Oscillation of a Kind of Two-Variables Functional Equation

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Abstract—In this paper, we consider the functional equation of the form

\[ aA(x, \sigma(y)) + bA(\tau(x), y) - cA(\tau(x), \sigma(y)) + P(x, y)A \left( \tau_{k+1}(x), \sigma_{l+1}(y) \right) = 0, \]

where \( a, b, c \) are positive constants, \( k, l \) are positive integers, function \( P : I \times I \rightarrow \mathbb{R}^+ = (0, \infty) \), \( I \subset (0, \infty) \) is an unbounded set, \( \tau, \sigma : I \rightarrow I \), \( \tau(x) \neq x \), \( \sigma(y) \neq y \), and \( \lim_{x \to \infty} \tau(x) = \infty \), \( \lim_{y \to \infty} \sigma(y) = \infty \), \( x, y \in I \). Some oscillation criteria of this equation are obtained. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Oscillation, Functional equations, Positive solutions.

1. INTRODUCTION

Consider the two-variables functional equation of the form

\[ aA(x, \sigma(y)) + bA(\tau(x), y) - cA(\tau(x), \sigma(y)) + P(x, y)A \left( \tau_{k+1}(x), \sigma_{l+1}(y) \right) = 0, \quad (1) \]

where \( a, b, c \) are positive constants, \( k \geq 1 \) and \( l \geq 1 \) are positive integers, function \( P : I \times I \rightarrow \mathbb{R}^+ = (0, \infty) \), \( I \subset (0, \infty) \) is an unbounded set, \( \tau, \sigma : I \times I \), \( \tau(x) \neq x \), \( \sigma(y) \neq y \), and \( \lim_{x \to \infty} \tau(x) = \infty \), \( \lim_{y \to \infty} \sigma(y) = \infty \), \( x, y \in I \). Define

\[ \tau^0(x) = x, \quad \tau^{i+1}(x) = \tau(\tau^i(x)), \quad i = 0, 1, \ldots, \quad x \in I \]

and

\[ \sigma^0(y) = y, \quad \sigma^{i+1}(y) = \sigma(\sigma^i(y)), \quad i = 0, 1, \ldots, \quad y \in I. \]

Let

\[ I_\alpha = [\alpha, \infty) \cap I, \quad \text{for } \alpha \in \mathbb{R}_+. \]
By a solution of (1), we mean a function $A : I \times I \rightarrow \mathbb{R}$ such that $\sup \{ |A(x,y)| : (x,y) \in I_{t_0} \times I_{t_0} \} > 0$ for any $x_0, y_0 \in I^1$ and $A$ satisfies (1) on $I \times I$. A solution $A(x,y)$ of (1) is said to be eventually positive if $A(x,y) > 0$ for all large $x$ and $y$, and eventually negative if $A(x,y) < 0$ for all large $x$ and $y$. It is said to be oscillatory if it neither eventually positive nor eventually negative.

Clearly, equation (1) includes delay partial difference equations as special cases. The oscillation of delay partial difference equation has been investigated in [1-8]. The oscillation of one-variable functional equations has been studied in [9-12]. In this paper, we shall obtain some sufficient conditions for all solutions of (1) to be oscillatory.

2. MAIN RESULTS

Define a set $E$ by

$$E = \{ \lambda > 0 \mid c - \lambda P(x,y) > 0, \text{ eventually} \}. \tag{2}$$

**Theorem 1.** Assume that

(i) $\limsup_{x,y \to \infty} P(x,y) > 0$, for $x, y \in I$;

(ii) there exist $X, Y \in I$ such that for $k > l$

$$\sup_{X \in E, x \in [X, Y]} \lambda \prod_{i=1}^{k} \left( c - \lambda P(\tau^i(x), \sigma^i(y)) \right) \prod_{j=1}^{k} \left( c - \lambda P(\tau^{i+j}(x), \sigma^i(y)) \right) < \left( \frac{2ab}{c} \right)^{k-l}, \tag{3}$$

and for $l > k$

$$\sup_{X \in E, x \in [X, Y]} \lambda \prod_{i=1}^{k} \left( c - \lambda P(\tau^i(x), \sigma^i(y)) \right) \prod_{j=1}^{l-k} \left( c - \lambda P(\tau^{k+j}(x), \sigma^{k+j}(y)) \right) < \left( \frac{2ab}{c} \right)^{k-l}. \tag{4}$$

Then every solution of (1) oscillates.

