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Stability of Markov Semigroups and Applications to Parabolic Systems*

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A new theorem for asymptotic stability of Markov semigroups is proved. This result is applied to semigroups generated by parabolic systems describing the evolution of densities of two-state diffusion processes.

1. INTRODUCTION

The use of semigroup methods for partial differential equations has had a long history starting with the works of Feller [7], Hille [10], and Yosida [24]. Markov semigroups play a special role in applications. These semigroups describe how the densities of initial states evolve in time. Consider, for example, the equation

$$\frac{\partial u}{\partial t} = Au, \tag{0.1}$$

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with the initial condition $u(x,0) = v(x)$, and define the semigroup $\{P(t)\}_{t \geq 0}$ by $P(t)v(x) = u(x,t)$. If A is the differential operator of the form

$$Au = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij}(x)u)}{\partial x_i \, \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i(x)u)}{\partial x_i}, \tag{0.2}$$

then $P(t)v$ is the density of some diffusion process X_t provided that v is the density of X_0. If all $a_{ij} \equiv 0$, then (0.1) is known as the Liouville equation and describes the evolution of densities under the flow generated by the system of ordinary differential equations $x' = b(x)$. If A is an infinitesimal generator of a Markov semigroup and P is a Markov operator, then for any $\lambda > 0$ the operator $A - \lambda I + \lambda P$ is also an infinitesimal generator of some Markov semigroup. In this way we obtain a broad class of equations which generate Markov semigroups. These equations appear in such diverse areas as astrophysics—fluctuations in the brightness of the Milky Way [4], population dynamics [14, 16], and in the theory of jump processes [17, 23].

The problem of asymptotic stability of Markov semigroups is strictly connected with the long-time behaviour of the solutions of (0.1). This problem has been recently investigated in many papers. The book of Lasota and Mackey [12] can be consulted for an excellent survey of many results on this subject.

The purpose of this paper is to provide new sufficient conditions for asymptotic stability of abstract Markov semigroups and apply them to some system of partial differential equations. The basic idea of our method is the following. First we check that if a Markov semigroup has a nontrivial integral part and has some "transitivity" properties (i.e., it spreads supports and possesses an invariant density), then this semigroup is asymptotically stable [21]. The main difficulty in this method is to prove the existence of an invariant density. Our theorem, which guarantees the existence of an invariant density, is similar in spirit to Hasminskii's result [9] on the existence of a stationary density for a diffusion process.

We apply results concerning the asymptotic stability of Markov semigroups to a system of partial differential equations. This system describes the evolution of the density of a two-state diffusion process. Such processes appear in transport phenomena in sponge-type structures [1, 3, 13]. In particular, we generalize a theorem on the asymptotic stability of a randomly flashing diffusion given in [13]. The main result can also be applied to the Fokker–Planck equation. In this case we obtain some earlier results on asymptotic stability [5, 19, 22].

The plan of the paper is as follows. In Section 1 we introduce a system of partial differential equations connected with two-state diffusion processes. Then we formulate a theorem on the asymptotic stability of this system and give a probabilistic interpretation of the result. The proof of the main result is given in Sections 2 and 3. In particular, in Section 2 we prove some results on the asymptotic stability of abstract Markov semigroups. Section 4 contains some examples and remarks.

1. MAIN RESULT

We consider the system of equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= -p(x)u_1 + q(x)u_2 + A_1 u_1, \\ \frac{\partial u_2}{\partial t} &= p(x)u_1 - q(x)u_2 + A_2 u_2. \end{aligned} \tag{1.1}$$

Throughout the paper we assume that $p\colon \mathbb{R}^d \to [0,\infty)$ and $q\colon \mathbb{R}^d \to [0,\infty)$ are continuous and bounded functions. The differential operators A_1, A_2 are given by

$$A_k f = \sum_{i,j=1}^{d} \frac{\partial^2 \big(a_{ij}(x,k) f\big)}{\partial x_i\, \partial x_j} - \sum_{i=1}^{d} \frac{\partial \big(b_i(x,k) f\big)}{\partial x_i}.$$

We assume that the coefficients $a_{ij}(\cdot,k)$ are $C_b^3(\mathbb{R}^d)$ functions and $b_i(\cdot,k)$ are $C_b^2(\mathbb{R}^d)$ functions, where $C_b^k(\mathbb{R}^d)$ is the space of k-times differentiable bounded functions whose partial derivatives of order $\le k$ are continuous and bounded. Moreover, we assume that for $k = 1,2$ the operator A_k is a differential operator of the first order $(a_{ij}(\cdot,k) \equiv 0$ for all $i,j)$ or A_k is an elliptic operator; that is, the following property holds:

$$\sum_{i,j=1}^{d} a_{ij}(x,k)\,\lambda_i \lambda_j \ge \alpha|\lambda|^2$$

for some $\alpha > 0$ and every $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. We admit two cases:

Case I. Both operators A_k are elliptic. Then we assume that there exists $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^d$ such that $p(x_1) > 0$ and $q(x_2) > 0$.

Case II. The operator A_1 is elliptic and A_2 is a first-order differential operator. Now we assume that $p(x) > 0$ and $q(x) > 0$ for all $x \in \mathbb{R}^d$.

