Hamilton cycle decompositions of the tensor products of complete bipartite graphs and complete multipartite graphs

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1. Introduction

A k-regular graph G has a Hamilton cycle decomposition if its edge set can be partitioned into \( \frac{k}{2} \) Hamilton cycles when k is even, or into \( (k-1)/2 \) Hamilton cycles plus a 1-factor (or a perfect matching) when k is odd. We write \( G = H_1 \oplus H_2 \oplus \cdots \oplus H_k \) if \( H_1, H_2, \ldots, H_k \) are edge-disjoint subgraphs of G and \( E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k) \). The complete graph on m vertices is denoted by \( K_m \) and its complement is denoted by \( \overline{K}_m \). \( C_m \) denotes the cycle of length m.

For two simple graphs G and H their wreath product, denoted by \( G \star H \), has vertex set \( V(G) \times V(H) \) in which \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent whenever \( g_1 g_2 \in E(G) \) or \( g_1 = g_2 \) and \( h_1 h_2 \in E(H) \). Similarly, \( G \times H \), the tensor product of the graphs G and H, has vertex set \( V(G) \times V(H) \) in which two vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent whenever \( g_1 g_2 \in E(G) \) and \( h_1 h_2 \in E(H) \). The tensor product is known to be commutative and distributive over an edge-disjoint union of subgraphs, that is, if \( G = H_1 \oplus H_2 \oplus \cdots \oplus H_k \), then \( G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H) \).

We shall use the following notation throughout the paper. Let G and H be simple graphs with \( V(G) = \{x_1, x_2, \ldots, x_m\} \) and \( V(H) = \{u_1, u_2, \ldots, u_n\} \). Then \( V(G \times H) = V(G) \times V(H) \). For our convenience, we write \( V(G) \times V(H) = \bigcup_{i=1}^{m} X_i \), where \( X_i \) stands for \( \{x_i\} \times V(H) \). Further, in the sequel, we shall denote the set of vertices of \( X_i \) as \( \{x_j \mid 1 \leq j \leq n\} \), where \( x_j \) stands for the vertex \( (x_i, u_j) \). \( X_i, 1 \leq i \leq m \), is called the \( i \)th layer of \( G \times H \). We shall call \( G \times H \) an \( m \)-partite graph with partite sets \( X_1, X_2, \ldots, X_m \). (We can also consider \( G \times H \) as an \( n \)-partite graph with partite sets \( U_i = V(G) \times \{u_i\}, 1 \leq i \leq n \).

Let G be a bipartite graph with bipartition \( (X, Y) \), where \( X = \{x_1, x_2, \ldots, x_n\} \), \( Y = \{y_1, y_2, \ldots, y_n\} \). If \( x_j y_j \in E(G) \), then \( x_j y_j \) is called an edge of \( G \times H \) from \( X \) to \( Y \). The distance \( i \) from \( X \) to \( Y \) if \( i \leq j \) or \( n - (j - i) \) if \( i > j \). The same edge is said to be of distance \( i \) from \( X \) to \( Y \) if \( i \geq j \) or \( n - (j - i) \), if \( i < j \). If \( G \) contains the set of edges \( F_i(X, Y) = \{x_j y_{j+i} \mid 1 \leq j \leq n\}, 0 \leq i \leq n - 1 \), where the addition in the subscript is taken modulo \( n \) with residues 1, 2, \ldots, \( n \), then we say that \( G \) has a 1-factor of distance \( i \) from \( X \) to \( Y \). Note that \( F_i(X, Y) = F_{i+n}(Y, X) = \{y_k x_{k+i} \mid 1 \leq k \leq n\}, 0 \leq i \leq n - 1 \). Clearly, if \( G = K_m \), then \( E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y) \).

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Let \( k \) be a positive integer and let \( L \) be a subset of \( \{1, 2, \ldots, \lfloor \frac{L}{2} \rfloor \} \). A circulant \( X = X(k; L) \) is a graph with vertex set \( V(X) = \{u_1, u_2, \ldots, u_L\} \) and the edge set \( E(X) = \{u_iu_{i+1} \mid i \in \{1, 2, \ldots, L\}, i \notin L \} \), where addition in the subscript of \( u \) is taken modulo \( k \) with residues \( 1, 2, \ldots, k \). The edge \( u_iu_{i+1} \), \( i \in L \), is said to be of distance \( i \), and \( L \) is called the edge distance set of the circulant \( X \). It is clear that if \( \gcd(k, l) = 1 \), then the circulant \( X(k; l) \) is a Hamilton cycle. We shall name the graph isomorphic to \( X(2r; \{1, r\}) \) as \( W_n \).

For a digraph \( D \), by \( \mathcal{A}(D) \) we mean the arc set of \( D \). Definitions which are not given here can be found in [4] or [7].

In this paper, we study the Hamilton cycle decomposition of \( K_{r, r} \times (K_m \ast K_n) \), \( r \geq 2, m \geq 3 \). The problem of finding Hamilton cycle decompositions of product graphs is not new. Hamilton cycle decompositions of various product graphs have been studied by many authors; see, for example, [1, 6, 8, 9, 11–14]. It has been conjectured [6] that if both \( G \) and \( H \) are Hamilton cycle decomposable graphs, then \( G \ast H \) is Hamilton cycle decomposable. In [13], Ng has obtained a partial solution to the following conjecture of Alspach et al. [1]: If \( D_1 \) and \( D_2 \) are directed Hamilton cycle decomposable digraphs, then \( D_1 \ast D_2 \) is directed Hamilton cycle decomposable. Jha [9] has advanced the following conjecture: If both \( G \) and \( H \) are Hamilton cycle decomposable graphs and \( G \ast H \) is connected, then \( G \ast H \) is Hamilton cycle decomposable. But this conjecture is disproved in [3]. Because of this, finding Hamilton cycle decompositions of the tensor product of Hamilton cycle decomposable graphs is considered to be difficult. In [2] it has been proved that \( K_r \times K_r \) is Hamilton cycle decomposable. In [11] \( K_r \times K_r \) is shown to be Hamilton cycle decomposable and in [12] it is shown that the tensor product of two regular complete multipartite graphs is Hamilton cycle decomposable. In this paper, we prove the following:

**Theorem 1.1.** If \( r \geq 2 \) and \( m \geq 3 \), then \( K_{r, r} \times (K_m \ast K_n) \) has a Hamilton cycle decomposition.

### 2. Proof of the main theorem

First we prove a few lemmas; using these lemmas we prove the main result of this paper.

Throughout the following lemma and its proof we assume that the subscripts of \( x_i \)'s and the superscripts of \( x_i \)'s are taken modulo \( m \) with residues \( 1, 2, \ldots, m \) and the subscripts of \( x_i \)'s are taken modulo \( n \) with residues \( 1, 2, \ldots, n \); the addition in the distance of \( 1 \)-factors is taken modulo \( n \) with residues \( 0, 1, 2, \ldots, n-1 \), and \( 0 \leq \alpha, \beta, \gamma \leq n-1, 3 \leq x \leq m \).

