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Color ferromagnetism of quark matter and quantum Hall states of gluons in SU(3) gauge theory

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Abstract

We show a possibility that a color ferromagnetic state exists in SU(3) gauge theory of quark matter with two flavors. Although the state involves three types of unstable modes of gluons, all of these modes are stabilized by forming a quantum Hall state of one of the modes. We also show that at large chemical potential, a color superconducting state (2SC) appears even in the ferromagnetic state. This is because Meissner effect by condensed anti-triplet quark pairs does not work on the magnetic field in the ferromagnetic state.

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One of the most intriguing phases possessed by dense quark matter is the color superconductivity [1] caused by the condensation of quark pairs with two flavors. The condensation breaks the gauge symmetry of SU(3) to SU(2). Consequently, some of gluons gain masses due to the condensation and Meissner effect operates in the magnetic field of the gluons. In the case of real superconductor, the condensation of Cooper pairs of electrons breaks U(1) gauge symmetry so that the magnetic field associated with the U(1) gauge symmetry is expelled from or squeezed in the superconductor. Therefore, both ferromagnetism and superconductivity are not realized simultaneously in real condensed matter. In the quark matter, however, both phenomena [2] can be realized simultaneously because magnetic fields in remaining SU(2) gauge symmetry are not affected by the condensation of the quark pair.

We have shown in the previous paper [2] that a color ferromagnetic state of quark matter arises in the SU(2) gauge theory, in which a color magnetic field is generated spontaneously. Although the magnetic field induces unstable modes of gluons [3,4], the modes have been shown to be stabilized by the formation of a quantum Hall state [5] of the unstable gluons. The quantum Hall state possesses a color charge. The charge is supplied by quark

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matter. Consequently, in the quark matter the stable ferromagnetic state in the SU(2) gauge theory exists along with the quantum Hall state of the gluons.

In this Letter we point out in the SU(3) gauge theory that a stable color ferromagnetic state [3] of quark matter also exists along with a quantum Hall state of gluons. In the state the diagonal color magnetic field $\vec{B} \propto \cos\theta\lambda_3 + \sin\theta\lambda_8$ is spontaneously generated. The magnetic field induces three types of unstable modes, each of which has different color charges. One of them condenses to form a quantum Hall state. The condensation makes the ferromagnetic state be stabilized. Furthermore, the state coexists with 2 flavor color superconducting state (2CS). Namely, in a ferromagnetic state with $B \propto \lambda_3$ ($\theta = 0$), an anti-triplet quark pair $\langle \epsilon^{3jk} q_j q_k \rangle$ can condense [1]. Since the magnetic field $B \propto \lambda_3$ does not couple with the quark condensation, the Meissner effect does not operate on the field. In this way the coexistence of both ferromagnetism and superconductivity is possible in the SU(3) gauge theory. This is owing to the fact that the SU(3) gauge group has the maximal Abelian subgroup of $U(1) \times U(1)$; each group of U(1) is associated with superconductivity or ferromagnetism, respectively. There is a critical chemical potential which separates the phase of only the ferromagnetic state without the superconductivity and the coexistence phase of both states.

We assume in the Letter that loop calculations or perturbative calculations give physically correct results in the ferromagnetic phase although the gauge coupling constant is not necessarily small in the energy regime (500–1000 MeV) of our consideration. (The existence of the nontrivial minimum of the effective potential in eB has been proved [6] beyond the loop approximation under the reasonable assumption that the gauge coupling constant allows the Landau singularity. Thus, our results depending on this existence is expected to hold even for large coupling constant.)

As is well known [3,4], the one-loop effective potential V of the constant color magnetic field in the SU(n_c) gauge theory with n_f flavors is given by $V_{\text{eff}} = \frac{11N}{96\pi^2} g^2 B^2 (\log(gB/\Lambda^2) - \frac{1}{2}) - \frac{i}{8\pi} g^2 B^2$, with an appropriate renormalization [6] of the gauge coupling g, where $N = n_c - 2n_f/11$. Hereafter we consider the SU(3) gauge theory with massless quarks of two flavors; $N = 29/11$. But, most of our results (e.g., the existence of the ferromagnetic phase) hold even for massive quarks. The potential implies the spontaneous generation of a color magnetic field. The direction of the magnetic field in real space can be arbitrarily chosen and the direction in color space can be taken in general such as B is in the maximal Abelian sub-algebra; $B = |B|(\cos\theta\lambda_3 + \sin\theta\lambda_8)$, where λ_i are Gell-Mann matrices and we restrict the value of θ such as $-\pi/6 \leqslant \theta \leqslant \pi/6$ due to the Weyl symmetry. In any case of their choices the spontaneous generation of the magnetic field breaks the spatial rotational symmetry and the SU(3) gauge symmetry into the gauge symmetry of $U(1) \times U(1)$. It is interesting to note that the hypothesis of "Abelian dominance" [7] holds exactly in this ferromagnetic phase since physics at long wave length is governed by only the gauge fields of maximal Abelian gauge group $U(1) \times U(1)$.