**Proof.** Suppose, to the contrary, we let $A(x,y)$ be an eventually positive solution. We define a subset $S$ of the positive numbers as follows:

$$S(A) = \{ \lambda > 0 \mid aA(x, \sigma(y)) + bA(\tau(x), y) - (c - \lambda P(x,y))A(\tau(x), \sigma(y)) \leq 0, \text{ eventually} \}. \tag{5}$$

From (1), we have

$$aA(x, \sigma(y)) + bA(\tau(x), y) < cA(\tau(x), \sigma(y)). \tag{6}$$

Hence,

$$A(\tau(x), \sigma(y)) < \frac{c}{b} A(\tau(x), \sigma^2(y)) < \cdots < \left( \frac{c}{b} \right)^{l} A(\tau(x), \sigma^{l+1}(y))$$

$$< \left( \frac{c}{b} \right)^{l} \left( \frac{a}{c} \right)^{k} A(\tau^{k+1}(x), \sigma^{l+1}(y)). \tag{7}$$

Substituting (5) into (1), we obtain

$$aA(x, \sigma(y)) + bA(\tau(x), y) - \left( c - \left( \frac{b}{c} \right)^{l} \left( \frac{a}{c} \right)^{k} P(x,y) \right) A(\tau(x), \sigma(y)) \leq 0,$$

which shows that $S(A)$ is nonempty. For $\lambda \in S$, we have eventually

$$c - \lambda P(x,y) > 0,$$

which implies that $S \subset E$. Due to Condition (i), the set $E$ is bounded, and hence $S(A)$ is bounded. From (5), we have

$$A(\tau(x), \sigma(y)) \leq \frac{c}{a} A(\tau^2(x), \sigma(y)) \quad \text{and} \quad A(\tau(x), \sigma(y)) \leq \frac{c}{b} A(\tau(x), \sigma^2(y)). \tag{8}$$
Let $\mu \in S$. Then

$$ \left( \frac{2ab}{c} \right) A(\tau(x), \sigma(y)) \leq aA(\tau(x), \sigma^2(y)) + bA(\tau^2(x), \sigma(y)) $$

$$ \leq (c - \mu P(\tau(x), \sigma(y))) A(\tau^2(x), \sigma^2(y)) .$$

If $k > l$, then

$$ A(\tau(x), \sigma(y)) \leq \left( \frac{2ab}{c} \right)^{-l} \prod_{i=1}^{l} (c - \mu P(\tau^i(x), \sigma^i(y))) A(\tau^{l+1}(x), \sigma^{l+1}(y)) \tag{8} $$

and

$$ A(\tau^{l+1}(x), \sigma^{l+1}(y)) \leq \frac{1}{a} (c - \mu P(\tau^{l+1}(x), \sigma^l(y))) A(\tau^{l+2}(x), \sigma^{l+1}(y)) $$

$$ \leq \cdots \leq \left( \frac{1}{a} \right)^{k-l} \prod_{j=1}^{k-l} (c - \mu P(\tau^{l+j}(x), \sigma^l(y))) A(\tau^{k+1}(x), \sigma^{l+1}(y)) .$$

Hence,

$$ A(\tau(x), \sigma(y)) \leq \left( \frac{2ab}{c} \right)^{-l} a^{-k-l} \prod_{i=1}^{l} (c - \mu P(\tau^i(x), \sigma^i(y))) $$

$$ \times \prod_{j=1}^{k-l} (c - \mu P(\tau^{l+j}(x), \sigma^l(y))) A(\tau^{k+1}(x), \sigma^{l+1}(y)) . \tag{9} $$

Similarly, if $l > k$, then

$$ A(\tau(x), \sigma(y)) \leq \left( \frac{2ab}{c} \right)^{-k} b^{-k-l} \prod_{i=1}^{k} (c - \mu P(\tau^i(x), \sigma^i(y))) $$

$$ \times \prod_{j=1}^{l-k} (c - \mu P(\tau^{k+j}(x), \sigma^k(y))) A(\tau^{k+1}(x), \sigma^{l+1}(y)) . \tag{10} $$

