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Multivariate order statistics via multivariate concomitants

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ABSTRACT

Let $\underline{X}_1, \ldots, \underline{X}_n$ denote a set of n independent identically distributed k-dimensional absolutely continuous random variables. A general class of complete orderings of such random vectors is supplied by viewing them as concomitants of an auxiliary random variable. The resulting definitions of multivariate order statistics subsume and extend orderings that have been previously proposed such as norm ordering and N-conditional ordering. Analogous concepts of multivariate record values and multivariate generalized order statistics are also described.

1. Introduction

Let $\underline{X}_1, \ldots, \underline{X}_n$ denote a set of n independent identically distributed k-dimensional absolutely continuous random variables. Several papers have recently appeared in which efforts were made to supply suitable definitions of multivariate order statistics in this context (e.g. [1–4]). In this paper we point out that all of these definitions can be subsumed by an ordering induced by considering the $\underline{X}_i$'s as concomitants of some continuous univariate random variable for which classical order statistics are, of course, well defined. Analogous definitions are available for generalized multivariate order statistics, multivariate record values and progressively censored multivariate variables in terms of multivariate concomitants.

The paper is organized as follows. In Section 2 the concomitants approach is presented. In Section 3 the distribution of the j'th multivariate order statistic is obtained. In Section 4 the conditionally ordered multivariate variables are presented as a special case. In Section 5 some extensions of the concomitants approach are analyzed. In Section 6 some examples are given to illustrate the proposed methods.

2. Multivariate concomitants

Let $(\underline{X}_i, Y_i)$, $i = 1, \ldots, n$, be independent identically distributed (i.i.d.) random variables. Assume that the $\underline{X}_i$'s are absolutely continuous k-dimensional random variables with common joint density $f_{\underline{X}}(\underline{x})$ and distribution function $F_{\underline{X}}(\underline{x})$. In addition, assume that the Y_i's are univariate random variables and have a common continuous distribution function $F_Y(y)$.

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Note that we do not require that the $(k+1)$-dimensional random variables $(\underline{X}_i, Y_i)$ be absolutely continuous. Since the Y_i's have a continuous distribution, their order statistics are unambiguously defined in the usual manner. For $j = 1, \dots, n$, the j'th k-dimensional concomitant of the Y_i's is defined to be the $\underline{X}$ variable associated with the j'th largest Y variable (i.e. with $Y_{j:n}$). It will be denoted by $\underline{X}_{[j:n]}$.

In most discussions of concomitants, we view the Y's as being ordered and view the $\underline{X}$'s as only being variables associated with the ordered Y's. The construction can however be used to define a total ordering on the $\underline{X}$'s as follows. The smallest $\underline{X}$ is the concomitant of $Y_{1:n}$, the second smallest $\underline{X}$ is the concomitant of $Y_{2:n}$, etc. This ordering of the $\underline{X}$'s depends on the conditional distribution of Y given $\underline{X}$. For any choice of the conditional distribution $F_{Y|\underline{X}}(y|\underline{x})$ a total ordering of the $\underline{X}$'s will be well defined.

Thus, instead of being confronted with a lack of total orderings of the $\underline{X}$'s, we are possessed of an embarrassment of riches, in terms of a plethora of total orderings associated with the many possible choices for $F_{Y|\underline{X}}(y|\underline{x})$. Typically we use the notation $\underline{X}_{[j:n]}$ to denote the j'th concomitant. In the present context, since we are using the concomitant structure to induce an ordering of the $\underline{X}$'s, we will use the notation $\underline{X}_{j:n}$ and speak of this random variable as the j'th smallest $\underline{X}$. The set $\underline{X}_{1:n}, \dots, \underline{X}_{n:n}$ will be called the (multivariate) order statistics of the random sample of $\underline{X}$'s, though to be precise we should speak of them as being the $(F_{Y|\underline{X}})$-order statistics.