System (1.1) can be rewritten as an evolution equation

$$\frac{\partial u}{\partial t} = \mathsf{A}u, \tag{1.2}$$

where $u = (u_1, u_2)$ and the operator A is given by

$$\mathsf{A}(u_1, u_2) = (-pu_1 + qu_2 + A_1u_1, pu_1 - qu_2 + A_2u_2).$$

Let $X = \mathbb{R}^d \times \{1,2\}$ and let $L^1(X)$ be the Banach space of integrable functions on X with the norm

$$\|v\| = \int_{\mathbb{R}^d} |v(x,1)|\, dx + \int_{\mathbb{R}^d} |v(x,2)|\, dx.$$

By Σ we denote the σ-algebra of Borel subsets of X and let μ be the product measure on Σ. We can identify the space $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ and $L^1(X)$ by $v_i(x) = v(x,i)$ for $i = 1,2$ and $v \in L^1(X)$. Let

$$D(X) = \left\{v \in L^1(X) \colon v \geq 0, \|v\| = 1\right\}.$$

We will check that if $u(t)$ is the solution of (1.2) with the initial condition $u(0) = v$, $v \in D(X)$, then $u(t) \in D(X)$ for all $t \geq 0$.

Let A_k^*, $k = 1,2$, be the linear operators given by

$$A_k^* f = \sum_{i,j=1}^{d} a_{ij}(x,k) \frac{\partial^2 f}{\partial x_i\, \partial x_j} + \sum_{i=1}^{d} b_i(x,k) \frac{\partial f}{\partial x_i}.$$

Let V_1 and V_2 be twice-differentiable real functions defined on $\mathbb{R}^d$ and let $V\colon \mathbb{R}^d \times \{1,2\} \to \mathbb{R}$ be given by $V(x,i) = V_i(x)$ for $i = 1,2$. Set

$$\mathsf{A}_1^* V = -pV_1 + pV_2 + A_1^* V_1 \qquad \text{and} \qquad \mathsf{A}_2^* V = qV_1 - qV_2 + A_2^* V_2.$$

Then $\mathsf{A}^* V = (\mathsf{A}_1^* V, \mathsf{A}_2^* V)$ is the adjoint operator of A.

By $B(r)$ we denote the closed ball in $\mathbb{R}^d$ with centre 0 and radius r. Let $C^k(\mathbb{R}^d)$ be the space of k-times differentiable functions whose partial derivatives of order $\leq k$ are continuous.

The main result of this paper is the following.

THEOREM 1. *Assume that there exist nonnegative functions* $V_1 \in C^2(\mathbb{R}^d)$ *and* $V_2 \in C^2(\mathbb{R}^d)$ *and* $r > 0$ *such that*

$$\sup_{x \notin B(r)} \mathsf{A}_k^* V(x) < 0 \qquad \text{for } k = 1,2. \tag{1.3}$$

Then there exists a stationary solution $v_* \in D(X)$ *of* (1.2) *and for every solution* $u(t)$ *of* (1.2) *such that* $u(0) \in D(X)$ *we have*

$$\lim_{t \to \infty} \| u(t) - v_* \| = 0. \tag{1.4}$$

The proof of Theorem 1 is given in Sections 2 and 3. Now we give a probabilistic interpretation of our result.

Let σ be a $d \times d$-dimensional matrix function defined on $\mathbb{R}^d \times \{1,2\}$ and let b be a d-dimensional vector function defined on $\mathbb{R}^d \times \{1,2\}$. Set $a_{ij} = \frac{1}{2}\sum_{k=1}^{d} \sigma_{ik}\sigma_{jk}$. Suppose that W_t is a d-dimensional Wiener process. Consider the stochastic differential equation:

$$dX_t = \sigma(X_t, Y_t)\, dW_t + b(X_t, Y_t)\, dt. \tag{1.5}$$

If the functions p and q are independent of x, then Y_t is a continuous-time Markov chain on the phase space $\{1,2\}$ independent of W_t with transition probabilities $1 \to 2$ and $2 \to 1$ in time Δt equal to $p\,\Delta t + o(\Delta t)$ and $q\,\Delta t + o(\Delta t)$. If p and q depend on x, then Y_t is also a stochastic process on the phase space $\{1,2\}$, but now the probability that the process Y_t changes its value depends on X_t. Namely,

$$\operatorname{Prob}\big(Y_{t+\Delta t} = 2 | Y_t = 1\big) = p(X_t)\,\Delta t + o(\Delta t),$$

$$\operatorname{Prob}\big(Y_{t+\Delta t} = 1 | Y_t = 2\big) = q(X_t)\,\Delta t + o(\Delta t).$$

Let X_0 be a d-dimensional random variable independent of W_t and Y_t. A solution of (1.5) is called a two-state diffusion process.

The process Y_t can be introduced in the following more formal way. Let $\Lambda_1(x) = p(x)$, $\Lambda_2(x) = q(x)$, and let Z_t be a continuous-time Markov chain on the phase space $\{1,2\}$ independent of W_t with transition probabilities $1 \to 2$ and $2 \to 1$ in time Δt equal to $\Delta t + o(\Delta t)$. Then $Y_t = Z_{\Gamma_t}$ and the process Γ_t satisfy the equation $d\Gamma_t = \Lambda_{Y_t}(X_t)\,dt$. If we add the last two equations to system (1.5), we obtain a system of stochastic equations for X_t, Y_t, and Γ_t.

The pair (X_t, Y_t) constitutes a Markov process on $\mathbb{R}^d \times \{1,2\}$. If there exist functions $u_1(x)$ and $u_2(x)$ such that

$$\operatorname{Prob}(X_t \in A, Y_t = k) = \int_A u_k(x,t)\, dx \qquad \text{for } k = 1,2,$$

then u_1 and u_2 satisfy system (1.1). Now we can formulate Theorem 1 in the following way. If there exist nonnegative functions V_1 and V_2 satisfying (1.3), then the densities of distributions of the process (X_t, Y_t) converge to some stationary density as $t \to \infty$. We will not use further this probabilistic interpretation. We have only given some general remarks about stochastic processes connected with system (1.1) to show the motivation to study the problem.

