1}$. Since $\theta \gamma \lambda (g) = \lambda (g)$ for all $g \in G_n$ if and only if $\gamma \in G_n^1$, and $\theta \gamma \pi (x) = \pi (x)$, for $x \in M$, $E_n$ is a faithful normal conditional expectation of $M$ onto $M_n$. By construction $E_m E_n = E_m$ if $m \leq n$.

Since $G_n^1$ has a finite index in $\hat{G}$ the Radon-Nikodym derivative $f_n = d\mu_n / d\mu \in L^1 (\hat{G}, \mu)$, and since $G_n^1 \supset G_n^2 \supset \cdots$ with $\bigcap_{n=1}^{\infty} G_n^1 = \{ e \}$ the sequence $(f_n)_{n \in \mathbb{N}}$ is an approximate unit for $L^1 (\hat{G}, \mu)$. But then

$$\| \phi \cdot E_n - \phi \| \to 0, \quad n \to \infty, \quad \text{for } \phi \in M_*.$$  

Q.E.D.

**Proposition 8.4.** Let $M$ be a factor of type $\text{III}_0$ with separable predual. Then the Main Theorem holds for $M$.

**Proof.** By Corollary 4.8 the Main Theorem holds for factors of type $\text{II}_\infty$ with separable preduals. By [12, 14.2.3] each von Neumann algebra of type $\text{II}_\infty$ with separable predual is the direct integral of factors of type $\text{II}_\infty$; hence the Main Theorem holds for such von Neumann algebras by Proposition 8.2. Finally, with $M$ as the proposition the Main Theorem holds for $M$ by Theorem 7.1 and Proposition 8.3.

**Proposition 8.5.** Let $M$ be a von Neumann algebra with separable predual. Then the Main Theorem holds for $M$.

**Proof.** We may consider the different types separately (see, e.g., the proof of Proposition 8.2). By Proposition 8.4 the Main Theorem holds for factors of type $\text{III}_0$, by Proposition 5.7 for factors of type $\text{III}_1$, $0 < \lambda < 1$, and by [4] for factors of type $\text{III}_1$, since in the latter case $N$ is a factor by [21, Cor. 9.7]. Thus by Proposition 8.2 and [12, 14.2.3] the Main Theorem holds for $M$ of type $\text{III}$. As in the proof of Proposition 8.4 the Main Theorem holds for $M$ of type $\text{II}$, and the isometry part holds for $M$ of type $I$ by Theorem 4.7 and Proposition 8.2. Q.E.D.

### 9. Proof of the Main Theorem in the General Case

In this section we complete the proof of the Main Theorem. We first prove it for $\sigma$-finite (= countably decomposable) von Neumann algebras and then extend it to general algebras. For the $\sigma$-finite case we shall use the martingale theorem, 7.1, and for this we need to construct an increasing net of algebras of the same type as our $M$.

**Proposition 9.1.** Let $M$ be a $\sigma$-finite von Neumann algebra of type $I$. Then there exist a countably generated von Neumann subalgebra $N \subset M$ and a faithful normal conditional expectation, $E_N : M \to N$, of $M$ onto $N$, such
that if $P$ is any von Neumann subalgebra of $M$ with $N \subset P \subset M$ and with a normal conditional expectation $E_P: M \to P$, such that $E_N = E_N E_P$, then $P$ is of type II$_1$ (resp. II$_\infty$) if $M$ is of type II$_1$ (resp. II$_\infty$).

Proof. We first assume $M$ is of type II$_1$. As for factors [6, Chap. III, Sect. 7, Thm. 1] $M$ contains a copy $N$ of the hyperfinite II$_1$-factor. If $e$ is a projection in $N$ then its central support in $M$ is 1. Since $M$ is of type II$_1$, there is a faithful normal conditional expectation $E_N$ of $M$ onto $N$. Let $P$ be a von Neumann subalgebra of $M$ containing $N$, and suppose there is a normal conditional expectation $E_P: M \to P$ such that $E_N E_P = E_N$. By a result of Tomiyama [17, Prop. 10.21] $P$ is semifinite. Since $E_N |_P$ is a conditional expectation of $P$ onto $N$ and $N$ is of type II, $P$ is of type II by [17, Prop. 10.21]. Since $M$ is finite, so is $P$.

To prove the proposition in the II$_\infty$-case, write $M = M_0 \otimes B(H)$, where $M_0$ is of type II$_1$ and $H$ is a separable Hilbert space, and choose $N_0 \subset M_0$ such that the conclusions of the proposition hold for the pair $(M_0, N_0)$. Let $N = N_0 \otimes B(H)$ and $E_N = E_{N_0} \otimes 1$. The same proof as in the II$_1$-case now applies to $M$ and $N$. Q.E.D.

PROPOSITION 9.2. Let $M$ be a $\sigma$-finite von Neumann algebra of type III. Then there exists a countably generated von Neumann subalgebra $N \subset M$ such that any von Neumann subalgebra $P$ with $N \subset P \subset M$ is of type III.

The proof of this proposition is divided into some lemmas. Recall that $U(M)$ denotes the unitary group in $M$ and $S_n(M)$ the set of normal states of $M$.

LEMMA 9.3. Let $M$ be a $\sigma$-finite von Neumann algebra of type III. If $\phi, \psi \in S_n(M)$ coincide on the center $Z(M)$ of $M$ then

$$\psi \in \text{conv}\{u\phi u^*: u \in U(M)\}^-, \text{ (norm closure).}$$

Proof. Recall from [12, 6.3.4] that in a $\sigma$-finite von Neumann algebra of type III two projections are equivalent if and only if they have the same central support. Let $\phi, \psi \in S_n(M)$, and assume that

$$\psi \notin \text{conv}\{u\phi u^*: u \in U(M)\}^-.$$

By the Hahn–Banach theorem there exists a self-adjoint operator $a \in M$ such that

$$\psi(a) > \sup\{\phi(u^*au): u \in U(M)\}.$$
Put \( \varepsilon = \psi(a) - \sup \{ \phi(u^*au) : u \in U(M) \} > 0 \). By spectral theory we can choose an operator \( b \) of the form

\[
b = \sum_{i=1}^{n} \lambda_i p_i, \quad \lambda_1 > \lambda_2 \cdots > \lambda_n,
\]

where \( p_1, \ldots, p_n \) are orthogonal projections in \( M \) with sum 1, such that \( \|a - b\| < \varepsilon/2 \). Since \( \varphi \) and \( \psi \) are states

\[
\psi(b) - \sup \{ \varphi(u^*bu) : u \in U(M) \} > \varepsilon - \|a - b\|,
\]

whence

(2) \( \psi(b) > \sup \{ \varphi(u^*bu) : u \in U(M) \} \).

Put \( q_i = p_i + \cdots + p_1, \ i = 1, \ldots, n \). Then

\[
b = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) q_i + \lambda_n 1.
\]

Put

\[
c = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) c(q_i) + \lambda_n 1,
\]

where \( c(p) \) is the central support of a projection \( p \). Then \( c \in Z(M) \), and \( c \geq b \). The operators

\[
c_1 = c(q_1), \quad c_i = c(q_i) - c(q_{i-1}), \quad i = 2, \ldots, n,
\]

are orthogonal central projections with sum 1, and we have

\[
c = \sum_{i=1}^{n} \lambda_i c_i.
\]

Since \( c_i = (c(p_1) \vee \cdots \vee c(p_i)) - (c(p_1) \vee \cdots \vee c(p_{i-1})) \) we have \( c(p_i) \geq c_i \), \( i = 1, \ldots, n \), and therefore \( c(p_i c_i) = c_i \). This implies that

\[
p_i c_i \sim c_i, \quad i = 1, \ldots, n.
\]

Choose partial isometries \( v_i \in M \) such that \( v_i^* v_i = c_i \), \( v_i v_i^* = c_i \). Then \( v = \sum_{i=1}^{n} v_i \) is an isometry, and \( v v^* = \sum_{i=1}^{n} c_i \). Therefore

\[
v^* b v = v^* \left( \sum_{i=1}^{n} c_i p_i \right) v = v^* \left( \sum_{i=1}^{n} \lambda_i c_i p_i \right) v
\]

\[
= \sum_{i=1}^{n} \lambda_i v^* c_i p_i v = \sum_{i=1}^{n} \lambda_i c_i = c.
\]
Since any isometry in $M$ can be approximated in the strong operator topology by unitaries in $M$,

$$c \in \{u^*bu: u \in U(M)\}^- \quad (\sigma\text{-weak closure}).$$

Since also $c \geq b$,

$$\psi(c) \geq \psi(b) > \sup\{\varphi(u^*bu): u \in U(M)\} \geq \varphi(c);$$

hence $\varphi$ and $\psi$ do not coincide on $Z(M)$.

Q.E.D.

**Lemma 9.4.** Let $M$ be a $\sigma$-finite von Neumann algebra of type III. Then for all $\varphi \in S_n(M)$, $\varphi$ belongs to the set

$$S_\varphi = \operatorname{conv}\{\frac{1}{2}(v\varphi v^* + w\varphi w^*): v, w \in M, v^*v = w^*w = 1, vv^* + ww^* = 1\}^-.$$ 

**Proof.** Let $\varphi \in S_n(M)$. Since $M$ is properly infinite, we can choose two isometries $v, w \in M$ such that $vv^* + ww^* = 1$. Put

$$\chi = \frac{1}{2}(v\varphi v^* + w\varphi w^*) \in S_n(M).$$

It is clear that $\varphi$ and $\chi$ coincide on $Z(M)$, so by Lemma 9.3

$$\varphi \in \operatorname{conv}\{u\chi u^*: u \in U(M)\}^-.$$ 

But for $u \in U(M)$

$$u\chi u^* = \frac{1}{2}(v_1\varphi v_1^* + w_1\varphi w_1^*),$$

where $v_1 = uv$, $w_1 = uw$. Clearly $v_1^*v_1 = w_1^*w_1 = 1$, and $v_1v_1^* + w_1w_1^* = 1$.

Q.E.D.

**Lemma 9.5.** Let $M$ be a $\sigma$-finite and semifinite von Neumann algebra. Let $\varphi \in S_n(M)$ and $S_\varphi$ as in Lemma 9.4. Then $\varphi \notin S_\varphi$.

**Proof.** We can assume that $M$ is properly infinite; otherwise $S_\varphi$ is empty. Let $\tau$ be a faithful normal semifinite trace on $M$ and put $h = d\varphi/d\tau \in L^1(M, \tau)$. Assume first that $h$ is bounded. If

$$\psi = \frac{1}{2}(v\varphi v^* + w\varphi w^*),$$

where $v, w \in M$, $v^*v = w^*w = 1, vv^* + ww^* = 1$, then

$$\frac{d\psi}{d\tau} = \frac{1}{2} (vhv^* + whw^*).$$
Since \( vv^* \perp ww^* \) it follows that \( \|d\psi/dt\| = \frac{1}{2} \|h\| \). Hence for every \( \chi \in S_\phi \)

\[
\left\| \frac{d\chi}{dt} \right\| \leq \frac{1}{2} \|h\| = \frac{1}{2} \left\| \frac{d\phi}{dt} \right\|.
\]

Thus \( \phi \) cannot belong to \( S_\phi \), proving the lemma in case \( h \) is bounded.

