Characterization of the Domain of Fractional Powers of a Class of Elliptic Differential Operators with Feedback Boundary Conditions

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1. INTRODUCTION

We consider in this paper a system of linear differential operators \( \mathcal{L} \) in a bounded domain \( \Omega \) of \( \mathbb{R}^m \) with the boundary \( \Gamma \) which consists of a finite number of smooth components of \((m-1)\)-dimension. Actually, let \( \mathcal{L} \) denote a uniformly elliptic differential operator of order 2 in \( \Omega \) defined by

\[
\mathcal{L}u = - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u,
\]

where \( a_{ij}(x) = a_{ji}(x) \) for \( 1 \leq i, j \leq m, x \in \Omega \), and for some positive \( \delta \)

\[
\sum_{i,j=1}^{m} a_{ij}(x) \xi_i \xi_j \geq \delta |\xi|^2, \quad \forall \xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m, \quad \forall x \in \Omega.
\]

Associated with \( \mathcal{L} \) is a boundary operator \( \tau_1 \) of the Dirichlet type (case I) or \( \tau_2 \) of the generalized Neumann type (case II) defined by

\[
\tau_1 u = u|\Gamma,
\]

and

\[
\tau_2 u = \frac{\partial u}{\partial \nu} + \sigma(\xi) u = \sum_{i,j=1}^{m} a_{ij}(\xi) \frac{\partial u}{\partial x_j}|\Gamma + \sigma(\xi) u|\Gamma,
\]

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respectively, where \((v_1(\xi), ..., v_m(\xi))\) denotes the unit outer normal at \(\xi \in \Gamma\). Necessary regularity on \(\Omega\) and on \(\Gamma\) of coefficients of \(\mathcal{L}\) and \(\tau_2\) is assumed tacitly (see, e.g., [1, 4, 8, 9, 14]). Let us define the linear operators \(L_1\) and \(L_2\) in \(L^2(\Omega)\) by
\[
L_1 u = \mathcal{L} u, \quad u \in \mathcal{D}(L_1) = \{ u \in H^2(\Omega) ; \tau_1 u = 0 \}
\]
and
\[
L_2 u = \mathcal{L} u, \quad u \in \mathcal{D}(L_2) = \{ u \in H^2(\Omega) ; \tau_2 u = 0 \},
\]
respectively. The operators \(L_1\) and \(L_2\) are classical and very standard. Among the well known properties, their fractional powers are of our special interest. In [3, 5], a concrete characterization of the domain of fractional powers of \(L_1\) and \(L_2\) is obtained. A part of these results played an important role in some problems of boundary control systems [10, 11, 12]: The boundary control problem is reduced to a distributed control problem, i.e., a problem with a homogeneous boundary condition, by a simple transformation of the state. However, they do not provide us a satisfactory means, for example, in stability analysis of boundary feedback control systems [13]. The study of \(\mathcal{L}\) with feedback boundary condition and its fractional powers then becomes necessary. The objective of this paper is to develop the study of fractional powers of linear operators \(M_1\) and \(M_2\) introduced just below. As far as the author’s knowledge, basic properties of \(M_1\) and \(M_2\) are not well known, in contrast to the case of \(L_1\) and \(L_2\).

Let us define the linear operators \(M_1\) and \(M_2\) in \(L^2(\Omega)\) by
\[
M_1 u = \mathcal{L} u, \quad u \in \mathcal{D}(M_1) = \{ u \in H^2(\Omega) ; \tau_1 u = \sum_{k=1}^{p} \langle u, w_k \rangle \_\Omega h_k \text{ on } \Gamma \}, (1.1)
\]
and
\[
M_2 u = \mathcal{L} u, \quad u \in \mathcal{D}(M_2) = \{ u \in H^2(\Omega) ; \tau_2 u = \sum_{k=1}^{p} \langle u, w_k \rangle \_\Gamma h_k \text{ on } \Gamma \}, (1.2)
\]
respectively. Here, \(\langle \cdot, \cdot \rangle \_\Omega\) and \(\langle \cdot, \cdot \rangle \_\Gamma\) denote the inner products in \(L^2(\Omega)\) and \(L^2(\Gamma)\), respectively, \(p\) a positive integer depending on the control problems under consideration, and necessary regularities for the functions \(w_k\) and \(h_k\) are assumed in the following sections. Thus, the boundary conditions for \(M_1\) and \(M_2\) are described as a feedback type. The boundary control system corresponding to, for example, \(M_1\) is described by
\[
\frac{du}{dt} + M_1 u = 0, \quad t > 0, \quad u(0) = u_0 \quad (1.3)
\]
in \( L^2(\Omega) \), or
\[
\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in} \quad \Omega,
\]
\[
u|_{\Gamma} = \sum_{k=1}^{p} \langle u, w_k \rangle_{\Omega} h_k \quad \text{on} \quad \Gamma,
\]
\[
u(0, \cdot) = u(\cdot) \quad \text{in} \quad \Omega.
\]

The operators \( M_1 \) and \( M_2 \) are not a standard type in the sense that the boundary conditions are composed of terms of local nature (\( \tau_1 \) and \( \tau_2 \)) and those of global nature (\( \langle \cdot, \cdot \rangle_{\Omega} \) and \( \langle \cdot, \cdot \rangle_{\Gamma} \)). A particular difference between \( M_1 \) and \( M_2 \) lies in their \textit{accretiveness}. In fact, it is easily shown that \( M_2 \) (or its right shift \( M_2 + c, c > 0, \) if necessary) is \( m \)-accretive, while \( M_1 \) is not! Thus, different approaches are necessary for \( M_1 \) and \( M_2 \).

Throughout the paper, all norms will denote \( L^2(\Omega) \)- or \( \mathcal{L}(L^2(\Omega)) \)-norms. In Section 2, some well known facts are reviewed and preliminary results for \( M_1 \) and \( M_2 \) are developed, where basic assumptions and notations are introduced. In Section 3, the main results and their proofs are stated, where the domains of fractional powers for \( M_1 \) and \( M_2 \) are characterized in terms of Sobolev spaces. Since \( m \)-accretiveness for \( M_1 + c \) is not expected, the reader will find a considerable difference between \( M_1 \) and \( M_2 \) in studying their structures. The results turns out to be a striking extension of Fujiwara's and Grisvard's characterization [3, 5] stated in Section 2. Finally the concluding remarks are stated in Section 5, where we discuss versions of the main results occuring due to the replacement of some parameters in \( M_1 \) and \( M_2 \).

2. PRELIMINARY RESULTS

Let us begin with reviewing the well known spectral property for \( L_1 \) and \( L_2 \). There is a sector \( \Sigma_{-\pi} = \Sigma - \pi, \pi > 0 \), such that \( \Sigma_{-\pi} \) is contained in the resolvent sets \( \rho(L_i), i = 1, 2 \) and that the following estimates hold:
\[
\| (\lambda - L_i)^{-1} \| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma_{-\pi}, \quad i = 1, 2
\]
(2.1)
where \( \Sigma = \{ \lambda \in \mathbb{C}; 0 < \theta_0 < |\arg \lambda| < \pi \} \), \( 0 < \theta_0 < \pi/2 \), and the upper bar means the closure of a set. Choose a positive constant \( c(> \pi) \), and set \( L_{ic} = L_i + c \), \( i = 1, 2 \). Then, fractional powers of the operators \( L_{1c} \) and \( L_{2c} \) are well defined. In order to characterize the domains of \( L_{1c}^\alpha \) and \( L_{2c}^\alpha \), \( 0 \leq \theta \leq 1 \), it
is assumed in the rest of the paper that \( \sigma(\xi) \) appearing in the boundary operator \( \tau_2 \) has a suitable smooth extension to \( \hat{\Omega} \). The distance from \( x \in \mathbb{R}^m \) to \( \Gamma \) is denoted by \( \zeta(x) \). Then we have the following two fundamental theorems of [3, 5]:

**Theorem 2.1** (Case I. The Dirichlet boundary condition). The domain of the fractional powers \( L_{1c}^\alpha \) is characterized as follows:

(i) \( \mathcal{D}(L_{1c}^\alpha) = H^{2\alpha}(\Omega), \quad 0 \leq \alpha < \frac{1}{4} \);

(ii) \( \mathcal{D}(L_{1c}^{3/4}) = \left\{ u \in H^{3/2}(\Omega); \left( \int_{\Omega} \frac{1}{\zeta(x)} |u|^2 \, dx \right)^{1/2} < \infty \right\} \); and

(iii) \( \mathcal{D}(L_{1c}^{1/2}) = H_{1c}^{3/2}(\Omega), \quad \frac{1}{4} \leq \alpha \leq 1 \),

where the space \( H_{1c}^{3/2}(\Omega) \) is defined by

\[
H_{1c}^{3/2}(\Omega) = \{ u \in H^{3/2}(\Omega); u|_\Gamma = 0 \text{ on } \Gamma \}, \quad \alpha > \frac{1}{2}, \tag{2.2}
\]

The proof of Theorems 2.1 and 2.2 is carried out by transforming first a class of functions in a neighborhood of \( \Gamma \) into functions on the half space \( \mathbb{R}^m_{+} \) and then introducing operators of extension to the whole space \( \mathbb{R}^m \), e.g., a reflection operator with respect to the hypersurface \( \{ y_m = 0 \} \) and operators of restriction to \( \mathbb{R}^m_{+} \).

**Theorem 2.2** (Case II. The generalized Neumann boundary condition). The domain of the fractional powers \( L_{1c}^\alpha \) is characterized as follows:

(i) \( \mathcal{D}(L_{2c}^\alpha) = H^{2\alpha}(\Omega), \quad 0 \leq \alpha < \frac{3}{4} \);

(ii) \( \mathcal{D}(L_{2c}^{3/4}) = \left\{ u \in H^{3/2}(\Omega); \left( \int_{\Omega} \frac{1}{\zeta(x)} |\tau_2 u|^2 \, dx \right)^{1/2} < \infty \right\} \); and

(iii) \( \mathcal{D}(L_{2c}^{1/2}) = \{ u \in H^{3/2}(\Omega); \tau_2 u = 0 \text{ on } \Gamma \}, \quad \frac{3}{4} \leq \alpha \leq 1 \),

where \( \tau_2 \) is a first order differential operator given by

\[
\tau_2 u = \frac{\partial u}{\partial \zeta} + \sigma(x) u. \tag{2.3}
\]
When $L_1$ and $L_2$ are replaced by $M_1$ and $M_2$, respectively, it is natural to expect that the feedback boundary condition would appear in the above theorems. In fact, this expectation is true, and the corresponding results are stated in Section 3. We develop here some basic properties of $M_1$ and $M_2$. Most fundamental is the existence of the resolvents and their decay estimates. Henceforth $c$ denotes a various positive constant independent of arguments under consideration unless otherwise indicated. Our first result is stated as follows:

**Theorem 2.3.** (i) **Case I** (The Dirichlet boundary condition). Let us suppose that $w_k$'s and $h_k$'s in $M_1$ satisfy the assumption

$$w_k \in L^2(\Omega), \quad h_k \in H^{1/2}(\Gamma), \quad 1 \leq k \leq p.$$  

Then the domain $\mathcal{D}(M_1)$ is dense. There is a sector $\Sigma_{-\beta} = \{\lambda \in \mathbb{C} : \beta > \gamma, \gamma > \alpha\}$ such that $\Sigma_{-\beta}$ is contained in the resolvent set $\rho(M_1)$ and that the following estimate holds:

$$\|(\lambda - M_1)^{-1}\| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma_{-\beta}. \quad (2.5)$$