**Lemma 2.1.** Let the vertex set of the \( m \)-partite graph \( G \) be \( \{x_1', x_2', \ldots, x_n'\} \), \( 1 \leq i \leq m \). Let \( G \) be a spanning subgraph of \( G_m \times K_n \) containing the edges of \( F_1 \cup F_2 \), where \( F_1 = F_{\alpha}(x_1, x_2) \cup F_{\alpha}(x_2, x_3) \cup \bigcup_{j=3}^{n} F_{\alpha}(x_j, x_{j+1}) \), \( F_2 = F_{\alpha+1}(x_1, x_2) \cup F_{\alpha+1}(x_2, x_3) \cup \bigcup_{j=3}^{n} F_{\alpha+1}(x_j, x_{j+1}) \), \( \beta \neq \gamma \), and \( F_{\alpha}(x_i, x_j) \) denotes the 1-factor of distance \( k \) from \( x_i \) to \( x_j \); then

(a) if \( \gcd(2x + \sum_{i=1}^{m} \beta_i, n) = 1 \) and \( \gcd(2\alpha + 2 + \sum_{i=1}^{m} \gamma_i, n) = 2 \), then \( F_1 \cup F_2 \) can be decomposed into two Hamilton cycles of \( G \);

(b) if \( \gcd(2x + \sum_{i=1}^{m} \beta_i, n) = 2 \) and \( \gcd(2\alpha + 2 + \sum_{i=1}^{m} \gamma_i, n) = 1 \), then \( F_1 \cup F_2 \) can be decomposed into two Hamilton cycles of \( G \);

(c) if \( \gcd(2x + \sum_{i=1}^{m} \beta_i, n) = 2 \) and \( \gcd(2\alpha + 2 + \sum_{i=1}^{m} \gamma_i, n) = 2 \), then \( F_1 \cup F_2 \) can be decomposed into two Hamilton cycles of \( G \);

(d) if \( n \geq 8 \), \( \gcd(2\alpha + \sum_{i=1}^{m} \beta_i, n) = 4 \) and \( \gcd(2\alpha + 2 + \sum_{i=1}^{m} \gamma_i, n) = 4 \), then \( F_1 \cup F_2 \) can be decomposed into two Hamilton cycles of \( G \).

**Proof.** **Proof of (a).** Clearly, \( F_1 \) is a Hamilton cycle and \( F_2 \) is a 2-factor of \( G \) consisting of two cycles \( C' \) and \( C'' \) of equal length.

The vertices \( x_1', x_2', \ldots, x_{n-1}' \) are contained in a single cycle of \( F_2 \), say, \( C' \), and the vertices \( x_1', x_2', \ldots, x_{n-1}' \) are contained in the other cycle \( C'' \) of \( F_2 \). Now we decompose \( F_1 \cup F_2 \) into two Hamilton cycles \( H' \) and \( H'' \) of \( G \) as follows:

\[
H' = (F_1 - \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\}) \cup \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\} \quad \text{and} \quad H'' = (F_2 - \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\}) \cup \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\}. \]

The fact that \( H' \) is a Hamilton cycle of \( G \) can be seen by letting \( a = x_1', b = x_2'^2 + c = x_2'^2 + d = x_2'^2 + F = F_1 \) and \( H = H' \) in the graphs of Fig. 1.

The fact that \( H'' \) is a Hamilton cycle of \( G \) can be seen by letting \( a = x_1', b = x_2'^2, c = x_2'^2, d = x_2'^2 + F = F_2 \) and \( H = H'' \) in the graphs of Fig. 2.

**Proof of (b).** Clearly, \( F_2 \) is a Hamilton cycle and \( F_1 \) is a 2-factor of \( G \) consisting of two cycles \( C' \) and \( C'' \) of equal length. The vertices \( x_1', x_2', \ldots, x_{n-1}' \) are contained in a single cycle of \( F_1 \), say, \( C' \), and the vertices \( x_1', x_2', \ldots, x_{n-1}' \) are contained in the other cycle \( C'' \) of \( F_1 \). Now we decompose \( F_1 \cup F_2 \) into two Hamilton cycles \( H' \) and \( H'' \) of \( G \) as follows:

\[
H' = (F_1 - \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\}) \cup \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\} \quad \text{and} \quad H'' = (F_2 - \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\}) \cup \{x_1'^2 + x_2'^2 + x_3'^2 + \ldots + x_{n-1}'^2\}. \]

The fact that \( H' \) is a Hamilton cycle of \( G \) can be seen by letting \( a = x_1', b = x_2'^2 + c = x_2'^2 + d = x_2'^2 + F = F_2 \) and \( H = H'' \) in the graphs of Fig. 1. The fact that \( H'' \) is a Hamilton cycle of \( G \) can be seen by letting \( a = x_1', b = x_2'^2 + c = x_2'^2 + d = x_2'^2 + F = F_1 \) and \( H = H' \) in the graphs of Fig. 2.

**Proof of (c).** Clearly each \( F_i, i = 1, 2 \), is a 2-factor of \( G \) consisting of two cycles \( C_i' \) and \( C_i'' \) of equal length. The vertices \( x_1', x_2', \ldots, x_{n-1}' \) are contained in a single cycle of \( F_i \), say, \( C_i' \), and the vertices \( x_1', x_2', \ldots, x_{n-1}' \) are contained
in the other cycle $C''_3$ of $F_i$. Now we decompose $F_i \cup F_2$ into two Hamilton cycles $H'$ and $H''$ of $G$ as follows: $H' = (F_1 - \{x_1^2, x_2^2, x_3^2\} \cup \{x_4^2, x_5^2, x_6^2\})$ and $H'' = (F_2 - \{x_1^2, x_2^2, x_3^2\} \cup \{x_4^2, x_5^2, x_6^2\}).$

The fact that $H'$ is a Hamilton cycle of $G$ can be seen by letting $a = x_1, b = x_2, c = x_3, d = x_4,$ and $H = H'$ in the graphs of Fig. 2. The fact that $H''$ is a Hamilton cycle of $G$ can be seen by letting $a = x_1, b = x_2, c = x_3, d = x_4,$ and $H = H''$ in the graphs of Fig. 2.

Proof of (d). Clearly, each $F_i, i = 1, 2$, is a 2-factor of $G$ consisting of four cycles, say, $C_1^2, C_2^2, C_3^2$ and $C_4^2$, of equal length. By the choice of $F_i$, we suppose $x_1^2, x_2^2, x_3^2 \in C_1^2, 1 \leq k \leq 4, 0 \leq l \leq \frac{n}{4} - 1$. Now we decompose $F_1 \cup F_2$ into two Hamilton cycles $H'$ and $H''$ as follows: $H' = (F_1 - \{x_1^2, x_2^2, x_3^2\} \cup \{x_4^2, x_5^2, x_6^2\})$ and $H'' = (F_2 - \{x_1^2, x_2^2, x_3^2\} \cup \{x_4^2, x_5^2, x_6^2\}).$ Indeed, $H'$ and $H''$ are Hamilton cycles of $G$; see Fig. 3. □

Remark 2.2. From the construction of the Hamilton cycles $H'$ and $H''$ of the above lemma, it is clear that for every vertex in the $i$th layer, $1 \leq i \leq m$, except in the second layer, the preceding and succeeding vertices of any vertex in the Hamilton cycles $H'$ and $H''$ are in $X_{i-1}$ and $X_{i+1}$ and for some vertices in the second layer both the preceding and succeeding vertices are in $X_1$ or $X_2$. Consequently, in the subgraph obtained by the intersection of $H'$ (resp. $H''$) with the subgraph induced by the first three layers of $G$ all the vertices in $X_1$ and $X_2$ are of degree 1 in the subgraph. Further, the intersection of $H'$ (resp. $H''$) with the subgraph of $G$ induced by $X_i \cup X_{i+1}$, $3 \leq i \leq m$, is a 1-factor (of the induced subgraph). This fact will be used later. □

For the rest of the results, except Theorem 1.1, proved in this paper we assume the following: let $m \geq 4$ be even and let $\{u_1, u_2, \ldots, u_m\}$ be the vertex set of $K_m \ast K_n$. Place the vertices of $K_m \ast K_n$ in the circular order $\{u_1, u_2, \ldots, u_m\}$. Let the
Ifr directed Hamilton cycle decomposition of \( G \) and then we decompose can be described as sets of the 2
\[
W \quad \text{with residues} 1, \ldots, 2r.
\]

**Proof.**

**Lemma 2.3.** Let \( F \) be the set of edges of distance \( l \) \( X \times X \). In the construction of \( H' \) (resp. \( H^* \))

Fig. 3.