Although the magnetic field is spontaneously generated, this state is known to be unstable due to the presence of the imaginary part in V_{eff}. Namely, there are unstable modes [4,6] of gluons in the ferromagnetic state. These modes generate the imaginary part in V_{eff}. In general, unstable modes are excited to lead to a stable state by their condensation. It is, however, non-trivial to find the stable state of gluons in the gauge theory.

As we have shown in the SU(2) gauge theory [2], such unstable gluons occupy the lowest Landau level and condense to form a fractional quantum Hall state. Quantum Hall states of electrons [5] are known to have a finite gap like BCS states and be stable. Similarly, the quantum Hall state of the gluons has been shown to be stable; unstable modes disappear in the quantum Hall state. (According to numerical simulations [8], Laughlin states representing fractional quantum Hall states are shown to arise even for the bosons just like gluons.)

The condensed gluons leading to the quantum Hall state possess a color charge, in other words, the color charge condenses in the state. Such a color charge of the condensed gluons must be supplied by others in a neutral system. Quark matter is such a supplier. Therefore, the stable ferromagnetic state involving the quantum Hall state of gluons can arise in the dense quark matter. (Since the energy density of the quarks in the presence of the magnetic field is lower [2] than that of the quarks without the magnetic field, the spontaneous generation of the magnetic field is favored also in the quark sector.) In contrast, the state does not arise as a spatially uniform vacuum state of the gauge theory. There are no suppliers of the color charges in the vacuum. The unstable modes cannot form quantum

Hall states. Instead, their large fluctuations would lead to "spaghetti vacuum" [6,9] in which quark confinement would be realized.

Now, we wish to study unstable modes of gluons in the SU(3) gauge theory and to see how they are stabilized by forming a quantum Hall state. In the SU(3) gauge theory there are three types of unstable modes in general. They are identified as modes having an imaginary part in their energy spectra. In general, spectra of the gluons in the magnetic field B are given by $E^2 = 2g_i B(n + 1/2 \pm 1) + k_3^2$ where g_i is the coupling strength with the magnetic field. n, ± 1 and k_3 denote Landau level, spin and momentum parallel to $\vec{B}$, respectively. The imaginary part in E comes from the contribution of their anomalous magnetic moments, which corresponds to the negative term in E^2. Obviously, the unstable modes are those occupying the lowest Landau level ($n = 0$) with spin parallel (-1) to $\vec{B}$ and with small momentum $k_3^2 < g_i B$. Hence, the identification of the unstable modes can be done by looking for such modes with the anomalous magnetic moments. In general, there are 6 gluons A'_μ which can couple with B, i.e., $[A'_\mu, B] \neq 0$. Thus, 3 complex fields can be composed of the 6 real fields of the gluon. They are color charged vector fields coupled with B and possible unstable modes. Actually, unstable modes are easily identified from the SU(3) gauge fields as,

$$\Phi_1 = (A_1 + iA_2)/\sqrt{2}, \quad \Phi_2 = (A_4 + iA_5)/\sqrt{2} \quad \text{and} \quad \Phi_3 = (A_6 - iA_7)/\sqrt{2}, \tag{1}$$

where A_a is defined as a spatial component of the gauge field with particular polarization; $A_a^\mu = e^\mu A_a$ with $e^\mu = (0, 1, i, 0)$. These modes occupy the lowest Landau level with spin parallel to the magnetic field. They have conserved charges of $U(1) \times U(1)$ such that $(Q_3(\Phi_1) = 1,\ Q_8(\Phi_1) = 0)$, $(Q_3(\Phi_2) = 1/2,\ Q_8(\Phi_2) = \sqrt{3}/2)$ and $(Q_3(\Phi_3) = 1/2,\ Q_8(\Phi_3) = -\sqrt{3}/2)$. To clarify that these modes are really unstable, we see a Lagrangian for these modes by extracting them from the action of the SU(3) gauge field; we simply neglect the other stable modes,