Assume now \( h \) is unbounded. Put

\[
f(\lambda) = \inf \left\{ \|\phi - \omega\| : \omega \in M^+_\chi, \frac{d\omega}{dt} \leq \lambda 1 \right\}.
\]

Since \( x \rightarrow \|x\| = \tau(|x|) \) is \( \sigma \)-weakly lower continuous on \( M \), a simple compactness argument shows that for \( \lambda > 0 \) there exists \( \omega \in M^+_\chi \) such that

\[
f(\lambda) = \|\phi - \omega\|, \quad \frac{d\omega}{dt} \leq \lambda 1.
\]

Hence \( f(\lambda) > 0 \) for all \( \lambda > 0 \). It is also clear that \( f(\lambda) \) is a decreasing function on \( \mathbb{R}^+ \) and \( f(\lambda) \to 0 \) as \( \lambda \to \infty \). Hence we can choose \( \lambda_0 > 0 \) such that

\[
f(\lambda_0/2) > f(\lambda_0).
\]

Choose \( \omega \in M^+_\chi \) such that

\[
f(\lambda_0) = \|\phi - \omega\|, \quad \frac{d\omega}{dt} \leq \lambda_0 1.
\]

Let \( v, w \in M, v^*v = w^*w = 1, vv^* + ww^* = 1 \), and put

\[
\psi = \frac{1}{2}(v\phi v^* + w\phi w^*),
\]

\[
\omega' = \frac{1}{2}(v\omega v^* + w\omega w^*).
\]

Clearly \( \|\psi - \omega'\| \leq \|\phi - \omega\| = f(\lambda_0) \). Moreover,

\[
\frac{d\omega'}{dt} = \frac{1}{2} \left( v \frac{d\omega}{dt} v^* + w \frac{d\omega}{dt} w^* \right).
\]

Since \( vv^* \perp ww^* \) we have

\[
\left\| \frac{d\omega'}{dt} \right\| \leq \frac{1}{2} \left\| \frac{d\omega}{dt} \right\| \leq \frac{\lambda_0}{2} 1.
\]

If \( \chi \) belongs to the convex set

\[
S^0_\phi = \text{conv}\{ \frac{1}{2}(v\phi v^* + w\phi w^*): v, w \in M, v^*v = w^*w = 1, vv^* + ww^* = 1 \},
\]
then by letting $\omega''$ be the corresponding convex combination of the $(\omega')$'s we see that

$$\|\chi - \omega''\| \leq f(\lambda_0) \quad \text{and} \quad \frac{d\omega''}{dt} \leq \frac{\lambda_0}{2} 1.$$ 

If $\varphi \in S_\varphi$ then we can choose $\chi_n \in S_\varphi^0$ such that $\|\varphi - \chi_n\| \to 0$ and $\omega_n \in M_\varphi^+$ such that

$$\|\chi_n - \omega_n\| \leq f(\lambda_0), \quad \frac{d\omega_n}{dt} \leq \frac{\lambda_0}{2} 1.$$ 

This shows

$$\inf \left\{ \|\varphi - \omega\| : \omega \in M_\varphi^+, \frac{d\omega}{dt} \leq \frac{\lambda_0}{2} 1 \right\} \leq f(\lambda_0),$$

which contradicts the choice of $\lambda_0$ that $f(\lambda_0) < f(\lambda_0^2)$; hence $\varphi \notin S_\varphi$. Q.E.D.

**Corollary 9.6.** Let $M$ be a von Neumann algebra with a faithful normal state $\varphi$. Then $M$ is of type III if and only if $\varphi$ belongs to the convex set

$$\text{conv}\left\{ \frac{1}{2}(v\varphi v^* + w\varphi w^*) : v, w \in M, v^*v = w^*w = 1, vv^* + ww^* = 1 \right\}^{-}.$$

**Proof.** By assumption $M$ is $\sigma$-finite. If $M$ is of type III the conclusion follows from Lemma 9.4. If $M$ is not of type III, let $e$ be the largest central projection in $M$ such that $eM$ is semilinite. Put $\varphi' = \|e\varphi\|^{-1} e\varphi$. If $\varphi \in S_\varphi$ then $\varphi' \in S_\varphi^0$, which is impossible by Lemma 9.5. Q.E.D.

**Proposition 9.7.** Let $M$ be a $\sigma$-finite von Neumann algebra of type III, and let $\varphi$ be a faithful normal state on $M$. By Lemma 9.4 we can choose sequences $(v_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ of isometries in $M$ such that $v_nv_n^* + w_nw_n^* = 1$ and such that

$$\varphi \in \text{conv}\left\{ \frac{1}{2}(v_n\varphi v_n^* + w_n\varphi w_n^*) : n \in \mathbb{N} \right\}^{-}.$$ 

Let $N$ be the von Neumann subalgebra of $M$ generated by the $v_n$'s and the $w_n$'s. If $P$ is a von Neumann algebra with $N \subset P \subset M$ put $\varphi' = \varphi|_P$. Then $\varphi'$ is a faithful normal state on $P$, and

$$\varphi' \in \text{conv}\left\{ \frac{1}{2}(v_n\varphi' v_n^* + w\varphi' w_n^*) : n \in \mathbb{N} \right\}^{-}.$$ 

Hence by Corollary 9.6, $P$ is of type III. Q.E.D.

**Proposition 9.7.** Let $M$ be a $\sigma$-finite von Neumann algebra. Then the Main Theorem holds for $M$. 

---

**EQUIVALENCE OF NORMAL STATES**

---
Proof. Let $\omega$ be a faithful normal state on $M$. Note that if $N \subset M$ is a countably generated von Neumann algebra then the $\sigma^\omega$-invariant von Neumann algebra

$$
\left( \bigcup_{i \in \mathbb{R}} \sigma_i^\omega(N) \right)'' = \left( \bigcup_{i \in \mathbb{Q}} \sigma_i^\omega(N) \right)''
$$

is also countably generated. We can thus write $M$ as an increasing union $M = \left( \bigcup_{x \in A} M_x \right)^{-}$, where $A$ is an index set such that $x \leq y$ implies $M_x \subset M_y$, and each $M_x$ is countably generated and $\sigma^\omega$-invariant. By [20] or [17, Thm. 10.1] there are faithful normal conditional expectations $E_x: M \to M_x$ of $M$ onto $M_x$ for all $x \in A$ such that $x \leq y$ implies $E_x E_y = E_y$. Thus the first part of the Main Theorem holds for $M$ by Theorem 7.1 and Proposition 8.5.

To show the second part we may assume $M$ is of type $\text{II}_1$, $\text{II}_\infty$, or $\text{III}$. If $M$ is of type $\text{III}$ we can by Proposition 9.2 choose $x_0 \in A$ such that $x_0 \leq x$ implies $M_x$ is of type $\text{III}$. If $M$ is of type $\text{II}_1$ choose $N$ as in Proposition 9.1 and let $w$ be a faithful normal trace on $N$. Then $N$ is $\sigma^\omega$-invariant and Proposition 9.1 is applicable, so we can assume all $M_x$ are of type $\text{II}_1$. If $M$ is of type $\text{II}_\infty$ then we choose $N$ in Proposition 9.1 to be $N_0 \otimes B(H)$ where $M = M_0 \otimes B(H)$, $N_0$ being isomorphic to the hyperfinite $\text{II}_1$-factor. Let $\tau_0$ be a faithful normal trace on $M_0$ and $\varphi$ a faithful normal state on $B(H)$. If $\omega = \tau_0 \otimes \varphi$ then $\omega$ is a faithful normal state on $M$ such that $N$ is $\sigma^\omega$-invariant. Thus Proposition 9.1 shows that we can choose all the $M_x$ of type $\text{II}_\infty$. The conclusion of the proposition now follows from Theorem 7.1 and Proposition 8.5. Q.E.D.

We next proceed to prove the Main Theorem for a general von Neumann algebra. The proof will be a reduction to the $\sigma$-finite case. The first two results describe the basic techniques for this reduction.

**Lemma 9.8.** Let $P$ be a von Neumann algebra with a faithful normal semifinite weight $w$. Let $H$ be a Hilbert space and $e$ a finite nonzero projection in $B(H)$. Let $\tau_0$ be the normal trace on $B(H)$ such that $\tau_0(e) = 1$, and let $\chi$ be the normal state $\chi(x) = \tau_0(ex)$ on $B(H)$. If $M = P \otimes B(H)$ then there is a natural isomorphism

$$
\gamma: Z(M \times_{\sigma^\omega \otimes \tau_0} \mathbb{R}) \to Z(P \times_{\sigma^\omega} \mathbb{R})
$$

with adjoint map $\gamma^*$ such that if $\varphi \in P^+$ then

$$
\gamma^*(\varphi) = (\varphi \otimes \chi)^{\wedge}.
$$

**Proof.** Since $\sigma^{\tau_0} = 1$ it is immediate that

$$
M \times_{\sigma^\omega \otimes \tau_0} \mathbb{R} \cong (P \times_{\sigma^\omega} \mathbb{R}) \otimes B(H).
$$
Thus there is a natural isomorphism
\[ \gamma: Z(M \times_{\sigma^00} \mathbb{R}) \to Z(P \times_{\sigma^00} \mathbb{R}) \]
given by \( \gamma(x \otimes 1) = x \) with adjoint map \( \gamma^* \), whose restriction map is a bijection
\[ \gamma^*: Z(P \times_{\sigma^00} \mathbb{R})^+ \to Z(M \times_{\sigma^00} \mathbb{R})^+ \].

Now for each \( \zeta \in \mathbb{R} \) the dual automorphism \( \theta_\zeta \) on \( M \times_{\sigma^00} \mathbb{R} \) is the tensor product \( \theta_\zeta \otimes 1 \), where \( \theta_\zeta \) is the dual automorphism on \( P \times_{\sigma^00} \mathbb{R} \). Thus if \( \tau \) is the canonical trace on \( P \times_{\sigma^00} \mathbb{R} \) then \( \tau \otimes \tau_0 \) is the canonical trace on \( M \times_{\sigma^00} \mathbb{R} \). In particular, if \( \varphi \in P_+ \) and \( e_\varphi \) is the projection in \( P \times_{\sigma^00} \mathbb{R} \) used in the definition of \( \bar{\phi} \) (see Definition 3.2), then by definition of the state \( \chi \), \( e_\varphi \otimes \chi = e_\varphi \otimes e \). It follows that \( (\varphi \otimes \chi)^\wedge = \gamma^*(\bar{\phi}) \).

**PROPOSITION 9.9.** Let \( P \) be a \( \sigma \)-finite properly infinite von Neumann algebra and \( H \) a Hilbert space. Let \( M = P \otimes B(H) \). Then there is an isometry \( \Pi \) of \( M_+^+ / \sim \) onto \( P_+^+ / \sim \) and an isometry \( \delta \) of \( \{ \hat{\psi}: \psi \in M_+^+ \} \) onto \( \{ \hat{\phi}: \varphi \in P_+^+ \} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
M_+^+ / \sim & \xrightarrow{\Pi} & P_+^+ / \sim \\
\downarrow & & \downarrow \\
\hat{\psi}: \{ \psi \in M_+^+ \} & \xrightarrow{\delta} & \{ \hat{\phi}: \varphi \in P_+^+ \}
\end{array}
\]

where the vertical maps are the canonical ones.