ith partite set of \( K_m \times \overline{K}_n \) be \( U_i = \{ u_{i+jm} \mid 0 \leq j \leq n - 1 \}, 1 \leq i \leq m \). Thus \( K_m \times \overline{K}_n \) is isomorphic to the circulant
\( X(mn; \{ 1, 2, \ldots, mn \}) \). When considering the graph \( W_{2r} \times (K_m \times \overline{K}_n) \), we assume that the graph
\( K_m \times \overline{K}_n \) is given in its circulant form. Hence the edge \( x_1x_j \) in \( W_{2r} \) and the edge \( u_1u_{k+1} \) in \( K_m \times \overline{K}_n \) give rise to the two edges
\( x_1^l x_{k+1}^l \) and \( x_k^l x_{k+1}^l \) in \( W_{2r} \times (K_m \times \overline{K}_n) \). Hence, corresponding to the edge \( x_1x_j \) in \( W_{2r} \), the edge set of the subgraph induced by
\( x_1 \cup x_j \) in \( W_{2r} \times (K_m \times \overline{K}_n) \) is \( \bigcup_{i \in L} F_i(x_1, x_j) \), where \( L = \{ 1, 2, \ldots, mn \} \). Let the partite sets of the 2r-partite graph \( W_{2r} \times (K_m \times \overline{K}_n) \) be \( X_i = \{ x_1^l, x_2^l, \ldots, x_m^l \}, 1 \leq i \leq 2r \). Then the edge set of \( W_{2r} \times (K_m \times \overline{K}_n) \) can be described as
\( \bigcup_{i=1}^{2r} \bigcup_{j \in L} F_i(x_1, x_j) \). Let \( L = \{ 1, 2, \ldots, mn \} \). Then we decompose \( G_t \) into Hamilton cycles. This isomorphic decomposition is achieved by obtaining the following directed Hamilton cycle decomposition of \( W_{2r} \), the digraph arises out of \( W_{2r} \) by replacing each one of its edges by

**Lemma 2.3.** If \( r \geq 3 \) is odd, \( m \geq 4 \) and \( n \geq 2 \) are even, then \( W_{2r} \times (K_m \times \overline{K}_n) \) is Hamilton cycle decomposable.

**Proof.**

Throughout the proof of this lemma the subscripts of \( x_i^l \)’s and \( x_j^l \)’s and the superscripts of \( x_i^l \)’s are taken modulo 2r
with residues 1, 2, \ldots, 2r and the subscripts of \( x_j^l \)’s are taken modulo \( mn \) with residues 1, 2, \ldots, \( mn \). Let the vertex set of
\( W_{2r} \) be \( \{ x_1, x_2, x_3, \ldots, x_{2r} \} \). Then its edge set can be described as \( \{ (x_i, x_{i+1}) \mid 1 \leq i \leq 2r \} \). Let the partite sets of the 2r-partite graph \( W_{2r} \times (K_m \times \overline{K}_n) \) be \( X_i = \{ x_1^l, x_2^l, x_3^l, \ldots, x_m^l \}, 1 \leq i \leq 2r \). Then the edge set of \( W_{2r} \times (K_m \times \overline{K}_n) \) can be described as
\( \bigcup_{i=1}^{2r} \bigcup_{j \in L} F_i(x_1, x_j) \). Let \( L = \{ 1, 2, \ldots, mn \} \). Then we decompose \( G_t \) into Hamilton cycles. This isomorphic decomposition is achieved by obtaining the following directed Hamilton cycle decomposition of \( W_{2r} \), the digraph arises out of \( W_{2r} \) by replacing each one of its edges by
Table 1

<table>
<thead>
<tr>
<th>$F_i$</th>
<th>$0 \leq j \leq \frac{n}{2} - 1$, $2 \leq i \leq m - 1$, $F_{jm+i}$</th>
<th>$1 \leq j \leq \frac{n}{2} - 1$, $F_{jm+i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$jm + i$</td>
<td>$jm + 1$</td>
</tr>
<tr>
<td>1</td>
<td>$jm + i$</td>
<td>$jm + 1$</td>
</tr>
<tr>
<td>$\frac{m}{2} - 1$</td>
<td>$\frac{m}{2} - jm - i$</td>
<td>$\frac{m}{2} - jm - 1$</td>
</tr>
<tr>
<td>$\frac{m}{2} - 1$</td>
<td>$\frac{m}{2} - jm - i$</td>
<td>$\frac{m}{2} - jm - 1$</td>
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<tr>
<td>1</td>
<td>$jm + i$</td>
<td>$jm + 1$</td>
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<tr>
<td>1</td>
<td>$jm + i$</td>
<td>$jm + 1$</td>
</tr>
<tr>
<td>$\frac{m}{2} - 1$</td>
<td>$\frac{m}{2} - jm - i$</td>
<td>$\frac{m}{2} - jm - 1$</td>
</tr>
<tr>
<td>$\frac{m}{2} - 1$</td>
<td>$\frac{m}{2} - jm - i$</td>
<td>$\frac{m}{2} - jm - 1$</td>
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<td>.</td>
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<tr>
<td>1</td>
<td>$jm + i$</td>
<td>$jm + 1$</td>
</tr>
<tr>
<td>1</td>
<td>$jm + i$</td>
<td>$jm + 1$</td>
</tr>
<tr>
<td>$\frac{m}{2} - 1$</td>
<td>$\frac{m}{2} - jm - i$</td>
<td>$\frac{m}{2} - jm - 1$</td>
</tr>
<tr>
<td>$\frac{m}{2} - 1$</td>
<td>$\frac{m}{2} - jm - i$</td>
<td>$\frac{m}{2} - jm - 1$</td>
</tr>
</tbody>
</table>

a symmetric pair of arcs. The required directed Hamilton cycle decomposition $\{H_1, H_2, H_3\}$ of $W_{2r}$ is given by $H_1 = \bigcup_{i=1}^{r}(X_{i}, X_{i+1}), (X_{i+1}, X_{i}) \rightarrow H_2 = \bigcup_{i=1}^{r}(X_{2i-2}, X_{2i-1}), (X_{2i-1}, X_{2i+1}) \rightarrow H_3 = \bigcup_{i=1}^{r}(X_{i}, X_{i+1})$.

Clearly, $H_1, H_2$ and $H_3$ are arc-disjoint directed Hamilton cycles of $W_{2r}$. We define $G_t$, $t = 1, 2, 3$, as follows:

Let $L' = \{1, 2, \ldots, mn/2\} - \{km \mid 1 \leq k \leq n/2\}$.

Then $G_t = \bigcup_{(x_i, x_j) \in A_k} \bigcup_{l \in L'} F_t(x_i, x_j)$, $t = 1, 2, 3$. Using the fact that $F_t(x_i, x_j) = F_{mn-i}(x_i, x_j)$, we can check that $W_{2r} \times (K_m \ast \overrightarrow{K}_n) = \bigcup_{i=1}^{r} G_t$. Clearly, $G_t$, $t = 1, 2, 3$, is isomorphic to $G$, where $G = \bigcup_{i=1}^{r} \bigcup_{l \in L'} F_t(x_i, x_j)$, $t = 1, 2, 3$. Thus to prove the existence of a Hamilton cycle decomposition of $G_t$, $t = 1, 2, 3$, it is enough to decompose $G$ into Hamilton cycles. We divide the proof into two cases.

**Case 1.** $mn \equiv 0 \pmod{8}$.

We initially obtain a 2-factorization of $G$ and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose each of them into two Hamilton cycles of $G$. We obtain a 2-factorization of $G$ by describing the 2-factors $F_i$, $i \in \{1, 2, \ldots, mn/2\} - \{km \mid 1 \leq k \leq n/2\}$, as given in Table 1.

In Table 1, the 1 (resp. $jm + i, jm + 1$) in the first row denotes the distance of the 1-factor from $X_1$ to $X_2$ that we have chosen for the construction of the 2-factor $F_1$ (resp. $F_{jm+i}, F_{jm+1}$). Similarly, the 1 (resp. $jm + i, jm + 1$) in the second row denotes the distance of the 1-factor from $X_2$ to $X_3$ that we have chosen for the construction of $F_1$ (resp. $F_{jm+i}, F_{jm+1}$) and so on. That is, an $s$ in the $s$th row of the table denotes the distance of the 1-factor from $X_3$ to $X_{s+1}$ that we have chosen for the construction of $F_1$ or $F_{jm+i}$ or $F_{jm+1}$ according to whether $s$ is in column 1 or 2 or 3. Further, in the table every successive four rows, except the first six rows, are identical, in order, and the four entries in each of the three columns described by these four rows add up to a multiple of $mn$.