$$L_{\text{unstable}} = \sum_{s=1,2,3} \left(\left| \left(i\partial_\mu - g_s A_\mu^B \right) \Phi_s \right|^2 + 2g_s B |\Phi_s|^2 \right) - V(\Phi), \tag{2}$$

where $g_1 = g\cos\theta$, $g_2 = g(\cos\theta + \sqrt{3}\sin\theta)/2$, $g_3 = g(\cos\theta - \sqrt{3}\sin\theta)/2$ and the potential, $V(\Phi) = g^2((\sum_{s=1}^{3} |\Phi_s|^2)^2 - 3|\Phi_2\Phi_3|^2)$. A_μ^B represents a gauge potential of the magnetic field B. We have used the fact that the modes Φ_s occupy the lowest Landau level. We should note that all of g_i are positive or zero in the range of $-\pi/6 \leqslant \theta \leqslant \pi/6$.

We can see the negative mass terms $(-2g_s B|\Phi_s|^2)$ of the modes, Φ_s. It implies the instability of the state, $\langle \Phi_s \rangle = 0$. Thus, they are unstable modes. This situation is very similar to a model of complex scalar field with double-well potential. Namely, in the model with a potential such as $-m^2|\Phi|^2 + \lambda|\Phi|^4/2$, the state such that $\langle \Phi \rangle = 0$ is unstable. Thus, a spatially uniform unstable mode (the mode is given by the fluctuation, $\delta\Phi$ with zero momentum, $\vec{k} = 0$, around the state $\langle \Phi \rangle = 0$) is excited to condense and form a stable state $\langle \Phi \rangle = m/\sqrt{\lambda}$. In this simple model the spatially uniform unstable mode $\delta\Phi(\vec{k} = 0)$ is unique. On the other hand, there is no such a solution as $\langle \Phi_s \rangle = \text{const} \neq 0$ in the gauge theory described by Eq. (2). This is because the gauge potential A_μ^B has a spatial dependence. Physically, not only there are no spatially uniform modes but also there are infinite number of degenerate unstable modes in the magnetic field. That is, the unstable modes Φ_s in the lowest Landau level have such a form of the wave functions as $\exp[-ik_2x_2 - \frac{1}{2}g_s B(x_1 - k_2/g_s B)^2]$, where k_2 denotes a momentum perpendicular to $\vec{B} = (0, 0, B)$. Here we have taken only modes with $k_3 = 0$ uniform in the direction parallel to B. Obviously, all of the modes labeled by the parameter k_2 are infinitely degenerate. The parameter indicates the location of the modes in the coordinate of x_1. We can see that none of these unstable modes is spatially uniform. Since each type of the modes involves infinitely degenerate nonuniform states, it is not trivial to find a uniform stable state formed by the condensation of the modes. Actually, it was a very difficult task to find such a stable state formed by the unstable modes. We need to take into account repulsive interactions $V(\Phi)$ between the modes in order to find the state. Although a reasonable variational state has been proposed [6], the state has no spatial uniformness and has not yet been shown to be really stable.

In the case of the SU(2) gauge theory, only one type of the unstable mode exist with a repulsive interaction term such as $|\Phi|^4$. We have shown that the mode forms a quantum Hall state, in which the instability disappears. On the other hand, in the SU(3) gauge theory we have three types of the unstable modes with their repulsive interaction terms $V(\Phi_s)$ more complicated than the term $|\Phi|^4$ in the case of the SU(2) gauge theory. But we can show that one (Φ_1) of the modes condenses to form a stable quantum Hall state. Consequently the instability of the mode disappears just as in the SU(2) gauge theory. The other modes ($\Phi_{2,3}$) gain sufficiently large positive mass terms from the condensation of Φ_1. Since such positive masses are larger than the bare negative masses, the modes obtain positive mass terms in consequence. In this way all of the unstable modes are stabilized.