Substituting (9) and (10) into (1), we have, respectively,

$$ aA(x, \sigma(y)) + bA(\tau(x), y) - cA(\tau(x), \sigma(y)) $$

$$ + P(x, y) \left( \frac{2ab}{c} \right)^{l} a^{k-l} \prod_{i=1}^{l} (c - \mu P(\tau^i(x), \sigma^i(y))) $$

$$ \times \prod_{j=1}^{k-l} (c - \mu P(\tau^{l+j}(x), \sigma^l(y))) A(\tau(x), \sigma(y)) \leq 0, \quad \text{for } k > l \tag{11} $$

and

$$ aA(x, \sigma(y)) + bA(\tau(x), y) - cA(\tau(x), \sigma(y)) $$

$$ + P(x, y) \left( \frac{2ab}{c} \right)^{k} b^{l-k} \prod_{i=1}^{k} (c - \mu P(\tau^i(x), \sigma^i(y))) $$

$$ \times \prod_{j=1}^{l-k} (c - \mu P(\tau^{k+j}(x), \sigma^k(y))) A(\tau(x), \sigma(y)) \leq 0, \quad \text{for } l > k . \tag{12} $$
Hence,
\[
\begin{align*}
&\frac{\alpha A(x, \sigma(y)) + bA(\tau(x), y)}{\alpha A(x, \sigma(y)) + bA(\tau(x), y)} + \frac{a}{b} \left( \frac{2ab}{c} \right)^l a^{k-l} \sup_{x \in I_x, y \in I_y} \left[ \left( \prod_{i=1}^{k-l} \left( c - \mu P(\tau^i(x), \sigma^i(y)) \right) \right) \right] \\
&\times \prod_{j=1}^{k-l} \left( c - \mu P(\tau^{l+j}(x), \sigma^j(y)) \right) ^{-1} \right] \end{align*}
\]
A(\tau(x), \sigma(y)) \leq 0, \quad \text{for } k \geq l,
\]
and
\[
\begin{align*}
&\frac{\alpha A(x, \sigma(y)) + bA(\tau(x), y)}{\alpha A(x, \sigma(y)) + bA(\tau(x), y)} + \frac{a}{b} \left( \frac{2ab}{c} \right)^k b^{l-k} \sup_{x \in I_x, y \in I_y} \left[ \left( \prod_{i=1}^{k-l} \left( c - \mu P(\tau^i(x), \sigma^i(y)) \right) \right) \right] \\
&\times \prod_{j=1}^{l-k} \left( c - \mu P(\tau^{k+k+j}(x), \sigma^{k+j}(y)) \right) ^{-1} \right] \end{align*}
\]
A(\tau(x), \sigma(y)) \leq 0, \quad \text{for } l > k.
\]
From (13) and (14), we obtain
\[
\left( \frac{2ab}{c} \right)^l a^{k-l} \sup_{x \in I_x, y \in I_y} \left[ \left( \prod_{i=1}^{k-l} \left( c - \mu P(\tau^i(x), \sigma^i(y)) \right) \right) \right] \\
\times \prod_{j=1}^{k-l} \left( c - \mu P(\tau^{l+j}(x), \sigma^j(y)) \right) ^{-1} \right] \in S, \quad \text{for } k \geq l
\]
and
\[
\left( \frac{2ab}{c} \right)^k b^{l-k} \sup_{x \in I_x, y \in I_y} \left[ \left( \prod_{i=1}^{k-l} \left( c - \mu P(\tau^i(x), \sigma^i(y)) \right) \right) \right] \\
\times \prod_{j=1}^{l-k} \left( c - \mu P(\tau^{k+k+j}(x), \sigma^{k+j}(y)) \right) ^{-1} \right] \in S, \quad \text{for } l > k.
\]
On the other hand, (3) implies that there exists \( \alpha \in (0, 1) \) such that for \( k > l \)
\[
\sup_{x \in I_x, y \in I_y} \lambda \prod_{i=1}^{l} \left( c - \lambda P(\tau^i(x), \sigma^i(y)) \right) \prod_{j=1}^{k-l} \left( c - \lambda P(\tau^{l+j}(x), \sigma^j(y)) \right) < \alpha \left( \frac{2ab}{c} \right)^l a^{k-l}, \quad (17)
\]
and (4) implies that there exists \( \alpha \in (0, 1) \) such that for \( l > k \)
\[
\sup_{x \in I_x, y \in I_y} \lambda \prod_{i=1}^{l} \left( c - \lambda P(\tau^i(x), \sigma^i(y)) \right) \prod_{j=1}^{l-k} \left( c - \lambda P(\tau^k(x), \sigma^{k+j}(y)) \right) \\
< \alpha \left( \frac{2ab}{c} \right)^k b^{l-k}. \quad (18)
\]
Hence, for \( k > l \),
\[
\sup_{x \in I_x, y \in I_y} \prod_{i=1}^{l} \left( c - \mu P(\tau^i(x), \sigma^i(y)) \right) \prod_{j=1}^{k-l} \left( c - \mu P(\tau^{l+j}(x), \sigma^j(y)) \right) < \alpha \left( \frac{2ab}{c} \right)^l a^{k-l}, \quad (19)
\]
and for \( l > k \)