(ii) **Case II** (The generalized Neumann boundary condition). Let us suppose that $w_k$'s and $h_k$'s in $M_2$ satisfy the assumption

$$w_k \in L^2(\Gamma), \quad h_k \in H^{1/2}(\Gamma), \quad 1 \leq k \leq p.$$  

Then the domain $\mathcal{D}(M_2)$ is dense. There is a sector $\Sigma_{-\gamma} = \{\lambda \in \mathbb{C} : \gamma > \beta, \beta > \alpha\}$ such that $\Sigma_{-\gamma}$ is contained in the resolvent set $\rho(M_2)$ and that the following estimate holds:

$$\|(\lambda - M_2)^{-1}\| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma_{-\gamma}. \quad (2.7)$$

From the control theoretic viewpoint, it is interesting to investigate how the feedback terms affect the adjoint structures of $M_1$ and $M_2$. In fact, we have the following results:

**Proposition 2.4.** We assume that the conditions (2.4) and (2.6) are satisfied in Case I and Case II, respectively.

(i) The adjoint operator of $M_1$ is described by

$$M_1^*v = \mathcal{D}^*v + \sum_{k=1}^{p} \left< \frac{\partial v}{\partial n}, h_k \right> \cdot \frac{w_k}{\|w_k\|^2},$$  

$$v \in \mathcal{D}(M_1^*) = H^2(\Omega) \cap H_0^1(\Omega) = \mathcal{D}(L_1), \quad (2.8)$$

From the control theoretic viewpoint, it is interesting to investigate how the feedback terms affect the adjoint structures of $M_1$ and $M_2$. In fact, we have the following results:
where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$:

$$
\mathcal{L}^* u = - \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - \sum_{i=1}^m \frac{\partial}{\partial x_i} (b_i(x) u) + c(x) u.
$$

(ii) Assume that $w_k$'s in $M_2$ belong to $H^{1/2}(\Gamma)$ in addition. The adjoint operator of $M_2$ is then described by

$$
\mathcal{M}_2^* v = \mathcal{L}^* v,
$$

where $(\mathcal{L}^*, \mathcal{M}_2^*)$ denotes the formal adjoint of $(\mathcal{L}, \mathcal{M}_2)$, and $\mathcal{M}_2^* = \partial^2 / \partial v + (\sigma(\xi) + \sum_{i=1}^m b_i(\xi) v_i(\xi))$.

The rest of the section is devoted to the proof of these results.

**Proof of Theorem 2.3**

Although Cases I and II look similar, we need different approaches. In fact, a sesquilinear form is available in Case II, while it is not in Case I.

Case I. Let us consider the boundary value problem

$$(\lambda - \mathcal{L}) u = 0 \quad \text{in } \Omega \quad \text{and} \quad \tau_1 u = u|_{\Gamma} = f \quad \text{on } \Gamma$$

for any given $f \in H^{1/2}(\Gamma)$. There is a unique solution $u \in H^2(\Omega) \cap H^1(\Omega)$ if $\lambda$ is in $\rho(L_1)$, and the solution $u$ is denoted by $N_1(\lambda) f$. The solution $u$ is expressed, for example, as

$$
u = N_1(\lambda) f = R_1 f - (\lambda - L_1)^{-1} (\lambda - \mathcal{L}) R_1 f,
$$

where $R_1$ denotes a linear operator belonging to $\mathcal{L}(H^1(\Gamma); H^2(\Omega))$ such that

$$
R_1 f|_{\Gamma} = f, \quad \text{and} \quad \frac{\partial}{\partial v} R_1 f|_{\Gamma} = 0. \quad (2.10)
$$

The operator $R_1$ is not uniquely determined. We need the following lemma regarding the behavior of $N_1(\lambda)$, the proof of which is to be given later:

**Lemma 2.5.** Assumption (2.4) implies that

$$
\langle N_1(\lambda) h_j, w_k \rangle \to 0 \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda \in \rho(L_1).
$$
For a given \( f \in L^2(\Omega) \), let us consider the problem

\[
    u = (\lambda - L_1)^{-1} f + \sum_{k=1}^{p} \langle u, w_k \rangle_\Omega N_1(\lambda) h_k. \tag{2.11}
\]

If the problem has a solution \( u \in H^2(\Omega) \), this solves the boundary value problem

\[
    (\lambda - M_1) u = f.
\]

Suppose for a moment that (2.11) admits a solution \( u \). Then it is immediately seen that, for a sufficiently large \( |\lambda| \)

\[
    \langle u, w \rangle_\Omega = (1 - \Phi(\lambda))^{-1} \langle (\lambda - L_1)^{-1} f, w \rangle_\Omega, \tag{2.12}
\]

where \( \langle \cdot, w \rangle_\Omega \) denotes a \( p \times 1 \) column vector whose \( k \)th component is given by \( \langle \cdot, w_k \rangle_\Omega \), and \( \Phi(\lambda) \) the \( p \times p \) matrix given by

\[
    \Phi(\lambda) = \begin{bmatrix} \langle N_1(\lambda) h_j, w_k \rangle_\Omega : \ j \rightarrow 1, \ldots, p \end{bmatrix}.
\]

Note that \( (1 - \Phi(\lambda))^{-1} \) exists when \( |\lambda| \) goes to \( \infty \), due to the estimate in Lemma 2.5. By substituting this into (2.11), \( u \) must have the expression:

\[
    u = (\lambda - L_1)^{-1} f + \sum_{k=1}^{p} [(1 - \Phi(\lambda))^{-1} \langle (\lambda - L_1)^{-1} f, w_k \rangle_\Omega ] k N_1(\lambda) h_k. \tag{2.13}
\]

Conversely, it is easily seen that \( u \) given by (2.13) satisfies the relation (2.12), which immediately leads to the equation (2.11). Uniqueness of solutions to \( (\lambda - M_1) u = f \) is almost immediate. The estimate (2.5) with some \( \beta > \alpha \) is derived from the above expression (2.13) and Lemma 2.5.

**Denseness of \( D(M_1) \).** Let us choose a \( \lambda \in \Sigma_{-\beta} \). We only have to show that the relation

\[
    \langle (\lambda - M_1)^{-1} f, \varphi \rangle_\Omega = 0 \quad \text{for all} \quad f \in L^2(\Omega)
\]

implies that \( \varphi = 0 \). We see from (2.13) that

\[
    0 = \langle (\lambda - L_1)^{-1} f, \varphi \rangle_\Omega + \sum_{k=1}^{p} [(1 - \Phi(\lambda))^{-1} \langle (\lambda - L_1)^{-1} f, w_k \rangle_\Omega ] k \langle N_1(\lambda) h_k, \varphi \rangle_\Omega \n\]

\[
    = \langle (\lambda - L_1)^{-1} f, \varphi \rangle_\Omega + \sum_{k=1}^{p} a_k \langle (\lambda - L_1)^{-1} f, w_k \rangle_\Omega \n\]

\[
    = \left\langle f, (\lambda - L_1)^{-1} \left( \varphi + \sum_{k=1}^{p} a_k w_k \right) \right\rangle_\Omega ,
\]
that is
\[(\bar{\lambda} - L^*_1)^{-1}\left(\varphi + \sum_{k=1}^{r} a_k w_k\right) = 0, \quad \text{or} \quad \varphi + \sum_{k=1}^{r} a_k w_k = 0,\]
where
\[(a_1 \cdots a_r) = (\langle \Lambda_1(\bar{\lambda}) h_1, \varphi \rangle \cdots \langle \Lambda_1(\bar{\lambda}) h_p, \varphi \rangle)(1 - \Phi(\bar{\lambda}))^{-1}.\]

Thus we see that
\[0 = \langle \Lambda_1(\bar{\lambda}) h_j, \varphi + \sum_{k=1}^{r} a_k w_k\rangle = \langle \Lambda_1(\bar{\lambda}) h_j, \varphi \rangle + \sum_{k=1}^{r} a_k \langle \Lambda_1(\bar{\lambda}) h_j, w_k \rangle \quad (1 \leq j \leq p),\]
or
\[(0 \cdots 0) = (\langle \Lambda_1(\bar{\lambda}) h_1, \varphi \rangle \cdots \langle \Lambda_1(\bar{\lambda}) h_p, \varphi \rangle) + (a_1 \cdots a_r) \Phi(\bar{\lambda})
= (\langle \Lambda_1(\bar{\lambda}) h_1, \varphi \rangle \cdots \langle \Lambda_1(\bar{\lambda}) h_p, \varphi \rangle)(1 - \Phi(\bar{\lambda}))^{-1}
= (a_1 \cdots a_r).\]

We have shown that \(\varphi = 0.\)

**Proof of Lemma 2.5.** Abbreviate the subscripts of \(h_j\) and \(w_k\) for simplicity. Consider first the case where \(w\) belongs to \(\mathcal{D}(\Omega).\) Since both \(\Lambda_1(\bar{\lambda}) h\) and \(\Lambda_1(\bar{\lambda}) w\) belong to \(H^2(\Omega)\) and \(w\) belongs to \(\mathcal{D}(L^*_1),\) we see that
\[(\lambda + c)\langle \Lambda_1(\bar{\lambda}) h, w \rangle = \langle \mathcal{L}_1 \Lambda_1(\bar{\lambda}) h, w \rangle = \langle \mathcal{L}_1 \Lambda_1(\bar{\lambda}) h - \Lambda_1(\bar{\lambda}) (-c) h, w \rangle 
= \langle L_{1_1} \Lambda_1(\bar{\lambda}) h - \Lambda_1(\bar{\lambda}) (-c) h, w \rangle 
= \langle \Lambda_1(\bar{\lambda}) h - \Lambda_1(\bar{\lambda}) (-c) h, L_{1_1}^* w \rangle.\]

On the other hand, the expression of \(\Lambda_1(\bar{\lambda})\) via \(R_1\) easily implies the estimate
\[\|\Lambda_1(\bar{\lambda})\|_{\mathcal{L}(H^2(\Omega) \to L^2(\Omega))} \lesssim \text{const}, \quad \bar{\lambda} \in \rho(L_1).\]

Thus we finally find that
\[\langle \Lambda_1(\bar{\lambda}) h, w \rangle \lesssim \text{const}(1 + |\bar{\lambda}|)^{-1} \to 0 \quad \text{as} \quad |\bar{\lambda}| \to \infty.\]

Second note that a general \(w \in L^2(\Omega)\) is arbitrarily approximated (in \(L^2(\Omega)\)-topology) by an element of \(\mathcal{D}(\Omega),\) say \(w_n.\) Combining this with the
above argument and the boundedness of \(\|N_i(h)\|\), we establish the decay estimate. This completes the proof of Lemma 2.5.