It is important to note that the relevant unstable modes should be uniform in a coordinate of x_3. Namely, the mode with $k_3 = 0$ has stronger instability than those with $k_3 \neq 0$ since it grows up more rapidly in time; the wave function of the mode with $E(k_3) = i\sqrt{g_s B - k_3^2}$ grows up just as $e^{|E(k_3)|t}$. Thus, we take only such a mode with $k_3 = 0$. Accordingly, the gluons of the field Φ_s are spatially 2-dimensional ones just like electrons in 2-dimensional quantum wells, which may form quantum Hall states under external magnetic field. Consequently, we can use a Chern–Simons gauge theory in spatial 2 dimensions for the discussion of the unstable modes

It is well known that the Chern–Simons gauge theory [5] is very useful for analysis of quantum Hall states. That is, we consider so-called composite gluons which are bosons with fictitious magnetic flux. The flux is expressed by the Chern–Simons gauge field. The point for the use of such composite gluons is that only when the fictitious flux cancels with the magnetic field B, the bosons can condense to form quantum Hall states with spatially uniformness. This way of understanding the quantum Hall state is well known in condensed matter physics; the picture of "composite electrons" is used as very powerful tool for the analysis of quantum Hall states. Thus, we apply the method to the physics of the gluons.

Using Chern–Simons gauge fields, we write down spatially 2-dimensional Lagrangian of the composite bosons representing the unstable modes in the following way,

$$L_a = \left|\left(i\partial_\mu - g_1 A_\mu^B + a_\mu\right)\phi_1\right|^2 + 2g_1 B|\phi_1|^2 - \frac{\epsilon^{\mu\nu\lambda}}{4\alpha} a_\mu \partial_\nu a_\lambda + \sum_{s=2,3} \left(\left|\left(i\partial_\mu - g_s A_\mu^B\right)\phi_s\right|^2 + 2g_s B|\phi_s|^2\right) - V_a(\phi), \tag{3}$$

with $V_a = V/l_3$ (l_3 being the length scale of quark matter in 3 direction) and $\phi_s = \Phi_s\sqrt{l_3/2}$ for $s = 2, 3$. We have introduced a Chern–Simons gauge field only for the unstable mode, ϕ_1, which has the largest negative mass term and is expected to compose quantum Hall states. This Lagrangian describes the composite bosons ϕ_1 attached by Chern–Simons flux $\epsilon^{ij}\partial_i a_j$ (fictitious magnetic flux). α should be chosen to be $2\pi\ \times$ integer in order to guarantee that the field ϕ_1 describes boson. Hereafter, we take $\alpha = 2\pi$ for simplicity. The equivalence of this Lagrangian, L_a and L_{unstable} has been demonstrated in the operator formalism [10] although the equivalence had been known in the path integral formalism using the world lines of the ϕ_a particles.

In order to see that the unstable modes denoted by ϕ_1 form a stable quantum Hall state, we derive equations of motion for the field ϕ_1,

$$\phi_1^\dagger i\partial_0\phi_1 + \text{c.c.} + 2a_0|\phi_1|^2 = \frac{1}{4\pi}\epsilon_{ij}\partial_i a_j, \tag{4}$$

$$\phi_1^\dagger\left(i\partial_i - g_1 A_i^B + a_i\right)\phi_1 + \text{c.c.} = \frac{1}{4\pi}\epsilon_{ij}(\partial_0 a_j - \partial_j a_0), \tag{5}$$

$$(i\partial_0 + a_0)^2\phi_1 - \left(i\vec{\partial} - g_1\vec{A}^B + \vec{a}_1\right)^2\phi_1 + 2g_1 B\phi_1 = \frac{\partial}{\partial\phi_1^\dagger}V_a. \tag{6}$$

We can easily find a solution of a spatially uniform state $\langle\phi_1\rangle = v_1$ ($\langle\phi_2\rangle = \langle\phi_3\rangle = 0$) only when the Chern–Simons flux cancels with B, that is, $g_1\vec{A}^B = \vec{a}$. This cancellation is only possible for a_0 and v_1 satisfying

$$2a_0v_1^2 = \frac{g_1B}{4\pi} \quad \text{and} \quad a_0^2v_1 + 2g_1Bv_1 = \frac{\partial V_a}{\partial\phi_1^\dagger} = \frac{2g^2}{l_3}v_1^3 \tag{7}$$

or in dimensionless notations

$$2\bar{a}_0\bar{v}_1^2 = \frac{\cos\theta}{4\pi} \quad \text{and} \quad \bar{a}_0^2 + 2\cos\theta = \frac{2g^2}{\bar{l}_3}\bar{v}_1^2, \tag{8}$$

where $\bar{a}_0(\theta)$, $\bar{v}_1^2(\theta)$ and $1/\bar{l}_3$ are dimensionless quantities normalized in unit of $(gB)^{1/2}$.