2. STABILITY AND SWEEPING OF MARKOV SEMIGROUPS

Let (X, Σ, μ) be a σ-finite measure space. By $D = D(X, \Sigma, \mu)$ we denote the subset of $L^1(X) = L^1(X, \Sigma, \mu)$ which contains all *densities*, that is,

$$D = \{f \in L^1(X): f \geq 0, \|f\| = 1\},$$

which $\|\cdot\|$ stands for the norm in $L^1(X)$. A linear mapping P: $L^1(X) \to L^1(X)$ is called a *Markov operator* if $P(D) \subset D$.

A density f_* is *invariant* if $Pf_* = f_*$. A Markov operator P is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{n \to \infty} \|P^n f - f_*\| = 0 \qquad \text{for } f \in D.$$

A semigroup $\{P(t)\}_{t \geq 0}$ of linear operators on $L^1(X)$ is said to be a *continuous semigroup of Markov operators* if $P(t)$ is a Markov operator for every $t \geq 0$ and if for every $f \in L^1(X)$ the function $t \mapsto P(t)f$ is continuous. We consider the equation

$$u'(t) = Au, \tag{2.1}$$

with the initial condition $u(0) = v$. We assume that this equation generates a continuous semigroup $\{P(t)\}_{t \geq 0}$ of Markov operators on $L^1(X)$ given by $P(t)v = u(t)$.

A density $f_* \in D$ is called *invariant* under the semigroup $\{P(t)\}_{t \geq 0}$ if $P(t)f_* = f_*$ for every $t \geq 0$. The semigroup $\{P(t)\}_{t \geq 0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t \to \infty} \|P(t)f - f_*\| = 0 \qquad \text{for } f \in D.$$

A Markov operator P is called *sweeping* [11] with respect to the set $Z \in \Sigma$ if

$$\lim_{n \to \infty} \int_Z P^n f \, d\mu = 0 \qquad \text{for } f \in D.$$

The semigroup $\{P(t)\}_{t\geq 0}$ is *sweeping* with respect to Z if

$$\lim_{t\to\infty}\int_Z P(t)f\,d\mu = 0 \qquad \text{for } f\in D.$$

Let a family $\mathcal{Z}\subset\Sigma$ be given. We say that a Markov operator P (the semigroup of Markov operators $\{P(t)\}_{t\geq 0}$) is sweeping with respect to $\mathcal{Z}$ if P ($\{P(t)\}_{t\geq 0}$) is sweeping with respect to each set $Z\in\mathcal{Z}$. It is easy to check that the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable (sweeping) if there exists a $t_0>0$ such that the operator $P(t_0)$ is asymptotically stable (sweeping).

Let $\mathsf{R}=\mathsf{R}(1,\mathsf{A})=(I-\mathsf{A})^{-1}$ be the resolvent of the operator A. It is easy to show that R is a Markov operator on the space $L^1(X)$. The operator R is also defined by the formula

$$\mathsf{R}f=\int_0^\infty e^{-t}P(t)f\,dt. \tag{2.2}$$

LEMMA 1. *If the semigroup $\{P(t)\}_{t\geq 0}$ is sweeping with respect to a measurable set Z, then the resolvent R is sweeping with respect to Z.*

Proof. Let $f\in D$. From (2.2) it follows that

$$\mathsf{R}^n f=\int_0^\infty r_n(t)P(t)f\,dt,$$

where $r_n(t)=t^n e^{-t}/n!$. Let ε be a given constant. Then there exists $t_0>0$ such that

$$\int_Z P(t)f(x)\,\mu(dx)<\varepsilon/2 \qquad \text{for } t\geq t_0.$$

Since $\int_0^{t_0} r_n(t)\,dt\to 0$ as $n\to\infty$, there exists a positive integer n_0 such that $\int_0^{t_0} r_n(t)\,dt<\varepsilon/2$ for $n\geq n_0$. Hence

$$\begin{aligned}\int_Z \mathsf{R}^n f(x)\,\mu(dx) &\leq \int_0^{t_0} r_n(t)\,dt+\int_{t_0}^\infty r_n(t)\left(\int_Z P(t)f(x)\,\mu(dx)\right)dt\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\int_{t_0}^\infty r_n(t)\,dt<\varepsilon\end{aligned}$$

for $n\geq n_0$, which completes the proof. ∎

Let P be a Markov operator on $L^1(X)$ and let $Z_0\in\Sigma$. Let V: $X\to[0,\infty)$ be a measurable function. By D_V we denote the set

$$D_V=\left\{f\in D\colon \int_X f(x)V(x)\,\mu(dx)<\infty\right\}.$$

Since μ is a σ-finite measure the set D_V is nonempty. Assume that there exists a measurable function $W\colon X \to \mathbb{R}$ such that

(i) there exists an $M > 0$ such that $W(x) \le M$ for $x \in X$,

(ii) there exists an $\varepsilon > 0$ such that $W(x) \le -\varepsilon$ for $x \notin Z_0$,

(iii) for every $f \in D_V$ we have

$$\int_X V(x) Pf(x)\, d\mu(x) \le \int_X V(x) f(x)\, d\mu(x) + \int_X W(x) Pf(x)\, d\mu(x). \tag{2.3}$$

Then the function V will be called a *Hasminskii function* for the operator P and the set Z_0. If V is a Hasminskii function, then for any real constant c the function $\overline{V} = V + c$ is also a Hasminskii function and $D_V = D_{\overline{V}}$. The sweeping is an opposite property to the asymptotic stability. The following lemma shows that we can use a Hasminskii function to exclude the sweeping.