**Case II.** The domain \(\mathcal{D}(M_2)\) is clearly dense, since \(\mathcal{D}(\Omega) = C_0^\infty(\Omega)\) is contained in \(\mathcal{D}(M_2)\). Let us consider the boundary value problem

\[(\lambda - \mathcal{L}) u = 0 \quad \text{in } \Omega \quad \text{and} \quad \tau_2 u = \frac{\partial u}{\partial v} + \sigma(\xi) u = f \quad \text{on } \Gamma\]

for any given \(f \in H^{1/2}(\Gamma)\). There is a unique solution \(u \in \mathcal{H}^2(\Omega)\) for \(\lambda \in \rho(L_2)\), and the solution \(u\) is denoted by \(N_2(\lambda)f\), where \(N_2(\lambda) \in \mathcal{L}(H^{1/2}(\Gamma);\ H^2(\Omega))\). By introducing an operator \(R_2\) such that [8]

\[R_2 f|_\Gamma = 0, \quad \text{and} \quad \frac{\partial}{\partial v} R_2 f|_\Gamma = f, \quad \forall f \in H^{1/2}(\Gamma), \quad (2.14)\]

the solution \(N_2(\lambda)f\) is expressed as

\[N_2(\lambda)f = R_2 f - (\lambda - L_2)^{-1} (\lambda - \mathcal{L}) R_2 f.\]

In order to consider the boundary value problem

\[(\lambda - M_2) u = f, \quad (2.15)\]

a sesquilinear form is available in our case. The sesquilinear form associated with \(M_2\) is the form on \(H^1(\Omega)\) given by

\[B[u, \varphi] = \sum_{i,j=1}^m \left\langle a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right\rangle_\Omega + \sum_{i=1}^m \left\langle b_i(x) \frac{\partial u}{\partial x_i}, \varphi \right\rangle_\Omega + \left\langle c(x) u, \varphi \right\rangle_\Omega + \left\langle \sigma(\xi) u, \varphi \right\rangle_\Gamma - \sum_{k=1}^{p} \left\langle u, w_k \right\rangle_\Gamma \left\langle h_k, \varphi \right\rangle_\Gamma.\]

By setting \(B[u, \varphi] = B[u, \varphi] + c \langle u, \varphi \rangle_\Omega\) for a sufficiently large constant \(c > 0\), a standard argument [9] shows that

\[\Re B[u, u] \geq \text{const} \|u\|^{2}_{H^1(\Omega)}, \quad u \in H^1(\Omega) \quad \text{and} \quad \|B[u, \varphi]\| \leq \text{const} \|u\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}.\]

Thus, for any \(f \in L^2(\Omega)\), there exists a unique \(u \in H^1(\Omega)\) such that

\[B[u, \varphi] = \langle f, \varphi \rangle_\Omega, \quad \forall \varphi \in H^1(\Omega).\]
Let \( v \in H^2(\Omega) \) be the unique solution to the problem

\[
\mathcal{L}_\tau v = f, \quad \tau_2 v = \sum_{k=1}^{p} \langle u, w_k \rangle_T h_k.
\]

The solution \( v \) is expressed as

\[
v = L_{\tau_2}^{-1} f + \sum_{k=1}^{p} \langle u, w_k \rangle_T N_2(-c) h_k.
\]

Green's formula implies that, for any \( \varphi \in H^1(\Omega) \),

\[
\langle f, \varphi \rangle_\Omega = \langle \mathcal{L}_\tau v, \varphi \rangle_\Omega = \left\langle \sigma v - \sum_{k} \langle u, w_k \rangle_T h_k, \varphi \right\rangle_\Omega + \sum_{i,j} \left( a \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right) \Omega + \sum_{i} \left( b_i \frac{\partial v}{\partial x_i} \varphi \right)_\Omega + \left\langle (c(x) + c) v, \varphi \right\rangle_\Omega.
\]

Thus we see that

\[
\tilde{B}_\tau [v - u, \varphi] = 0 \quad \text{for} \quad \forall \varphi \in H^1(\Omega),
\]

where \( \tilde{B}_\tau \) denotes the sesquilinear form associated with \( L_{\tau_2} \) (\( \tilde{B}_\tau \) is a special \( B \), in the case where \( w_k = 0 \) or \( h_k = 0 \), \( 1 \leq k \leq p \)). Since \( c > 0 \) is large enough, we see that

\[
\text{Re} \tilde{B}_\tau [g, g] \geq \text{const} \| g \|^2_{H^1(\Omega)}
\]

for all \( g \in H^1(\Omega) \). This shows that

\[
u = v \in H^2(\Omega) \quad \text{and} \quad M_{\tau_2} u = f.
\]

Uniqueness of the solution \( u \) will be immediate, due to coerciveness of \( B \).

The operator \( M_{\tau_2} \) is a continuous bijection from \( \mathcal{D}(M_{\tau_2}) \) onto \( L^2(\Omega) \). Thus the inverse \( M_{\tau_2}^{-1} \) belongs to \( \mathcal{L}(L^2(\Omega); \mathcal{D}(M_{\tau_2})) \), or

\[
\| u \|_{H^2(\Omega)} \leq \text{const} \| M_{\tau_2} u \|, \quad \text{for} \quad \forall u \in H^2(\Omega).
\]

Let us go back to (2.15). The problem (2.15) is equivalent to the solvability of the problem

\[
((\lambda + c) M_{\tau_2}^{-1} - 1) u = M_{\tau_2}^{-1} f
\]
in $L^2(\Omega)$. Since $M_1^{-1} \in \mathcal{D}(L^2(\Omega))$ is compact, we only have to seek the region of $\lambda$ in which uniqueness of solutions to (2.15) holds (the Riesz–Schauder theory [9, 15]). Now it is straightforward to find out this region and to obtain the estimate (2.7) in some sector $\Sigma$ (see, e.g., [1, 9]). Since the calculation is very elementary but tedious, we omit the rest of the proof. The proof of Theorem 2.3 is thereby completed. Q.E.D.

Let us turn to the proof of Proposition 2.4.

Proof of Proposition 2.4. Since both cases (i) and (ii) are essentially the same, we only consider the case (i). The proof is merely a version of Green’s formula. In fact, let $M_1^\gamma$ denote the operator given by the right-hand side of (2.8) with $\mathcal{D}(M_1^\gamma) = \mathcal{D}(L^1)$. It is clear that $\rho(M_1^\gamma)$ exists in a sector $\Sigma_{-\gamma}$ for some positive $\gamma$. In fact, $\sum_{k=1}^\infty \langle \partial_1 v, h_k \rangle w_k$ is subordinate to some fractional power of $L_1^\gamma$ with exponent less than 1. Green’s formula implies that, for $u \in \mathcal{D}(M_1)$ and $v \in \mathcal{D}(M_1^\gamma)$

$$\langle M_1^\gamma u, v \rangle = \langle u, L^\gamma v \rangle + \left\langle u, \sum_{k=1}^\infty \left( \frac{\partial_1 v}{\delta x}, h_k \right) w_k \right\rangle = \langle u, M_1^\gamma v \rangle,$$

which shows that $M_1^\gamma \in M_1^\dagger$. Conversely, choose a $c > 0$ such that $-c \in \rho(M_1) \cap \rho(M_1^\gamma)$. It is clear that

$$(M_1^\gamma)^{-1} = (M_1^{-1})^* \in \mathcal{D}(L^2(\Omega)).$$

For any $u \in \mathcal{D}(M_1^\gamma)$, set $v = M_1^\gamma u \in L^2(\Omega)$. Then, setting $w = M_1^\gamma v$, we see that $v = M_1^\gamma v$. This implies that

$$M_1\gamma(u - w) = 0, \quad \text{or} \quad u = w \in \mathcal{D}(M_1).$$

We have shown that $\mathcal{D}(M_1^\gamma) = \mathcal{D}(M_1)$, that is, $M_1^\gamma = M_1^\gamma$. This completes the proof of Proposition 2.4. Q.E.D.

3. MAIN RESULTS

In Theorem 2.3, we have shown that, if $c > 0$ is chosen large enough, a sector obtained as a suitable right shift of $\Sigma$ is contained in the resolvent sets $\rho(M_{1c})$ and $\rho(M_{2c})$, and the decay estimates for the resolvents $(\lambda - M_{1c})^{-1}$ and $(\lambda - M_{2c})^{-1}$ are guaranteed in that sector. Thus fractional powers for $M_{1c}$ and $M_{2c}$ are well defined. In this section, we extend Theorems 2.1 and 2.2 to the case of $M_{1c}$ and $M_{2c}$, respectively. Let us recall
here the definition of the operators \( R_1 \) and \( R_2 \) given by (2.10) and (2.14), respectively:

\[
R_1 \in \mathcal{D}(H^{1/2}(\Gamma); H^2(\Omega)); \quad R_1 f \big|_{\Gamma} = f, \quad \text{and} \quad \frac{\partial}{\partial n} R_1 f \big|_{\Gamma} = 0, \quad \forall f \in H^{1/2}(\Gamma),
\]

\[
R_2 \in \mathcal{D}(H^{1/2}(\Gamma); H^2(\Omega)); \quad R_2 f \big|_{\Gamma} = 0, \quad \text{and} \quad \frac{\partial}{\partial n} R_2 f \big|_{\Gamma} = f, \quad \forall f \in H^{1/2}(\Gamma).
\]

Our main results are Theorems 3.1 and 3.2 stated as follows:

**Theorem 3.1** (Case I. The Dirichlet boundary condition). Suppose that \( w_k, 1 \leq k \leq p, \) belong to \( H^1(\Omega) \) for an arbitrarily small \( \epsilon > 0. \) Then the domain of the fractional powers \( M_{\epsilon}^\theta \), \( 0 \leq \theta \leq 1, \) is characterized as follows:

(i) \( \mathcal{D}(M_{\epsilon}^\theta) = H^{2\theta}(\Omega), \quad 0 \leq \theta < \frac{1}{4}; \)

(ii) \( \mathcal{D}(M_{\epsilon}^{1/2}) \)

\[
= \left\{ u \in H^{1/2}(\Omega); \left[ \frac{1}{\alpha(x)} \left| u - \sum_{k=1}^{p} \langle u, w_k \rangle \right| \tau_k h_k \right] dx < \infty \right\}; \quad \text{and}
\]

(iii) \( \mathcal{D}(M_{\epsilon}^\theta) = H^{2\theta}_{f_1}(\Omega), \quad \frac{1}{4} < \theta \leq 1, \)

where \( H^{2\theta}_{f_1}(\Omega) \) denotes the space defined by

\[
H^{2\theta}_{f_1}(\Omega) = \left\{ u \in H^{2\theta}(\Omega); u \big|_{\Gamma} = \sum_{k=1}^{p} \langle u, w_k \rangle \tau_k h_k \text{ on } \Gamma \right\}, \quad 2\theta > \frac{1}{2};
\]

Moreover, we have the interpolation relation

\[
\mathcal{D}(M_{\epsilon}^\theta) = \left[ \mathcal{D}(M_{\epsilon}^1), L^2(\Omega) \right]_{1-\theta}, \quad 0 \leq \theta \leq 1,
\]

where \( [\cdot, \cdot]_{1-\theta} \) denotes an intermediate space lying between two spaces, one of which is densely embedded in the other.