3. The distribution of the *j*'th multivariate order statistic

For $j \in \{1, \dots, n\}$ and for a fixed choice of $F_{Y|\underline{X}}$, consider the distribution of $\underline{X}_{j:n}$ (which we recall is, in fact, equal to $\underline{X}_{[j:n]}$, the j'th concomitant). For any k-dimensional Borel set B we have

$$\begin{aligned}
\Pr(\underline{X}_{j:n} \in B) &= \Pr(\underline{X}_{[j:n]} \in B) \\
&= \sum_{k=1}^{n} \Pr(\underline{X}_k \in B \quad \text{and } Y_k \text{ is the } j\text{th largest among } (Y_1, \dots, Y_n)) \\
&= n \Pr(\underline{X}_1 \in B \quad \text{and } Y_1 \text{ is the } j\text{th largest among } (Y_1, \dots, Y_n)) \\
&= n \binom{n-1}{j-1} \Pr(\underline{X}_1 \in B, Y_1 > Y_l, \quad l = 2, \dots, j, Y_1 < Y_l,\ l = j+1, \dots, n) \\
&= n \binom{n-1}{j-1} \int \cdots \int_B f_{\underline{X}_1}(\underline{x}) \Pr(Y_2, \dots, Y_j < Y_1 \quad \text{and } Y_{j+1}, \dots, Y_n > Y_1 | \underline{X} = \underline{x}) \mathrm{d}\underline{x} \\
&= n \binom{n-1}{j-1} \int \cdots \int_B \int_{-\infty}^{\infty} f_{\underline{X}_1}(\underline{x}) \Pr(Y_2, \dots, Y_j < y \text{ and } Y_{j+1}, \dots, Y_n > y | Y_1 = y, \underline{X} = \underline{x}) \mathrm{d}\underline{x} \mathrm{d}F_{Y|\underline{X}}(y|\underline{x}) \\
&= n \binom{n-1}{j-1} \int \cdots \int_B \int_{-\infty}^{\infty} f_{\underline{X}_1}(\underline{x}) [F_Y(y)]^{j-1} [1 - F_Y(y)]^{n-j} \mathrm{d}\underline{x} \mathrm{d}F_{Y|\underline{X}}(y|\underline{x}).
\end{aligned}$$

From this we obtain the density of $\underline{X}_{j:n}$ in the form

$$f_{\underline{X}_{j:n}}(\underline{x}) = n \binom{n-1}{j-1} f_{\underline{X}}(\underline{x}) \int_{-\infty}^{\infty} [F_Y(y)]^{j-1} [1 - F_Y(y)]^{n-j} \mathrm{d}F_{Y|\underline{X}}(y|\underline{x}). \tag{3.1}$$

By analogous arguments, we may obtain the joint density of several of the order statistics of the $\underline{X}$'s. Thus for $1 \le j_1 < \cdots < j_m \le n$ we have

$$\begin{aligned}
&f_{\underline{X}_{j_1:n}, \dots, \underline{X}_{j_m:n}}(\underline{x}_1, \dots, \underline{x}_m) \\
&\quad = n! \prod_{l=1}^{m} f_{\underline{X}}(\underline{x}_l) \int_{-\infty}^{\infty} \prod_{l=1}^{m+1} \frac{[F_Y(\underline{y}_l) - F_Y(\underline{y}_{l-1})]^{j_l - j_{l-1} - 1}}{(j_l - j_{l-1} - 1)!} I(y_1 < \cdots < y_m) \mathrm{d}F_{Y|\underline{X}}(y_1|\underline{x}_1) \cdots \mathrm{d}F_{Y|\underline{X}}(y_m|\underline{x}_m)
\end{aligned}$$

where we have used the notational convention that $y_0 = -\infty, y_{m+1} = \infty, j_0 = 0$ and $j_{m+1} = n+1$.