LEMMA 2. *Let P be a Markov operator and $Z_0 \in \Sigma$. Assume that there exists a Hasminskii function for the operator P and the set Z_0. Then P is not sweeping with respect to Z_0.*

Proof. Let $f_0 \in D_V$. Then from (2.3) it follows that $P^n f_0 \in D_V$ for every integer $n \ge 1$. Suppose, contrary to our claim, that the operator P is sweeping with respect to Z_0. Then

$$\lim_{n\to\infty} \int_{Z_0} P^n f_0(x)\, d\mu(x) = 0.$$

This implies that there exists a positive integer $n_0 = n_0(f_0)$ such that

$$\int_{Z_0} P^{n+1} f_0(x)\, d\mu(x) < \frac{\varepsilon}{4M}, \qquad \int_{X\setminus Z_0} P^{n+1} f_0(x)\, d\mu(x) > \frac{1}{2} \tag{2.4}$$

for $n \ge n_0$. From (2.3) it follows that

$$\begin{aligned}
&\int_X V(x) P^{n+1} f_0(x)\, d\mu(x) \\
&\quad \le \int_X V(x) P^n f_0(x)\, d\mu(x) + \int_X W(x) P^{n+1} f_0(x)\, d\mu(x) \\
&\quad = \int_X V(x) P^n f_0(x)\, d\mu(x) + \int_{Z_0} W(x) P^{n+1} f_0(x)\, d\mu(x) \\
&\qquad + \int_{X\setminus Z_0} W(x) P^{n+1} f_0(x)\, d\mu(x).
\end{aligned}$$

Now, conditions (i), (ii), and (2.4) give

$$\int_X V(x) P^{n+1} f_0(x)\, d\mu(x) \le \int_X V(x) P^n f_0(x)\, d\mu(x) - \frac{\varepsilon}{4}. \quad (2.5)$$

Since $\int_X V(x) P^n f_0(x)\, d\mu(x) < \infty$ for every positive integer n, we have

$$\lim_{n\to\infty} \int V(x) P^n f_0(x)\, d\mu(x) = -\infty,$$

which is impossible. The proof is completed. ∎

The *support* of an $f \in L^1(X)$ is defined up to a set of measure zero by the formula

$$\operatorname{supp} f = \{x \in X: f(x) \neq 0\}.$$

We say that a Markov operator P *spreads supports* if for every set $A \in \Sigma$ with $\mu(A) < \infty$ and for every $f \in D(X)$ we have $\lim_{n\to\infty} \mu(\operatorname{supp} P^n f \cap A) = \mu(A)$.

A Markov operator P is called *partially integral* if it can be written in the form

$$Pf(x) = \int h(x, y) f(y)\, \mu(dy) + Rf(x), \quad (2.6)$$

where R is a positive contraction on $L^1(X)$ and the kernel h is a measurable nonnegative function such that

$$\int_X \int_X h(x, y)\, \mu(dy)\, \mu(dx) > 0. \quad (2.7)$$

The following proposition provides a tool for checking the asymptotic stability of Markov operators.

PROPOSITION 1. *Let P be a partially integral Markov operator. If P spreads supports and has an invariant density, then P is asymptotically stable.*

The proof of Proposition 1 is given in [21, Corollary 2] and is based on the book by Foguel [8]. Special cases when P is an integral operator were considered in [2, 15, 18, 20]. From now on, we restrict our investigation to Markov operators when (X, ρ) is a metric space and Σ is a σ-algebra which contains Borel subsets of X. We assume that $\mathcal{Z}$ is the family of compact subsets of X. We assume that P is a Markov operator on $L^1(X)$ and P has the form (2.6).

We need the following proposition (see [21, Theorem 2]).

PROPOSITION 2. *Assume that a Markov operator P spreads supports and has no invariant density. Suppose that P has the following property*:

(*I*) *For every* $y_0 \in X$ *there exist an* $\varepsilon > 0$ *and a measurable function* $\eta \geq 0$ *such that* $\int \eta(x)\mu(dx) > 0$ *and*

$$h(x, y) \geq \eta(x)\mathbf{1}_{B_\varepsilon(y_0)}(y), \tag{2.8}$$

where $B_\varepsilon(y_0) = \{y \in X: \rho(y, y_0) \leq \varepsilon\}$.
Then P is sweeping with respect to $\mathcal{Z}$.

Now we can state the main result of this section.

THEOREM 2. *Let* $\{P(t)\}_{t \geq 0}$ *be a Markov semigroup generated by* (2.1) *and* $P = P(t_0)$ *for some* $t_0 > 0$. *Assume that the operator P spreads supports and satisfies* (I). *Let* Z_0 *be a compact set. Assume that there exists a Hasminskii function for the resolvent* $\mathcal{R}$ *of the semigroup* $\{P(t)\}_{t \geq 0}$ *and the set* Z_0. *Then the semigroup* $\{P(t)\}_{t \geq 0}$ *is asymptotically stable.*

Proof. From (I) it follows immediately that P is a partially integral operator. We claim that P has an invariant density. Conversely, suppose that P has no invariant density. Then from Proposition 2 it follows that P is sweeping with respect to $\mathcal{Z}$. Hence the semigroup $\{P(t)\}_{t \geq 0}$ is sweeping with respect to $\mathcal{Z}$. According to Lemma 1, the resolvent $\mathcal{R}$ is also sweeping with respect to $\mathcal{Z}$. But Lemma 2 implies that the resolvent $\mathcal{R}$ is not sweeping with respect to Z_0. This contradicts our assumption and, consequently, P has an invariant density. Now, since P is a partially integral operator which spreads supports and has an invariant density, the operator P is asymptotically stable. This implies that the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable and completes the proof. ∎