Clearly, all the 2-factors defined above, except the $n/2$ 2-factors, $F_1$ and $F_{jm+1}$, $1 \leq j \leq \frac{n}{2} - 1$, are Hamilton cycles. These $n/2$ 2-factors are combined with $n/2$ Hamilton cycles, namely, $F_{jm+2}$, $0 \leq j \leq \frac{n}{2} - 1$, in pairs, to obtain a Hamilton cycle decomposition of $G$ as follows: $F_{jm+1} \cup F_{jm+2}$, $0 \leq j \leq \frac{n}{2} - 1$, can be decomposed into two Hamilton cycles, by Lemma 2.1.

**Case 2.** $mn \equiv 4 \pmod{8}$.

As in the above case, we obtain a 2-factorization of $G$ and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of $G$. We obtain a 2-factorization of $G$ by describing the 2-factors $F_i$, $i \in \{1, 2, \ldots, mn/2\} - \{km \mid 1 \leq k \leq n/2\}$ of $G$ as given in Table 2. In Table 2, every successive four rows, except the first six rows, are identical, in order, and the four entries in each of the six columns described by these four rows add up to a multiple of $mn$.

Clearly, all the 2-factors of $G$ defined above, except the $n/2$ 2-factors, $F_1$ and $F_{jm+3}$, $1 \leq j \leq \frac{n}{2} - 1$, are Hamilton cycles. Now we combine these $n/2$ 2-factors with the $n/2$ Hamilton cycles, $F_2$ and $F_{jm+2}$, $1 \leq j \leq \frac{n}{2} - 1$, in pairs, to obtain a Hamilton cycle decomposition of $G$ as follows: $F_1 \cup F_2$ and $F_{jm+2} \cup F_{jm+3}$, $1 \leq j \leq \frac{n}{2} - 1$, can be decomposed into two Hamilton cycles, by Lemma 2.1. This completes the proof. □

For the rest of the paper, except the proof of Theorem 1.1, we assume the following:

1. $m \geq 4$ is even, $n \geq 3$ is odd.
2. A denotes the set \( \{1, 2, 3, \ldots, \frac{mn}{2} - 1\} - \{im \mid 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq n - 1\} \), \( B \) denotes the set \( \{\frac{mn}{2} + 1, \frac{mn}{2} + 2, \frac{mn}{2} + 3, \ldots, mn - 1\} - \{\frac{n}{2} \left( \frac{n}{2} + 1 \right) \} \), \( D = B \cup C \) and \( E = A \cup D \).

3. Let the vertex set of \( W_6 \) be \( \{y_1, y_2, \ldots, y_6\} \). For odd \( r \geq 5 \), let the vertex set of \( W_{2r} \) be \( \{x_1, x_2, x_3, \ldots, x_{2r}\} \); consequently, the 6-partite graph \( W_6 \times (K_m \ast K_n) \) has the vertex set \( Y_1 = \{y_1, y_2, y_3, \ldots, y_{m'}\} \), \( 1 \leq i \leq 6 \), and the vertex set of \( W_{2r} \times (K_m \ast K_n) \) is \( X_i = \{x_1, x_2, x_3, \ldots, x_{m'}\} \), \( 1 \leq i \leq 2r \). (For the later part of the paper, we need to have two different kinds of vertex sets of \( W_{2r} \times (K_m \ast K_n) \) when \( r = 3 \) and \( r \geq 5 \).)

4. The subscripts of \( y_i \)'s and \( y_i \)'s and the superscripts of \( x_i \)'s are taken modulo 6 with residues 1, 2, \ldots, 6. The subscripts of \( x_i \)'s and \( x_i \)'s and the superscripts of \( x_i \)'s are taken modulo 2r with residues 1, 2, \ldots, 2r \( r \geq 5 \) and the subscripts of \( x_i \)'s and \( x_i \)'s are taken modulo mn with residues 1, 2, \ldots, mn.

5. Clearly, the edge set of \( W_6 \) is \( \{y_iy_{i+1} \mid 1 \leq i \leq 6\} \cup \{y_iy_{i+3} \mid 1 \leq i \leq 3\} \) and, for odd \( r \geq 5 \), the edge set of \( W_{2r} \) is \( \{x_ix_{i+1} \mid 1 \leq i \leq 2r\} \cup \{x_ix_{i+r} \mid 1 \leq i \leq r\} \).

6. The edge set of the 6-partite graph \( W_6 \times (K_m \ast K_n) \) is given by \( \left( \bigcup_{i=1}^{m} \left( \bigcup_{j \in E} F_j(Y_i, Y_{i+1}) \right) \right) \cup \left( \bigcup_{i=1}^{3} \left( \bigcup_{j \in E} F_j(Y_i, Y_{i+3}) \right) \right) \) and the edge set of the 2r-partite graph \( W_{2r} \times (K_m \ast K_n) \) is given by \( \left( \bigcup_{i=1}^{2r} \left( \bigcup_{j \in E} F_j(X_i, X_{i+1}) \right) \right) \cup \left( \bigcup_{i=1}^{r} \left( \bigcup_{j \in E} F_j(X_i, X_{i+r}) \right) \right) \).

7. \( G_1 \) and \( G_2 \) denote the spanning subgraphs of \( W_6 \times (K_m \ast K_n) \), where \( G_1 = \bigcup_{i=1}^{6} \left( \bigcup_{j \in E} F_j(Y_i, Y_{i+1}) \right) \) and \( G_2 = \bigcup_{i=1}^{6} \left( \bigcup_{j \in E} F_j(Y_i, Y_{i+3}) \right) \).

Lemma 2.4. If \( m \geq 4 \) is even and \( n \geq 3 \) is odd, then \( W_6 \times (K_m \ast K_n) = G' \oplus G'' \oplus G''' \oplus F \), where \( G' \cong G'' \cong G_1 \), \( G''' \cong G_2 \) and \( F \) is a 1-factor of \( W_6 \times (K_m \ast K_n) \), where \( G_1 \) and \( G_2 \) are as described above.

Proof. Let \( G' = \left( \bigcup_{j \in C} F_j(Y_1, Y_2) \right) \cup \left( \bigcup_{j \in C} F_j(Y_3, Y_5) \right) \cup \left( \bigcup_{j \in C} F_j(Y_3, Y_6) \right) \), \( G'' = \left( \bigcup_{j \in C} F_j(Y_2, Y_3) \right) \cup \left( \bigcup_{j \in C} F_j(Y_4, Y_5) \right) \cup \left( \bigcup_{j \in C} F_j(Y_4, Y_6) \right) \) and \( G''' = \left( \bigcup_{j \in C} F_j(Y_1, Y_{i+1}) \right) \) and let \( F = \bigcup_{j=1}^{mn/2} Y_{mn/2} \). Clearly, \( G' \cong G'' \cong G_1 \), \( G''' \cong G_2 \) and \( F \) is a 1-factor of \( W_6 \times (K_m \ast K_n) \). From the above lemma, it is clear that to prove that \( W_6 \times (K_m \ast K_n) \) has a Hamilton cycle decomposition, it is enough to prove that the even regular graphs \( G_1 \) and \( G_2 \) have Hamilton cycle decompositions, which we prove below.

Lemma 2.5. \( G_1 \) is Hamilton cycle decomposable.

Proof. We prove this lemma in two cases.

Case 1. \( m \equiv 4 \) (mod 8).

We prove this case in two subcases.
Table 3

<table>
<thead>
<tr>
<th>F_1</th>
<th>F_2</th>
<th>F_3</th>
<th>F_4</th>
<th>F_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2n-1</td>
<td>2n-3</td>
<td>2n-4</td>
<td>2n-5</td>
<td>2n-7</td>
</tr>
<tr>
<td>2n-1</td>
<td>2n-3</td>
<td>2n-4</td>
<td>2n-5</td>
<td>2n-7</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

Subcase 1.1. m = 4.

If n = 3, then G_1 can be decomposed into four 2-factors F_1, F_2, F_3 and F_5 as shown in Table 3.