\[
\sup_{x \in I_\varepsilon, y \in I_\varepsilon} \prod_{i=1}^{\frac{k}{l-k}} \left( c - \mu P \left( \tau^i(x), \sigma^i(y) \right) \right) \prod_{j=1}^{l-k} \left( c - \mu P \left( \tau^k(x), \sigma^{k+l}(y) \right) \right) < \frac{\alpha}{\mu} \left( \frac{2ab}{c} \right)^k b^{-k}. \quad (20)
\]

From (15) and (19) for \( k > l \), (16) and (18) for \( l > k \), we have that \( \mu/\alpha \in S \). Repeating the above procedure, we conclude that \( \mu(1/\alpha)^n \in S, n = 1, 2, \ldots \), which contradicts the boundedness of \( S \). The proof is complete.

**COROLLARY 1.** In addition to (i) of Theorem 1, assume that, for \( k > l \),

\[
\liminf_{l \geq k, y \to \infty} P(x, y) = \liminf_{l \geq k, y \to \infty} \left( \frac{2ab}{c} \right)^{-l} a^{-k} \frac{k^k}{(k+1)^{k+1}}. \quad (21)
\]

and for \( l \geq k \),

\[
\liminf_{l \geq k, y \to \infty} P(x, y) = \liminf_{l \geq k, y \to \infty} \left( \frac{2ab}{c} \right)^{-k} b^{k+l} \frac{l^k}{(l+1)^{l+1}}. \quad (22)
\]

Then every solution of (1) oscillates.

**PROOF.** We see that

\[
\max_{M/l > \lambda > 0} \lambda(M - \lambda y)^k = M^k + y^k. \quad (23)
\]

Hence, (21) and (22) imply that (3) and (4) hold. By Theorem 1, every solution of (1) oscillates. The proof is complete.

**THEOREM 2.** In addition to (i) of Theorem 1, assume that there exist \( X, Y \subseteq I \) such that for \( k > l \)

\[
\sup_{x \in I_\varepsilon, y \in I_\varepsilon} \lambda \left[ \prod_{i=1}^{k-l} \prod_{j=1}^{l} \left( c - \lambda P \left( \tau^i(x), \sigma^i(y) \right) \right) \right]^{1/(k-l)} < \left( \frac{2ab}{c} \right)^l \left( \frac{a}{c} \right)^{(1/2)(l+1)} \quad (23)
\]

and for \( l > k \)

\[
\sup_{x \in I_\varepsilon, y \in I_\varepsilon} \lambda \left[ \prod_{j=1}^{l-k} \prod_{i=1}^{k} \left( c - \lambda P \left( \tau^i(x), \sigma^i(y) \right) \right) \right]^{1/(l-k)} < \left( \frac{2ab}{c} \right)^k \left( \frac{c}{b} \right)^{(1/2)(l-k+1)} \quad (24)
\]

Then every solution of (1) oscillates.

**PROOF.** If \( k > l \), from (12)

\[
A(\tau(x), \sigma(y)) \leq \left( \frac{2ab}{c} \right)^{-l} \prod_{i=1}^{l} \left( c - \mu P \left( \tau^i(x), \sigma^i(y) \right) \right) A \left( \tau^{i+1}(x), \sigma^{i+1}(y) \right). \quad (25)
\]