Solving Eqs. (7) or (8) for v_1 and a_0, and inserting v_1 into the potential $V(\phi)$, we can see that the modes ϕ_2 and ϕ_3 gain a sufficiently large positive mass ($= 2g^2v_1^2/l_3 = 2g_1B + a_0^2$) caused by the condensation of the mode ϕ_1. Therefore, the masses of $\phi_{2,3}$ become positive owing to this Higgs mechanism because $2g_1B + a_0^2$ is larger than their original negative masses, $-2g_2B$ and $-2g_3B$, respectively; note that $g_1(\theta) > g_2(\theta), g_3(\theta)$ for $-\pi/6 < \theta < \pi/6$. Hence, the instability of the state, $\langle\phi_{2,3}\rangle = 0$ is removed. As in the SU(2) gauge theory, the state $\langle\phi_1\rangle = v_1$ is also stable. In this way, the condensation of the field, ϕ_1 leads to the stable ferromagnetic state.

It is well known in the picture of the composite boson that the state $\langle\phi_1\rangle = v_1$ represents a quantum Hall state of the ϕ_1 particles with so-called filling factor, $\nu = 2\pi\rho_{g1}/g_1B$, being equal to 1/2; ρ_{g1} ($= \phi_1^\dagger i\partial_0\phi_1 + \text{c.c.} + 2a_0|\phi_1|^2$) denotes a number density of ϕ_1 particles. It is easy to show [11] that Hall conductivity of the state is given by $2\pi\nu/g_1$ as expected in usual quantum Hall states. Similarly, we can show that the state has a finite gap, E, by solving a small fluctuation around $\phi_1 = v_1$ in Eq. (4); $E = 2\sqrt{a_0^2 + g_1^2v_1^2/l_3}$. Therefore, all of the unstable modes become harmless owing to the formation of a quantum Hall state of the mode ϕ_1. Obviously, the result holds irrespective of quark mass.

Next, we wish to determine the direction θ of B in the color space. The determination is not trivial. In order to determine θ explicitly, we need to take account of the physical conditions, that is, color neutrality of quark matter and the minimum of its energy including the energy of the gluon condensate; both the energies of the quark and the condensate, $\langle\phi_1\rangle$, of the unstable modes depend on the direction.

We first impose the color neutrality conditions, $\langle\lambda_3\rangle = \langle\lambda_8\rangle = 0$, where the average should be taken over all of the quarks and the gluon condensate; we should note that owing to the generation of B, the SU(3) gauge symmetry is broken into the gauge symmetry of $U(1)\times U(1)$ so that only conserved color charges are those associated with the group generators of $\lambda_{3,8}$. It is easy to see that the condition implies that number density ρ_i of quarks with color type i satisfies the following equations, $\rho_1 + \rho_{g1} = \rho_2$ and $\rho_1 + \rho_2 = \rho_3$. (We denote the color types of quarks such that $q_1 = (1,0,0)$, $q_2 = (0,1,0)$ and $q_3 = (0,0,1)$.)