Clearly, F_1 and F_5 are Hamilton cycles of G_1, and F_2 and F_3 are 2-factors, each having two components, since the greatest common divisor of the sum of the distances of the 1-factors from Y_1 to Y_(i+1) in the construction of F_2 and F_3) and mn (= 12) is 2. But F_2 and F_3 can be decomposed into two Hamilton cycles, by Lemma 2.1.

Hence we assume that n ≥ 5. First we obtain a 2-factorization of G_1 and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of G_1. A 2-factorization of G_1 is obtained by defining the 2-factors F_1, i ∈ A, as shown in Table 4.

The 2-factors F_1, F_2, F_5, F_7 and F_{4j+2}, F_{4j+3}, 2 ≤ j ≤ [r/2] − 1, are combined as F_1 ∪ F_2, F_5 ∪ F_7 and F_{4j+2} ∪ F_{4j+3}, 2 ≤ j ≤ [r/2] − 1, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_1 from Table 4.

Subcase 1.2. m ≥ 12.

As above, we obtain a 2-factorization of G_1 and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of G_1. A 2-factorization of G_1 is obtained by defining the 2-factors F_1, i ∈ A, as shown in Table 5.

The 2-factors F_1, F_2, F_{jm+\frac{r}{2}}, F_{jm+\frac{r}{2}+1}, 0 ≤ j ≤ [r/2] − 1 and F_{jm+\frac{r}{2}+2i}, F_{jm+\frac{r}{2}+2i+1}, 1 ≤ i ≤ \frac{m}{4} − 1, 0 ≤ j ≤ [r/2] − 1, are combined as F_1 ∪ F_2, F_{jm+\frac{r}{2}} ∪ F_{jm+\frac{r}{2}+1}, 0 ≤ j ≤ [r/2] − 1, and F_{jm+\frac{r}{2}+2i} ∪ F_{jm+\frac{r}{2}+2i+1}, 1 ≤ i ≤ \frac{m}{4} − 1, 0 ≤ j ≤ [r/2] − 1, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_1 from Table 5.

Case 2. m ≡ 0, 2, 6 (mod 8).

As in the previous case, we obtain a 2-factorization of G_1. A 2-factorization of G_1 is obtained by defining the 2-factors F_1, i ∈ A, as shown in Table 6.

If m ≡ 0 (mod 8), then the 2-factors F_1, F_2, F_{jm+\frac{r}{2}}, F_{jm+\frac{r}{2}+1}, 0 ≤ j ≤ [r/2] − 1, and F_{jm+\frac{r}{2}+2i}, F_{jm+\frac{r}{2}+2i+1}, 1 ≤ i ≤ \frac{m}{4} − 1, 0 ≤ j ≤ [r/2] − 1, are combined as F_1 ∪ F_2, F_{jm+\frac{r}{2}} ∪ F_{jm+\frac{r}{2}+1}, 0 ≤ j ≤ [r/2] − 1, and F_{jm+\frac{r}{2}+2i} ∪ F_{jm+\frac{r}{2}+2i+1}, 1 ≤ i
Lemma 2.1. \(F \) decomposes them into two Hamilton cycles of \(G\).

First we obtain a 2-factorization of \(G\), 2-factors are readily seen to be Hamilton cycles of \(G\), by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of \(G\), from Table 6.

If \(m \equiv 2, 6 \pmod{8}\), then the 2-factors \(F_1, F_2, F_{2i+1}, F_{2i+2}, 1 \leq i \leq \frac{m-6}{4}, F_{jm+i}, F_{jm+2i}, 1 \leq i \leq \frac{m-2}{4}, 1 \leq j \leq \frac{m}{2} - 1, F_{jm+\frac{m}{4}}, F_{jm+\frac{m}{4}+1}, 0 \leq j \leq \frac{m}{2} - 1, \) and \(F_{jm+\frac{m}{2}}, F_{jm+\frac{m}{2}+i}, 1 \leq i \leq \frac{m-2}{4}, 1 \leq j \leq \frac{m}{2} - 1, F_{jm+\frac{m}{2}} \cup F_{jm+\frac{m}{2}+i}, 0 \leq j \leq \frac{m}{2} - 1, \) and \(F_{jm+\frac{m}{2}+i-1} \cup F_{jm+\frac{m}{2}+1}, 1 \leq i \leq \frac{m-2}{4}, \frac{m}{2} \leq j \leq \frac{m}{2} - 1, \) and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of \(G\) from Table 6.

Lemma 2.2. \(G_2\) is Hamilton cycle decomposable.

Proof. We prove this lemma in three cases.

Case 1. \(m \equiv 4 \pmod{8}\).

First we obtain a 2-factorization of \(G_2\) and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of \(G_2\). A 2-factorization of \(G_2\) is obtained by defining the 2-factors \(F_i, i \in D\), as shown in Table 7.

The 2-factors \(F_{jm+2i-1}, F_{jm+2i}, 1 \leq i \leq \frac{m-4}{4}, \frac{m}{2} \leq j \leq n - 1, \) and \(F_{jm+\frac{m}{4}}, F_{jm+\frac{m}{4}+1}, \frac{m}{2} \leq j \leq n - 1, \) and \(F_{jm+\frac{m}{4}} \cup F_{jm+\frac{m}{4}+1}, \frac{m}{2} \leq j \leq n - 1, \) and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of \(G_2\) from Table 7.

Case 2. \(m \equiv 0 \pmod{8}\).

As in the previous case, we obtain a 2-factorization of \(G_2\). A 2-factorization of \(G_2\) is obtained by defining the 2-factors \(F_i, i \in D\), as shown in Table 8.
From the construction of every Hamilton cycle

Lemma 2.1

Remark 2.7

For our future reference, we label the Hamilton cycles of the Hamilton cycle decomposition of $G_2$ from Table 8.

Table 9

<table>
<thead>
<tr>
<th>$F_{\frac{m}{2}}$</th>
<th>$\frac{m}{2} + 1 \leq i \leq \left(\frac{m}{2} + 1\right)m - 1, \quad \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \quad \frac{m}{2} + 1 \leq j \leq n - 1, \quad \frac{m}{2} + 1 \leq j \leq n - 1, \quad \frac{m}{2} + 1 \leq i \leq m - 1,</th>
<th>$F_i$</th>
<th>$F_{jm+i}$</th>
<th>$F_{jm+n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{\frac{m}{2}}$</td>
<td>$i$</td>
<td>$jm + i$</td>
<td>$jm + \frac{m}{2}$</td>
<td>$jm + i$</td>
</tr>
<tr>
<td>$\frac{m}{2}$</td>
<td>$i$</td>
<td>$jm + i$</td>
<td>$jm + \frac{m}{2}$</td>
<td>$jm + i$</td>
</tr>
<tr>
<td>$mn - 1$</td>
<td>$\frac{mn}{2} - i - 1$</td>
<td>$\frac{mn}{2} - jm - i$</td>
<td>$\frac{mn}{2} - jm - \frac{m}{2}$</td>
<td>$\frac{mn}{2} - jm - i$</td>
</tr>
<tr>
<td>$mn - 1$</td>
<td>$\frac{mn}{2} - i - 1$</td>
<td>$\frac{mn}{2} - jm - i$</td>
<td>$\frac{mn}{2} - jm - \frac{m}{2}$</td>
<td>$\frac{mn}{2} - jm - i$</td>
</tr>
<tr>
<td>$\frac{m}{2}$</td>
<td>$i$</td>
<td>$jm + i$</td>
<td>$jm + \frac{m}{2}$</td>
<td>$jm + i$</td>
</tr>
<tr>
<td>$\frac{m}{2}$</td>
<td>$i$</td>
<td>$jm + i$</td>
<td>$jm + \frac{m}{2}$</td>
<td>$jm + i$</td>
</tr>
</tbody>
</table>

The 2-factors $F_{jm+2i-1}, F_{jm+2i+1}, 1 \leq i \leq \frac{m-n}{4}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}$, are combined as $F_{jm+2i-1} \cup F_{jm+2i+1}, 1 \leq i \leq \frac{m-n}{4}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of $G_2$ from Table 8.