By (7) and (25),

\[
A \left( \tau^{i+1}(y), \sigma(y) \right) \leq \left( \frac{2ab}{c} \right)^{-l} \prod_{i=1}^{l} \left( c - \mu P \left( \tau^i(x), \sigma^i(y) \right) \right) A \left( \tau^{i+1+j}(x), \sigma^{i+1}(y) \right) \leq \left( \frac{2ab}{c} \right)^{-l} \prod_{i=1}^{l} \left( c - \mu P \left( \tau^{i+j}(x), \sigma^i(y) \right) \right) A \left( \tau^{k-l}(x), \sigma^{i+1}(y) \right), \quad (26)
\]

for \( j = 1, 2, \ldots, k - l \).
Hence,

\[ A^{k-l}(\tau(x), \sigma(y)) \leq \prod_{j=1}^{k-l} \left( \frac{e}{a} \right)^j A^{i+j}(\tau^{i+1}(x), \sigma^{i+1}(y)) \]

\[ \leq \prod_{j=1}^{k-l} \left\{ \left( \frac{e}{a} \right)^j \left( \frac{2ab}{c} \right)^{-\frac{l}{2}} \prod_{i=1}^{l} \left[ c - \mu P(\tau^{i+j}(x), \sigma^i(y)) \right] A^{i+j}(\tau^{i+1}(x), \sigma^{i+1}(y)) \right\} \]

\[ = \left( \frac{2ab}{c} \right)^{-l(k-l)} \left( \frac{e}{a} \right)^{(1/2)(k-l+1)(k-l)} \times \prod_{j=1}^{k-l} \prod_{i=1}^{l} \left[ c - \mu P(\tau^{i+j}(x), \sigma^i(y)) \right] A^{k-l}(\tau^{k+1}(x), \sigma^{k+1}(y)). \]

That is,

\[ A(\tau(x), \sigma(y)) \leq \left( \left( \frac{2ab}{c} \right)^{-l(k-l)} \left( \frac{e}{a} \right)^{(1/2)(k-l+1)(k-l)} \prod_{j=1}^{k-l} \prod_{i=1}^{l} \left[ c - \mu P(\tau^{i+j}(x), \sigma^i(y)) \right] \right)^{1/(k-l)} \times A(\tau^{k+1}(x), \sigma^{k+1}(y)). \]

Similarly, if \( l > k \), then

\[ A(\tau(x), \sigma(y)) \leq \left( \left( \frac{2ab}{c} \right)^{-l(l-k)} \left( \frac{e}{b} \right)^{(1/2)(l-k+1)(l-k)} \prod_{j=1}^{l-k} \prod_{i=1}^{k} \left[ c - \mu P(\tau^{i+j}(x), \sigma^{i+j}(y)) \right] \right)^{1/(l-k)} \times A(\tau^{k+1}(x), \sigma^{k+1}(y)). \]

The rest of the proof is similar to that of Theorem 1, and thus is omitted.

**Corollary 2.** Assume that for \( k > l \),

\[ \lim_{t \to \infty, x, y \to \infty} \frac{1}{|k-l|} \sum_{j=1}^{k-l} \sum_{i=1}^{l} P(\tau^{i+j}(x), \sigma^{i_j}(y)) > \frac{c^{l+1}l}{(l+1)^2} \left( \frac{2ab}{c} \right)^{-l} \left( \frac{e}{a} \right)^{(1/2)(k-l+1)} \]

and for \( l > k > 0 \),

\[ \lim_{t \to \infty, x, y \to \infty} \frac{1}{|l-k|} \sum_{j=1}^{l-k} \sum_{i=1}^{k} P(\tau^{i+j}(x), \sigma^{i+j}(y)) > \frac{c^{l-k}k^k}{(k+1)^{k+1}} \left( \frac{2ab}{c} \right)^{-k} \left( \frac{e}{b} \right)^{(1/2)(k-l+1)} \]

Then every solution of (1) oscillates.

**Proof.** Since

\[ \max_{\lambda > 0, \lambda > 0} \lambda(c - \lambda M)^l = \frac{c^{l+1}l}{M(l+1)^{l+1}}. \]

Let

\[ M = \frac{1}{(k-l)!} \sum_{j=1}^{k-l} \sum_{i=1}^{l} P(\tau^{i+j}(x), \sigma^{i+j}(y)). \]
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Then

\[
\lambda \left[ \prod_{j=1}^{k-l} \prod_{i=1}^{l} \left( c - \lambda P\left( \tau^{i+j}(x), \sigma^i(y) \right) \right) \right]^{1/(k-l)}
\]

\[
\leq \frac{\lambda}{(k-l)!} \left[ \sum_{j=1}^{k-l} \sum_{i=1}^{l} \left( c - \lambda P\left( \tau^{i+j}(x), \sigma^i(y) \right) \right) \right]
\]

\[
\leq \lambda \left[ c - \frac{\lambda}{(k-l)!} \sum_{j=1}^{k-l} \sum_{i=1}^{l} P\left( \tau^{i+j}(x), \sigma^i(y) \right) \right]
\]

\[
\leq \frac{\lambda}{(k-l)!} \left[ \sum_{j=1}^{k-l} \sum_{i=1}^{l} P\left( \tau^{i+j}(x), \sigma^i(y) \right) \right]^{-1}
\]

\[
\leq \left( \frac{2ab}{c} \right)^k \left( \frac{c}{d} \right)^{(1/2)(k-l+1)}
\]