Example 1: Suppose that (X_1, X_2, Y) has a trivariate normal distribution with (for simplicity) mean vector $(0, 0, 0)$ and variance–covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}.$$

It follows that the conditional distribution of Y given $(X_1, X_2) = (x_1, x_2)$ is univariate normal with mean $\rho(x_1 + x_2)/(1+\rho) \triangleq \delta(x_1 + x_2)$ and variance $(1 + \rho - 2\rho^2)/(1+\rho) \triangleq \tau^2$ while the marginal distribution of Y is normal $(0, 1)$. It follows that the distribution of $\underline{X}_{j:n}$ will be of the form

$$f_{\underline{X}_{j:n}}(\underline{x}) = n \binom{n-1}{j-1} f_{\underline{X}}(\underline{x}) \int_{-\infty}^{\infty} [\Phi(y)]^{j-1} [1 - \Phi(y)]^{n-j} \frac{1}{\sqrt{2\pi}\,\tau} \exp -\frac{1}{2} \frac{(y - \delta(x_1 + x_2))^2}{\tau^2} \mathrm{d}y.$$

We remark that it is clearly possible to define record values of the $\underline{X}$'s as concomitants of the record values of the Y's, and to consider progressively censored $\underline{X}$'s as concomitants of progressively censored Y's. See [5] for detailed discussion of progressive censoring of ordinary order statistics. Some more details on these constructions will be provided in Section 5. In the next section we discuss the relationship between the present concomitant based definition of order statistics and the concept of conditionally ordered multivariate variables, introduced by Bairamov [6].

4. Conditionally ordered multivariate variables as a special case

Bairamov [6] (extending initial work by Bairamov and Gebizlioglu [1]) introduced a concept of conditional ordering of k-dimensional random variables. We will verify that this may be viewed as a special case of concomitant based ordering.

Bairamov begins with $\underline{X}_1, \ldots, \underline{X}_n$, a set of i.i.d. absolutely continuous k-dimensional random variables. He then considers a mapping $N : \mathbb{R}^k \to \mathbb{R}$. The function $N(\underline{x})$ is assumed to be a continuous function. It is also assumed that $N(\underline{x}) \geq 0$, for every $\underline{x} \in \mathbb{R}^k$ with $N(\underline{x}) = 0$ iff $\underline{x} = \underline{0}$. With these assumptions, it is asserted that the random variables $N(\underline{X}_1), \ldots, N(\underline{X}_n)$ will be i.i.d. continuous random variables and consequently will have unambiguously ordered values (no ties). He then defines an ordering on $\mathbb{R}^k$ by $\underline{x} <_N \underline{y}$ if $N(\underline{x}) < N(\underline{y})$. The conditionally N-ordered statistics are then defined to be $\underline{X}^{(N)}_{1:n}, \ldots, \underline{X}^{(N)}_{n:n}$ where $\underline{X}^{(N)}_{j:n}$ is the j'th largest of the $\underline{X}$'s with respect to the ordering $<_N$. It is apparent that it is not necessary to restrict N to be a non-negative function and it would appear that, to avoid ties among the $N(\underline{X}_j)$'s, we need to assume that, for every $c \in \mathbb{R}$,

$$\mu^{(k)}(\{x : N(\underline{x}) = c\}) = 0, \tag{4.1}$$

where $\mu^{(k)}$ denotes k-dimensional Lebesgue measure. For a suitable choice of the function $N(\underline{x})$, Bairamov [6] defines the structural function $h_N(\underline{x})$ by

$$h_N(\underline{x}) = \Pr(N(\underline{X}) \leq N(\underline{x})).$$

With this notation, the density of $\underline{X}^{(N)}_{j:n}$, the j'th N-conditionally ordered statistic, is given by

$$f_{\underline{X}^{(N)}_{j:n}}(\underline{x}) = n\binom{n-1}{j-1}[h_N(\underline{x})]^{j-1}[1 - h_N(\underline{x})]^{n-j} f_{\underline{X}}(\underline{x}). \tag{4.2}$$

This should be compared with (3.1), which provides the density of the j'th concomitant ordered statistic.