**Theorem 3.2** (Case II. The generalized Neumann boundary condition). The domain of the fractional powers \( M_{\epsilon}^\theta \), \( 0 \leq \theta \leq 1, \) is characterized as follows:

(i) \( \mathcal{D}(M_{\epsilon}^\theta) = H^{2\theta}(\Omega), \quad 0 \leq \theta < \frac{3}{4}; \)

(ii) \( \mathcal{D}(M_{\epsilon}^{3/4}) \)

\[
= \left\{ u \in H^{3/2}(\Omega); \left[ \frac{1}{\alpha(x)} \left| \tau_h u - \sum_{k=1}^{p} \langle u, w_k \rangle \tau_k R_2 h_k \right| \right] dx < \infty \right\}; \quad \text{and}
\]
The following result discusses algebraic similarity of $M_1$ and $M_2$ to operators with homogeneous boundary conditions: $\tau_1 u = 0$ and $\tau_2 u = 0$, respectively. Originally it comes from a control theoretic study of $M_1$ and $M_2$:

**Theorem 3.3.** (i) For any $\theta \in \mathbb{R}^1$, $M^0_{1r}$ is algebraically similar to $(L_{1r} - F_1)^\theta$ in the sense that

$$M^0_{1r} = L_{1r}^{1/4+\varepsilon}(L_{1r} - F_1)^\theta L_{1r}^{-1/4-\varepsilon}, \quad \rho(M^0_{1r}) = \rho((L_{1r} - F_1)^\theta),$$

where $0 < \varepsilon < 1/4$ and the operator $F_1$ is defined by

$$F_1 u = \sum_{k=1}^{p} \langle L_{1r}^{1/4+\varepsilon} u, w_k \rangle_R L_{1r}^{-1/4-\varepsilon} N_1(-c) h_k.$$ 

(ii) For any $\theta \in \mathbb{R}^1$, $M^0_{2r}$ is algebraically similar to $(L_{2r} - F_2)^\theta$ in the sense that

$$M^0_{2r} = L_{2r}^{1/4+\varepsilon}(L_{2r} - F_2)^\theta L_{2r}^{-1/4-\varepsilon}, \quad \rho(M^0_{2r}) = \rho((L_{2r} - F_2)^\theta),$$

where $0 < \varepsilon < 1/2$ and the operator $F_2$ is defined by

$$F_2 u = \sum_{k=1}^{p} \langle L_{2r}^{1/4+\varepsilon} u, w_k \rangle_R L_{2r}^{-1/4-\varepsilon} N_2(-c) h_k.$$ 

The rest of the section is devoted to the proof of the above theorems. As we have seen in Section 2, the approach to $M_1$ in this section is also quite different from the one to $M_2$.

**Proof of Theorem 3.1**

The structure of the proof is involved, and thus divided into several steps for the reader's convenience.

**First Step (Operator $T_1$).** A serious difficulty is that $M_1$ is no more an accretive operator. So, our strategy is to introduce, instead, another operator $K$ defined below (Second Step) via $T_1$, where $T_1$ denotes an operator formally defined by

$$v = T_1 u = u - \sum_{k=1}^{p} \langle u, w_k \rangle_R R_i h_k. \quad (3.1)$$
It turns out that the operator $K$ is accretive if an additional regularity assumption on $w_i$'s is added (see Proposition 3.4, (ii)).

By definition, operator $T_1$ clearly belongs to $\mathcal{L}(L^2(\Omega)) \cap \mathcal{L}(\mathcal{D}(M); \mathcal{D}(L))$, where both $\mathcal{D}(M)$ and $\mathcal{D}(L)$ are equipped with the topology of $H^2(\Omega)$. Let us examine its inverse. Set $T_1 u = 0$. Then

$$
\langle u, w_j \rangle_{\alpha} = \sum_{k=1}^{p} \langle u, w_k \rangle_{\alpha} \langle R_1 h_k, w_j \rangle_{\alpha}, \quad 1 \leq j \leq p, \quad \text{or} \\
\langle u, w \rangle_{\alpha} = \Psi \langle u, w \rangle_{\alpha},
$$

where $\Psi$ means the $p \times p$ matrix defined by

$$
\Psi = \begin{bmatrix} \langle R_1 h_k, w_j \rangle_{\alpha} & k \rightarrow 1, \ldots, p \end{bmatrix}.
$$

Since $R_1$ admits a great deal of freedom of choice, we first assume that $\det(1 - \Psi) \neq 0$. In fact, we only have to make a slight modification of $R_1$, if necessary. A general $R_1$ assuming only (2.10) is considered later in the Fourth Step of the proof. Under this assumption, we see that

$$
\langle u, w \rangle_{\alpha} = 0, \quad \text{or} \quad u = 0.
$$

Thus $T_1$ is injective (namely, its formal inverse $T_1^{-1}$ exists), and the inverse $T_1^{-1}$ is calculated as

$$
u = T_1^{-1} v = v + \sum_{k=1}^{p} [(1 - \Psi)^{-1} \langle v, w \rangle_{\alpha}]_k R_1 h_k. \quad (3.2)
$$

The operator $T_1^{-1}$ belongs to $\mathcal{L}(L^2(\Omega))$. Moreover, $u = T_1^{-1} v$ satisfies the relation

$$
\langle u, w \rangle_{\alpha} = (1 - \Psi)^{-1} \langle v, w \rangle_{\alpha}.
$$

Thus $T_1^{-1}$ maps $\mathcal{D}(L_1)$ onto $\mathcal{D}(M_1)$ and belongs to $\mathcal{L}(\mathcal{D}(L_1); \mathcal{D}(M_1))$. The well known interpolation theory [8] implies that

$$
T_1 \in \mathcal{L}([\mathcal{D}(M), L^2(\Omega)]_{1-\theta}; \mathcal{D}(L^p)), \quad \text{and} \\
T_1^{-1} \in \mathcal{L}(\mathcal{D}(L^p); [\mathcal{D}(M), L^2(\Omega)]_{1-\theta}), \quad 0 \leq \theta \leq 1.
$$

(3.3)

Here we have used the fact that $[\mathcal{D}(L_1), L^2(\Omega)]_{1-\theta}$ is equal to $\mathcal{D}(L^p)$ due to the $m$-accretiveness of $L_{1c}$. 
Second Step (Operator $K$). Owing to the First Step, we are able to introduce a new operator $K$ by

$$K = T_1 M_1 T_1^{-1}, \quad \mathcal{D}(K) = \mathcal{D}(L_1) = H^2(\Omega) \cap H_0^1(\Omega).$$  (3.4)

The operator $K$ plays a role of connecting $M_1$ with $L_1$ (see the diagram at the end of the Second Step). If $\lambda$ is in $\rho(M_1)$, then $\lambda - K$ has a bounded inverse, and

$$(\lambda - K)^{-1} = T_1 (\lambda - M_1)^{-1} T_1^{-1} \in \mathcal{D}(L^2(\Omega)).$$

In view of the decay estimate (2.5), the sector $\Sigma_{-\beta}$ is contained in $\rho(K)$ and

$$\| (\lambda - K)^{-1} \| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma_{-\beta}.$$

Thus, if $c$ is larger than $\beta$, fractional powers of $K_c = K + c$ are well defined. The operator $K_c^{-\theta}$ is by definition calculated as follows:

$$K_c^{-\theta} = \frac{-1}{2\pi i} \int_{\gamma} \lambda^{-\theta}(\lambda - K_c)^{-1} d\lambda = \frac{-1}{2\pi i} \int_{\gamma} \lambda^{-\theta} T_1 (\lambda - M_1)^{-1} T_1^{-1} d\lambda$$

$$= T_1 M_1^{-\theta} T_1^{-1}, \quad \theta \geq 0,$$  (3.5)

where $i = \sqrt{-1}$, and $\gamma$ denotes the boundary of a suitable right shift of the sector $\Sigma$, oriented according to increasing $\text{Im} \lambda$. The operator $K$ enjoys nice properties. For example, relation (3.5) immediately implies that

$$T_1 \in \mathcal{S}(\mathcal{D}(M_{1c}^\theta); \mathcal{D}(K_c^\theta)) \text{ and } T_1^{-1} \in \mathcal{S}(\mathcal{D}(K_c^\theta); \mathcal{D}(M_{1c}^\theta)), \quad 0 \leq \theta \leq 1. \quad (3.6)$$

The following proposition forms a key result of the theorem, the proof of which is stated in the Last Step:

**Proposition 3.4.** (i) If $c$ is large enough, the equivalence relation

$$\mathcal{D}(K_c^\theta) = \mathcal{D}(L_{1c}^\theta), \quad 0 \leq \theta \leq 1$$  (3.7)

holds algebraically and topologically.

(ii) If $w_k$, $1 \leq k \leq p$, belong to $H^1_0(\Omega)$ in addition, then $K_c$ is $m$-accretive, namely

$$\text{Re} \langle K_c u, u \rangle_\Omega \geq \text{const} \| u \|^2, \quad u \in \mathcal{D}(K).$$  (3.8)

**Remark.** The above (i) is proved independent of (ii). In the case where $w_k$’s belong to $H^1_0(\Omega)$, however, (ii) immediately implies the assertion (i),
once we observe the equivalence relation: $\mathcal{D}(K) = \mathcal{D}(L_1)$. In fact, now that both $K$ and $L_1$ are $m$-accretive in this case, a generalization of the Heinz inequality [6] is now applied to show the equivalence relation (3.7).

According to this proposition, relation (3.6) is rewritten as

**Lemma 3.5.** The operator $T_1$ is a continuous bijection from $\mathcal{D}(M^\theta_1)$ onto $\mathcal{D}(L^\theta_1)$ for each $0 \leq \theta \leq 1$, and thus,

$$T_1 \in \mathcal{D}(\mathcal{D}(M^\theta_1); \mathcal{D}(L^\theta_1)) \text{ and } T_1^{-1} \in \mathcal{D}(\mathcal{D}(L^\theta_1); \mathcal{D}(M^\theta_1)), \ 0 \leq \theta \leq 1. \quad (3.9)$$

Although the $m$-accretiveness of $M_1$ is never expected and thus a generalization of the Heinz inequality [6] cannot be applied, relation (3.3) combined with Lemma 3.5 yields the last assertion of the theorem:

$$\mathcal{D}(M^\theta_1) = [\mathcal{D}(M_1), L^2(\Omega)]_{1-\theta}, \ 0 \leq \theta \leq 1.$$  

The above relations are summarized as the following diagram:

$$[\mathcal{D}(M_1); L^2(\Omega)]_{1-\theta} \xrightarrow{T_1} \mathcal{D}(L^\theta_1) \xleftarrow{T_1^{-1}} \mathcal{D}(M^\theta_1)$$

**Third Step.** We are in a position to prove the characterization: (i) to (iii) of the theorem.

**Proof of (i)** (the case where $0 \leq 2\theta < 1/2$). Note that both $T_1$ and $T_1^{-1}$ belong to $\mathcal{D}(H^2(\Omega)), \ 0 < \theta < 1/4$. Since $T_1$ also belongs to $\mathcal{D}(M^\theta_1); H^2(\Omega))$ and $T_1^{-1}$ to $\mathcal{D}(H^2(\Omega); \mathcal{D}(M^\theta_1))$ by Lemma 3.5 and Theorem 2.1, (i), the assertion of (i) is now immediate.