Case 3. $m \equiv 2, 6 \pmod{8}$

As above, first we obtain a 2-factorization of $G_2$. A 2-factorization of $G_2$ is obtained by defining the 2-factors $F_i, i \in D$, as shown in Table 9.

The 2-factors $F_{\frac{m}{2}}, F_{\frac{m}{2}+1}, F_{jm+2i-1}, F_{jm+2i+1}, 1 \leq i \leq \frac{m-n}{4}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}$, are combined as $F_{\frac{m}{2}} \cup F_{\frac{m}{2}+1}, F_{jm+2i-1} \cup F_{jm+2i+1}, 1 \leq i \leq \frac{m-n}{4}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}, \frac{m}{2} + 1 \leq j \leq n - 1, 1 \leq i \leq \frac{m}{2}$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of $G_2$ from Table 9.

Remark 2.7. From the construction of every Hamilton cycle $H$ of the Hamilton cycle decomposition of $G_1$ (resp. $G_2$) obtained in Lemma 2.5 (resp. Lemma 2.6), the vertices in $Y_1$ and $Y_2$ are of degree 1 in the subgraph $G_1[Y_1 \cup Y_2 \cup Y_3] \cap H$ (resp. $G_2[Y_1 \cup Y_2 \cup Y_3] \cap H$), obtained by taking the intersection of $H$ with the subgraph induced by $Y_1 \cup Y_2 \cup Y_3$ in $G_1$ (resp. $G_2$). This is true as we use the Hamilton cycle decomposition described in Lemma 2.1; see Remark 2.2. Similarly, the vertices in $Y_1$ and $Y_2$ of degree 1 in the subgraph $G_1[Y_1 \cup Y_4 \cup Y_5] \cap H$ (resp. $G_2[Y_1 \cup Y_4 \cup Y_5] \cap H$). Also, the vertices in $Y_1$ and $Y_2$ are of degree 1 in the subgraph $G_1[Y_1 \cup Y_5 \cup Y_6 \cup Y_7] \cap H$ (resp. $G_2[Y_1 \cup Y_5 \cup Y_6 \cup Y_7] \cap H$). This fact will be used in the proof of Lemma 2.9.

Remark 2.8. Let $G_{1,1}, G_{1,2}$ and $G_{1,3}$ be the subgraphs of $G_1$ induced by $Y_1 \cup Y_2 \cup Y_3, Y_3 \cup Y_4 \cup Y_5, Y_5 \cup Y_6 \cup Y_1, \text{respectively.}$ From each Hamilton cycle $H_i, i \in A, \text{of} \ G_1 \text{that we have obtained in Lemma 2.5, we form a new graph $H_i$ as follows: let $G_{1,1}, G_{1,2}$ and $G_{1,3}$ denote the subgraphs of $G_1 \cap H, G_1 \cap H_1$ and $G_1 \cap H_3$ of $H_i$, then $G_1 \cap H_i$ is obtained by considering the disjoint copies of $G_{1,1}, G_{1,2}$ and $G_{1,3}$ and adding certain disjoint paths between $G_{1,1}, G_{1,2}$ and $G_{1,3}$, and $G_{1,2}$ and $G_{1,3}$. From the construction of the Hamilton cycles $H_i, i \in A$, of $G_1$, the vertices in $Y_1$ and $Y_2$ (resp. $Y_3$ and $Y_5$, and $Y_5$ and $Y_1$) are of degree 1 in $G_{1,1}$, $G_{1,2}$ and $G_{1,3}$; see Remark 2.7. Corresponding vertices of $Y_5$ in $G_{1,2}$ and $Y_5$ in $G_{1,3}$ are connected by vertex-disjoint paths of length, say, $k_1$ (all of whose internal vertices are new). Similarly, corresponding vertices of $Y_5$ (resp. $Y_1$) in $G_{1,2}$ (resp. $G_{1,3}$) and $Y_5$ (resp. $Y_1$) in $G_{1,3}$ (resp. $G_{1,1}$) are connected by vertex-disjoint paths, say, $k_2$ (resp. $k_3$) (all of whose internal vertices are new). Call the resulting graph $H_i$. Then $H_i$ is indeed a cycle. This is easy to see, because if we delete the internal vertices of the paths of length $k_1, k_2$ and $k_3$ (that we have used to connect the graphs $G_{1,1}, G_{1,2}$ and $G_{1,3}$) and identify the vertices of $Y_1$ (resp. $Y_5$) in $G_{1,1}$ (resp. $G_{1,2}, G_{1,3}$) with their corresponding vertices in $Y_5$ (resp. $Y_1$) of $G_{1,2}$ (resp. $G_{1,3}, G_{1,1}$), then what we get is $H_i$, the Hamilton cycle of $G_1$. We shall use this remark in the proof of Lemma 2.9.

Lemma 2.9. If $r \geq 5$ and $n \geq 3$ are odd and $m \geq 4$ is even, then $W_{2r} \times (K_m \ast \overline{K}_n)$ has a Hamilton cycle decomposition.

Proof. First we obtain a digraph $D'$ from $W_{2r}$ as follows: replace the edge $x_ix_{i+r}, 1 \leq i \leq r$, of $W_{2r}$ by a symmetric pair of arcs and replace the edge $x_ix_{i+1}, 1 \leq i \leq 2r$, of $W_{2r}$ by two directed arcs from $x_i$ to $x_{i+1}$, that is, with the same tail and head. Thus we have a 3-regular digraph, that is, $d^+ = 3 = d^-$. Now we decompose $D'$ into three directed Hamilton cycles $H_1, H_2$ and $H_3$, as follows:

$H_1 = \{(x_{i+2i-1}, x_{2i-1}), (x_{2i-1}, x_{2i}) \mid 1 \leq i \leq r\}, H_2 = \{(x_{2i-2}, x_{2i-1}), (x_{2i-1}, x_{2i+2i-1}) \mid 1 \leq i \leq r\}, H_3 = \{(x_i, x_{i+1}) \mid 1 \leq i \leq 2r\}$. Clearly, $H_1, H_2$ and $H_3$ are arc-disjoint directed Hamilton cycles of $D'$. 

We colour the arcs (not a proper colouring) of $H_i$, $i = 1, 2, 3$, with two colours $a$ and $b$. Using this colouring and the Hamilton cycle decompositions of $G_1$ and $G_2$ described in Lemmas 2.5 and 2.6, we decompose $W_{2r} \times (K_m \ast \overline{K}_n)$ into Hamilton cycles.

First we colour arcs of $H_1$ as follows: assign colour $a$ to the arcs $(x_{r+1}, x_1)$, $(x_1, x_2)$, $(x_{r+3}, x_3)$, $(x_3, x_4)$, $(x_{r+5}, x_5)$ and $(x_5, x_6)$. Colour the remaining arcs as follows: colour the arc $(x_{2i-1}, x_{2i})$ (resp. $(x_{2i+1}, x_{2i+1})$), $4 \leq i \leq r$, with $a$ (resp. $b$) if $2i - 1 \equiv 3 \pmod{4}$ or with $b$ (resp. $a$) if $2i - 1 \equiv 1 \pmod{4}$.

Colour the arcs of $H_2$ as follows: assign the colour $a$ to the arcs $(x_{2r}, x_1)$, $(x_1, x_{r+1})$, $(x_{r+3}, x_3)$, $(x_3, x_{r+3})$, $(x_4, x_5)$ and $(x_5, x_{r+5})$ and colour the remaining arcs as described below: colour $(x_{2i-2}, x_{2i-1})$ (resp. $(x_{2i+2}, x_{2i+1})$), $4 \leq i \leq r$, with $a$ (resp. $b$) if $2i - 1 \equiv 3 \pmod{4}$ or with $b$ (resp. $a$) if $2i - 1 \equiv 1 \pmod{4}$.