3. PROOF OF THEOREM 1

First we give a semigroup representation of solutions of (1.2). The operators A_1 and A_2 generate continuous semigroups of Markov operators $\{T^1(t)\}_{t \geq 0}$ and $\{T^2(t)\}_{t \geq 0}$ on the space $L^1(\mathbb{R}^d)$, that is, for any $f \in L^1(\mathbb{R}^d)$,

$$u_k(t) = T^k(t)f, \qquad k = 1, 2,$$

are the solutions of the evolution equations $u_k' = A_k u_k$ with the initial conditions $u_k(0) = f$. In the proof of Theorem 1 we use some auxiliary results concerning the semigroups $\{T^1(t)\}_{t \geq 0}$ and $\{T^2(t)\}_{t \geq 0}$. If A_k is an elliptic operator, then the semigroup $\{T^k(t)\}_{t \geq 0}$ is an integral semigroup; that is, for every $t > 0$ there exists a Borel measurable function $r_k(x, y, t)$

such that

$$T^k(t)f(x) = \int_{\mathbb{R}^d} r_k(x,y,t)f(y)\,dy. \tag{3.1}$$

The kernel $r_k(x,y,t)$ is positive and continuous with respect to $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$. Hence, for any nonnegative $g \in L^1(\mathbb{R}^d)$, $g \not\equiv 0$, we have $T^k(t)g(x) > 0$ for every $x \in \mathbb{R}^d$, $t > 0$. If $\{T^2(t)\}_{t\geq 0}$ is generated by the equation

$$\frac{\partial u_2}{\partial t} = -\sum_{i=1}^{d} \frac{\partial(b_i(x,2)u_2)}{\partial x_i}, \tag{3.2}$$

then the semigroup $\{T^2(t)\}_{t\geq 0}$ can be given explicitly. Namely, let $\pi(t,\bar{x})$ be the solution of the system $x'(t) = b(x(t),2)$ with the initial condition $x(0) = \bar{x}$, $\bar{x} \in \mathbb{R}^d$. If $f \in L^1(\mathbb{R}^d)$ and u_2 is the solution of (3.2) with the initial condition $u_2(x,0) = f(x)$, then

$$T^2(t)f(x) = u_2(x,t) = f(\pi(-t,x))\det\left(\frac{\partial\pi(-t,x)}{\partial x}\right). \tag{3.3}$$

This implies that if $\operatorname{supp} f = \mathbb{R}^d$ then $\operatorname{supp} T^2(t)f = \mathbb{R}^d$.

Now, we denote by B the differential operator

$$B(u_1,u_2) = (A_1u_1, A_2u_2).$$

The operator B generates a semigroup $\{S(t)\}_{t\geq 0}$ of Markov operators on the space $L^1(X)$ given by

$$S(t)(u_1,u_2) = \big(T^1(t)u_1, T^2(t)u_2\big), \qquad t \geq 0. \tag{3.4}$$

Let λ be a constant such that

$$\lambda > \max_{x\in\mathbb{R}^d}\{p(x) + q(x)\}.$$

We define the operator T by

$$\mathsf{T}(u_1,u_2) = \lambda^{-1}\big((\lambda - p)u_1 + qu_2, pu_1 + (\lambda - q)u_2\big). \tag{3.5}$$

It is easily seen that T is a Markov operator on $L^1(X)$. Equation (1.2) can be rewritten as the evolution equation

$$u'(t) = \lambda\,\mathsf{T}u - \lambda u + Bu, \qquad \text{where } u = (u_1,u_2). \tag{3.6}$$

From the Phillips perturbation theorem [6], (3.6) with the initial condition $u'(0) = v$ generates a continuous semigroup $\{P(t)\}_{t \geq 0}$ of Markov operators on $L^1(X)$ given by

$$P(t)v = u(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n S_n(t)v, \tag{3.7}$$

where $S_0(t) = S(t)$ and

$$S_{n+1}(t)v = \int_0^t S_0(t-s)\mathcal{T} S_n(s)v\,ds, \qquad n \geq 0. \tag{3.8}$$

The semigroup $\{P(t)\}_{t \geq 0}$ satisfies the integral equation

$$P(t)v = e^{-\lambda t}S(t)v + \lambda \int_0^t e^{-\lambda s} S(s)\mathcal{T} P(t-s)v\,ds. \tag{3.9}$$

Observe that condition (1.4) is equivalent to the asymptotic stability of the semigroup $\{P(t)\}_{t \geq 0}$.

The conjugated semigroup $\{P^*(t)\}_{t \geq 0}$ defined on $L^\infty(X)$ is not continuous. But if we consider the Banach space

$$C_0(X) = \left\{ f \in C_b(X) \colon \lim_{\|x\| \to \infty} f(x,k) = 0, \text{for } k = 1,2 \right\}$$

with the supremum norm, then $\{P^*(t)\}_{t \geq 0}$ is a continuous semigroup of contractions on $C_0(X)$. Indeed, let $C_0(\mathbb{R}^d)$ be the subspace of $C_b(\mathbb{R}^d)$ of all functions which vanish at infinity. Then $\{T^{k*}(t)\}_{t \geq 0}$, $k = 1,2$, are continuous semigroups of contractions on $C_0(\mathbb{R}^d)$. Consequently, $\{S^*(t)\}_{t \geq 0}$ is a continuous semigroup of contractions on $C_0(X)$. The operator $\mathcal{T}^*$ is also a contraction on $C_0(X)$. The semigroup $\{P^*(t)\}_{t \geq 0}$ is generated by the evolution equation

$$u'(t) = \lambda \mathcal{T}^* u - \lambda u + B^* u. \tag{3.10}$$

From the Phillips perturbation theorem it follows that $\{P^*(t)\}_{t \geq 0}$ is also a continuous semigroup of contractions on $C_0(X)$. This implies that if $\mathcal{R}^* = (I - \mathcal{A}^*)^{-1}$ is the resolvent of the semigroup $\{P^*(t)\}_{t \geq 0}$, then $\mathcal{R}^*$ is a contraction on the space $C_0(X)$.