**Proof of (ii)** (the case where $2\theta = 1/2$). Suppose that

$$u \in H^{1/2}(\Omega), \quad \text{and} \quad \int_{\Omega} \frac{1}{1-\sqrt{x}} |T_1u|^2 \, dx < \infty. \quad (3.10)$$

This relation is immediate: For any $u \in \mathcal{D}(K)$, we estimate as

$$|Ku| \leq \|L^{-1/2}u\| \leq \|T_1^{-1}u\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq \|L_1u\|.$$  

As to the converse inequality, recall that $M_1$ is a continuous bijection from $\mathcal{D}(M_1)$ ($\in H^{\theta}(\Omega)$) onto $L^2(\Omega)$. Then,

$$|L_1u| \leq \|u\|_{H^\theta(\Omega)} \leq \|T_1^{-1}u\|_{H^{1/2}(\Omega)} \leq \|T_1M_1^{-1}u\| = \|u\|_{H^\theta(\Omega)}.$$
Then $v = T_1 u$ belongs to $\mathcal{D}(L_{1,2}^1)$, due to Theorem 2.1, (ii). We see from Lemma 3.5 that $u = T_1^{-1} v$ belongs to $\mathcal{D}(M_{1,2}^1)$. Conversely, if $u$ is in $\mathcal{D}(M_{1,2}^1)$, then $v = T_1 u$ belongs to $\mathcal{D}(L_{1,2}^1)$, again due to Lemma 3.5. Thus $u$ satisfies (3.10).

Proof of (iii) (the case where $1/2 < 2\theta \leq 2$). It is easily seen from Theorem 2.1, (iii) that

$$T_1 \in \mathcal{D}(H_0^{2\theta}(\Omega); \mathcal{D}(L_0^n)) \quad \text{and} \quad T_1^{-1} \in \mathcal{D}(\mathcal{D}(L_0^n); H_0^{2\theta}(\Omega)),$$

which, again combined with Lemma 3.5, shows that

$$\mathcal{D}(M_{1,2}^n) = H_0^{2\theta}(\Omega), \quad 1/2 < 2\theta \leq 2.$$

Fourth Step (Operator $R_1$). We have assumed so far that $\det(1 - \Psi) \neq 0$ in the First Step. Let us consider a general $R_1$, say $\tilde{R}_1$ assuming only (2.10). Then

$$(\tilde{R}_1 - R_1) h_k |_{y = 0} = 0, \quad 1 \leq k \leq p.$$

It is enough to consider the behavior of these functions in a neighborhood of $\Gamma$. Introducing a partition of unity of $\Gamma$, we can move to the half space $\mathbb{R}^{n-1}_+ = \{ y = (y_1, ..., y_m) \in \mathbb{R}^m, y_m > 0 \}$. In a neighborhood of $\{ y_m = 0 \}$, the transformed $\zeta(x)$ behaves like $y_m$. The transformed $(\tilde{R}_1 - R_1) h_1$, still denoted as the same symbol, belong to $H_0^2(\mathbb{R}^{n-1}_+)$; are absolutely continuous in $y_m$ for almost all $y' = (y_1, ..., y_{m-1})$; and satisfy

$$(\tilde{R}_1 - R_1) h_k(y', y_m) = \left[ y_m \frac{\partial}{\partial y_m} (\tilde{R}_1 - R_1) h_k(y', t) \right] dt$$

for almost all $y' \in \mathbb{R}^{n-1}$. Thus, by going back to the original coordinates, it is immediately seen that

$$\int_\Omega \frac{1}{\zeta(x)} |(\tilde{R}_1 - R_1) h_k|^2 \, dx < \infty,$$

which shows that the expression (ii) does not depend on a particular choice of $R_1$.

Last Step. In order to complete the theorem, let us turn to the proof of the auxiliary results mentioned above.
Proof of Proposition 3.4. (i) Let us examine what the form of the operator \( K = T_1 M T_1^{-1} \) is. Note that \( w_k \)'s belong to \( H^2(\Omega) = \mathcal{D}(L_1^{1,\gamma}) \), \( 0 < \varepsilon < 1/4 \). Then \( K \) is, by definition, written as

\[
K u = L_1 u - \sum_{k=1}^p \langle L_1, u, w_k \rangle \delta R_1 h_k + \sum_{k=1}^p \left[ \frac{1}{(1 - \nu)} \langle u, w \rangle \right] \delta R_1 h_k
\]

\[
= L_1 u - \sum_{k=1}^p \langle L_1^{1,\gamma} u, L_1^{1,\gamma} w_k \rangle \delta R_1 h_k
\]

\[
+ \sum_{k=1}^p \left[ \frac{1}{(1 - \nu)} \langle u, w \rangle \right] \delta R_1 h_k + c \sum_{k=1}^p K w_k R_1 h_k
\]

\[
= L_1 u + Du, \quad u \in \mathcal{D}(K).
\]

Here, \( D \) is an operator subordinate to \( L_1^{1,\gamma} \), namely

\[
\|Du\| \leq \text{const} \|L_1^{1,\gamma} u\|, \quad u \in \mathcal{D}(L_1^{1,\gamma}).
\]

Since \( \mathcal{D}(K_\omega) \) is equal to \( \mathcal{D}(L_1) \) anyway (see the footnote below Proposition 3.4), we see that the relations

\[
\mathcal{D}(K^{\omega}_\beta) \subset \mathcal{D}(L_1^{1,\omega}), \quad \text{and} \quad \mathcal{D}(L_1^{\alpha}) \subset \mathcal{D}(K^{\omega}_\beta), \quad 0 \leq \alpha < \beta \quad (3.11)
\]

hold algebraically and topologically [7]. Note that

\[
K_\omega^{\gamma} - L_1^{\gamma} = -\frac{1}{2\pi i} \int_C \lambda^{-\gamma} \left( \lambda - K_\omega \right)^{-1} D(\lambda - L_1^{1,\gamma})^{-1} d\lambda
\]

\[
= -\frac{1}{2\pi i} \int_C \lambda^{-\gamma} \left( \lambda - L_1^{1,\gamma} \right)^{-1} D(\lambda - K_\omega)^{-1} d\lambda, \quad 0 \leq \omega \leq 1,
\]

where \( C \) denotes a contour of a suitable right shift of \( \partial \Sigma \) oriented according to increasing \( \text{Im} \lambda \). For any given \( u \in \mathcal{D}(K^{\omega}_\beta) \), \( 0 \leq \omega \leq 1 \), there is a unique \( \varphi \in L_1^{1,\gamma} \) such that \( u = K_\omega^{\gamma} \varphi \). Thus,

\[
K_\omega^{\gamma} \varphi = L_1^{1,\gamma} \varphi - \frac{1}{2\pi i} \int_C \lambda^{-\gamma} \left( \lambda - L_1^{1,\gamma} \right)^{-1} D(\lambda - K_\omega)^{-1} \varphi d\lambda.
\]

According to (3.11) and the moment inequality for \( K_\omega \), the integrand is estimated as follows:

\[
\|D(\lambda - K_\omega)^{-1} \varphi\| \leq \text{const} \|L_1^{1,\gamma}(\lambda - K_\omega)^{-1} \varphi\|
\]

\[
\leq \text{const} \|L_1^{1,\gamma} K_\omega^{-\gamma}\| \|K_\omega^{\gamma}(\lambda - K_\omega)^{-1} \varphi\|
\]

\[
\leq \frac{\text{const}}{(1 + |\lambda|)^{\gamma - \omega}} \|\varphi\|,
\]
where $1 - \varepsilon < \eta < 1$. Thus we see that
\[
\|\lambda^{-\alpha} L_n^\alpha \lambda^{-\alpha} (\lambda - L_{ir})^{-1} D(\lambda - K_n)^{-1} \varphi\| \leq \frac{\text{const}}{(1 + |\lambda|)^{2-\eta}} \|\varphi\|,
\]
the last term of which is integrable on $C$. This means that $\mathcal{D}(K_n^\alpha)$ is contained in $\mathcal{D}(L_n^\alpha)$, and that
\[
L_n^\alpha u = \varphi - \frac{1}{2\pi i} \int_C \lambda^{-\alpha} L_n^\alpha (\lambda - L_{ir})^{-1} D(\lambda - K_n)^{-1} \varphi \, d\lambda,
\]
and
\[
\|L_n^\alpha u\| \leq \text{const} \|\varphi\| = \text{const} \|K_n^\alpha u\|.
\]
The converse relation
\[
\mathcal{D}(L_n^\alpha) \subset \mathcal{D}(K_n^\alpha), \quad \text{and} \quad \|K_n^\alpha u\| \leq \text{const} \|L_n^\alpha u\|
\]
is similarly proved. This finishes the proof of (i).

(ii) We first note that the adjoint operator of $T_1$ in $L^2(\Omega)$ is given by
\[
T_1^* u = u - \sum_{k=1}^p \langle u, R_1 h_k \rangle_{\Omega} w_k. \tag{3.12}
\]
According to the assumption (2.4) on $w_k$, we see that
\[
T_1^* u|_{\delta} = 0, \quad \text{for} \quad u \in H^\theta(\Omega), \quad \theta > \frac{1}{2}.
\]
Applying Green’s formula, we calculate as
\[
\langle K, u \rangle_{\Omega} = \langle T_1 M_1, T_1^{-1} u, u \rangle_{\Omega} = \langle L, T_1^{-1} u, T_1^* u \rangle_{\Omega}
\]
\[
= \left\langle \frac{\partial}{\partial y} T_1^{-1} u, T_1^* u \right\rangle_{\Gamma} + B_1[ T_1^{-1} u, T_1^* u]
\]
\[
= B_1[ T_1^{-1} u, T_1^* u], \quad u \in \mathcal{D}(K_1),
\]
where $B_1[ \cdot, \cdot ] = B_1[ \cdot, \cdot ] + c \langle \cdot, \cdot \rangle_{\Omega}$ denotes the sesquilinear form on $H^1(\Omega)$, and
\[
B_1[u, \varphi] = \sum_{i,j=1}^m \left( a_{ij}(x) \frac{\partial u}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right)_\Omega
\]
\[
+ \sum_{i,j=1}^m \left( b_{ij}(x) \frac{\partial u}{\partial x_j}, \varphi \right)_\Omega + \left( c(x) u, \varphi \right)_\Omega.
\]
Note that $B_1$ is a special case of the sesquilinear form $B$ associated with $M_2$ (see, e.g., Section 2). Thus, if $c$ is large enough, we have the inequalities
\[ \Re B_1[u, u] \geq \delta \|u\|^2_{H^1(\Omega)}, \quad u \in H^1(\Omega) \]
and
\[ |B_1[u, \varphi]| \leq \gamma \|u\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}, \]
for some positive $\delta$ and $\gamma$. Thus we estimate as

\[ \Re \langle K, u, u \rangle_\Omega = \Re B_1[T_1^{-1}u, T_1^\ast u] \\
= \Re B_1[T_1^\ast u, T_1^\ast u] + \Re B_1[T_1^{-1}u - T_1^\ast u, T_1^\ast u] \\
\geq \|T_1^\ast u\|_{H^1(\Omega)} \left( \delta \|T_1^\ast u\|_{H^1(\Omega)} - \gamma \|T_1^{-1}u - T_1^\ast u\|_{H^1(\Omega)} \right). \quad (3.13) \]

According to the expression (3.2) of $T_1^{-1}$, we note that
\[
\|T_1^{-1}u - T_1^\ast u\|_{H^1(\Omega)} \\
= \left\| \sum_{k=1}^n \left( (1 - \varphi)^{-1} \langle u, w \rangle_\Omega \right) R_k h_k + \sum_{k=1}^n \left( \langle u, R_k h_k \rangle_\Omega \right) w_k \right\|_{H^1(\Omega)} \\
\leq C \|u\|.
\]

Here, $C > 0$ denotes some constant. It is significant that the above left-hand side is bounded from above by the $L^2(\Omega)$-norm of $u$. Substituting the above inequality into (3.13), we see that, for any $\varepsilon > 0$,
\[
\Re \langle K, u, u \rangle_\Omega \geq \left( \delta - \frac{C\varepsilon}{2} \right) \|T_1^\ast u\|^2_{H^1(\Omega)} - \frac{C\gamma}{2\varepsilon} \|u\|^2.
\]
Choosing $\varepsilon$ small enough and then $d > C\gamma/2\varepsilon$, we obtain the desired estimate
\[
\Re \langle K_{e+d}u, u \rangle_\Omega \geq \Re \langle K, u, u \rangle_\Omega + d \langle u, u \rangle_\Omega \geq \text{const} \|u\|^2, \quad u \in \mathcal{D}(K).
\]

Thus, by replacing $c$ by a larger constant $c + d$, the $m$-accretiveness of $K$ has been proved. This finishes the proof of (ii). The proof of Theorem 3.1 is thereby complete.