Colour the arcs of $H_3$ as follows: assign the colour $b$ to the arcs $(x_{2r}, x_1)$, $(x_1, x_{r+1})$, $(x_{r+3}, x_3)$, $(x_3, x_{r+3})$, $(x_4, x_5)$ and $(x_5, x_{r+5})$ and colour the remaining arcs $(x_{2i-2}, x_{2i-1})$ and $(x_{2i-1}, x_{2i})$, $4 \leq i \leq r$, with $b$ if $2i - 1 \equiv 3 \pmod{4}$ or with $a$ if $2i - 1 \equiv 1 \pmod{4}$.

Properties of the colouring

By the colouring described above, we can observe the following:

1. The arcs $(x_{2i-1}, x_{2i}) \in H_1$ and $(x_{2i-1}, x_{2i}) \in H_3$, $1 \leq i \leq r$, are assigned different colours.
2. The arcs $(x_{2i-2}, x_{2i-1}) \in H_2$ and $(x_{2i-2}, x_{2i-1}) \in H_3$, $1 \leq i \leq r$, are assigned different colours.
3. The arcs $(x_{r+1}, x_{2i-1}) \in H_1$ and $(x_{r+1}, x_{2i-1}) \in H_2$, $1 \leq i \leq r$, are assigned the same colour.
4. The arcs $(x_{r+1}, x_{2i-1})$ and $(x_{2i-1}, x_{2i})$, $4 \leq i \leq r$, of $H_1$ are assigned different colours.
5. The arcs $(x_{r+1}, x_{2i-1})$ and $(x_{2i-1}, x_{2i-1})$, $4 \leq i \leq r$, of $H_2$ are assigned different colours.
6. The arcs $(x_{r+1}, x_{2i-1})$ and $(x_{2i-1}, x_{2i})$, $4 \leq i \leq r$, of $H_3$ are assigned the colour $b$ if $2i - 1 \equiv 3 \pmod{4}$ or $a$ if $2i - 1 \equiv 1 \pmod{4}$.

Next we decompose $W_{2r} \times (K_m \ast \overline{K}_n)$ into Hamilton cycles using the colourings of the arcs of $H_i’s, i = 1, 2, 3$, and the Hamilton cycle decompositions of $G_1$ and $G_2$.

From the construction of $H_1$ in $D’$, it is clear that the pairs of arcs ${x_{r+1}, x_1}$, $(x_1, x_2)$, $(x_{r+3}, x_3)$, $(x_3, x_4)$ and $(x_{r+5}, x_5)$, $(x_5, x_6)$ describe three arc-disjoint directed paths of length 2 along $H_1$; let these paths be $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$, respectively. $P_{1,2}$ occurs after $P_{1,1}$, and $P_{1,3}$ occurs after $P_{1,2}$, where we assume that $x_{r+1}$ is the origin of $H_1$. We shall use these $P_{1,i}’s$ in the following construction.

Using the Hamilton cycles $H_i, i \in A$, of $G_1$ and $H_1$ of $D’$ we now construct a graph $H_{1,1}$ (which is actually a cycle) as follows: consider the directed Hamilton cycle $H_1$ of $D’$. Arrange the $X_i’s$ of $W_{2r} \times (K_m \ast \overline{K}_n)$ according to the order of occurrence of the vertices $x_i$ in $H_1$; see Fig. 4.

We shall add some edges between the consecutive layers, in the above order, using some rule described below, and prove that the resulting graph $H_{1,1}$ is a Hamilton cycle of $W_{2r} \times (K_m \ast \overline{K}_n)$.

The corresponding edges of the Hamilton cycle $H_i$, $i \in A$, of $G_1$, joining the vertices of $Y_1$ and $Y_2$ are added between the layers $X_{r+1}$ and $X_1$ with the preservation of the subscripts of the vertices, that is, if $y_1^r y_2^r$ is an edge of $H_1$ joining the vertex $y_1^r$ of $Y_1$ and the vertex $y_2^r$ of $Y_2$ in $G_1$, then the edge added between $x_{r+1}$ and $x_1$ is $x_{r+1}^r x_1^r$. Similarly, the corresponding edges of the Hamilton cycle $H_i$, $i \in A$, of $G_1$ joining the vertices of $Y_2$ and $Y_3$ are added between the layers $X_1$ and $X_2$.

Similarly, add the corresponding edges of $H_i, i \in A$, of $G_1$ joining the vertices of $Y_3$ (resp. $Y_5$) and $Y_4$ (resp. $Y_6$) between the layers $X_{r+3}$ (resp. $X_{r+5}$) and $X_3$ (resp. $X_5$) and also add the corresponding edges of $H_i, i \in A$, joining the vertices of $Y_4$ (resp. $Y_6$) and $Y_5$ (resp. $Y_7$) between the layers $X_3$ (resp. $X_5$) and $X_4$ (resp. $X_6$). Further, for all arcs $(x_p, x_q)$ of $H_1$, other than the arcs in $P_{1,1}, P_{1,2}$ and $P_{1,3}$, add the edges of the 1-factor (of the subgraph of $W_{2r} \times (K_m \ast \overline{K}_n)$ induced by $X_p \cup X_q$) of distance 1 (resp. $mn - 1$) from $x_p$ to $x_q$ if the colour assigned to the arc $(x_p, x_q)$ of $H_1$ is $a$ (resp. $b$). Now we have fixed a 1-factor (of the subgraph of $W_{2r} \times (K_m \ast \overline{K}_n)$ induced by the consecutive layers) between two consecutive layers corresponding to the ends of the arcs of $H_1$, other than the arcs in $P_{1,1}, P_{1,2}$ and $P_{1,3}$. Call the resulting graph $H_{1,1}$. The subgraph of $H_{1,1}$ induced by $X_{r+1} \cup X_1 \cup X_2$ contains a disjoint union of paths of length 2 covering all of its vertices, wherein the vertices of $X_{r+1}$ and $X_2$ are of degree 1; see Remark 2.7. Similarly, in the subgraph of $H_{1,1}$ induced by $X_{r+3} \cup X_3 \cup X_4$ (resp. $X_{r+5} \cup X_5 \cup X_6$) the vertices of $X_{r+3}$ (resp. $X_{r+5}$) and $X_4$ (resp. $X_6$) are of degree 1; see Remark 2.7.

From the property (4) of the colouring described for the arcs of $H_1$ and the fact that $r$ is odd, it can be checked that among the arcs in the segment of $H_1$ beginning from the terminus $x_2$ (resp. $x_4, x_6$) of $P_{1,1}$ (resp. $P_{1,2}, P_{1,3}$) and going to the origin $x_{r+3}$ (resp. $x_{r+5}, x_{r+1}$) of $P_{1,2}$ (resp. $P_{1,3}, P_{1,1}$), the number of arcs receiving the colour $a$ is the same as the number of arcs receiving the colour $b$. Consequently, from the kth vertex of $X_2$ (resp. $X_4, X_6$) to the kth vertex of $X_{r+3}$ (resp. $X_{r+5}, X_{r+1}$) there is a path in $H_{1,1}$. Hence the graph $H_{1,1}$ is just like the construction described in the Remark 2.8 and hence it is a Hamilton
Fig. 4.

Order of occurrence of the layers $X_i$ corresponding to $H_1$

cycle. Thus $H_{i,1}$ is a Hamilton cycle of $W_{2r} \times (K_m \ast \overline{K}_n)$. Thus corresponding to the Hamilton cycles $H_i$ of $G_1$ and $H_1$ of $D'$, we have obtained a Hamilton cycle $H_{i,1}$ of $W_{2r} \times (K_m \ast \overline{K}_n)$.

From the construction of $H_2$, it is clear that the pairs of arcs $\{(x_{2r}, x_1), (x_1, x_{r+1})\}$, $\{(x_2, x_3), (x_3, x_{r+3})\}$ and $\{(x_4, x_5), (x_5, x_{r+5})\}$ describe three arc-disjoint directed paths of length 2 along $H_2$; let these paths be $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$, respectively. $P_{2,2}$ occurs after $P_{2,1}$, and $P_{2,3}$ occurs after $P_{2,2}$, where we assume that $x_{2r}$ is the origin of $H_2$. These paths $P_{2,j}$ are used in the following construction. We shall construct, like $H_{i,1}$, another graph $H_{i,2}$ (which is also a cycle) using the Hamilton cycle $H_i$, $i \in A$, of $G_1$ and the directed Hamilton cycle $H_2$ of $D'$.