Similarly, we have

\[
\lambda \left[ \prod_{j=1}^{k-l} \prod_{i=1}^{l} \left( c - \lambda P\left( \tau^{i+j}(x), \sigma^i(y) \right) \right) \right]^{1/(l-k)}
\]

\[
\leq \left( \frac{2ab}{c} \right)^k \left( \frac{c}{d} \right)^{(1/2)(l-k+1)}
\]

By Theorem 2, every solution of (1) oscillates. The proof is complete.

**THEOREM 3.** In addition to (i) of Theorem 1, assume that there exist \( X, Y \in I \) such that for \( k = l \)

\[
\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \prod_{i=1}^{k} \left( c - \lambda P\left( \tau^i(x), \sigma^i(y) \right) \right) < \left( \frac{2ab}{c} \right)^k .
\]

Then every solution of (1) oscillates.

**PROOF.** Let \( \mu \in S \). Then from (7)

\[
A(\tau(x), \sigma(y)) \leq \left( \frac{2ab}{c} \right)^{-k} \prod_{i=1}^{k} \left( c - \mu P\left( \tau^i(x), \sigma^i(y) \right) \right) \left( \tau^{k+1}(x), \sigma^{k+1}(y) \right).
\]

The rest of the proof is similar to that of Theorem 1, and thus is omitted.

Since

\[
\max_{M/\eta > \lambda > 0} \lambda(M - \lambda q)^k = \frac{M^{k+1} q^k}{q(k+1)^{k+1}},
\]

hence, we have the following result.

**COROLLARY 3.** Assume that \( k = l \) and that

\[
\lim_{x, y \to \infty} P(x, y) = q > \frac{c^{k+1} q^k}{(k+1)^{k+1} \left( \frac{2ab}{c} \right)^{-k}} .
\]

Then every solution of (1) oscillates.

**THEOREM 4.** In addition to (i) of Theorem 1, assume that there exist \( X \geq x_0, Y \geq y_0 \) such that for \( k, l > 0 \),

\[
\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \prod_{i=1}^{k} \left( c - \lambda P\left( \tau^i(x), y \right) \right) \prod_{j=1}^{l} \left( c - \lambda P\left( \tau^j(x), \sigma^j(y) \right) \right) < a^k b^l .
\]

Then every solution of (1) oscillates.
PROOF. Let $\mu \in S$. Then
\[
A(\tau(x), \sigma(y)) \leq \frac{1}{a} (c - \mu P(\tau(x), y)) A(\tau^2(x), \sigma(y))
\]
\[
\leq \left( \frac{1}{a} \right)^k \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), y) \right) A(\tau^{k+1}(x), \sigma(y))
\]
\[
\leq \left( \frac{1}{a} \right)^k \left( \frac{1}{b} \right) \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), y) \right) \prod_{j=1}^{l} \left( c - \mu P(\tau^j(x), \sigma^j(y)) \right)
\times A(\tau^{k+1}(x), \sigma^{l+1}(y)).
\]
The rest of the proof is similar to that of Theorem 1, and thus is omitted.

Since
\[
\max_{M/q > 0} \lambda(M - \lambda q)^{k+l} = \frac{M^{k+l+1} (k+l)^{k+l}}{q(k+l+1)^{k+l+1}}
\]
and (34), we have the following result.

COROLLARY 4. Assume that $k, l > 0$ and that
\[
\liminf_{x, y \to \infty} P(x, y) = q > 1 \quad \text{and} \quad (31)
\]
Then every solution of (1) oscillates.

THEOREM 5. In addition to (i) of Theorem 1, assume that there exist $X, Y \in I$ such that for $k, l > 0$
\[
\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \left[ \prod_{j=1}^{k+1} \prod_{i=1}^{l} \left( c - \lambda P(\tau^j(x), \sigma^i(y)) \right) \right]^{1/(k+l)} < a \left( \frac{b}{c} \right)^{(1/2)(l+1)}
\]
or
\[
\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \left[ \prod_{i=1}^{k} \prod_{j=1}^{l} \left( c - \lambda P(\tau^i(x), \sigma^j(y)) \right) \right]^{1/(k+l)} < b \left( \frac{a}{c} \right)^{(1/2)(k+1)}.
\]
Then every solution of (1) oscillates.