Q.E.D.

Proof of Theorem 3.2

The proof is somewhat simpler than the proof of Theorem 3.1, since the operator $M_{2c}$ is $m$-accretive in our case. An operator similar to $T_1$ appears later in the Third Step. In order to apply this operator, however, we must introduce the operator $L_2 - F_2$ similar to $M_{2c}$ in the First Step.
First Step (Operator $L_2 - F_2$). We shall see that the operator $F_2$ in Theorem 3.3 naturally appears in the following context: Let us consider the following differential equation in $L^2(\Omega)$:

$$\frac{du}{dt} + M_2 u = 0, \quad u(0) = u_0 \in L^2(\Omega). \tag{3.14}$$

Problem (3.14) is well posed and generates an analytic semigroup $\exp(-tM_2)$, $t > 0$, due to Theorem 2.3, (ii), and a unique solution $u$ is given by $u(t) = \exp(-tM_2) u_0$. For any fixed $\theta$, $1/4 < \theta < 3/4$, set $v(t) = L_{2\theta}^{-\theta} u(t)$. According to Theorem 2.2, (i), $v(t)$ belongs to $\mathcal{D}(L_2)$ and satisfies the differential equation

$$\frac{dv}{dt} + (L_2 - F_2) v = 0, \quad v(0) = v_0 = L_{2\theta}^{-\theta} u_0, \tag{3.15}$$

where $F_2$ is defined in Theorem 3.3 as

$$F_2 v = \sum_{k=1}^{p} \langle L_{2\theta}^{\mu} v, w_k \rangle \tau L_{2\theta}^{1-\theta} N_{2\theta}(-c) h_k, \quad \mathcal{D}(F_2) \ni \mathcal{D}(L_2).$$

In fact, (3.14) is rewritten as

$$0 = \frac{du}{dt} - cu + \tau \left( u - \sum_{k=1}^{p} \langle u, w_k \rangle \tau N_{2\theta}(-c) h_k \right)$$

$$= \frac{du}{dt} - cu + L_{2\theta} \left( u - \sum_{k=1}^{p} \langle u, w_k \rangle \tau N_{2\theta}(-c) h_k \right).$$

By applying $L_{2\theta}^{-\theta}$ to the both sides, equation (3.15) for $v$ is obtained. Since $\theta$ is less than $3/4$, the following lemma is immediate.

**Lemma 3.6.** The operator $L_2 - F_2$ has a compact resolvent. There is a $\delta > 0$ such that $\Sigma_{-\delta}$ is contained in $\rho(L_2 - F_2)$, and that

$$\|(\lambda - L_2 + F_2)^{-1}\| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma_{-\delta}.$$
or in other words

$$(\lambda - M_2)^{-1} = L_2^\theta (\lambda - L_2 + F_2)^{-1} L_2^{-\theta} \quad (3.16)$$

for $\Re \lambda < -\delta$. The right-hand side of Eq. (3.16) is analytic in $\lambda \in \rho(L_2 - F_2)$. Thus, $(\lambda - M_2)^{-1}$ has an extension to an operator analytic in $\lambda \in \rho(L_2 - F_2)$. The extension is, however, nothing but the resolvent of $M_2$ [2]. This shows that $\rho(L_2 - F_2)$ is contained in $\rho(M_2)$ and that Eq. (3.16) holds for $\lambda \in \rho(L_2 - F_2)$.

**Second Step (Proof of (i)).** Choose a constant $c \geq \max(\delta, \gamma)$ so that fractional powers for $M_2$ and $L_2 c - F_2$ are well defined. According to (3.16), we calculate as

$$L_2^{-\theta} M_2^{-\theta} = -\frac{1}{2\pi i} \int_C \lambda^{-\theta} L_2^{-\theta} (\lambda - M_2)^{-1} d\lambda$$

$$= -\frac{1}{2\pi i} \int_C \lambda^{-\theta} (\lambda - L_2 + F_2)^{-1} L_2^{-\theta} d\lambda = (L_2 - F_2)^{-\theta} L_2^{-\theta}$$

where $C$ denotes a contour of a suitable right shift of $\delta \Sigma$. Thus,

$$M_2^{-\theta} = L_2^\theta (L_2 c - F_2)^{-\theta} L_2^{-\theta} \quad (3.17)$$

We need to characterize the domain of $(L_2 - F_2)^\theta$. In view of the definition of the operator $F_2$, $F_2$ is subordinate to some power of $L_2 c$ with exponent larger than $1/2$. So the $m$-accretiveness of $L_2 - F_2$ is not expected. Nevertheless, we have the following result, the proof of which is stated later in the Last Step.

**Proposition 3.7.** The equivalence relation $\mathcal{D}((L_2 - F_2)^\theta) = \mathcal{D}(L_2^\theta)$, $0 < \omega < 3/4 + \theta$ holds algebraically and topologically.

According to Proposition 3.7, we see that

$$L_2^\theta (L_2 c - F_2)^\theta L_2^{-2\theta} = L_2^\theta (L_2 c - F_2)^{-\theta} (L_2 c - F_2)^{2\theta} L_2^{-2\theta} \in \mathcal{D}(L^2(\Omega))$$

since $2\theta$ is less than $3/4 + \theta$. Thus the relation (3.17) implies that, for any $u \in \mathcal{D}(L_2^\theta)$,

$$M_2^{-\theta} (L_2^\theta (L_2 c - F_2)^\theta L_2^{-\theta} u) = u, \quad \text{or} \quad M_2^{-\theta} u = L_2^\theta (L_2 c - F_2)^\theta L_2^{-\theta} u$$

which shows that $\mathcal{D}(L_2^\theta)$ is contained in $\mathcal{D}(M_2^{-\theta})$, and that

$$\|M_2^{-\theta} u\| \leq \text{const} \|L_2^\theta u\|, \quad u \in \mathcal{D}(L_2^\theta).$$
As to the converse relation, set \( v = M^{\theta}_{2e} u \) for \( u \in \mathcal{D}(M^{\theta}_{2e}) \). Then,
\[
\begin{align*}
u &= L^{\alpha}_{2e}(L_{2e} - F_{2}) - 0 L^{\alpha}_{2e} v \\
&= L^{\alpha}_{2e} L^{2\theta}_{2e}(L_{2e} - F_{2}) - 0 (L_{2e} - F_{2})^{\theta} L^{\alpha}_{2e} v \in \mathcal{D}(L^{\alpha}_{2e}),
\end{align*}
\]
which shows that \( \mathcal{D}(M^{\theta}_{2e}) \) is contained in \( \mathcal{D}(L^{\alpha}_{2e}) \), and that
\[
\| L^{\alpha}_{2e} u \| \leq \text{const} \| M^{\theta}_{2e} u \|, \quad u \in \mathcal{D}(M^{\theta}_{2e}).
\]
Therefore, we have shown that \( \mathcal{D}(M^{\theta}_{2e}) = \mathcal{D}(L^{\alpha}_{2e}) \) with equivalent graph norms for any \( \theta, 1/4 < \theta < 3/4 \). We note that, since both \( M_{2e} \) and \( L_{2e} \) are \( m \)-accretive, the same is true for \( M^{\theta}_{2e} \) and \( L^{\alpha}_{2e} \). For a fixed \( \theta, 1/4 < \theta < 3/4 \), a generalization of the Heinz inequality [6] is applied to \( M^{\theta}_{2e} \) and \( L^{\alpha}_{2e} \) to derive that
\[
\mathcal{D}(M^{\theta}_{2e}) = \mathcal{D}((M^{\theta}_{2e})^{\omega \theta}) = \mathcal{D}((L^{\alpha}_{2e})^{\omega \theta}) = \mathcal{D}(L^{\alpha}_{2e}), \quad 0 \leq \omega \leq \theta
\]
with equivalent graph norms, which proves (i) of the theorem.

**Third Step (Operator \( T_{2} \)).** The proof of (ii) and (iii) is carried out as follows: As we have shown in the First Step, note that \( \mathcal{D}(M^{\frac{1}{2}}_{2}) \) is equal to \( H^{1}(\Omega) \). Following \( T_{1} \) in Theorem 3.1, let us define an operator \( T_{2} \) formally by
\[
v = T_{2} u = u - \sum_{k=1}^{n} \langle u, w_{k} \rangle_{\omega} R_{2} h_{k}. \quad (3.18)
\]
We can consider \( H^{1}(\Omega) \) as the basic space for \( T_{2} \). According to the choice of the operator \( R_{2} \), we note that
\[
v|_{\omega} = u|_{\omega} - \sum_{k=1}^{n} \langle u, w_{k} \rangle_{\omega} R_{2} h_{k}|_{\omega} = u|_{\omega}.
\]
Thus it is clear that \( T_{2} \) is injective, and \( T_{2}^{-1} \) is given by
\[
u = T_{2}^{-1} v = v + \sum_{k=1}^{n} \langle v, w_{k} \rangle_{\omega} R_{2} h_{k}.
\]
It is easy to see that
\[
T_{2} \in \mathcal{L}(\mathcal{D}(M_{2}); \mathcal{D}(L_{2})); \mathcal{D}(M^{\frac{1}{2}}_{2}); \mathcal{D}(L^{\frac{1}{2}}_{2})), \quad \text{and}
\]
\[
T_{2}^{-1} \in \mathcal{L}(\mathcal{D}(L_{2}); \mathcal{D}(M_{2})); \mathcal{D}(L^{\frac{1}{2}}_{2}); \mathcal{D}(M^{\frac{1}{2}}_{2})).
\]
Since both $M_{2\epsilon}$ and $L_{2\epsilon}$ are $m$-accretive, the interpolation theory implies that

$$T_{2} \in \mathcal{L}(\mathcal{D}(M_{2\epsilon}); \mathcal{D}(L_{2\epsilon}^{1-\theta/2})),$$

and

$$T_{2}^{-1} \in \mathcal{L}(\mathcal{D}(L_{2\epsilon}^{1-\theta/2}); \mathcal{D}(M_{2\epsilon}^{1-\theta/2})), \quad 0 \leq \theta \leq 1,$$

(see, for example, [8, Theorem 6.1]). Thus we see that, for any $u \in \mathcal{D}(M_{2\epsilon})$, $3/4 < \theta \leq 1$, $v = T_{2}u$ belongs to $H^{\theta}(\Omega)$ and $\tau_{2}v = 0$ by Theorem 2.2, (iii), and that $\tau_{2}u = \sum_{k \in \Lambda} \langle u, w_{k} \rangle L_{2\epsilon}^{1-\theta/2}h_{k}$. Therefore $u$ belongs to $H^{\theta}_{2\epsilon}(\Omega)$. Conversely, for any $u \in H^{\theta}_{2\epsilon}(\Omega)$, $v = T_{2}u$ belongs to $H^{\theta}(\Omega)$ and $\tau_{2}v = 0$, that is, $v$ belongs to $\mathcal{D}(L_{2\epsilon}^{\theta})$. Thus $u = T_{2}^{-1}v$ belongs to $\mathcal{D}(M_{2\epsilon}^{\theta})$ by (3.20), which proves (iii) of the theorem. Relation (iii) is similarly proved by means of the operator $T_{2}$ and thus omitted.