The link between the two expressions is provided by defining $Y = N(\underline{X})$, i.e. by considering a joint distribution for $(\underline{X}, Y)$ in which Y is a deterministic function of $\underline{X}$. In such a case, the conditional distribution of Y given $\underline{X} = \underline{x}$ is degenerate at $N(\underline{x})$ and (3.1) becomes

$$f_{\underline{X}^{(N)}_{j:n}}(\underline{x}) = n\binom{n-1}{j-1}[F_{N(\underline{X})}(N(\underline{x}))]^{j-1}[1 - F_{N(\underline{X})}(N(\underline{x}))]^{n-j} f_{\underline{X}}(\underline{x}),$$

which, recalling the definition of $h_N(\underline{x})$, coincides with (4.2).

In most of the examples considered by Bairamov [6] (and Bairamov and Gebizlioglu [1]), the function $N(\underline{x})$ was chosen to be a measure of the "size" or norm of the vector $\underline{x}$. In this fashion the $\underline{X}$'s are ordered with respect to their relative "sizes", in direct analogy to the univariate case. However, there is no need to restrict $N(\underline{x})$ to be of such a form. It is only necessary that (4.1) holds for the function $N(\underline{x})$ in order to define conditional N-ordering.

The idea of using a function such as $N(\underline{x})$ to induce an ordering on the $\underline{X}$'s had been discussed prior to the publication of Bairamov and Gebizlioglu. Reiss [7, p. 66] speaks of total ψ-ordering and refers to [8] for the definition. Later Kaufmann and Reiss [9] refer to the ordering as g-ordering.

Kaufmann and Reiss [9] point out that conditional ordering associated with the function $N(\underline{x}) = \|\underline{x} - \underline{x}_0\|$, for some specified point $\underline{x}_0$ in $\mathbb{R}^k$, occurs quite naturally in the context of nearest neighbor analysis.

5. Remarks

5.1. Record values

(i) Concomitant based definition. Let $\{(\underline{X}_i, Y_i)\}_{i=1}^{\infty}$ be a sequence of absolutely continuous i.i.d. random variables with the $\underline{X}$'s being of dimension k and the Y's of dimension 1. A convenient reference for the definition and elementary properties of record values is [10]. Let $T(0), T(1), \ldots$ denote the record times associated with the sequence $\{Y_i\}_{i=1}^{\infty}$ and let $\{Y_{(i)}\}_{i=0}^{\infty}$ denote the corresponding sequence of record values of the Y's. Thus $Y_{(i)} = Y_{T(i)}$, $1 = 0, 1, \ldots$. Note that, by convention, Y_1 is a record, the zeroth record, so $T(0) = 1$. The sequence of concomitant based record values of the $\underline{X}$'s is defined by identifying $\underline{X}_{(i)}$, the i'th record among the $\underline{X}$'s, to be the $\underline{X}$ concomitant of the i'th record $Y_{(i)}$. The density of this i'th record is then given by

$$f_{\underline{X}_{(i)}}(\underline{x}) = \int_{-\infty}^{\infty} f_{\underline{X}|Y}(\underline{x}|y) f_{Y_{(i)}}(y)\mathrm{d}y.$$

(ii) Conditional N-record values. In this case $Y = N(\underline{X})$ and the i'th record $\underline{X}_{(i)}$ is defined to be $\underline{X}_{T(i)}$ where $T(i)$ is the i'th record time for the sequence $\{N(\underline{X}_i)\}_{i=1}^{\infty}$. If we define $Y_i = N(\underline{X}_i)$ then we may obtain the following expression for the density of $\underline{X}_{(i)}$:

$$f_{\underline{X}_{(i)}}(\underline{x}) = f_{\underline{X}}(\underline{x}) \int_{-\infty}^{N(\underline{x})} \frac{f_{Y_{(n-1)}}(y)}{1 - F_{Y_{(n-1)}}(y)} \mathrm{d}y.$$

5.2. Progressive censoring and other generalized order statistics

Concomitants of generalized order statistics of the Y's can viewed as generalized order statistics of the $\underline{X}$'s. See [11] for details on generalized order statistics. This includes progressively censored samples. See [6] for some specific expressions for N-conditional progressively censored samples.