**Proof.** Let \( e \) be a 1-dimensional projection in \( B(H) \) and \( \text{Tr} \) the usual trace on \( B(H) \), so \( \text{Tr}(e) = 1 \). By Proposition 2.6, since \( P \) is properly infinite, there is a natural isometric bijection
\[ \Pi: M_+^+ / \sim \to P_+^+ / \sim \]
given by \( \Pi([\omega]) = [v \omega v^*]_p \), where \( p = 1 \otimes e \) and \( v \) is a partial isometry from \( \text{supp}(\omega) \) into \( p \). Furthermore, by Lemma 9.8, if \( \omega_0 \in \Pi([\omega]) \) and \( \chi \) denotes the state \( \chi(x) = \text{Tr}(ex) \) on \( B(H) \) then
\[ \hat{\omega} = (\omega_0 \otimes \chi)^\wedge = \gamma^*(\hat{\omega}_0) \].

If we let \( \delta \) be the inverse of \( \gamma^* \) restricted to the set \( \{ \hat{\psi}: \psi \in M_+^+ \} \) then \( \delta \) is the desired isometry onto the set \( \{ \hat{\phi}: \varphi \in P_+^+ \} \) making the diagram commutative. Q.E.D.

**LEMMA 9.10.** Let \( M = \sum_i \otimes_{\sigma^0_i} M_i \) be a direct sum of von Neumann algebras. If the Main Theorem holds for each \( M_i, \ i \in I \), then it holds for \( M \).
Proof. Let $\varphi, \psi \in M_*^+$. Then

$$\varphi = \sum_{i \in I} \varphi_i, \quad \psi = \sum_{i \in I} \psi_i, \quad \varphi_i, \psi_i \in (M_i)_*^+,$$

where only a countable number of the summands are nonzero. If we index them by $i \in \mathbb{N}$ and let $\varepsilon > 0$, we can for each $i$ and $\varphi'_i, \psi'_i \in (M_i)_*^+$, unitarily equivalent to $\varphi_i$ and $\psi_i$, respectively, such that

$$\|\varphi'_i - \psi'_i\| < d([\varphi], [\psi]) + 2^{-i\varepsilon}.$$ 

Let $\varphi' = \sum_{i \in \mathbb{N}} \varphi'_i, \psi' = \sum_{i \in \mathbb{N}} \psi'_i$. Then $\varphi'$ and $\psi'$ are unitarily equivalent to $\varphi$ and $\psi$, respectively, and

$$d([\varphi], [\psi]) \leq \sum_{i} \|\varphi'_i - \psi'_i\|$$

$$< \sum_{i} (d([\varphi], [\psi])) + 2^{-i\varepsilon}$$

$$= \sum_{i} \|\varphi'_i - \psi'_i\| + \varepsilon.$$

By arbitrariness of $\varepsilon$,

$$d([\varphi], [\psi]) \leq \sum_{i \in \mathbb{N}} \|\varphi'_i - \psi'_i\|.$$

Now the crossed product $N = M \times_{\sigma_\omega} \mathbb{R}$ splits into a direct sum

$$N = \sum_{i \in I} N_i = \sum_{i \in I} M_i \times_{\sigma_\omega} \mathbb{R},$$

where $\omega_i = \omega|_{N_i}$ ($\omega$ being a faithful normal semifinite weight on $M$). One checks easily that

$$\hat{\varphi} = \sum_{i} \varphi_i, \quad \hat{\psi} = \sum_{i} \psi_i,$$

and so by (3),

$$\|\hat{\varphi} - \hat{\psi}\| = \sum \|\varphi_i - \psi_i\| \geq d([\varphi], [\psi]).$$

Since the opposite inequality follows from Lemma 3.3 the first part of the Main Theorem is proved.

To prove the second part one notices that every $\chi \in Z(N)_*^+$ splits into a direct sum $\chi = \sum_{i \in I} \chi_i$, $\chi_i \in Z(N_i)_*^+$. Since the dual automorphism $\theta_i$ also splits into a direct sum $\sum_{i \in I} \theta^i_s$ of dual automorphisms of $N_i$ it is clear that if $\chi \circ \theta_i \geq e^{-s}\chi$, $s > 0$, then $\chi_i \circ \theta^i_s \geq e^{-s}\chi_i$. The proof of the surjectivity is...
therefore obvious. Since the canonical trace on \( M \) is the direct sum of the canonical traces on the \( M_i \), part (iii) on the \( \text{II}_1 \)-case is also easily proved. Q.E.D.

*End of Proof of the Main Theorem.* Assume first that \( M \) is a finite von Neumann algebra. Then \( M = \sum \oplus M_i \), where \( M_i \) is \( \sigma \)-finite and finite (and of type \( \text{II}_1 \) if \( M \) is). Thus the Main Theorem holds for \( M \) by Proposition 9.7 and Lemma 9.10.

Suppose next \( M \) is properly infinite. In this case

\[
M \cong \sum \oplus P_i \otimes B(H_i),
\]

where each \( P_i \) is \( \sigma \)-finite and either of type \( \text{I}_\infty \), \( \text{II}_\infty \), or \( \text{III} \), and \( (H_i)_{i \in I} \) is a family of Hilbert spaces. By Lemma 9.10 it is enough to prove the theorem for each of the summands \( P_i \otimes B(H_i) \). Since the Main Theorem holds for each \( P_i \) by Proposition 9.7 it holds for \( P_i \otimes B(H_i) \) by Proposition 9.9. This completes the proof of the Main Theorem. Q.E.D.

10. *The Quotient Space \( S_n(M)/\sim \)*

Let \( M \) be a von Neumann algebra, and let \( S_n(M) \) be the set of normal states of \( M \). By Lemma 2.1, the space

\[
S_n(M)/\sim = \{ [\varphi] \in M^*_+ / \sim : \varphi \in S_n(M) \}
\]

is a complete metric space with diameter bounded by 2. Note that if \( M \) is not a factor, then \( \text{diam}(S_n(M)/\sim) = 2 \), because \( d([\varphi], [\psi]) = 2 \) when \( \varphi \) and \( \psi \) are centrally disjoint. In [3] Connes and the present authors computed the diameter of \( S_n(M)/\sim \) for all factors \( M \). In the semifinite case it was easily computed (see the Introduction of [3]). In this section we will give a new proof of the diameter formula for factors of type \( \text{III} \) based on the Main Theorem. From the Main Theorem and Proposition 6.6 we have:

**Proposition 10.1.** If \( M \) is a properly infinite von Neumann algebra with no type I direct summand, then the map \([\varphi] \mapsto \phi\) is an isometry of \( S_n(M)/\sim \) onto

\[
P_1(M) = \{ \chi \in S_n(Z(N)) : \chi \circ \theta_s \geq e^{-s} \chi, s > 0 \}.
\]

Moreover, \( P_1(M) \) is a norm closed (in general noncompact) Choquet simplex.
Lemma 10.2. Let $A$ be a unital abelian $C^*$-algebra, and let $(\theta_s)_{s \in \mathbb{R}}$ be a one-parameter group of automorphisms of $A$, such that $s \mapsto \theta_s(f)$ is norm-continuous for all $f \in A$. Let $(\alpha_s)_{s \in \mathbb{R}}$ be the corresponding flow on $\hat{A}$, i.e.,

$$\alpha_s(\delta) = \delta \circ \theta_{-s}, \quad \delta \in \hat{A}.$$ 

Put

$$P_1 = \{\chi \in S(A) : \chi \circ \theta_s \geq e^{-s}\chi \text{ for } s > 0\}.$$ 

Then only the following three cases can occur:

(i) $\hat{A}$ has more than one $\alpha$-orbit.

(ii) $A \cong C(\mathbb{R}/t_0\mathbb{Z})$ for some $t_0 > 0$ and $\alpha_s$ corresponds to translation by $s$ on $\mathbb{R}/t_0\mathbb{Z}$.

(iii) $A = \mathbb{C}$.

In case (i) $\text{diam}(P_1) = 2$. In case (ii) $\text{diam}(P_1) = 2 \cdot ((1 - \exp(-t_0/2))/(1 + \exp(-t_0/2)))$. In case (iii) $\text{diam}(P_1) = 0$.

Proof. If we exclude case (i) and case (iii), then $\hat{A}$ has more than one point, but only one $\alpha$-orbit. Choose $\delta_0 \in \hat{A}$. Then

$$\beta : s \mapsto \alpha_s(\delta_0)$$

is a continuous map of $\mathbb{R}$ onto $\hat{A}$ and

$$H = \{s \in \mathbb{R} \mid \alpha_s(\delta_0) = \delta_0\}$$

is a closed subgroup of $\mathbb{R}$. Clearly $H \neq \mathbb{R}$. Assume $H = \{0\}$. Then $\beta$ is one-to-one. Moreover, $\beta$ maps Borel sets of $\mathbb{R}$ onto Borel sets of $\hat{A}$, because $\beta([s_1, s_2])$ is compact for all $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$. Since $\beta$ is continuous it follows that it is a Borel isomorphism. But since $\mathbb{R}$ is amenable, $\hat{A}$ has an $\alpha$-invariant probability measure, while $\mathbb{R}$ has no translation invariant probability measure. This gives a contradiction, i.e., $H \neq \{0\}$. Therefore

$$H = t_0\mathbb{Z}$$

for some $t_0 > 0$. Then $\beta$ can be factorized through $\mathbb{R}/t_0\mathbb{Z}$, and by compactness of $\mathbb{R}/t_0\mathbb{Z}$ the corresponding map $\hat{\beta} : \mathbb{R}/t_0\mathbb{Z} \to \hat{A}$ is a homomorphism. Hence we are in case (ii). This proves the first part of the lemma. By Lemma 6.4

$$P_1 = \left\{\chi_\omega = \int_{-\infty}^{\omega} e^{s\omega} \circ \theta_s \, ds : \omega \in S(A)\right\}.$$
Since \( \hat{A} \) is the extreme point of \( S(\mathcal{A}) \), and since the norm on \( A^* \) is \( w^* \)-lower semicontinuous it follows that

\[
\text{diam}(P_1) = \sup \{ \| \chi_{\delta_1} - \chi_{\delta_2} \| : \delta_1, \delta_2 \in \hat{A} \}.
\]

If \( \delta_1, \delta_2 \) are not contained in the same orbit, the measures on \( \hat{A} \) corresponding to \( \chi_{\delta_1} \) and \( \chi_{\delta_2} \) are concentrated on two disjoint orbits. Hence \( \| \chi_{\delta_1} - \chi_{\delta_2} \| = 2 \), proving that \( \text{diam}(P_1) = 2 \) in case (i). Consider next case (ii) and let \( \delta_1, \delta_2 \in \hat{A}, \delta_1 \neq \delta_2 \). Then \( \delta_2 = \alpha_t(\delta_1) \) for a unique \( 0 < t < t_1 \). Moreover, we can assume that \( A = C(\mathbb{R}/t_0 \mathbb{Z}) \) with \( \alpha_s = \text{translation by } s \), and that \( \delta_1 \) and \( \delta_2 \) are the Dirac measures concentrated in \( 0 \) and \( i \), respectively. Here \( s \to \hat{s} \) denotes the quotient map \( \mathbb{R} \to \mathbb{R}/t_0 \mathbb{Z} \). By the periodicity of \( \alpha_s \):

\[
\chi_{\delta_1} = \int_0^\infty e^{-s\hat{s}} ds = \left( \sum_{n=0}^\infty e^{-n t_0} \right) \int_0^{t_0} e^{-s\hat{s}} ds,
\]

\[
\chi_{\delta_2} = \int_0^\infty e^{-s\hat{s}+i} ds = \sum_{n=0}^\infty e^{n t_0} \left( \int_0^{t_0} e^{-s\hat{s}+i} ds \right).
\]

Thus \( \chi_{\delta_1} \) and \( \chi_{\delta_2} \) have densities \( f_1 \) and \( f_2 \) with respect to the Lebesgue measure on \([0, t_0]\) given by

\[
f_1(s) = (1 - e^{-t})^{-1} e^{-s}, \quad 0 \leq s < t
\]

\[
f_2(s) = \begin{cases} (1 - e^{-t})^{-1} e^{-(s+t_0-t)}, & 0 \leq s < t \\ (1 - e^{-t})^{-1} e^{-(s-t)}, & t \leq s < t_0. \end{cases}
\]

Hence

\[
\| \chi_{\delta_1} - \chi_{\delta_2} \| = \| f_1 - f_2 \|_1
\]

\[
= 2 \frac{(1 - \exp(-t))(1 - \exp(t-t_0))}{1 - \exp(-t_0)}.
\]

This function has maximum at \( t = t_0/2 \), and the maximum value is

\[
2(1 - \exp(-t_0/2))(1 + \exp(-t_0/2))^{-1}.
\]

Case (iii) is trivial. Q.E.D.