PROOF. Let $\mu \in S$. Then eventually
\[
aA(x, \sigma(y)) \leq (c - \mu P(x, y)) A(\tau(x), \sigma(y))
\]
and
\[
bA(\tau(x), y)) \leq (c - \mu P(x, y)) A(\tau(x), \sigma(y)).
\]
Using (34), we get
\[
A(\tau(x), \sigma(y)) \leq \frac{1}{a} (c - \mu P(\tau(x), y)) A(\tau^2(x), \sigma(y))
\]
\[
\leq \cdots \leq \left( \frac{1}{a} \right)^k \prod_{i=1}^{k} (c - \mu P(\tau^i(x), y)) A(\tau^{k+1}(x), y).
\]
Hence,

\[
A(\tau(x), \sigma^{j+1}(y)) \leq \frac{1}{a_k} \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) A(\tau^{k+1}(x), \sigma^{j+1}(y))
\]

\[
\leq \left[ \frac{1}{a_k} \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) \right] A(\tau^{k+1}(x), \sigma^{j+1}(y)), \quad j = 1, 2, \ldots, l,
\]

and so

\[
A'(\tau(x), \sigma(y)) \leq \prod_{j=1}^{l} \left( \frac{c}{b} \right)^j A(\tau(x), \sigma^{j+1}(y))
\]

\[
\leq \prod_{j=1}^{l} \left( \frac{c}{b} \right)^j \left[ \frac{1}{a_k} \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) \right] A(\tau^{k+1}(x), \sigma^{j+1}(y))
\]

\[
= \frac{1}{a_k} \left( \frac{c}{b} \right)^{(l+1)/2} \left[ \prod_{j=1}^{l} \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) \right] A'(\tau^{k+1}(x), \sigma^{j+1}(y)));
\]

i.e.,

\[
A(\tau(x), \sigma(y)) \leq \left[ \frac{1}{a_k} \left( \frac{c}{b} \right)^{(l+1)/2} \prod_{j=1}^{l} \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) \right]^{1/l} A(\tau^{k+1}(x), \sigma^{j+1}(y)).
\]

Similarly, we have

\[
A(\tau(x), \sigma(y)) \leq \left( \frac{1}{b} \right)^l \prod_{j=1}^{l} \left( c - \mu P(\tau, \sigma^j(y)) \right) A(\tau(x), \sigma^{j+1}(y)),
\]

and

\[
A^k(x, y) \leq \prod_{j=1}^{l} \left( \frac{c}{a} \right)^j A(x, \sigma^{j-1}(y)) \leq \frac{1}{b^k} \left( \frac{c}{a} \right)^{(l+1)/2} \left[ \prod_{j=1}^{l} \prod_{i=1}^{k} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) \right]
\]

\[
\times A^k(\tau^{k+1}(x), \sigma^{j+1}(y));
\]

i.e.,

\[
A(\tau(x), \sigma(y)) \leq \left[ \frac{1}{b^k} \left( \frac{c}{a} \right)^{(l+1)/2} \prod_{i=1}^{k} \prod_{j=1}^{l} \left( c - \mu P(\tau^i(x), \sigma^j(y)) \right) \right]^{1/k} A(\tau^{k+1}(x), \sigma^{j+1}(y)).
\]

The rest of the proof is similar to that of Theorem 1, and thus is omitted.

**Corollary 5.** Assume that

\[
\liminf_{l \to x, y \to \infty} \frac{1}{k+1} \sum_{j=1}^{l} \sum_{i=1}^{k} P(\tau^i(x), \sigma^j(y)) > a^{-k} \left( \frac{c}{b} \right)^{(l+1)/2} \frac{k^k}{(k+1)^{k+1}},
\]

or

\[
\liminf_{l \to x, y \to \infty} \frac{1}{k+1} \sum_{j=1}^{l} \sum_{i=1}^{k} P(\tau^i(x), \sigma^j(y)) > b^{-l} \left( \frac{c}{a} \right)^{(l+1)/2} \frac{l^l}{(l+1)^{l+1}}.
\]

Then every solution of (1) oscillates.
REFERENCES