5.3. Jones constructions

Beginning with a univariate density $f_X(x)$ one way to write the density of the i'th order statistic is as

$$f_{X_{i:n}}(x) = \frac{F_X(x)^{i-1}[1 - F_X(x)]^{n-i}}{B(i, n-i+1)} f_X(x). \tag{5.1}$$

Jones [12] suggested the construction of an extended family of distributions analogous to (5.1) of the form

$$f(x; \alpha, \beta) = \frac{F(x)^{\alpha-1}[1 - F(x)]^{\beta-1}}{B(\alpha, \beta)} f(x), \tag{5.2}$$

where $\alpha > 0$, $\beta > 0$. Perhaps the easiest way to confirm that (5.2) is indeed a valid density function (i.e. integrates to 1) is to consider a random variable $Y \sim \mathcal{B}(\alpha, \beta)$. It is then easy to verify that $F^{-1}(Y)$ has (5.2) as its density, confirming that (5.2) is a valid density. Our discussion of multivariate order statistics leads naturally to a k-dimensional extension of this construction. We have, rewriting (3.1), for the i'th multivariate order statistic,

$$f_{\underline{X}_{j;n}}(\underline{x}) = f_{\underline{X}}(\underline{x}) \frac{\int_{-\infty}^{\infty} [F_Y(y)]^{j-1}[1 - F_Y(y)]^{n-j} \mathrm{d}F_{Y|\underline{X}}(y|x)}{B(j, n-j+1)}.$$

It is natural to extend this, following Jones, and to consider

$$f_{\alpha,\beta}(\underline{x}) = f_{\underline{X}}(\underline{x}) \frac{\int_{-\infty}^{\infty} [F_Y(y)]^{\alpha-1}[1 - F_Y(y)]^{\beta-1} \mathrm{d}F_{Y|\underline{X}}(y|x)}{B(\alpha, \beta)}.$$

In the case of N-conditional ordering with a structural function $h(\underline{x})$ this simplifies to

$$f_{\alpha,\beta}(\underline{x}) = f_{\underline{X}}(\underline{x}) \frac{[h(\underline{x})]^{\alpha-1}[1 - h(\underline{x})]^{\beta-1}}{B(\alpha, \beta)}. \tag{5.3}$$

An even more general family of models would be one in which $h(\underline{x})$ is simply required to satisfy $0 \le h(\underline{x}) \le 1$ and is not required to be a structural function of some $N(\underline{x})$. Of course, in such a case, the required normalizing constant will not usually be $[B(\alpha, \beta)]^{-1}$. Some examples of bivariate densities of the form (5.3) may be found in Section 6.

5.4. Concomitant densities revisited

The expression that we have in Eq. (3.1) for the density of the j'th concomitant looks different from the one usually encountered in the fully absolutely continuous case (i.e. the case in which $(\underline{X}, Y)$ is absolutely continuous). Our expression in this case is

$$f_{\underline{X}_{[j:n]}}(\underline{x}) = \binom{n-1}{j-1} f_{\underline{X}}(\underline{x}) \int_{-\infty}^{\infty} [F_Y(y)]^{j-1}[1 - F_Y(y)]^{n-j} f_{Y|\underline{X}}(y|\underline{x}) \mathrm{d}y.$$