**Lemma 10.3.** Let \( Z \) be an abelian von Neumann algebra, and let \( (\theta_s)_{s \in \mathbb{R}} \) be a pointwise \( \sigma \)-weakly continuous one-parameter group of automorphisms of \( Z \). Moreover, let \( A \subseteq Z \) be the abelian \( C^* \)-algebra of elements \( z \in Z \), for which \( s \to \theta_s(z) \) is norm-continuous. Then

\[
\text{diam}\{ \chi \in S_n(Z) : \chi \circ \theta_s \geq e^{-s} \chi, s > 0 \} = \text{diam}\{ \chi \in S(A) : \chi \circ \theta_s \geq e^{-s} \chi, s > 0 \}.
\]
Proof. Since $A$ is $\sigma$-weakly dense in $Z$, $Z_*$ is naturally contained in $A^*$, so the inequality $\leq$ is obvious. To show the opposite inequality, let $\chi_i \in S(A)$, $\chi_i \circ \theta_s \geq e^{-s}\chi_i$, $i = 1, 2$. Choose by Lemma 6.4 $\omega_i \in S(A)$, such that

$$\chi_i = \int_{-\infty}^{0} e^{s} \omega_i \circ \theta_s \, ds.$$

Since the set of restrictions from $S_n(Z)$ is $w^*$-dense in $S(A)$ we can choose nets $(\omega_{i,x})$ in $S(A)$ such that $\omega_{i,x}$ has natural extensions in $S_n(Z)$ and $\omega_{i,x} \to \omega_i$ in $w^*$-topology. Let

$$\chi_{i,x} = \int_{-\infty}^{0} e^{s} \omega_{i,x} \circ \theta_s \, ds.$$

As in the proof of Lemma 6.4

$$\chi_{i,x} \circ \theta_s \geq e^{-s}\chi_{i,x}, \quad s > 0,$$

and $\chi_{i,x} \to \chi_i$ in the $w^*$-topology of $A^*$. Since the norm is lower semicontinuous with respect to $w^*$-topology, we have

$$\|\chi_1 - \chi_2\| \leq \limsup_{\alpha} \|\chi_{1,\alpha} - \chi_{2,\alpha}\| \leq \text{diam}\{\chi \in S_n(Z): \chi \circ \theta_s \geq e^{-s}\chi, s > 0\}.$$

Thus the opposite inequality follows. Q.E.D.

**Proposition 10.4 [3].** If $M$ is a factor of type $\text{III}_\lambda$, $0 \leq \lambda \leq 1$, with separable predual, then

$$\text{diam}(S_n(M)/\sim) = 2 \frac{1 - \lambda}{1 + \lambda^{1/2}}.$$

**Proof.** Let $(Z(N), \theta_s)$ be as in the Main Theorem, and let $A \subseteq Z(N)$ be the set of $z \in Z(N)$ for which $s \mapsto \theta_s(z)$ is norm continuous. By Proposition 10.1 and Lemma 10.3

$$\text{diam}(S_n(M)/\sim) = \text{diam}\{\xi \in S(A): \xi \circ \theta_s \geq e^{-s}\xi, s > 0\}.$$

If $M$ is of type $\text{III}_0$, $(\theta_s)_{s \in R}$ is not periodic on $Z(N)$ (cf. [21, Thm. 9.6]). Therefore $\theta_s|_A$ cannot be periodic. Hence by Lemma 10.2, $A$ has more than one orbit with respect to $\alpha_s(\delta) = \delta \circ \theta_{-s}$, $\delta \in \hat{A}$, and

$$\text{diam}(S_n(M)/\sim) = 2.$$
If $M$ is of type III$_\lambda$, $0 < \lambda < 1$,

$$\langle Z(N), \theta_s \rangle \cong \langle (L^\infty(\mathbb{R}/t_0\mathbb{Z})), \tau_s \rangle,$$

where $t_0 = -\log \lambda$, and $\tau_s$ = translation by $s$ (cf. [21, Sects. 9-10]). Hence $A$ corresponds to the subalgebra $C(\mathbb{R}/t_0\mathbb{Z})$ of $L^\infty(\mathbb{R}/t_0\mathbb{Z})$, so by Lemma 10.2(ii)

$$\text{diam}(S_n(M)/\sim) = 2 \frac{1 - \exp(-t_0/2)}{1 + \exp(-t_0/2)} = 2 \frac{1 - \lambda^{1/2}}{1 + \lambda^{1/2}}.$$ 

Finally, if $M$ is of type III$_1$, $Z(N) = \mathbb{C}$, so $\text{diam}(S_n(M)/\sim) = 0$ by Lemma 10.3(iii). (Note that this does not give a new proof of the homogeneity of one state space of III$_1$-factors [4], because [4] was used in the proof of the Main Theorem.) Q.E.D.

The next proposition is an extension of [1, Thm. 3.4]

**Proposition 10.5.** If $M$ is a factor of III$_\lambda$, $0 < \lambda < 1$, then $P_1(M)$ is a compact Bauer simplex naturally isomorphic to the set of probability measures on $\mathbb{R}/t_0\mathbb{Z}$, where $t_0 = -\log \lambda$. In particular $S(M)/\sim$ is compact.

**Proof.** Let $Z(N)$, $\theta_s$, and $A$ be as in the proof of Proposition 10.4. We will make the identifications

$$\langle Z(N), \theta_s \rangle = \langle (L^\infty(\mathbb{R}/t_0\mathbb{Z})), \tau_s \rangle$$

$$A = C(\mathbb{R}/t_0\mathbb{Z}).$$

Note that $S(A)$ is the set of probability measures on $\mathbb{R}/t_0\mathbb{Z}$. Consider now the map

$$\omega \rightarrow \chi_\omega = \int_{-\infty}^{0} e^{s(\omega \circ \tau_s)} \, ds$$

of $S(A)$ into $S(A)$. The computations in the proof of Lemma 10.3 show that for all characters $\delta$ on $A$,

$$\chi_\delta \leq (1 - e^{-s})^{-1} \, ds,$$

where $ds$ is the Lebesgue measure on $[0, t_0]$. Since $S(A)$ is the $w^*$-closed linear span of the characters, we get

$$\chi_\omega \leq (1 - e^{-t_0})^{-1} \, ds$$

for all $\omega \in S(A)$. Hence each $\chi_\omega$ can be extended to a normal state (also denoted $\chi_\omega$) of $Z(N) = L^\infty(\mathbb{R}/t_0\mathbb{Z})$. Thus, by Lemma 6.5 $\omega \rightarrow \chi_\omega$ is a bijection of $S(A)$ onto $P_1(M)$. 
Let \( \delta_1, \delta_2 \) be two characters on \( A \), and let \( t \) be the distance between the two corresponding points in \( \mathbb{R}/t_0 \mathbb{Z} \) (\( 0 \leq t \leq \frac{1}{2} t_0 \)). Then either \( \delta_2 = \delta_1 \circ \theta_- \), or \( \delta_1 = \delta_2 \circ \theta_- \), so by the proof of Lemma 10.2
\[
\| \chi_{\delta_1} - \chi_{\delta_2} \| = 2 \frac{(1 - \exp(-t))(1 - \exp(t - t_0))}{1 - \exp(-t_0)} \leq 2 t.
\]
Hence \( \delta \to \chi_\delta \) is \( w^* \)-norm-continuous from \( \hat{A} \) to \( P_1(M) \). It now follows from the theory of vector valued integration that the map \( \omega \to \chi_\omega \) is \( w^* \)-norm-continuous from \( S(A) \) to \( P_1(M) \). By compactness of \( S(A) \) the map is a homomorphism. Q.E.D.

**Proposition 10.6.** If \( M \) is a factor of type III\(_0\), the simplex \( P_1(M) \) has no extreme points. Particularly \( P_1(M) \) and \( S(M)/\sim \) are not compact.

**Proof.** Let \((Z(N), \theta_\tau)\) and \( A \) be as in the proof of Lemma 10.4. Since \( M \) is a factor, \( \theta_\tau \) acts ergodically on \( Z(N) \). This implies that \( Z(N) \) has a faithful normal state \( \chi_0 \). Indeed, if \( \chi \) is any normal state on \( Z(N) \) and \((q_n)_{n=1}^{\infty}\) is an enumeration of the rational numbers, then
\[
\chi_0 = \sum_{n=1}^{\infty} 2^{-n} \chi \circ \theta_{q_n},
\]
is a normal state for which support projection is \( \theta_\tau \)-invariant, so by the ergodicity \( \chi_0 \) is faithful. Let \( \mu_0 \) be the measure of \( \hat{A} \) which corresponds to \( \chi_0 \). Then we can make the following identifications
\[
Z(N) = L^\infty(\hat{A}, d\mu_0) \quad A = C(\hat{A}).
\]
Let \( \alpha_s \cdot \delta = \delta \circ \theta_{\alpha_s} \), \( \delta \in \hat{A} \). Then \( \mu_0 \) is quasi-invariant under the flow \((\alpha_s)_{s \in \mathbb{R}}\), and
\[
\theta_{\alpha_s}(f)(\delta) = f(\alpha_{-s}(\delta)), \quad f \in L^\infty(\hat{A}, d\mu_0).
\]
For \( \omega \in S(A) \), let \( \chi_\omega \in S(A) \) be the state
\[
\int_{-\infty}^{0} e^{s(\omega \circ \theta_s)} \, ds
\]
and let \( Q_1 \) be the set of states \( \omega \in S(A) \) for which \( \chi_\omega \) can be extended to a normal state on \( Z(N) \). By Lemma 6.5 \( \omega \to \chi_\omega \) is a bijection of \( Q_1 \) onto \( P_1(M) \). Thus, if \( \psi \in S(Z(N)) \) is an extreme point of \( P_1(M) \), then
\[
\psi |_A = \chi_\delta
\]
for some extreme point $\delta$ of $Q_1$. If

$$\delta = (1 - \lambda) \delta_1 + \lambda \delta_2, \quad 0 < \lambda < 1, \quad \delta_1, \delta_2 \in S(A),$$

then $\delta_1$ and $\delta_2$ are dominated by scalar multiples of $\delta$, which implies that $\chi_{\delta_1}$ and $\chi_{\delta_2}$ are dominated by scalar multiples of $\chi_{\delta}$. Therefore $\delta_1, \delta_2 \in Q_1$. Hence $\delta$ is an extreme point of $S(A)$, i.e., $\delta \in \hat{A}$. Let $\mathcal{O} \subseteq \hat{A}$ be the orbit

$$\mathcal{O} = \{\alpha_\delta(\delta) \mid \delta \in \mathbb{R}\}.$$

Then $\mathcal{O}$ is a Borel set and $\chi_{\delta}$ corresponds to a probability measure $\nu$ concentrated on $\mathcal{O}$. Since $\chi_{\delta}$ is the restriction to $\hat{A}$ of a normal state on $Z(N)$, $\nu$ is absolutely continuous with respect to $\mu_0$. Hence $\mu_0(\mathcal{O}) > 0$, so by the ergodicity of $(\theta_s)_{s \in \mathbb{R}}$, $\mu_0(\hat{A} \setminus \mathcal{O}) = 0$. Hence $(Z(N), \theta_s)$ is given by a transitive flow, which contradicts that $M$ is of type III (cf. [21, Thm. 8.12 and Thm. 9.6]). Therefore $P_1(M)$ has no extreme points. In particular the simplex $P_1(M)$ is not compact. Q.E.D.