Furthermore, we impose the condition of the energy minimum. In order to calculate the energy of quarks, we note that the coupling strength e_i of ith quark with the magnetic field is given by $e_1 = g(\cos\theta + \sin\theta/\sqrt{3})/2$, $e_2 = g|-\cos\theta + \sin\theta/\sqrt{3}|/2$, and $e_3 = g|-2\sin\theta/\sqrt{3}|/2$, where g is the gauge coupling constant. Then, the energy of ith massless quark in the magnetic field is given by $\sqrt{e_iB(2n+1\pm1)+k_3^2}$ where integer $n \geqslant 0$ denotes Landau level and ±1 does the quark spin; momentum k_3 is a component parallel to B. Minimizing the total energy density of the three types of quarks in θ is difficult in general. As a simple example, we calculate it in a case where all of quarks occupy the lowest Landau level ($n = 0$) with spin parallel (-1) and the condensation of gluons ρ_{g1} is negligible. Then, the energy density of ith quark is given by $\epsilon_i = n_\mathrm{f}e_iBk_{\mathrm{f}i}^2/4\pi^2 = \pi^2\rho_i^2/n_\mathrm{f}e_iB$ where $k_{\mathrm{f}i}$ denotes the Fermi momentum and $n_\mathrm{f} = 2$ denotes the number of flavors. Here we have used the expression of the number density of quarks, ρ_i, in terms of the Fermi momentum $k_{\mathrm{f}i}$; $\rho_i = n_\mathrm{f}e_iBk_{\mathrm{f}i}/2\pi^2$. Since all of the quark densities are identical, $\rho_i = \rho$, when $\rho_{g1} = 0$, we find that the total energy density, $\sum_{i=1-3}\epsilon_i = \pi^2\rho^2(1/e_1 + 1/e_2 + 1/e_3)/n_\mathrm{f}B$, takes the minimum value at $\theta = \pi/6$. This is a simple case of all quarks occupying the lowest Landau level (namely much small $\rho/(gB)^{3/2} \ll 1$) and of the negligible gluon contribution. But actually, the gluon contribution to the energy is dominant over the quark contribution since the energy density of the quarks occupying higher Landau levels depends very weakly on the direction θ of the color magnetic field.

In order to see it we calculate the energy of the gluon condensate depending on θ. Using the solutions of Eq. (8), we find that the energy density of the condensate normalized by $(gB)^2$ is given by

$$\bar{E}_v = \left(\bar{a}_0^2(\theta)\bar{v}_1^2(\theta) - 2\bar{v}_1^2(\theta)\cos\theta + \frac{g^2}{\bar{l}_3}\bar{v}_1^4(\theta)\right)\frac{1}{\bar{l}_3} = \left(2\bar{a}_0^2(\theta)\bar{v}_1^2(\theta) - \frac{g^2}{\bar{l}_3}\bar{v}_1^4(\theta)\right)\frac{1}{\bar{l}_3}, \tag{9}$$

where we have represented the dependence of θ explicitly. Note that we have the factor of $1/\bar{l}_3$ in the above equation in order to obtain the three-dimensional energy density, $\bar{E}_v$, from two-dimensional Lagrangian in Eq. (3). We have proved numerically that the energy of the condensate takes the minimum at $\theta = 0$. Actual values of $\bar{E}_v$ at $\theta = 0$ are as follows. $\bar{E}_v\bar{l}_3 = -0.02$, -0.08, -1, -10 for $g^2/\bar{l}_3$ $(= g^2/(l_3\sqrt{gB})) = 20$, 10, 1, 0.1, respectively. Smaller coupling constants, $g^2/\bar{l}_3$, give rise to negatively larger energies. We do not know how large the coupling parameter g^2 is. We tentatively assume that g^2 is of the order of 1 or less for the consistency of our calculation. Then, assuming $l_3 = 1$–3 fm in real quark matter produced by heavy ion collisions and assuming $\sqrt{gB} \sim 200$ MeV, we obtain $\bar{l}_3 = 1$–3 and the order of the magnitude of the coupling constant $g^2/\bar{l}_3$ being $O(0.1)$–$O(1)$. Hence, the energy of the condensate takes a value, at least, such as $\bar{E}_v \simeq -1$ at $\theta = 0$. On the other hand, the energy density, E_B, of the quarks normalized by $(gB)^2$ is approximately given by $E_{B=0} = (3\pi^{2/3}/2^{7/3})(\rho/(gB)^{3/2})^{4/3} \simeq 1.28(\rho/(gB)^{3/2})^{4/3}$, i.e., the energy density of the quarks in the absence of the magnetic field. Namely, E_B is nearly equal to $E_{B=0}$ for large quark number density ρ; $E_B = E_{B=0}$ in the limit of $\rho/(gB)^{3/2} \to \infty$. Note that $E_{B=0}$ does not depend on θ. This indicates that θ dependent part of E_B is much small. For instance, $E_B - E_{B=0}$ at $\theta = 0$ is about 0.09 (0.13) for $\rho/(gB)^{3/2} = 10$ (1000). Therefore, θ dependence of the energy is determined by $\bar{E}_v$. Thus, we conclude that the direction of the magnetic field B is chosen such as $\theta = 0$, which gives the minimum energy of the quarks and the gluons. Anyway, the condensation of ϕ_1 makes B point to the direction of λ_3 in maximal Abelian subalgebra.