We note that the relation (ii) does not depend on a particular choice of $R_{2}$. In fact, the proof is essentially the same as the proof (the Fourth Step) of Theorem 3.1.

Last Step. Let us turn to the proof of Proposition 3.7.

Proof of Proposition 3.7. Since $\mathcal{D}(L_{\epsilon})$ is equal to $\mathcal{D}(L_{2\epsilon} - F_{2})$, we see that the relations

$$\mathcal{D}((L_{2\epsilon} - F_{2})^{\beta}) \subset \mathcal{D}(L_{\epsilon}^{\beta}), \quad \text{and} \quad \mathcal{D}(L_{\epsilon}^{\alpha}) \subset \mathcal{D}((L_{2\epsilon} - F_{2})^{\alpha}), \quad 0 \leq \alpha < \beta$$

hold algebraically and topologically [7]. So, just as in the proof of Proposition 3.4, (i), we are able to show that

$$\mathcal{D}((L_{2\epsilon} - F_{2})^{\omega}) = \mathcal{D}(L_{\epsilon}^{\omega}), \quad 0 \leq \omega \leq 1.$$

The equivalence relation for $\omega$, $1 < \omega < 3/4 + \theta$ is proved as follows: Take any $u \in \mathcal{D}(L_{2\epsilon}^{1+\theta})$, $0 < \kappa < \theta - 1/4$. In view of the relation

$$(L_{2\epsilon} - F_{2})u = L_{2\epsilon}u - \sum_{k \in \Lambda} \langle L_{2\epsilon}u, w_{k} \rangle L_{2\epsilon}^{1-\theta}N_{2}(c)h_{k},$$

both terms of the right-hand side belong to $\mathcal{D}(L_{\epsilon}^{\kappa}) = \mathcal{D}((L_{2\epsilon} - F_{2})^{\kappa})$, since $1 - \theta + \kappa$ is less than $3/4$. Thus, $u$ belongs to $\mathcal{D}((L_{2\epsilon} - F_{2})^{1+\theta})$. Moreover,

$$\|L_{2\epsilon} - F_{2})^{1+\theta}u\| \leq \text{const} \|L_{2\epsilon}^{1+\theta}u\|, \quad u \in \mathcal{D}(L_{2\epsilon}^{1+\theta}).$$

The converse inclusion relation is similarly proved. This finishes the proof of the proposition. The proof of Theorem 3.2 is thereby complete.
Proof of Theorem 3.3

Since both cases are similarly proved, we consider only the case of (ii). Let us fix a \( \theta, 1/4 < \theta < 3/4 \). Taking the inverse of the both sides of (3.17), we see that

\[
M_{2r}^\theta = L_{2r}^\theta (L_{2r} - F_2)^\theta L_{2r}^{-\theta} \quad \text{on} \quad \mathcal{D}(M_{2r}^\theta) = \mathcal{D}(L_{2r}^\theta).
\]  

(3.21)

Since both resolvents \((\lambda - M_{2r}^\theta)^{-1}\) and \((\lambda - L_{2r} - F_2)^{-1}\) are compact (see, for example, the proof of Theorem 2.3, (ii)), the spectrums \(\sigma(M_{2r}^\theta)\) and \(\sigma((L_{2r} - F_2)^\theta)\) consist only of eigenvalues. It is easily seen from (3.21) that

\[
\sigma(M_{2r}^\theta) = \sigma((L_{2r} - F_2)^\theta).
\]

(3.22)

In fact, suppose that \((\lambda - (L_{2r} - F_2)^\theta) u = 0\) for some \(\lambda\) and \(u\) \((\neq 0)\). By setting \(v = L_{2r}^\theta u\) \((\neq 0)\), \(v\) belongs to \(L^2(\Omega)\) and satisfies

\[
L_{2r}^{-\theta} v = \lambda (L_{2r} - F_2)^{-\theta} L_{2r}^{-\theta} v = \lambda (L_{2r} - F_2)^{-2\theta} (L_{2r} - F_2)^\theta L_{2r}^{-\theta} v,
\]

the last term of which belongs to \(\mathcal{D}((L_{2r} - F_2)^{2\theta}) = \mathcal{D}(L_{2r}^{2\theta})\) by Proposition 3.7 (In fact, 20 is less than \(3/4 + \theta\)). Thus we see that \(v\) belongs to \(\mathcal{D}(M_{2r}^\theta)\) and \((\lambda - M_{2r}^\theta) v = 0\), or \(\lambda \in \sigma(M_{2r}^\theta)\). The converse relation is similarly proved. As is well known \([7]\), the resolvents of \(M_{2r}^\theta\) and \((L_{2r} - F_2)^\theta\) exist via analytic continuation in the set: \(\Sigma^\theta \cup \bar{B}_\delta\) for some positive \(\delta\), where \(\Sigma^\theta\) denotes the sector: \(\{ \lambda \in \mathbb{C}; \theta \delta \leq |\lambda| \leq \pi \}\) containing \(\Sigma\), and \(\bar{B}_\delta\) the closed ball of radius \(\delta\): \(\{ \lambda \in \mathbb{C}; |\lambda| \leq \delta \}\). The resolvents satisfy the following estimates:

\[
\| (\lambda - M_{2r}^\theta)^{-1} \|, \quad \| (\lambda - (L_{2r} - F_2)^\theta)^{-1} \| \leq \text{const}, \quad \lambda \in \Sigma^\theta \cup \bar{B}_\delta.
\]

For any \(\pi > 0\), we calculate as

\[
M_{2r}^{-\pi} = (M_{2r}^\theta)^{-\pi} = \frac{1}{2\pi i} \int_{C_1} \lambda^{-\pi} (\lambda - M_{2r}^\theta)^{-1} d\lambda.
\]

\[
= \frac{1}{2\pi i} \int_{C_1} \lambda^{-\pi} L_{2r}^\theta (\lambda - (L_{2r} - F_2)^\theta)^{-1} L_{2r}^{-\theta} d\lambda
\]

\[
= L_{2r}^\theta ((L_{2r} - F_2)^\theta)^{-\pi} L_{2r}^{-\theta}
\]

\[
= L_{2r}^\theta (L_{2r} - F_2)^{-\pi} L_{2r}^{-\theta},
\]

(3.23)
where $C_1$ denotes the boundary of $\bar{\Omega} \cup \tilde{B}_f$, oriented according to increasing $\text{Im}\, \lambda$. Taking the inverse of (3.23), we see that

$$M_2\sigma = (M_2\sigma)^{-1} = (L_2\sigma (L_2 - F_2)^{-\alpha} L_2\sigma)^{-1}$$

$$= L_2\sigma (L_2 - F_2)^{\alpha} L_2\sigma, \quad \alpha > 0.$$ 

This completes the proof of Theorem 3.3. Q.E.D.

4. APPLICATION

In this section we apply one of the main results to robustness analysis of a boundary feedback control system. The boundary control system is described by

$$\frac{du}{dt} + M_2 u = 0, \quad u(0) = u_0. \quad (4.1)$$

When the coefficients in $M_2$ are perturbed, the perturbed system is then described by

$$\frac{du}{dt} + \tilde{M}_2 u = 0, \quad u(0) = u_0, \quad (4.2)$$

where

$$\tilde{M}_2 = \mathcal{D} u, \quad u \in \mathcal{D}(\tilde{M}_2) = \left\{ u \in H^1(\Omega); \tilde{\tau}_2 u = \sum_{k=1}^p \langle u, w_k \rangle_r \text{ on } \Gamma \right\},$$

and

$$\mathcal{D} u = - \sum_{k,j=1}^m \frac{\partial}{\partial x_j} \left( (1 + \kappa(x)) a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{j=1}^m b_j(x) \frac{\partial u}{\partial x_j} + \bar{c}(x) u,$$

$$\tilde{\tau}_2 u = \frac{\partial u}{\partial \nu} + \bar{\tau}(\xi) u = \sum_{k,j=1}^m \left( 1 + \kappa(\xi) \right) a_{ij}(\xi) v_i(\xi) \frac{\partial u}{\partial x_j} \bigg|_r + \bar{\tau}(\xi) u \bigg|_r.$$ 

Thus the perturbation to the principal part of $\mathcal{D}$ is assumed uniform. Throughout the section we assume, in addition to (2.6), that $w_k$'s belong to $H^1(\Omega)$, so that the adjoint operator $M_2^* \bar{\tau}$ enjoys the structure similar to that of $M_2$ (see, e.g., Proposition 2.4, (ii)). The index measuring the difference between $M_2$ and $M_2$ is introduced as

$$\eta = \| \kappa \|_{C^2(\Gamma)} + \sum_{i,j=1}^m \| \tilde{\tau}_i - b_i \|_{C(\Gamma)} + \| \bar{\tau} - c \|_{C(\Gamma)} + \| \bar{\tau} - \sigma \|_{C(\Gamma)}. \quad (4.3)$$
The domain $\mathcal{D}(\tilde{M}_2)$ differs a little bit from $\mathcal{D}(M_2)$, and the comparison of the resolvent set $\rho(\tilde{M}_2)$ with $\rho(M_2)$ seems not very simple. However, we assert the following:

**Theorem 4.1.** If $\eta$ is chosen small enough, there is an operator $K_\eta$ subordinate to $M_2$ such that

$$\rho(\tilde{M}_2) = \rho(M_2 - K_\eta), \quad (4.4)$$

and

$$\begin{aligned}
(\lambda - \tilde{M}_2)^{-1} &= M_2^\eta (\lambda - M_2 + K_\eta)^{-1} M_2^{-\theta}, \quad \lambda \in \rho(M_2 - K_\eta), \\
\|K_\eta u\| &\leq c(\eta) \|M_2 u\|, \quad u \in \mathcal{D}(M_2), \quad c(\eta) \to 0 \quad as \quad \eta \to 0,
\end{aligned} \quad (4.5)$$

where $\theta = 1/4 + \varepsilon$, $0 < \varepsilon < 1/4$.