11. States with Centralizers of Type $\text{II}_1$

It was shown in [4] that if $M$ is a nontype I factor with separable predual then there is a faithful normal state $\phi$ on $M$. In the present section we show an improvement of this result. To avoid technicalities we state it for faithful states.

**Theorem 11.1.** Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of type I. Suppose $\phi$ is a faithful normal state on $\mathcal{M}$ Then there is a faithful normal state $\psi \in [\phi]$ whose centralizer is of type $\text{II}_1$.

**Proof.** Since $M$ has no direct summand of type I there is a von Neumann subalgebra $\mathcal{P}$ of $\mathcal{M}$ such that $M = \mathcal{P} \Join M_2(\mathbb{C})$. Let $\tau_0$ be the tracial state on $M_2(\mathbb{C})$. Let $\omega$ be a faithful normal semifinite weight on $\mathcal{P}$. Then by Lemma 9.8 there is an isometry

$$\gamma_* : Z\left(\mathcal{P} \times_{\phi^\omega} \mathbb{R}\right)_* \rightarrow Z\left(M \times_{\phi^\omega} \mathbb{R}\right)_*$$

such that $\gamma*(\rho) = (\rho \otimes \tau_0)^\wedge$ for $\rho \in P_+$. Let $\theta_s$ and $\theta_\psi$ be the dual automorphisms of $M \times_{\phi^\omega} \mathbb{R}$ and $\mathbb{P} \times_{\phi^\omega} \mathbb{R}$, respectively. Then $\theta_s = \theta_\psi \otimes 1$, so if $\chi \in Z\left(\mathcal{P} \times_{\phi^\omega} \mathbb{R}\right)^+$ then $\chi \circ \theta_\psi s > e^{-s} \chi$, $s > 0$, if and only if $\gamma^*(\chi) \circ \theta_s > e^{-s} \gamma^*(\chi)$, $s > 0$. Furthermore, if $\tau$ is the canonical trace on $\mathcal{P} \times_{\phi^\omega} \mathbb{R}$ then $\tau \otimes \tau_0$ is the canonical trace on $M \times_{\phi^\omega} \mathbb{R}$, and $\chi \leq \tau$ if and only if $\chi \otimes \tau_0 \leq \tau \otimes \tau_0$. Since we can consider the finite and properly infinite portions of $M$ separately, it follows from the Main Theorem and Proposition 9.9 that if $\phi$
is a faithful normal state on $M$ then there is $\varphi_0 \in S_n(P)$ such that $\gamma^*(\varphi_0) = \tilde{\phi}$. Since $\gamma^*(\varphi_0) = (\varphi_0 \otimes \tau_0)^{\wedge}$ we have

$$\tilde{\phi} = (\varphi_0 \otimes \tau_0)^{\wedge}.$$ 

Let $p = \text{supp}(\varphi_0) \in P$. Then $p \otimes 1 = \text{supp}(\varphi_0 \otimes \tau_0)$. Since $\varphi$ is faithful, $p \otimes 1 \sim 1$ in $M$ by Theorem 2.2. If $M$ is finite, $p \otimes 1 = 1$, and $p = 1$. If $M$ is properly infinite then $p \otimes 1$ is properly infinite, and hence so is $p$. Thus $p \sim 1$ in $P$. Choose an isometry $v \in P$ such that $vv^* = p$. Let

$$\psi_0 = (v^* \otimes 1)(\varphi_0 \otimes \tau_0)(v \otimes 1) = v^* \varphi_0 v \otimes \tau_0.$$ 

Then $\psi_0$ is faithful and belongs to $[\varphi_0 \otimes \tau_0]$ by Theorem 2.2. By the Main Theorem $[\varphi_0 \otimes \tau] = [\varphi]$, and hence $\psi_0 \in [\varphi]$. Furthermore, the centralizer $M_{\psi_0}$ of $\psi_0$ contains the $I_2$-factor $1 \otimes M_2(C)$. If $\varepsilon > 0$ is given we can multiply $v \otimes 1$ by a unitary operator in $M$ and assume $\|\psi_0 - \varphi\| < \varepsilon$; hence we have shown that there is a $I_2$-factor $K_1$ with normalized trace $\tau_1$ and a faithful normal state $\varphi_1 \in [\varphi]$ such that $\varphi_1 = (\varphi_1|_{K_1 \cap M}) \otimes \tau_1$ and

$$\|\varphi - (\varphi_1|_{K_1 \cap M}) \otimes \tau_1\| < \varepsilon.$$ 

Continuing this procedure we obtain a sequence $(\varphi_n, K_n)$, where $\varphi_n \in [\varphi]$ is faithful, and $K_1, K_2, \ldots$ are commuting $I_2$-subfactors of $M$ with $K_1, \ldots, K_n$ belonging to the centralizer $M_{\varphi_n}$ of $\varphi_n$. Moreover, we can assume $\|\varphi_n - \varphi_{n+1}\| < 2^{-n}$; hence the states converge in norm to a state $\psi_1 \in [\varphi]$. By construction the von Neumann algebra $K$ generated by the $K_n$ is contained in the centralizer $M_{\psi_1}$ of $\psi_1$. Since $\psi_1$ commutes with $K$ so does its support $q = \text{supp}(\psi_1)$. But $qK$ is isomorphic to the hyperfinite $\Pi_1$-factor, since $\psi_1$ is a faithful normal trace on $qK$, and $qK$ is generated by the commuting $I_2$-factors $qK_n$. Moreover $qK \subseteq qM_{\psi_1}$. Hence the finite von Neumann algebra $qM_{\psi_1}$ is of type $\Pi_1$.

Let $w$ be an isometry in $M$ such that $ww^* = q$, and let $\psi = w^*\psi_1 w$. Then $\psi$ is faithful, and by Theorem 2.2, $\psi \in [\varphi]$. It is straightforward to check that the centralizer of $\psi$ is equal to $w^* (qM_{\psi_1}) w$; hence it is itself of type $\Pi_1$.

Q.E.D.

12. EXTENSIONS OF AUTOMORPHISMS TO $M \times_{\varphi_0} \mathbb{R}$

Let $\omega$ be a fixed faithful normal semifinite weight on the von Neumann algebra $M$, and let $\tilde{\omega}$ be the dual weight on $N = M \times_{\varphi_0} \mathbb{R}$. Let $\tau$ be the canonical trace on $N$ (see Sect. 3). We denote by $\text{Aut}(M)$ the automorphism group of $M$, and similarly for $N$. Let $\pi$ be the canonical isomorphism $M$ into $N$. 
PROPOSITION 12.1. For every \( \alpha \in \text{Aut}(M) \) there is a unique automorphism \( \tilde{\alpha} \in \text{Aut}(N) \) such that

(i) \( \tilde{\alpha}(\pi(x)) = \pi(\alpha(x)), \quad \forall x \in M. \)

(ii) \( \tilde{\alpha}(\lambda(s)) = \pi((D\omega : \alpha^{-1} : D\omega)_s) \lambda(s), \quad \forall s \in \mathbb{R}. \)

Furthermore, the map \( \alpha \to \tilde{\alpha} \) is a homomorphism of \( \text{Aut}(M) \) into \( \text{Aut}(N) \).

Proof. Put \( \omega' = \omega \circ \alpha^{-1} \), and let \( (\pi(M), \lambda(\mathbb{R})) \) and \( (\pi'(M), \lambda(\mathbb{R})) \) be the generators of \( \mathcal{N} = M \times \sigma_\omega \mathbb{R} \) and \( M \times \sigma_\omega \mathbb{R} \), respectively. Explicitly

\[
\begin{align*}
(\pi(x)\xi)(t) &= \sigma^\omega_\alpha(x) \xi(t) \\
(\pi'(x)\xi)(t) &= \sigma'^\omega_\alpha(x) \xi(t) \\
(\lambda(s)\xi)(t) &= \xi(t-s)
\end{align*}
\]

Let \( \iota \) denote the identity map on \( B(L^2(\mathbb{R})) \). Since \( \sigma'^\omega_\alpha = \alpha \circ \sigma^\omega_\alpha \circ \alpha^{-1} \), we have

\[
(\alpha \otimes \iota)(\pi(x)) = \pi'(\alpha(x)), \quad \forall x \in M.
\]

\[
(\alpha \otimes \iota)(\lambda(s)) = \lambda(s), \quad \forall s \in \mathbb{R}.
\]

Hence the restriction \( \alpha' \) of \( \alpha \otimes \iota \) to \( M \times \sigma_\omega \mathbb{R} \) is an isomorphism of \( M \times \sigma_\omega \mathbb{R} \) onto \( M \times \sigma_\omega \mathbb{R} \). Now let \( \Phi \) be the isomorphism of \( M \times \sigma_\omega \mathbb{R} \) onto \( M \times \sigma_\omega \mathbb{R} \) constructed in [21, Prop. 3.5] using the cocycle Radon–Nikodym derivative \( u_s = (D\omega : D\omega')_s \). Then

\[
\Phi(\pi'(x)) = \pi(x), \quad \forall x \in M,
\]

\[
\Phi(\lambda(s)) = \pi((D\omega : D\omega')_s) \lambda(s) = \pi((D\omega' : D\omega)_s) \lambda(s), \quad \forall s \in \mathbb{R}.
\]

Define \( \tilde{\alpha} \in \text{Aut}(N) \) by

\[
\tilde{\alpha} = \Phi \circ \alpha'.
\]

Then

\[
\tilde{\alpha}(\pi(x)) = \pi(\alpha(x)), \quad \forall x \in M.
\]

\[
\tilde{\alpha}(\lambda(s)) = \pi((D\omega' : D\omega)_s) \lambda(s), \quad \forall s \in \mathbb{R},
\]

completing the proof of (i) and (ii). This proves also the uniqueness of \( \tilde{\alpha} \), since \( \pi(x), \forall x \in M \), and \( \lambda(s), \forall s \in \mathbb{R} \), generate \( \mathcal{N} \).