Proof of Theorem 1. Let M be a positive constant such that $\mathcal{A}_k^* V(x) \leq M$ for $x \in X$ and $k = 1,2$. Define $V(x,k) = V_k(x) + M$ and $W(x,k) = \mathcal{A}_k^* V(x)$ for $k = 1,2$ and $x \in \mathbb{R}^d$. We show that V is a Hasminskii function for the semigroup $\{P(t)\}_{t \geq 0}$ and the set $Z_0 = B(r) \times \{1,2\}$. Conditions (i) and (ii) follow immediately from (1.3) and the definition of W. Now we check (2.3). Since the set of densities with bounded supports is dense in D_V, it is sufficient to prove (2.3) for each density f with a bounded support. Set $g(x) = V(x) - \mathcal{A}^*V(x)$. Then $g \geq 0$. Since $\mathcal{R}f \in D(X)$, $V \geq M$, and $\mathcal{A}^*V = W \leq M$, we can rewrite inequality (2.3) in the following way:

$$\int_X g(x)\, \mathcal{R}f(x)\, d\mu(x) \leq \int_X V(x) f(x)\, d\mu(x). \tag{3.11}$$

As $g \geq 0$, we can find an increasing sequence (g_n) of smooth functions with bounded supports converging uniformly on compact sets to g. If

$$\int_X g(x)\, \mathcal{R}f(x)\, d\mu(x) < \infty, \tag{3.12}$$

then for any $\delta > 0$ we can choose n such that

$$\int_X g(x)\, \mathcal{R}f(x)\, d\mu(x) \leq \delta + \int_X g_n(x)\, \mathcal{R}f(x)\, d\mu(x). \tag{3.13}$$

Let $h_n = \mathcal{R}^* g_n = (I - \mathcal{A}^*)^{-1} g_n$. Then $g_n = h_n - \mathcal{A}^* h_n$. Using standard arguments based on the maximum principle, we check that $h_n \leq V$. Indeed, let $U = V - h_n$. Then the function U satisfies the system

$$U - \mathcal{A}^*U = g - g_n. \tag{3.14}$$

Since $g_n \in C_0(X)$, we have $h_n = \mathcal{R}^* g_n \in C_0(X)$. This implies that

$$\lim_{\|x\| \to \infty} h_n(x, k) = 0$$

for $k = 1, 2$, and, consequently,

$$\liminf_{\|x\| \to \infty} U(x, k) \geq 0 \qquad \text{for } k = 1, 2. \tag{3.15}$$

If U is not a nonnegative function, then U has a global minimum at some point $(x_0, k_0) \in \mathbb{R}^d \times \{1, 2\}$ and $U(x_0, k_0) < 0$. Since (x_0, k_0) is a local minimum we have $A_{k_0}^* U(x_0, k_0) \geq 0$, and since (x_0, k_0) is a global minimum we also have

$$\mathcal{A}_{k_0}^* U(x_0, k_0) \geq A_{k_0}^* U(x_0, k_0) \geq 0.$$

From (3.14) we obtain

$$U(x_0, k_0) \geq g(x_0, k_0) - g_n(x_0, k_0) + \mathcal{A}_{k_0}^* U(x_0, k_0) \geq 0,$$

which contradicts our assumption that $U(x_0, k_0) < 0$. Consequently, $V \geq h_n$. Now, since $h_n = \mathbf{R}^* g_n$, from (3.13) it follows that

$$\begin{aligned}\int_X g(x)\, \mathbf{R}f(x)\, d\mu(x) &\leq \delta + \int_X g_n(x)\, \mathbf{R}f(x)\, d\mu(x)\\ &= \delta + \int_X h_n(x) f(x)\, d\mu(x)\\ &\leq \delta + \int_X V(x) f(x)\, d\mu(x).\end{aligned}$$

Letting $\delta \to 0$, we obtain (3.11). If

$$\int_X g(x)\, \mathbf{R}f(x)\, d\mu(x) = \infty,$$

then we can choose an integer n such that

$$\int_X g_n(x)\, \mathbf{R}f(x)\, d\mu(x) > \int_X V(x) f(x)\, d\mu(x). \tag{3.16}$$

In the same manner as before we can see that

$$\begin{aligned}\int_X g_n(x)\, \mathbf{R}f(x)\, d\mu(x) &= \int_X h_n(x) f(x)\, d\mu(x)\\ &\leq \int_X V(x) f(x)\, d\mu(x),\end{aligned}$$

which contradicts (3.16) and completes the proof of (3.11).