The proof of Theorem 4.1 is carried out along the line of [13]. We briefly sketch the proof and see that $K_\eta$ is defined via the fractional power of $M_2$ (see (4.8) below). First of all, if $\eta$ is small enough, there is a sector $\Sigma_{-\mu}^\mu, \mu \in \mathbb{R}^1$, such that the resolvent $(\lambda - \tilde{M}_2)^{-1}$ exists in $\Sigma_{-\mu}$, and satisfies the estimate

$$\|\lambda - \tilde{M}_2\|^{-1} \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma_{-\mu}.$$

The proof of the above estimate is standard, and carried out via the sesquilinear form associated with $\tilde{M}_2$. Let us introduce the operators $N_{\tau_2}(\lambda)$, $P_{\eta}$, $Q_{\eta}$, and $n_\tau$ as follows: For any $g \in \mathcal{H}^{1/2}(\Gamma)$, consider the boundary value problem

$$(\lambda - \mathcal{L}) u = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \tau_2 u - \sum_{k=1}^p \langle u, w_k \rangle \frac{\partial h_k}{\partial n} = g \quad \text{on} \quad \Gamma.$$

If $\lambda$ is in $\rho(M_2)$, the problem admits a unique solution $u \in \mathcal{H}^2(\Omega)$ expressed by

$$u = N_{\tau_2}(\lambda) g = R_2 g - (\lambda - M_2)^{-1} (\lambda - \mathcal{L}) R_2 g,$$

where $N_{\tau_2}(\lambda)$ belongs to $\mathcal{L}(\mathcal{H}^{1/2}(\Gamma); \mathcal{H}^2(\Omega))$. The operators $P_{\eta}$ and $Q_{\eta}$ on $\Gamma$ are defined by

$$P_{\eta} u = \left( \sigma - \frac{\delta}{1 + \kappa} \right) u - \sum_{k=1}^p \langle u, w_k \rangle \frac{\partial h_k}{\partial n} \frac{\kappa}{1 + \kappa} h_k.$$
and

\[ Q_{p} u = \left( \tau_{2} \mathcal{K} - \frac{\kappa \bar{a}}{1 + \kappa} \right) u + \sum_{k=1}^{N} \left\{ \left\langle u, w_{k} \right\rangle_{\Gamma} \frac{\kappa}{1 + \kappa} - \left\langle ku, w_{k} \right\rangle_{\Gamma} \right\} h_{k}, \]

respectively. The operator \( n_{p} \) on \( \Omega \) is defined by

\[ n_{p} u = \sum_{i} \left( \sum_{j} a_{ij} \frac{\partial u}{\partial x_{j}} - b_{i,\kappa} + (\bar{b}_{i} - b_{i}) \right) \frac{\partial u}{\partial x_{i}} + \left( - \mathcal{L}_{i} \mathcal{K} + \bar{c}(x) - c(x) \right) u. \]

When \( \eta \) is small, the operators \( P_{p}, Q_{p}, \) and \( n_{p} \) are small, too, in respective operator topology.

Let \( u \) be a solution to the perturbed equation (4.2). It is easy to see that

\[ \frac{du}{dt} + \mathcal{D} u = -(\mathcal{D} - \mathcal{L}) u = -l_{q} u, \]

\[ \tau_{2} u - \sum_{k=1}^{N} \left\langle u, w_{k} \right\rangle_{\Gamma} h_{k} = P_{q} u. \]

Note that \( u - N_{f_{2}} (-c) P_{q} u \) belongs to \( \mathcal{D}(M_{2}) \) for \( t > 0 \). Then,

\[ \frac{du}{dt} + M_{2} u - N_{f_{2}} (-c) P_{q} u = c u - l_{q} u. \]

Multiplying both sides by \( M_{2}^{\alpha} \) and setting \( v(t) = M_{2}^{\alpha} u(t) \), we see that \( v \) satisfies the equation

\[ \frac{dv}{dt} + M_{2} v = M_{2}^{3 \alpha} N_{f_{2}} (-c) P_{q} M_{2}^{\alpha} v - M_{2}^{\alpha} l_{q} M_{2}^{\alpha} v \]

\[ = K_{q} v, \quad v(0) = M_{2}^{\alpha} u_{0}. \] (4.7)

A difficulty is that the second term of the right-hand side of (4.7) is not defined for all \( v \in \mathcal{D}(M_{2}) \) in its present form. By noting that \( u \) belongs to \( \mathcal{D}(\bar{M}_{2}) \), however, a further calculation shows that \( K_{q} \) is rewritten as

\[ K_{q} v = M_{2}^{3 \alpha} N_{f_{2}} (-c) P_{q} M_{2}^{\alpha} v - M_{2}^{3 \alpha} \kappa M_{2}^{\alpha} v \]

\[ + M_{2}^{3 \alpha} N_{f_{2}} (-c) Q_{q} M_{2}^{\alpha} v - M_{2}^{\alpha} n_{q} M_{2}^{\alpha} v, \] (4.8)
which is defined on \( \mathcal{D}(M_2) \). By applying the interpolation theory to each term of (4.8), the estimate in (4.5) is obtained.

Let us compare \( \tilde{M}_2 \) with \( M_2 - K_\eta \). If \( \eta \) is small enough, the estimate in (4.5) guarantees that

\[
\| (\lambda - M_2 + K_\eta)^{-1} \| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \tilde{\Sigma}_-^\mu.
\]

Thus \( -M_2 + K_\eta \) generates an analytic semigroup \( \exp(i(-M_2 + K_\eta)) \). Note that

\[
(\lambda - \tilde{M}_2)^{-1} u_0 = -\int_0^\infty e^{i\tau} e^{-i\Omega_2} u_0 d\tau, \quad \text{Re} \lambda < -\mu, \quad u_0 \in L^2(\Omega).
\]

Multiplying both sides by \( M_2^{-\theta} \), we see that

\[
M_2^{-\theta} (\lambda - \tilde{M}_2)^{-1} u_0 = -\int_0^\infty e^{i\tau} (v(t) = M_2^{-\theta} e^{-i\Omega_2} u_0)
\]

\[
= (\lambda - M_2 - K_\eta)^{-1} M_2^{-\theta} u_0,
\]

or

\[
(\lambda - \tilde{M}_2)^{-1} = M_2^\theta (\lambda - M_2 + K_\eta)^{-1} M_2^{-\theta}, \quad \text{Re} \lambda < -\mu. \tag{4.9}
\]

The right-hand side of (4.9) is clearly analytic in \( \lambda \in \rho(M_2 - K_\eta) \). By analytic continuation, \( (\lambda - \tilde{M}_2)^{-1} \) has an extension analytic in \( \rho(M_2 - K_\eta) \) which is nothing but the resolvent of \( \tilde{M}_2 \). Thus we have shown that

\[
\rho(M_2 - K_\eta) \subset \rho(\tilde{M}_2).
\]

In order to show the converse relation, note that the relation

\[
M_2^{-\theta} (\lambda - \tilde{M}_2)^{-1} M_2^\theta = (\lambda - M_2 + K_\eta)^{-1}
\]

holds on \( \mathcal{D}(M_2^\theta) \). The left-hand side has a unique extension to an operator in \( \mathcal{D}(L^2(\Omega)) \), which is denoted by \( S(\lambda) \). The operator \( S(\lambda) \) is analytic in \( \lambda \in \rho(\tilde{M}_2) \). This result is proved by taking the adjoint of \( M_2^{-\theta} (\lambda - \tilde{M}_2)^{-1} M_2^\theta \) and noting that \( \mathcal{D}(M_2^\theta) = H^2(\Omega) \). Thus \( (\lambda - M_2 + K_\eta)^{-1} \) has an extension analytic in \( \rho(\tilde{M}_2) \). This proves that \( \rho(\tilde{M}_2) \) is contained in \( \rho(M_2 - K_\eta) \). We have therefore obtained the desired relation (4.4). This finishes the proof of Theorem 4.1. Q.E.D.
5. CONCLUDING REMARKS

I. Theorem 3.1 has been proved on the assumption that \( w_k, 1 \leq k \leq p \), belong to \( H^r (\Omega) \) for an arbitrarily small \( \varepsilon > 0 \). If \( w_k \)'s merely belong to \( L^2 (\Omega) \), what can we assert? It seems difficult at present to show that

\[
\mathcal{D} (M_{\theta}^\alpha) = \left[ \mathcal{D} (M_1), L^2 (\Omega) \right]_{1-\theta}, \quad 0 \leq \theta \leq 1
\]

for general \( w_k \)'s in \( L^2 (\Omega) \). However, introducing the operator \( F_1 \) in Theorem 3.3, and applying the method in the First and the Second Steps in the proof of Theorem 3.2, we can show at least that

\[
\mathcal{D} (M_{\theta}^\alpha) = H^{2\theta} (\Omega), \quad 0 \leq \theta < \frac{1}{4}
\]

II. In our previous paper [13], we studied the operator \( \tilde{M}_2 \) and its fractional powers, where \( \tilde{M}_2 \) is defined by

\[
\tilde{M}_2 u = D u, \quad \mathcal{D} (\tilde{M}_2) = \left\{ u \in H^2 (\Omega); \tau_\delta u = \sum_{k=1}^p \langle u, w_k \rangle_{\alpha} h_k \text{ on } \Gamma \right\},
\]

and \( w_k \)'s belong to \( L^2 (\Omega) \). In Theorem 3.2, let us replace \( M_2 \) by \( \tilde{M}_2 \). Then \( \tilde{M}_2 = \tilde{M}_2 + \epsilon \) is \( m \)-accretive, too, if \( \epsilon > 0 \) is large enough, and fractional powers for \( \tilde{M}_2 \) are well defined. Characterization of the domain \( \mathcal{D} (\tilde{M}_2^\alpha) \), \( 0 \leq \theta \leq 1 \) is similar to Theorem 3.2:

(i) \( \mathcal{D} (\tilde{M}_2^\alpha) = H^{2\theta} (\Omega), \quad 0 \leq \theta < \frac{3}{4} \);

(ii) \( \mathcal{D} (\tilde{M}_2^{3/4}) = \left\{ u \in H^{3/2} (\Omega); \left| \frac{1}{\tau_\delta (x)} \right| \tau_\delta u - \sum_{k=1}^p \langle u, w_k \rangle_{\alpha} \tau_\delta R_2 h_k \right| dx < \infty \}; \)

and

(iii) \( \mathcal{D} (\tilde{M}_2^\theta) = \left\{ u \in H^2 (\Omega); \tau_\delta u = \sum_{k=1}^p \langle u, w_k \rangle_{\alpha} h_k \text{ on } \Gamma \right\}, \frac{3}{4} < \theta \leq 1 \).

For the proof, we introduce the operator \( \tilde{T}_2 \) by

\[
\tilde{T}_2 u = u - \sum_{k=1}^p \langle u, w_k \rangle_{\alpha} R_2 h_k.
\]
The basic space for $\hat{T}_2$ is simply $L^2(\Omega)$. We do not need the First Step in the proof of Theorem 3.2. It is easily seen that

$$\hat{T}_2 \in \mathcal{L}(\mathcal{D}(\hat{M}^0_\theta)); \mathcal{D}(L^2_\theta)), \quad \text{and} \quad \hat{T}_2^{-1} \in \mathcal{L}(\mathcal{D}(L^2_\theta)); \mathcal{D}(\hat{M}^0_\theta)), \quad 0 \leq \theta \leq 1.$$ 

By applying this property to the modified problem, the above characterization is obtained.

We note that, in [13], another approach is employed in the above (i) (it corresponds to the First and the Second Steps in the proof of Theorem 3.2). Thus the approach via $\hat{T}_2$ gives a simpler alternative approach.

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