The usual expression is

$$\begin{aligned} f_{\underline{X}_{[j:n]}}(\underline{x}) &= \int_{-\infty}^{\infty} f_{\underline{X}|Y}(\underline{x}|y) f_{Y_{j:n}}(y) \mathrm{d}y \\ &= n\binom{n-1}{j-1} \int_{-\infty}^{\infty} f_{\underline{X}|Y}(\underline{x}|y)[F_Y(y)]^{j-1}[1 - F_Y(y)]^{n-j} f_{Y|\underline{X}}(y|\underline{x}) f_Y(y) \mathrm{d}y. \end{aligned}$$

But of course these are identical since both can be written as

$$f_{\underline{X}_{[j:n]}}(\underline{x}) = n\binom{n-1}{j-1} \int_{-\infty}^{\infty} [F_Y(y)]^{j-1}[1 - F_Y(y)]^{n-j} f_{\underline{X},Y}(\underline{x}, y) \mathrm{d}y.$$

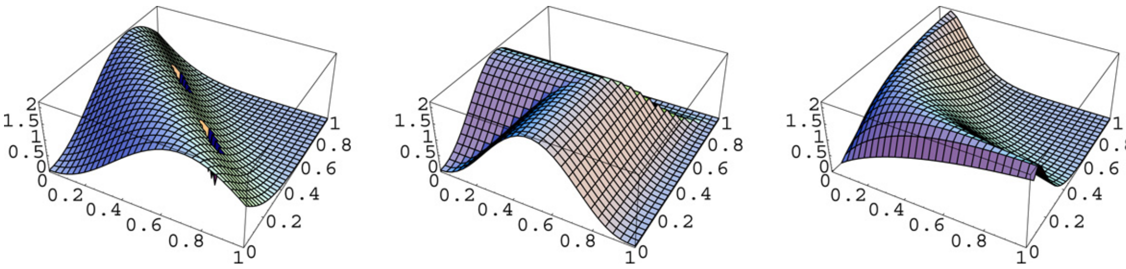

Fig. 1. Bivariate densities generated from the uniform family densities (6.2)–(6.4), from left to right (when $\alpha = 2$ and $\beta = 4$).

6. Examples

In Section 5, the following general class of multivariate distributions was developed, beginning with a basic density $f_{\underline{X}}(\underline{x})$ and an auxiliary function $h(\underline{x})$ satisfying $0 \le h(\underline{x}) \le 1$:

$$f_{\alpha,\beta,h}(\underline{x}) \propto f_{\underline{X}}(\underline{x})[h(\underline{x})]^{\alpha-1}[1-h(\underline{x})]^{\beta-1}. \tag{6.1}$$

If, when $\underline{X} \sim f_0(\underline{x})$, $h(\underline{X}) \sim \mathcal{U}(0,1)$, then it is readily verified that the required normalizing constant in (6.1) is $[B(\alpha,\beta)]^{-1}$. One way to assure this is to begin with an arbitrary function $g(\underline{x})$ and compute, assuming $X \sim f_0(\underline{x})$, $\Pr(g(\underline{X}) \le t) = F_g^*(t)$. Now define $h(\underline{x}) = F_g^*(g(\underline{X}))$. By construction we will have $h(\underline{X}) \sim \mathcal{U}(0,1)$ when $\underline{X} \sim f_0(\underline{x})$.

Example 1 (*A Uniform Case*). For simplicity assume $m = 2$ and assume that $f_0(\underline{x}) = I(\underline{x} \in [0,1]^2)$.