To show the map \( \alpha \to \tilde{\alpha} \) is a homomorphism let \( \alpha, \beta \in \text{Aut}(M) \). Let \( \gamma = \alpha \beta \). Clearly

\[
\tilde{\alpha} \tilde{\beta}(\pi(x)) = \pi(\alpha \beta(x)) = \tilde{\gamma}(\pi(x)), \quad \forall x \in M.
\]
If \( s \in \mathbb{R} \) we have
\[
\hat{\alpha} \hat{\beta}(\lambda(s)) = \hat{\alpha}(\pi((D\omega \circ \beta^{-1} : D\omega)_s) \lambda(s))
\]
\[
= \pi(\alpha((D\omega \circ \beta^{-1} : D\omega)_s)) \pi((D\omega \circ \alpha^{-1} : D\omega)_s) \lambda(s)
\]
\[
= \pi((D\omega \circ \beta^{-1} \circ \alpha^{-1} : D\omega \circ \alpha^{-1})_s (D\omega \circ \alpha^{-1} : D\omega)_s) \lambda(s)
\]
\[
= \pi((D\omega \circ \beta^{-1} \circ \alpha^{-1} : D\omega)_s) \lambda(s)
\]
\[
= \hat{\gamma}(\lambda(s)),
\]
because \((D\varphi \circ \alpha^{-1} : D\psi \circ \alpha^{-1})_s = \alpha((D\varphi : D\psi)_s)\) for all faithful normal semifinite weights \( \varphi, \psi \) on \( M \).

Q.E.D.

**Proposition 12.2.** With \( \alpha \) and \( \hat{\alpha} \) as in Proposition 12.1 we have

(i) \( \theta_s \circ \hat{\alpha} = \hat{\alpha} \circ \theta_s, s \in \mathbb{R} \).

(ii) \( \tau \circ \hat{\alpha} = \tau \).

(iii) \( \hat{\varphi} \circ \hat{\alpha} = (\varphi \circ \alpha)^{-1} \) for all normal semifinite weights \( \varphi \) on \( M \).

(iv) \( \hat{\varphi} \circ \hat{\alpha}|_{Z(N)} = (\varphi \circ \alpha)^{-1}, \varphi \in M_+^* \).

(v) If \( M \) is properly infinite with no direct summand of type \( I \), then the map \( \alpha \to \hat{\alpha}|_{Z(N)} \) is a continuous homomorphism of \( \text{Aut}(M) \) into \( \text{Aut}(Z(N)) \), when the automorphism groups have the topologies of simple norm convergence on the preduals.

**Proof.** We have that the dual action \( \theta \) of \( \sigma_x \) is given by \( \theta_x(\pi(x)) = \pi(x), \theta_x(\lambda(t)) = e^{-ixt}\lambda(t), x \in M, s, t \in \mathbb{R} \). Thus by Proposition 12.1 it is immediate that \( \theta_s \circ \hat{\alpha} = \hat{\alpha} \circ \theta_s \), proving (i).

From (i) it follows that \( \hat{\alpha} \) commutes with the operator valued weight \( T = \int_{-\infty}^{\infty} \theta_s ds \). Hence for every normal semifinite weight \( \varphi \) on \( M \) we have
\[
(\varphi \circ \alpha)^{-1} = \varphi \circ (\alpha \circ \pi^{-1}) \circ T = \varphi \circ (\pi^{-1} \circ \hat{\alpha}) \circ T = \varphi \circ \pi^{-1} \circ T \circ \hat{\alpha}
\]
\[
= \hat{\varphi} \circ \hat{\alpha},
\]
proving (iii). In particular, with \( \omega' = \omega \circ \alpha^{-1}, \hat{\varphi}' \circ \hat{\alpha} = (\omega' \circ \alpha)^{-1} = \hat{\omega} \). By spectral theory \( \hat{\alpha} \) extends to an automorphism of the unbounded self-adjoint operators affiliated with \( N \), still denoted by \( \hat{\alpha} \). Then
\[
\hat{\alpha} \left( \frac{d\hat{\omega}}{d\tau} \right) - \frac{d\hat{\omega}'}{d\tau}.
\]
Thus we have
\[
\frac{d\hat{\omega}}{d\tau} = \hat{\alpha}^{-1} \left( \frac{d\hat{\omega}'}{d\tau} \right) = \frac{d(\hat{\omega}' \circ \hat{\alpha})}{d(\tau \circ \hat{\alpha})} = \frac{d\hat{\omega}}{d(\tau \circ \hat{\alpha})}.
\]
hence \( \tau = \tau \circ \hat{\alpha} \), proving (ii).
To show (iv) let $h_\varphi = d\tilde{\varphi}/d\tau$ for $\varphi$ a normal semifinite weight on $M$. Then by (ii)

$$\tilde{\alpha}(h_\varphi) = \tilde{\alpha}\left(\frac{d\tilde{\varphi}}{d\tau}\right) = \frac{d(\tilde{\varphi} \circ \tilde{\alpha}^{-1})}{d(\tau \circ \tilde{\alpha}^{-1})} = \frac{d(\tilde{\varphi} \circ \tilde{\alpha}^{-1})}{d\tau} = h_{\varphi \circ \alpha^{-1}}.$$ 

Therefore, with $e_\varphi = \chi_{(1,\infty)}(h_\varphi)$ as in Section 3

$$\tilde{\alpha}(e_\varphi) = e_{\varphi \circ \alpha^{-1}}.$$ 

Hence for $z \in Z(N)$,

$$\varphi \circ \alpha^{-1}(z) = \tau(e_{\varphi \circ \alpha^{-1}z}) = \tau(\tilde{\alpha}(e_\varphi)z) = \tau(e_\varphi \tilde{\alpha}^{-1}(z)) = \tilde{\varphi} \circ \tilde{\alpha}^{-1}(z).$$

Since $\tilde{\alpha}^{-1} = (\alpha^{-1})^*$ we get (iv).

To show (v) let $(\alpha_k)$ be a net in $\text{Aut}(M)$ which converges to $\alpha \in \text{Aut}(M)$. Since $M^*_\varphi$ spans $M_\varphi$ linearly, this means that $\|\varphi \circ \alpha_k - \varphi \circ \alpha\| \to 0$ for all $\varphi \in M^*_\varphi$. Since the map $\varphi \to \tilde{\varphi}$ is norm decreasing by Lemma 3.3 it follows from (iv) that

$$\tilde{\varphi} \circ \tilde{\alpha}_k |_{Z(N)} = (\varphi \circ \alpha_k)^* \to (\varphi \circ \alpha)^* = \tilde{\varphi} \circ \tilde{\alpha} |_{Z(N)}.$$ 

By our assumptions on $M$ the Main Theorem together with Proposition 6.3 implies that $P(M) = \left\{ \tilde{\varphi} : \varphi \in M^*_\varphi \right\}$ spans a norm dense subspace of $Z(N)_\varphi$. Thus

$$\chi \circ \tilde{\alpha}_k |_{Z(N)} \to \chi \circ \tilde{\alpha} |_{Z(N)} \quad \text{for all } \chi \in Z(N)_\varphi,$$

so that $\tilde{\alpha}_k |_{Z(N)} \to \tilde{\alpha} |_{Z(N)}$, proving (v). Q.E.D.

If $\alpha \in \text{Aut}(M)$ then its adjoint map restricted to $M_\varphi$ acts by

$$\alpha^*_\varphi(\varphi) = \varphi \circ \alpha^{-1}.$$ 

The map $\alpha \to \alpha^*_\varphi$ is an isomorphism of $\text{Aut}(M)$ into the isometries on $M_\varphi$. If $u$ is a unitary operator in $M$ then for $x \in M$, $\varphi \in M^*_\varphi$, 

$$\alpha^*_\varphi(\varphi u \varphi^*) = \varphi(u^* \alpha^{-1}(x)u) = \varphi \circ \alpha^{-1}(x(u)^* xu(u)) = \alpha(u) \alpha^*(\varphi)(\alpha(u)^*)(x).$$

Thus $\alpha^*_\varphi(\varphi u \varphi^*)$ is unitarily equivalent to $\alpha^*_\varphi(\varphi)$. In particular $\alpha^*_\varphi$ defines a map

$$\tilde{\alpha}^*: M^*_\varphi/\sim \to M^*_\varphi/\sim.$$
Note that we clearly have for \( u \in U(M) \),
\[
(\alpha \circ \text{Ad}(u))_* = (\text{Ad}(u) \circ \alpha)_* = \tilde{\alpha}_*.
\]
We next study the relationship between \( \tilde{\alpha}_* \) and \( \tilde{\alpha}|_{Z(N)} \). For this we make the following definition.

**Definition 12.3.** Let \( \alpha \in \text{Aut}(M) \). We say \( \alpha \) is **pointwise inner** if for each \( \varphi \in M_*^+ \) there is \( u = u(\varphi) \) in the unitary group \( U(M) \) of \( M \) such that
\[
\alpha_* (\varphi) = \varphi \circ \alpha^{-1} = u \varphi u^*.
\]
We say \( \alpha \) is **approximately pointwise inner** if for each \( \varphi \in M_*^+ \) and \( \varepsilon > 0 \) there is \( u = u(\varphi, \varepsilon) \in U(M) \) such that
\[
\| \varphi \circ \alpha^{-1} - u \varphi u^* \| < \varepsilon.
\]
The following facts are easily verified

1. Every inner automorphism is pointwise inner.
2. If \( \alpha \) is approximately pointwise inner then \( \alpha|_{Z(M)} = 1 \).
3. The pointwise inner and the approximately pointwise inner automorphisms form normal subgroups of \( \text{Aut}(M) \).
4. \( \alpha \) is approximately pointwise inner if and only if \( \alpha_* = 1 \).

We let \( \text{Aut}(M) \) have the topology of simple norm convergence on \( M_*^+ \). As before \( N = M \times \sigma^0 \mathbb{R} \) for some faithful normal semifinite weight on \( M \).

**Theorem 12.4.** Let \( M \) be a von Neumann algebra. Then the approximately pointwise inner automorphisms form a closed normal subgroup of \( \text{Aut}(M) \) consisting of exactly those automorphisms \( \alpha \) for which \( \tilde{\alpha}|_{Z(N)} = 1 \).

**Proof.** Since \( \pi(Z(M)) \subset Z(N) \), if \( \tilde{\alpha}|_{Z(M)} = 1 \) then \( \alpha|_{Z(M)} = 1 \). Combining this with (2) it follows that if \( M \) is of type I then the approximately pointwise inner automorphisms are exactly the inner automorphisms and at the same time those for which \( \tilde{\alpha}|_{Z(M)} = 1 \). In particular the theorem holds for type I algebras.