Fix $t_0 > 0$ and set $P = P(t_0)$. According to Theorem 2, it is sufficient to prove that P spreads supports and satisfies (I). First we check that $\operatorname{supp} Pf = X$ for every density $f \in L^1(X)$. Let $f_1(x) = f(x,1)$ and $f_2(x) = f(x,2)$. If $\operatorname{supp} f_2 \neq \emptyset$, then in both Cases I and II we have $\operatorname{supp} qT^2(s)f_2 \neq \emptyset$ for all $s > 0$. From (3.7) and (3.8) it follows that

$$P(t)f(x,1) \geq e^{-\lambda t}\left(T^1(t)f_1 + \int_0^t T^1(t-s)\big(qT^2(s)f_2\big)\, ds\right). \tag{3.17}$$

For any $g \in L^1(\mathbb{R}^d)$, $g \geq 0$, $g \not\equiv 0$, we have $T^1(t)g(x) > 0$ for all $x \in \mathbb{R}^d$ and $t > 0$. This and (3.17) imply $P(t)f(x,1) > 0$ for all $x \in \mathbb{R}^d$ and $t > 0$. From (3.9) it follows that

$$P(t)f(x,2) \geq \int_0^t e^{-\lambda s} T^2(s)\big(p(x)P(t-s)f(x,1)\big)\, ds. \tag{3.18}$$

Let $w(x) = p(x)P(t-s)f(x,1)$. If A_2 is an elliptic operator, then $\operatorname{supp} w \neq \emptyset$ and $\operatorname{supp} T^2(s)w = \mathbb{R}^d$ for $s \in (0,t)$. If A_2 is a first-order differential operator, then $\operatorname{supp} w = \mathbb{R}^d$ and, consequently, $\operatorname{supp} T^2(s)w = \mathbb{R}^d$. From (3.18) it follows that $P(t)f(x,2) > 0$ for all $x \in \mathbb{R}^d$ and $t > 0$. Thus $\operatorname{supp} P = \operatorname{supp} P(t_0) = X$.

Now we check that P satisfies (I). If the operator A_k is elliptic, then

$$Pf(x,k) \geq \exp(-\lambda t_0)S(t_0)f(x,k) \geq \exp(-\lambda t_0)\int_{\mathbb{R}^d} r_k(x,y,t)f_k(y)\,dy$$

for $k \in \{1,2\}$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$. This implies that

$$h((x,k),(y,k)) \geq \exp(-\lambda t_0)r_k(x,y,t_0). \tag{3.19}$$

If A_2 is a first-order differential operator, then from (3.1) and (3.3) it follows that

$$\begin{aligned}
Pf(x,1) &\geq \exp(-\lambda t_0)\lambda S_1(t_0)f(x,1) \\
&\geq \exp(-\lambda t_0)\int_0^{t_0} T^1(t_0-s)\big(qT^2(s)f_2\big)(x,1)\,ds \\
&\geq \int_0^{t_0}\int_{\mathbb{R}^d} \exp(-\lambda t_0)r_1(x,\pi(s,y),t_0-s)q(\pi(s,y))f_2(y)\,dy\,ds.
\end{aligned}$$

Thus

$$h((x,1),(y,2)) \geq \int_0^{t_0} \exp(-\lambda t_0)r_1(x,\pi(s,y),t_0-s)q(\pi(s,y))\,ds.$$

Since, in this case, the functions q and r_1 are continuous and positive, for any y_0 we can find $\varepsilon > 0$ and $c > 0$ such that

$$h((x,1),(y,2)) \geq c\mathbf{1}_{B_\varepsilon(y_0)}(x)\mathbf{1}_{B_\varepsilon(y_0)}(y). \tag{3.20}$$

From (3.19) and (3.20) we conclude that the operator P satisfies (I). The proof is completed. ∎

4. EXAMPLES AND REMARKS

EXAMPLE 1. Assume that A is an elliptic operator and $A_1 = A_2 = A$. If there exist $\varepsilon > 0$, $r > 0$, and a nonnegative C^2-function $\overline{V}$ such that $A^*\overline{V}(x) \leq -\varepsilon$ for $\|x\| \geq r$, then system (1.1) is asymptotically stable. In this case the Hasminskii function is the following: $V(x,k) = \bar{V}(x)$ for $x \in \mathbb{R}^d$, $k = 1, 2$. If the pair (u_1, u_2) is a solution of (1.1), then $u = u_1 + u_2$ is a solution of the equation

$$\frac{\partial u}{\partial t} = Au. \tag{4.1}$$

Consequently, (4.1) is asymptotically stable if there exists a nonnegative C^2-function $\bar{V}$ such that $A^*\bar{V}(x) \le -\varepsilon$ for $\|x\| \ge r$.

In some cases we are able to find an invariant density, which implies that (4.1) is asymptotically stable. For example, in the one-dimensional case, $Af = (af)'' - (bf)'$, an invariant density exists if and only if

$$\int_{-\infty}^{\infty} \exp B(x)\, dx < \infty, \qquad \text{where } B(x) = \int_0^x \frac{b(y)}{a(y)}\, dy.$$

The invariant density is given by

$$u_*(x) = \frac{g(x)}{\|g\|}, \qquad \text{where } g(x) = \frac{1}{a(x)} \exp B(x).$$

Remark 1. In [5, 12, Theorem 11.9.1], it is proved that if there exists a nonnegative C^2-function satisfying some growth conditions and such that $\lim_{|x|\to\infty} V(x) = \infty$ and $A^*V \le -\alpha V + \beta$ for some $\alpha > 0$, $\beta > 0$, then (4.1) is asymptotically stable. Our condition on the Hasminskii function is less restrictive and, in fact, it is also a necessary condition for asymptotic stability. Indeed, Hasminskii [9] proved that (4.1) has an invariant density if and only if there exists a nonnegative function V such that $A^*V(x) = -1$ for $\|x\| \ge r$.

EXAMPLE 2. In [13] the one-dimensional system (1.1) was considered with $A_1 f = (af)'' - (bf)'$ and $A_2 f = -(bf)'$. Now, we generalize the main result of this paper. Let

$$B(x) = \int_0^x \frac{b(y)}{a(y)}\, dy$$

and we assume that

(i) there exist $x_0 > 0$ and $\delta > 0$ such that $xb(x) < 0$ and $p(x) \ge \delta$ for $|x| \ge x_0$,

(ii) $\gamma = \liminf_{|x|\to\infty} |b(x)| e^{-B(x)} > 0$.