Consider the function $g_1(\underline{x}) = x_1 + x_2$. If $\underline{X} \sim f_0(\underline{x})$, i.e. X_1, X_2 are i.i.d. $\mathcal{U}(0,1)$, then $g_1(\underline{X}) = X_1 + X_2$ has distribution function given by

$$F_{g_1}^*(t) = \begin{cases} \dfrac{t^2}{2}, & 0 < t < 1 \\ \dfrac{4t - t^2 - 2}{2}, & 1 < t < 2. \end{cases}$$

We then define $h_1(\underline{x}) = F_{g_1}^*(g_1(\underline{x})) = F_{g_1}^*(x_1 + x_2)$ and our model (6.1) becomes

$$\begin{aligned} f_{\alpha,\beta,h_1}(\underline{x}) = {} & \left\{ \left[\frac{(x_1+x_2)^2}{2}\right]^{\alpha-1} \left[1 - \frac{(x_1+x_2)^2}{2}\right]^{\beta-1} I(x_1+x_2<1) \right. \\ & \left. + \left[1 - \frac{(2-x_1-x_2)^2}{2}\right]^{\alpha-1} \left[\frac{(2-x_1-x_2)^2}{2}\right]^{\beta-1} I(x_1+x_2>1) \right\} [B(\alpha,\beta)]^{-1} I \quad (\underline{x} \in [0,1]^2). \end{aligned} \tag{6.2}$$

If instead we use $g_2(\underline{x}) = \max(x_1, x_2)$ we obtain the family

$$f_{\alpha,\beta,h_2}(\underline{x}) = \frac{(\max(x_1,x_2))^{2(\alpha-1)}[1-(\max(x_1,x_2))^2]^{\beta-1}}{B(\alpha,\beta)} I \quad (\underline{x} \in [0,1]^2). \tag{6.3}$$

As a third example, we use the function $g_3(\underline{x}) = x_1 x_2$. Using it we are led to the family

$$f_{\alpha,\beta,h_3}(\underline{x}) = \frac{(x_1x_2(1-\log(x_1x_2)))^{\alpha-1}(1-x_1x_2(1-\log(x_1x_2)))^{\beta-1}}{B(\alpha,\beta)} I \quad (\underline{x} \in [0,1]^2). \tag{6.4}$$

In Fig. 1, the densities (6.2)–(6.4) are displayed for when $\alpha = 2$ and $\beta = 4$.

Example 2 (*An Exponential Case*). Again assume $m = 2$ and this time assume that $f_0(\underline{x}) = \lambda^2 e^{-\lambda(x_1+x_2)} I(x_1 > 0, x_2 > 0)$. Thus when $\underline{X} \sim f_0(\underline{x})$, X_1 and X_2 are independent exponential random variables. If we use the function $g_1(\underline{x}) = x_1 + x_2$ in our construction we find that the distribution function of $g_1(\underline{X}) = X_1 + X_2$ is given by

$$F_{g_1}^*(t) = 1 - (1+\lambda t)e^{-\lambda t}.$$

So we define $h_1(\underline{x}) = F_{g_1}^*(x_1 + x_2)$ and our model (6.1) assumes the form

$$f_{\alpha,\beta,h_1}(\underline{x}) = \lambda^2 e^{-\lambda(x_1+x_2)} \frac{(1-e^{-\lambda(x_1+x_2)}(1+\lambda(x_1+x_2)))^{\alpha-1}(e^{-\lambda(x_1+x_2)}(1+\lambda(x_1+x_2)))^{\beta-1}}{B(\alpha,\beta)} \times I \quad (x_1 > 0, x_2 > 0). \tag{6.5}$$