Suppose \( M \) is of type II\(_1\). We show that if \( \alpha \in \text{Aut}(M) \) satisfies \( \alpha|_{Z(M)} = 1 \) then \( \alpha \) is approximately pointwise inner, and \( \tilde{\alpha}|_{Z(M)} = 1 \). Hence the theorem follows as in the type I case. Suppose then \( \alpha|_{Z(M)} = 1 \), and let \( \varphi \in M_*^+ \). Since \( \text{supp}(\varphi) \) is \( \sigma \)-finite there exists a finite normal trace \( \tau \) on \( M \) such that \( \text{supp}(\varphi) \leq \text{supp}(\tau) \). Let \( h = d\varphi/d\tau \). By spectral theory \( h \) can be
approximated in $L^1(M, \tau)$ by operators of the form $k = \sum_{i=1}^{n} \lambda_i e_i$ with $\lambda_i > 0$ and $e_1, \ldots, e_n$ mutually orthogonal projections in $M$. Since $\alpha|_{Z(M)} = 1$, $\tau \circ \alpha^{-1} = \tau$, and $\alpha(e) \sim e$ for all projections $e$ in $M$. Thus there is a unitary operator $u$ in $M$ such that $\alpha(k) = uku^*$, and $\alpha_\tau(\tau(k \cdot)) = u\tau(k \cdot) u^*$. By approximation then it is clear that given $\epsilon > 0$ we can find $u \in U(M)$ such that $\|\alpha_\tau(\phi) - u\phi u^*\| < \epsilon$; hence $\alpha$ is approximately pointwise inner. To show $\alpha|_{Z(N)} = 1$ note that $\sigma^\omega$ is inner for any faithful normal semifinite weight $\omega$. Let $\omega$ be a faithful normal semifinite trace on $M$ whose restriction to $Z(M)$ is semifinite. Then $\omega \circ \alpha^{-1} = \omega$, so by Proposition 12.1, $\tilde{\alpha}(\lambda(s)) = \lambda(s)$, $s \in \mathbb{R}$. Now $N = \pi(M) \otimes L^\infty(\mathbb{R})$, so $Z(N) = \pi(Z(M)) \otimes L^\infty(\mathbb{R})$. Since the $\lambda(s)$ generate $L^\infty(\mathbb{R})$ and $\alpha|_{Z(M)} = 1$, $\tilde{\alpha}|_{Z(N)} = 1$, proving the theorem if $M$ is of type $\text{II}_1$.

Finally assume $M$ is properly infinite with no direct summand of type I. Let $\phi \in M_+$ and $\alpha \in \text{Aut}(M)$. Put $\alpha_0 = \tilde{\alpha}|_{Z(N)}$. By Proposition 12.2 $\phi \circ \alpha_0^{-1} = (\phi \circ \alpha^{-1})^\wedge$. Thus by the Main Theorem $\tilde{\alpha}_*(\{\phi\}) = \{\phi\}$ if and only if $\phi \circ \alpha_0^{-1} = \phi$. By Proposition 6.3 the cone $P(M) = \{\phi : \phi \in M^*_+\}$ spans a dense subspace of $Z(N)_\wedge$. Thus $\tilde{\alpha}_* = 1$ if and only if $\alpha_0 = 1$. By Proposition 12.2 the map $\alpha \mapsto \alpha_0$ is a continuous homomorphism of $\text{Aut}(M)$ into $\text{Aut}(Z(N))$. By the above and (4) its kernel is the approximately pointwise inner automorphisms. This completes the proof of the theorem.

Q.E.D.

The group of approximately pointwise inner automorphisms will be investigated further in the next section. It is not clear what the group of pointwise inner automorphisms looks like. Even if $M$ is semifinite this problem is nontrivial. We are greatly indebted to V. Jones for pointing out to us that in the separable case its solution is a consequence of a result of Popa [14]:

**Proposition 12.5.** Let $M$ be a countably generated semifinite von Neumann algebra. Then each pointwise inner automorphism is inner.

**Proof.** Let $\alpha$ be a pointwise inner automorphism of $M$. Let $G$ denote the subgroup of $\text{Aut}(M)$ generated by $\alpha$ and the inner automorphisms $\text{Int}(M)$. Then $G/\text{Int}(M)$ is countable, so by [14, Thm. 4.2] there exists a maximal abelian subalgebra $A$ of $M$ generated by finite projections such that if $\beta \in G$ and $\beta|_A = 1$, then $\beta = Ad(v)$ with $v$ unitary in $A$. Let $\tau$ be a faithful normal semifinite trace on $M$, and let $e \in A$ be a finite projection in $M$. For $x \in M_+$,

$$\tau(exe) = \tau(x^{1/2}ex^{1/2}) \leq \tau(x).$$

Therefore $\tau$ is also semifinite on $eMe$. A normal trace on a finite von Neumann algebra is semifinite iff its restriction to the center is semifinite. Hence $\tau$ is semifinite on $Z(eMe)$. Since $eA$ is maximal abelian in $eMe$,
Thus $\tau$ is semifinite on $eA$, and since $A$ is generated by finite projections, $\tau$ is semifinite on $A$. Using now the separability of $A_*$ it follows that $A$ is the von Neumann algebra generated by a positive operator $h$ for which $\tau(h) < \infty$. Let $\varphi \in M_+^+$ be defined by $\varphi(x) = \tau(hx)$. Since $\varphi$ is pointwise inner there is $u \in U(M)$ such that $\varphi \circ \varphi^{-1} = wu^\ast$, whence $\tau(\varphi(h)x) = \tau(hx^{-1}(x)) = \varphi(\varphi^{-1}(x)) = \varphi(u^\ast xu) = \tau(uhu^\ast x)$, for all $x \in M$. Therefore $\varphi(h) = uhu^\ast$, and $\varphi|_A = Ad(u)|_A$. In particular $\varphi \circ Ad(u^\ast)|_A = 1$. Since $\varphi \circ A du^\ast \in G$, there is by the properties of $A$ a unitary operator $v \in A$ with $\varphi \circ Ad(u^\ast) = Ad(v)$. Thus $\varphi = Ad(vu) \in \text{Int}(M)$.

Q.E.D.

We remark that one can show that the above result is false in the non-separable case. Our next result shows that in the type III case all modular automorphisms are pointwise inner.

**Proposition 12.6.** Let $M$ be a von Neumann algebra and $\varphi$ a faithful normal semifinite weight on $M$. Then the modular automorphisms $\sigma_t^\varphi$, $t \in \mathbb{R}$, are pointwise inner.

**Proof.** Let $\psi \in M_+^+$ and put $e = \text{supp}(\psi)$. Choose a faithful normal weight $\psi'$ on $(1 - e)M(1 - e)$, and put $\psi = \psi + \psi'$. Then $\sigma_t^\psi(e) = e$, $t \in \mathbb{R}$. Therefore

$$\psi \circ \sigma_t^\psi = \psi.$$ 

Put $u_t = (D\varphi : D\psi)$, and $v_t = \sigma_t^\psi(u_t)$. Then for $x \in M$

$$\psi \circ \sigma_t^\psi(x) = \psi(u_t, \sigma_t^\psi(x) u_t^\ast)$$

$$= \psi \circ \sigma_t^\psi(v_t, xv_t^\ast) = \psi(v_t, xv_t^\ast),$$

i.e., $\psi \circ \sigma_t^\psi = v_t^\ast \psi v_t$. Hence $\sigma_t^\psi$ is pointwise inner for every $t \in \mathbb{R}$. Q.E.D.

**13. The Connes–Takesaki Module of an Automorphism**

In [5] Connes and Takesaki defined a homomorphism $\text{mod}$ of $\text{Aut}(M)$ into the automorphism group of "the flow of weights," called the fundamental homomorphism. We shall now show how the results of the previous section apply to yield new information on this homomorphism. We first recall some of the results and terminology from [5].

Two normal semifinite weights $\varphi$ and $\psi$ on the $\sigma$-finite properly infinite von Neumann algebra $M$ are equivalent ($\varphi \sim \psi$) if there is a partial isometry $u \in M$ such that

$$u^\ast u = \text{supp}(\varphi), \quad uu^\ast = \text{supp}(\psi), \quad u\varphi u^\ast = \psi.$$
Moreover, one writes \( \varphi < \psi \) if \( \varphi \) is equivalent to subweight \( \psi_e \) of \( \psi \), \( \psi_e \) being defined as
\[
\psi_e(x) = \psi(xe)
\]
for \( e \) a projection in the centralizer of \( \psi \). \( \varphi \) is said to have infinite multiplicity if its centralizer is properly infinite.

By a slight reformulation of [5, Thm. I.1.11] there is a unique pair \((p_M, \mathcal{F}_M)\) consisting of an abelian von Neumann algebra \( \mathcal{P}_M \) and a bijection \( p_M \) from the set of normal semifinite weights of infinite multiplicity onto the set of \( \sigma \)-finite projections in \( \mathcal{P}_M \), such that
\[
p_M(\varphi) \leq p_M(\psi) \iff \varphi < \psi.
\]
(In this formulation \( p_M \) is simply the restriction of \( p_M \) in [5, Thm. I.1.11] to the set of weights of infinite multiplicity.) For \( \lambda > 0 \) the map \( p_M(\varphi) \rightarrow p_M(\lambda \varphi) \) extends uniquely to an automorphism \( \mathcal{F}_M(\lambda) \) of \( \mathcal{P}_M \). The couple \((\mathcal{P}_M, (\mathcal{F}_M(\lambda))_{\lambda > 0})\) is called the global flow of weight.

Assume now that \( M \) is properly infinite with separable predual, and let \( \omega \) be a dominant weight on \( M \) (see [5, Chap. II]). Put \( d_M = p_M(\omega) \). Since any two dominant weights are equivalent, \( d_M \) is independent of the choice of \( \omega \). Moreover, \( \mathcal{F}_M(d_M) = d_M, \lambda > 0 \). Put \( P_M = d_M \mathcal{P}_M \), and let \( F_M^\lambda \) be the restriction of \( \mathcal{F}_M(\lambda) \) to \( P_M \). Then \( p_M \) maps the set of integrable weights (see [5, Chap. II]) of infinite multiplicity onto the set of projections in \( P_M \). The pair \((P_M, (F_M^\lambda)_{\lambda > 0})\) is called the smooth flow of weights.

Following [5, Chap. IV] one can to any \( \alpha \in \text{Aut}(M) \) associate a (unique) automorphism \( \text{mod}(\alpha) \in \text{Aut}(P_M, F_M) \) such that
\[
\text{mod}(\alpha)(p_M(\varphi)) = p_M(\varphi \circ \alpha^{-1})
\]
for every integrable weight \( \varphi \) of infinite multiplicity. The homomorphism \( \alpha \rightarrow \text{mod}(\alpha) \) is called the fundamental homomorphism.

As before let \( N = M \times_{\sigma \varphi} \mathbb{R} \) with \( \varphi \) a faithful normal semifinite weight on \( M \). If \( \alpha \in \text{Aut}(M) \) let \( \tilde{\alpha} \in \text{Aut}(N) \) be the extension found in Proposition 12.1.

**Proposition 13.1.** Let \( M \) be a properly infinite factor with separable predual. Then there is an isomorphism \( \beta: P_M \rightarrow Z(N) \) such that for each \( \alpha \in \text{Aut}(M) \),
\[
\tilde{\alpha}|_{Z(N)} = \beta \circ \text{mod}(\alpha) \circ \beta^{-1}.
\]

For the proof of the proposition it is necessary to go into the proof of Takesaki's duality theorem for crossed products [21, Thm. 4.5]. Let \( P \) be a von Neumann algebra acting on a Hilbert space \( H \), and suppose \( G \) is a locally compact abelian group with a continuous action \( \sigma \) as
automorphisms of $P$. Let $Q = P \times_{\sigma} G$ be the crossed product. Then $Q$ is generated by operators $\pi_{\sigma}(x)$, $x \in P$, and $\lambda(g)$, $g \in G$, defined on $L^2(G, H)$ by

$$(\pi_{\sigma}(x)\xi)(g) = \sigma_{-\sigma}(x)\xi(g), \quad x \in P,$$

$$(\lambda(h)\xi)(g) = \xi(g-h), \quad h \in G.$$ 

As an immediate application of [21, Props. 3.4 and 4.2] we have the following result.

**Lemma 13.2.** With the above notation let $\alpha \in \text{Aut}(P)$ commute with $\sigma$, i.e., $\alpha \sigma_g = \sigma_g \alpha$, $g \in G$. Then there is $\tilde{\alpha} \in \text{Aut}(Q)$ satisfying the following three conditions:

(i) $\tilde{\alpha}(\pi_{\sigma}(x)) = \pi_{\sigma}(\alpha(x))$, $x \in P$.