We check that system (1.1) is asymptotically stable by constructing a proper Hasminskii function. Set $\lambda = \min(1, \gamma/2)$, $M = \sup_{x\in\mathbb{R}} q(x)$, $\varepsilon =$

$\lambda(M\delta^{-1} + 1)^{-1}$, and $c = \varepsilon/\delta$. If $V_2(x) = |\int_0^x \exp(-B(y))\,dy|$ and $V_1(x) = c + V_2(x)$ for $|x| \geq 1$, then $V(x, k) = V_k(x)$ is the Hasminskii function.

EXAMPLE 3. Now, we assume that

$$\lim_{\|x\| \to \infty} \sum_{i=1}^{d} b_i(x, k) x_i = -\infty \qquad \text{for } k = 1, 2.$$

Then system (1.1) is asymptotically stable. In this case, $V_1(x) = V_2(x) = \|x\|^2$.

Remark 2. One might expect that if both equations $\partial u/\partial t = A_1 u$ and $\partial u/\partial t = A_2 u$ are asymptotically stable then system (1.1) is also asymptotically stable. But it is not true. Consider the following example. Let φ: $[0, 6] \to \mathbb{R}$ be a C^2-function such that $\varphi(x) = x$ for $x \in [0, 1]$, $\varphi(x) > 0$ for $x \in (1, 2)$, $\varphi(x) = 3 - x$ for $x \in [2, 4]$, $\varphi(x) < 0$ for $x \in (4, 5)$, $\varphi(x) = x - 6$ for $x \in [5, 6]$, and

$$\alpha = \int_0^3 \varphi(x)\,dx < \int_3^6 |\varphi(x)|\,dx = \beta. \tag{4.2}$$

Let b_1: $\mathbb{R} \to \mathbb{R}$ be a function such that $b_1(x) = -b_1(-x)$ for $x \in \mathbb{R}$ and $b_1(x + 6n) = \varphi(x)$ for any integer $n \geq 0$ and $x \in [0, 6]$. Set $b_2(x) = b_1(x + 3)$ for $x \geq 0$ and $b_2(x) = -b_2(-x)$ for $x \leq 0$. Let $A_k f = f'' - (b_k f)'$ and $B_k(x) = \int_0^x b_k(y)\,dy$. Then from (4.2) it follows that

$$\lim_{|x| \to \infty} \frac{B_k(x)}{|x|} = \frac{\alpha - \beta}{6} < 0.$$

This implies that $\int_{-\infty}^{\infty} \exp(B_k(x))\,dx < \infty$ and, consequently, the equations $\partial u/\partial t = A_1 u$ and $\partial u/\partial t = A_2 u$ are asymptotically stable. Now, set $u_1(x) = b_2(x)/(b_2(x) - b_1(x))$ and $u_2(x) = b_1(x)/(b_1(x) - b_2(x))$ when $b_2(x) \neq b_1(x)$ and $u_1(x) = u_2(x) = \frac{1}{2}$ if $b_2(x) = b_1(x)$. From the definition of b_1 and b_2 it follows immediately that $u_1 \in C_b^2(\mathbb{R})$, $u_2 \in C_b^2(\mathbb{R})$, and there exists $\delta > 0$ such that $u_1(x) \geq \delta$, $u_2(x) \geq \delta$ for $x \in \mathbb{R}$. Let c be a constant such that $|A_1 u_1(x)| \leq c$ for $x \in \mathbb{R}$. Let $q \equiv 2c/\delta$ and $p = (qu_2 + A_1 u_1)/u_1$. Then $p \in C_b(\mathbb{R})$, $q \in C_b(\mathbb{R})$ and $p(x) \geq \varepsilon$, $q(x) \geq \varepsilon$ for some $\varepsilon > 0$ and all $x \in \mathbb{R}$. It is easy to check that the functions $u_1(x)$ and $u_2(x)$ are stationary solutions of system (1.1). This implies that the semigroup $\{P(t)\}_{t \geq 0}$ generated by (1.1) has a positive fixed point u which is not integrable. From the monotonicity of the semigroup $\{P(t)\}_{t \geq 0}$, it follows immediately that this semigroup is not asymptotically stable.

Remark 3. It is interesting that system (1.1) can be asymptotically stable when both equations $\partial u/\partial t = A_1 u$ and $\partial u/\partial t = A_2 u$ are not stable. We sketch such an example. First we construct functions $b_1 \in C_b^2(\mathbb{R})$ and $b_2 \in C_b^2(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} \exp(B_1(x))\, dx = \int_{-\infty}^{\infty} \exp(B_2(x))\, dx = \infty, \tag{4.3}$$

$$\int_{-\infty}^{\infty} \exp([B_1(x) + B_2(x)]/2)\, dx = \alpha < \infty. \tag{4.4}$$

If $A_k f = f'' - (b_k f)'$, then from (4.3) it follows that equations $\partial u/\partial t = A_k u$ are not stable. Let

$$u_1(x) = u_2(x) = \frac{1}{2}\alpha^{-1} \exp([B_1(x) + B_2(x)]/2).$$

Then $A_1 u_1 + A_2 u_2 \equiv 0$. It is easy to check that we can find positive functions $p \in C_b(\mathbb{R})$ and $q \in C_b(\mathbb{R})$ such that $(q - p)u_1 + A_1 u_1 \equiv 0$. Then the function $u(x, k) = u_k(x)$ is a stationary density of the semigroup $\{P(t)\}_{t \geq 0}$. Consequently, the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable.

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