A comment is in order. To make a quantum Hall state, we can make all of the fields ϕ_s condense by introducing different Chern–Simons gauge fields, $\vec{a}_s$ to each ϕ_s. Then, we have such a state as $\langle\phi_s\rangle = v_s \neq 0$ for $s = 1$–3. But we can show that the state is not stable because the fluctuations $\delta\phi_s$ have an imaginary frequency in their spectra. On the other hand, the formation of the quantum Hall state of only the field ϕ_1 makes the state $\langle\phi_1\rangle \neq 0$, $\langle\phi_2\rangle = 0$ and $\langle\phi_3\rangle = 0$ stable as we have explained above. Similarly we can show that a quantum Hall state such as $\langle\phi_1\rangle = 0$, $\langle\phi_2\rangle \neq 0$ and $\langle\phi_3\rangle \neq 0$, is also stable. In the state two fields $\phi_{2,3}$ condense. Thus, the state possibly exists although we consider only the case of the condensed state of ϕ_1 in this Letter.

Up to now, we have shown that there are three types of the unstable modes of gluons induced by the color magnetic field generated spontaneously. These unstable modes are stabilized by the formation of the quantum Hall state of the unstable mode ϕ_1. Owing to the generation of the magnetic field, the gauge symmetry SU(3) is broken into the gauge symmetry $\mathrm{U}(1) \times \mathrm{U}(1)$. Furthermore, the formation of the quantum Hall state ($\langle\phi_1\rangle = v_1$) breaks the gauge symmetry into the gauge symmetry of U(1). This is the case for the color magnetic field whose direction, θ, is general. On the other hand, one of the unstable modes disappears at specific values of θ. For example, g_3 vanishes for $\theta = \pi/6$ so that the negative mass $(-2g_3B)$ of the field ϕ_3 vanishes. ϕ_3 becomes massless. Thus, the mode of ϕ_3 is stable. In this case, the gauge symmetry SU(3) is broken into the gauge symmetry $\mathrm{SU}(2) \times \mathrm{U}(1)$. Namely, there are 4 massless real fields; gauge fields of the maximal Abelian gauge group and $A_\mu^{6,7}$ corresponding to ϕ_3. Subsequent formation of a quantum Hall state with $\langle\phi_1\rangle = v_1$ breaks the symmetry into the gauge symmetry of U(1). Stability of the state $\langle\phi_{2,3}\rangle = 0$ is guaranteed by the positivity of their masses produced by the condensation of the field ϕ_1; ϕ_3 becomes massive as well as ϕ_2. In any case, stable ferromagnetic states accompanied by quantum Hall states are realized in the SU(3) gauge theory. Since the gauge symmetry is broken, the state is not a real vacuum of QCD, but a state of quark matter.

We should note that the color charge $Q_3(\phi_1)$ of the condensate $\langle\phi_1\rangle$ must be supplied by the quark matter produced by heavy ion collisions. Unless the chemical potential of the quark matter is large enough to supply the charge, such a condensation cannot occur, so that the stable ferromagnetic state cannot arise. Thus, we wish to estimate the minimum chemical potential of the quark matter needed for the formation of the condensate. We note that two-dimensional color charge density of the gluon condensation is given by $\rho_{g1} = g_1 B/4\pi$, namely the number density of the ϕ_1 particles. When the radius of the quark matter is l_3, the three-dimensional density, ρ_g, of the condensed gluons is approximately given such that $\rho_g = \rho_{g1}/l_3$. On the other hand, the color charge density of u and d quarks is $\frac{1}{2} \times 2 \times k_{\mathrm{f}}^3/3\pi^2$ where k_{f} denotes a Fermi momentum; the factors 1/2 and 2 come from the coupling strength in unit of g and the number of the flavor, respectively. The chemical potential, μ, of the quark at zero temperature is given by $\sqrt{k_{\mathrm{f}}^2 + m_q^2}$ where we have taken account of the quark mass, $m_q \sim 300$ MeV. Hence, by equating both densities we obtain the lower bound of the chemical potential such that

$$\mu = \sqrt{\left((3\pi g_1 B/4l_3)^{1/3}\right)^2 + m_q^2}$$
$$\sim 350\ \text{MeV}\sqrt{\left(\frac{180}{350}\right)^2 \frac{(\langle gB\rangle/0.04\ \text{GeV}^2)^{2/3}}{(l_3/3\ \text{fm})^{2/3}} + \left(\frac{300}{350}\right)^2 (m_q/300\ \text{MeV})^2}, \tag{10}$$

where we have referred to a typical scale of QCD as $\langle gB\rangle$. Therefore, the ferromagnetic state of the quark matter may arise in heavy ion collisions at chemical potentials larger than 350 MeV.