(ii) $\tilde{\alpha}(\lambda(g)) = \lambda(g)$, $g \in G$.

(iii) $\tilde{\alpha}\sigma_p = \sigma_p \tilde{\alpha}$, $p \in \hat{G}$ where $\tilde{\alpha}$ is the dual action of $\sigma$ on $Q$ (i.e., $\tilde{\alpha}\sigma_p(\pi_{\sigma}(x)) = \pi_{\sigma}(x)$, $\tilde{\alpha}(\lambda(g)) = \langle g, p \rangle \lambda(g)$).

**Lemma 13.3.** Let notation be as above. Let $\alpha \in \text{Aut}(P)$ commute with $\sigma$ and let $\tilde{\alpha}$ be the extension of $\tilde{\alpha}$ to $Q \times_{\sigma} \hat{G}$ given by Lemma 13.2. Then there exists an isomorphism

$$\gamma : P \otimes B(L^2(G)) \to Q \times_{\sigma} \hat{G}$$

such that

$$\tilde{\alpha} = \gamma \circ (\alpha \otimes 1) \circ \gamma^{-1}.$$

The proof of this lemma can be read out of the proof of [21, Thm. 4.5]. In order to define $\gamma$ we indicate the main points in Takesaki's proof. As in [21, Eq. 4.8] we define an operator $F$ on the continuous $H$-valued functions on $G \times \hat{G}$ with the compact support by

$$(F\xi)(g, h) = \int_{\hat{G}} \langle h, p \rangle \xi(g, p) \, dp.$$ 

$F$ extends to a unitary operator on $L^2(G \times \hat{G}, H)$ onto $L^2(\hat{G} \times G, H)$ which we still denote by $F$. We denote by $R$ the von Neumann algebra

$$R = F(Q \times_{\sigma} \hat{G}) F^*.$$ 

The operators $u(g) = F\pi_{\sigma}(\lambda(g)) F^*$, $g \in G$, and $u(p) = F\lambda(p) F^*$, $p \in \hat{G}$, satisfy the canonical commutation relations and therefore generate a von
Neumann algebra $B$, which is isomorphic to $B(L^2(G))$. Hence there is an isomorphism

$$\delta: R \to (R \cap B') \otimes B$$

given by $\delta(xy) = x \otimes y$, $x \in R \cap B'$, $y \in B$.

Now let $\alpha \in \text{Aut}(P)$ commute with $\sigma$, and let $\tilde{\alpha}$ be its extension to $Q \times \hat{G}$.

Define $\alpha_0 \in \text{Aut}(R)$ by

$$\alpha_0 = (A \ d F) \circ \tilde{\alpha} \circ (A \ d F^*) .$$

Then $\alpha_0 | _B = 1$.

If $x \in P$ we define an operator $\Pi(x)$ on $L^2(G \times G, H)$ by

$$(\Pi(x) \xi)(g, h) = \sigma_{-h}^{-1}(x) \xi(g, h).$$

By [21, Lem. 4.3 and 4.4] $\Pi$ is a normal isomorphism of $P$ onto $R \cap B'$; hence $\pi \otimes i$ is an isomorphism

$$\Pi \otimes i: P \otimes B(L^2(G)) \to (R \cap B') \otimes B,$$

where we identify $B$ and $B(L^2(G))$.

It follows from the proof of [21, Lem. 4.3] that

$$\alpha_0(\Pi(x)) = \Pi(\alpha(x)), \quad x \in P,$$

and since $\alpha_0 | _B = 1$, $\alpha_0(xy) = \alpha_0(x) y$, $x \in R \cap B'$, $y \in B$. Thus we find

$$\delta \circ \alpha_0 \circ \delta^{-1}(x \otimes y) = \alpha_0(x) \otimes y.$$ 

It follows that

$$\delta \circ \alpha_0 \circ \delta^{-1} \circ (\Pi \otimes i) = (\Pi \otimes i) \circ (\pi \otimes i) ;$$

hence if

$$\gamma = \text{Ad}(F^*) \circ \delta^{-1} \circ (\Pi \otimes i),$$

then $\gamma$ is the desired isomorphism of $P \otimes B(L^2(G))$ onto $Q \times \hat{G}$ such that

$$\tilde{\alpha} = \gamma \circ (\pi \otimes i) \circ \gamma^{-1}. \quad \text{Q.E.D.}$$

**Lemma 13.4.** Let notation be as above. Let $Z = Z(Q \times \hat{G})$. Then there is an isomorphism

$$\gamma_0: Z(P) \to Z$$

such that

$$\tilde{\alpha} |_Z = \gamma_0 \circ \alpha |_{Z(P)} \circ \gamma_0^{-1}.$$
Proof. For $x \in Z(P)$ let
\[ \gamma_0(x) = \gamma(x \otimes 1), \]
with $\gamma$ as in Lemma 13.3. By the lemma $\gamma_0$ is an isomorphism of $Z(P)$ onto $Z$. Furthermore, if $x \in Z(P)$ then by Lemma 13.3
\[ \gamma_0 \circ \alpha(x) = \gamma(\alpha(x) \otimes 1) = \tilde{\alpha} \circ \gamma(x \otimes 1) = \tilde{\alpha} \circ \gamma_0(x). \]
Q.E.D.

Proof of Proposition 13.1. Let $\omega$ be a dominant weight on $M$, and let $N_0$ be the centralizer of $\omega$. There is a unique trace $\tau_0$ on $N_0$ such that $\omega = \tau_0 \circ E$, where $E$ is the operator valued weight from $M$ to $N_0$ given by
\[ E(x) = \int_{-\infty}^{\infty} \sigma_t^\omega(x) \, dt, \quad x \in M^+. \]
(cf. [5, Lem. II.2.7]). Moreover,
\[ M \cong N_0 \times \theta_0 \mathbb{R} \]
for some action $\theta_0 : \mathbb{R} \to \text{Aut}(N_0)$ satisfying
\[ \tau_0 \circ (\theta_0)_s = e^{-s} \tau_0, \quad s \in \mathbb{R} \]
(cf., e.g., [5, Thm. II.1.3]). Furthermore, by [21, Thm. 8.3] $\sigma^\omega$ is the dual action of $\theta_0$. In particular $N = M \times \sigma_0$ equals $(N_0 \times \theta_0 \mathbb{R}) \times \theta_0 \mathbb{R}$. Now let $\alpha \in \text{Aut}(M)$. Since $\omega \circ \alpha$ is a dominant weight there exists $u \in U(M)$ such that $\omega \circ \alpha = \omega \circ \text{Ad}(u^*)$ [5, Thm. II.1.1]; hence $\alpha \circ \text{Ad}(u)$ leaves $\omega$ invariant. By the homomorphic property of the map $\alpha \to \tilde{\alpha}$ and the fact that $(\text{Ad}(u))|_{Z(N)} = 1$ we have
\[ \left(\alpha \circ \text{Ad}(u)\right)^\sim|_{Z(N)} = \tilde{\alpha}|_{Z(N)}. \]
This, together with the fact that $\text{mod}(\alpha) = \text{mod}(\alpha \circ \text{Ad}(u))$, shows that we can replace $\alpha$ by $\alpha \circ \text{Ad}(u)$, and thus assume $\omega \circ \alpha = \omega$. The crossed product $M = N_0 \times \sigma_0 \mathbb{R}$ is generated by $N_0$ and a one-parameter group of unitaries $(u(s))_{s \in \mathbb{R}}$. From the arguments of [5, p. 569], there exists $b \in U(N_0)$, such that
\[ \alpha(u(s)) = b^* u(s) b. \]
If we replace $\alpha$ by $\text{Ad}(b) \circ \alpha$, we may assume $\omega \circ \alpha = \omega$ and $\alpha(u(s)) = u(s)$, $s \in \mathbb{R}$. Now apply Lemma 13.4 to $G = \mathbb{R}$, $P = N_0$, $Q = M$, $Q \times \sigma \tilde{G} = N$, and the automorphism $\alpha|_{N_0}$. Then $(\alpha|_{N_0})^\sim = \alpha$ and $(\alpha|_{N_0})^\sim = \tilde{\alpha}$. Hence it follows from Lemma 13.5 that there is an isomorphism $\gamma_0 : Z(N_0) \to Z(N)$, such that
\[ (1) \quad \tilde{\alpha}|_{Z(N)} = \gamma_0 \circ \alpha|_{Z(N_0)} \circ \gamma_0^{-1}. \]
By [5, Thm. I.1.11] the map $e \rightarrow p_M(e)$, $e$ a projection in $Z(N_0)$, can be extended to an isomorphism $p_\omega$ of $Z(N_0)$ onto $P_M$. Moreover,

$$F^M_\lambda \circ p_\omega = p_\omega \circ (\theta_0)_{-\log \lambda}, \quad \lambda \in \mathbb{R}^+$$

(cf. [5, Cor. II.2.5]). Furthermore, by [5, p. 554]

$$p_\omega^{-1} \circ \text{mod}(\alpha) \circ p_\omega = \alpha|_{Z(N_0)}.$$  

(2) $p_\omega^{-1} \circ \text{mod}(\alpha) \circ p_\omega = \alpha|_{Z(N_0)}$.

Let $\beta = \gamma_0 \circ p_\omega^{-1}$. Then $\beta$ is an isomorphism from $P_M$ onto $Z(N)$ which by (1) and (2) satisfies

$$\tilde{\beta}|_{Z(N)} = \beta \circ \text{mod}(\alpha) \circ \beta^{-1}.$$ Q.E.D.

**Corollary 13.5.** Let $M$ be a properly infinite factor with separable predual. Then the kernel of the fundamental homomorphism mod is the set of all approximately pointwise inner automorphisms.

**Proof.** This follows immediately from Theorem 12.4 and Proposition 13.1.

**Corollary 13.6.** Let $M$ be a factor with separable predual, and let $\alpha \in \text{Aut } M$. Then we have

(i) If $M$ is of type $\text{II}_1$ then $\alpha$ is approximately pointwise inner.

(ii) If $M$ is of type $\text{II}_\infty$ with trace $\tau$ then $\alpha$ is approximately pointwise inner if and only if $\tau \circ \alpha = \tau$.

(iii) If $M$ is of type $\text{III}_1$ then $\alpha$ is approximately pointwise inner.

(iv) If $M$ is of type $\text{III}_\lambda$, $0 < \lambda < 1$, then $\alpha$ is approximately pointwise inner if and only if $\omega \circ \alpha \sim \omega$ whenever $\omega$ is a generalized trace.

**Proof.** (i) Follows from the proof of Theorem 12.4.

(ii) Follows from Theorem 12.4 together with Corollary 13.5 and [5, Prop. IV.1.2].

(iii) By [4] if $M$ is of type $\text{III}_1$ then $\varphi \sim \psi$ iff $\varphi(1) = \psi(1)$, $\varphi, \psi \in M_\tau^+$, so $\bar{\alpha}_* = 1$.

(iv) Follows by Theorem 12.4 together with Corollary 13.5 and [5, Prop. IV.1.3]. Q.E.D.

**References**

1. J. Bion-Nadal, Espace des états normaux d'une facteur de type $\text{III}_\lambda$, $0 < \lambda < 1$, et d'une facteur de type $\text{III}_0$, *Canad. J. Math.* 36 (1984), 830–882.