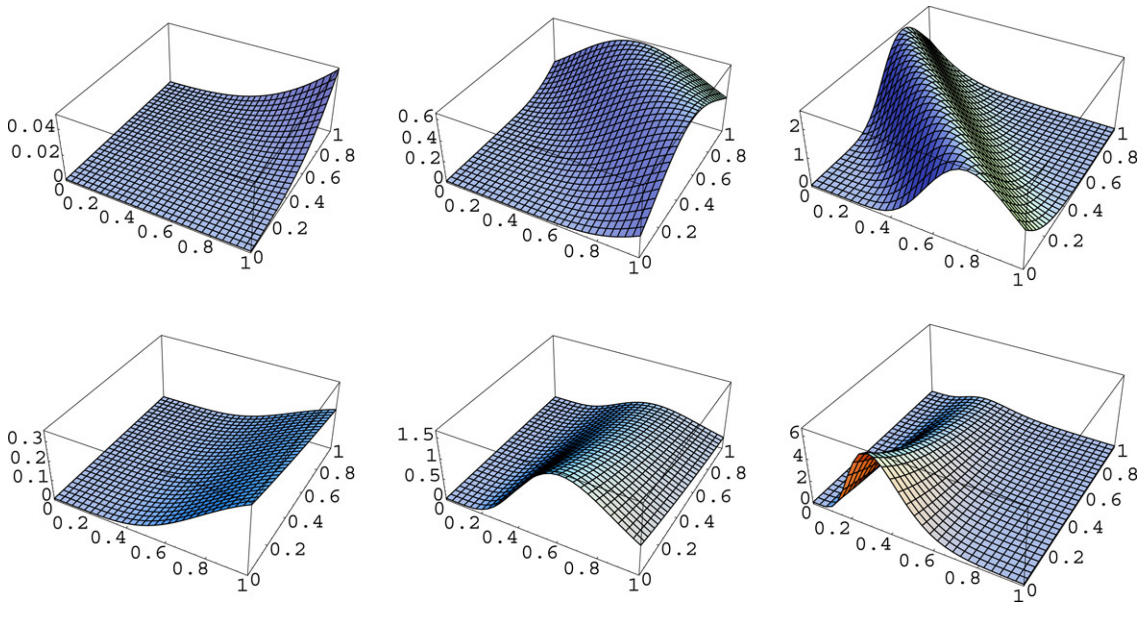

Fig. 2. Bivariate densities generated from the exponential family densities (6.5) (upper figures) and (6.6) (lower figures) with $\alpha = \beta = 8$ and $\lambda = 0.5$, 1 and 2 (from left to right).

If instead we use the function $g_2(\underline{x}) = x_1$, we eventually obtain, from (6.1), the family of densities

$$f_{\alpha,\beta,h_2}(\underline{x}) = \lambda^2 \mathrm{e}^{-\lambda(x_1+x_2)} \frac{(1-\mathrm{e}^{-\lambda x_1})^{\alpha-1}\mathrm{e}^{-\lambda(\beta-1)x_1}}{B(\alpha,\beta)} I \quad (x_1 > 0, x_2 > 0). \tag{6.6}$$

Sample densities of these forms are illustrated in Fig. 2 from the exponential family densities (6.6) (upper figures) and (6.6) (lower figures) with $\alpha = \beta = 8$ and $\lambda = 0.5$, 1 and 2 (from left to right).

7. More general conditional distributions

Motivated by the discussion provided in [9] for conditional ordering the following results may be verified for $(F_{Y|\underline{X}})$-order statistics.

Begin with $(\underline{X}_i, Y_i)$, $i = 1, \ldots, n$, as i.i.d. random variables where the Y_i's have a common continuous distribution. Define $\underline{X}_{j:n}$ to be the jth smallest $\underline{X}$ (actually the concomitant of $Y_{j:n}$), $j = 1, 2, \ldots, n$. It follows that the conditional distribution of $\underline{X}_{j_1+1:n}, \ldots, \underline{X}_{j_2-1:n}$ given that $Y_{j_1:n} = y_1$ and $Y_{j_2:n} = y_2$ is identical to the distribution of the $j_2 - j_1 - 1$ $\underline{X}$-concomitants of a sample from the joint distribution of $(\underline{X}, Y)$ truncated to the set $\{(\underline{x}, y) : y_1 < y < y_2\}$.

Point process distributions analogous to these described in [9] may also be formulated in the more general context in which Y is a non-deterministic function of $\underline{X}$, as contrasted with the case in which Y is a deterministic function of $\underline{X}$ as described in [6,9,8].

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