We have discussed the ferromagnetic phase of the SU(3) gauge theory. The phase appears at chemical potentials larger than a critical one, below which the hadronic phase is present. We already know the presence of a color superconducting phase (2SC) in much large chemical potential. Thus, we wish to ask which state arises in the quark matter, the color superconducting state or the color ferromagnetic state. Although both states do not appear simultaneously in the ordinary matter owing to Meissner effects, they are compatible in the SU(3) gauge theory without any contradictions.

In order to analyze the possibility, we note that the direction of the color magnetic field is pointed such as $\theta = 0$. Thus, an anti-triplet quark pair condensation such as $\epsilon^{ijk}\langle q_j q_k\rangle = (0, 0, u_3)$ may arise because the magnetic field $B \propto \lambda_3$ is not inhibited by the Meissner effect; the magnetic field does not couple with the condensation. Therefore, both states may arise simultaneously in a dense quark matter.

Explicit calculations have been done [12] within a NJL model where the effects of the color magnetic field, $B \propto \lambda_3$ on the chiral condensation and the quark pair condensation have been discussed. The authors of the paper considered such a model in order to see the effect of vacuum fluctuations of the color magnetic field, $\langle B^2\rangle \neq 0$ on the quark matter. On the other hand, as we have shown, such a color magnetic field is generated spontaneously. Using their results, we find that the magnetic field induces the chiral condensation at any chemical potentials less than a critical one, beyond which the quark pair condensation arises; 2CS appears. It is interesting that the chiral condensation is not compatible with the quark pair condensation. Thus, in the 2CS, the chiral symmetry is not broken. The point we learn from the Letter is that the ferromagnetic phase can coexist with the color superconducting phase.

It seems apparently that the pointing of B to λ_3 is necessary for the existence of the superconducting state. We remember that our conclusion of $\theta = 0$ has been obtained by minimizing the energy of the quark matter under the assumption of g^2 being the order of 1. Without the assumption the conclusion of $\theta = 0$ cannot be obtained. However, the magnetic field naturally orients to λ_3 when the superconducting state arises in the ferromagnetic state. This is because the presence of such a state is energetically more favored than the absence of the state; the state with $\theta = 0$ involving the superconducting state is energetically favored at large chemical potentials than a state with $\theta \neq 0$ involving no superconducting state.

To summarize, we have found that the phase of the quark matter at zero temperature has the following structure. At small chemical potentials the hadronic phase with the broken chiral symmetry is present. It is unclear whether or not the ferromagnetic phase arises before the chiral symmetry is restored when we increase the chemical potential. But it is clear that the hadronic phase is changed into the ferromagnetic phase at a certain chemical potential. In the phase the color magnetic field, $\vec{B} \propto \cos\theta\lambda_3 + \sin\theta\lambda_8$, is generated spontaneously whose direction in the color space is pointed such as $\theta = 0$. A quantum Hall state of gluons, ϕ_1, is also formed in the phase. As a result, SU(3) gauge symmetry is broken into the U(1) gauge symmetry and the chiral symmetry is also broken. Further increase of the chemical potential makes a quark pair condensation $\epsilon^{ijk}\langle q_i q_j\rangle = (0, 0, u)$ arise so that the color superconducting phase (2CS) appears. This state does not expel the color magnetic field because the field does not couple with the quark pair condensate. In the phase of the coexistence of both states (ferromagnetic and superconducting states), the chiral symmetry is restored, but the U(1) gauge symmetry is broken due to the quark pair condensation. This is a brief picture of the phase of the quark matter in the SU(3) gauge theory with two flavors. At much larger chemical potentials strange quarks may play a role and they may form color flavor locking superconducting phase with u and d quarks. In the phase the color ferromagnetic state disappears.

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