For the construction of $H_{i,2}$ consider the directed Hamilton cycle $H_2$ of $D'$. Arrange the $X_i$'s of $W_{2r} \times (K_m \ast \overline{K}_n)$ according to the order of occurrence of the vertices of $X_i$'s in $H_2$. The corresponding edges of the Hamilton cycle $H_i$, $i \in A$, of $G_1$, joining the vertices of $Y_1$ and $Y_2$ are added between the layers $X_{2r}$ and $X_1$ with the preservation of the subscripts of the vertices. Also, the corresponding edges of the Hamilton cycle $H_i$, $i \in A$, of $G_1$ joining the vertices of $Y_2$ and $Y_3$ are added between the layers $X_1$ and $X_{r+1}$. Similarly, add edges between the layers $X_2$ and $X_3$, and $X_3$ and $X_{r+3}$ (resp. $X_4$ and $X_5$, and $X_5$ and $X_{r+5}$) corresponding to the edges of $H_i$, $i \in A$, of $G_1$ between $Y_3$ and $Y_4$, and $Y_4$ and $Y_5$ (resp. $Y_5$ and $Y_6$, and $Y_6$ and $Y_1$). For the arcs $(x_p, x_q)$ of $H_2$ not in $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$, add the edges of the 1-factor (of the subgraph of $W_{2r} \times (K_m \ast \overline{K}_n)$ induced by $X_p \cup X_q$) of distance $i$ (resp. $mn - i$) from $X_p$ to $X_q$ if the colour assigned to the arc $(x_p, x_q)$ is a (resp. $b$). Call the resulting graph $H_{i,2}$.
The subgraph of $H_{1, 2}$ induced by $X_{2r} \cup X_1 \cup X_{r+1}$ (resp. $X_2 \cup X_3 \cup X_{r+3}, X_4 \cup X_5 \cup X_{r+5}$) contains only a disjoint union of paths of length 2 covering all of its vertices, wherein the vertices of $X_{2r}, X_{r+1}$ (resp. $X_2 \cup X_{r+3}, X_4 \cup X_{r+5}$) are of degree 1; see Remark 2.7. The proof of the fact that $H_{1, 2}$ is a Hamilton cycle of $W_{2r} \times (K_m \ast \overline{K_n})$ is similar to the proof for $H_{1, 1}$. Thus corresponding to the Hamilton cycles $H_1$ of $G_1$ and $H_2$ of $D'$, we have obtained a Hamilton cycle $H_{1, 2}$ of $W_{2r} \times (K_m \ast \overline{K_n})$.

From the construction of $H_{1, 2}$, it is clear that the arcs $(x_2r, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5)$ and $(x_5, x_6)$ describe a directed path $P_{3, 1}$ of length 6 along $H_{1, 2}$, where we assume that $x_{2r}$ is the origin of $H_{1, 2}$. Using the Hamilton cycles $H'_i, i \in D$, of $G_2$ and $H_i$ of $D'$ we shall construct a graph $H_{1, 3}$ (which is actually a cycle) as follows: consider the directed Hamilton cycle $H_3$ of $D'$. Arrange the $X'_i$'s of $W_{2r} \times (K_m \ast \overline{K_n})$ according to the order of occurrence of the vertices $x_i$ in $H_3$. The corresponding edges of the Hamilton cycle $H'_i = f_i \in i \in D$, of $G_2$ joining the vertices of $Y_1$ and $Y_2$ are added between the layers $X_2$ and $X_1$. Similarly, the corresponding edges of $H'_i, i \in D$, of $G_2$ joining the vertices of $Y_2$ (resp. $Y_3, Y_4, Y_5, Y_6$) and $Y_3$ (resp. $Y_4, Y_5, Y_6, Y_1$) are added between the layers $X_1$ (resp. $X_2, X_3, X_4, X_5$) and $X_2$ (resp. $X_3, X_4, X_5, X_6$).

Again, for all the arcs $(x_r, x_q)$ of $H_{1, 2}$ other than the arcs in $P_{3, 1}$, add the edges of the 1-factor (of the subgraph of $W_{2r} \times (K_m \ast \overline{K_n})$ induced by $X_r \cup X_q$) of distance $i$ (resp. $mn - i$) from $x_r$ to $x_q$ if the colour assigned to the arc $(x_r, x_q)$ of $H_{1, 2}$ is $b$ (resp. $a$). Call the resulting graph $H_{1, 3}$. The subgraph of $H_{1, 3}$ induced by $X_{2r} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6$ contains only a disjoint union of paths covering all of its vertices, wherein the vertices of $X_{2r} \cup X_6$ are of degree 1.

From the property (6) of the colouring described for the arcs of $H_3$ and the fact that $r$ is odd, it follows that among the arcs from the terminus $x_6$ of $P_{3, 1}$ to the origin $x_{2r}$ of $P_{3, 1}$ (taken along $H_{1, 2}$) the number of arcs receiving the colour $a$ is same as the number of arcs receiving the colour $b$. Consequently, for the $k$th vertex of $X_6$ to the $k$th vertex of $X_{2r}$ there is a path in $H_{1, 3}$ (taken along the direction of $H_3$). We can check that $H_{1, 3}$ is a Hamilton cycle of $W_{2r} \times (K_m \ast \overline{K_n})$. Let $F = \bigcup_{i=1}^{r} F_{mn/2}(X_{2i-1}, X_{2i+1})$. Clearly, $F$ is a 1-factor of $W_{2r} \times (K_m \ast \overline{K_n})$.

Next we shall show that $\{H_{1, 1} \mid i \in A\}, \{H_{1, 2} \mid i \in A\}, \{H_{1, 3} \mid i \in D\}$ and $F$ partitions the edge set of $W_{2r} \times (K_m \ast \overline{K_n})$.

The arcs $(x_{2i-1}, x_{2i-1}) \in H_1$ and $(x_{2i-1}, x_{2i-1}) \in H_2$. $1 \leq i \leq r$, are assigned the same colour by property (3) and hence the Hamilton cycles in $\{H_{1, i} \mid i \in A\}$ contain the edges of the 1-factors of distances in $A \cup B$ from $X_{2i-1}$ to $X_{2i+1}$, using the fact that $F(X_i, X_{i-1}) = F_{mn/2}(X_i, X_{i-1})$, $A = \{mn - i \mid i \in B\}$ and $B = \{mn - i \mid i \in A\}$. $F$ contains the 1-factor of distance in $C = \{mn/2\}$ from $X_{2i-1}$ to $X_{2i+1}$. Hence $H_{1, i}, H_{2} \mid i \in A \cup F$ contains all the edges of $W_{2r} \times (K_m \ast \overline{K_n})$ joining the vertices of $X_{2i-1}$ and $X_{2i+1}$.

The arcs $(x_{2i-1}, x_{2i}) \in H_1$ and $(x_{2i-1}, x_{2i}) \in H_3$, $1 \leq i \leq r$, are assigned different colours $a$ and $b$. If $(x_{2i-1}, x_{2i}) \in H_1$ is assigned the colour $a$ (resp. $b$) and $(x_{2i-1}, x_{2i}) \in H_3$ is assigned the colour $b$ (resp. $a$), then the Hamilton cycles in $\{H_{1, i} \mid i \in A\} \cup \{H_{1, i} \mid i \in D\}$ contain all the edges of the 1-factors of distances in $E = A \cup D$, where $D = B \cup C$, where $A \cup C = \{mn - i \mid i \in D\}$ from $X_{2i-1}$ to $X_{2i}$. Hence the Hamilton cycles in $\{H_{1, i} \mid i \in A\} \cup \{H_{1, i} \mid i \in D\}$ contain all the edges of $W_{2r} \times (K_m \ast \overline{K_n})$ joining the vertices of $X_{2i-1}$ and $X_{2i}$.

Similarly, the arcs $(x_{2i-2}, x_{2i-1}) \in H_2$ and $(x_{2i-2}, x_{2i-1}) \in H_3$, $1 \leq i \leq r$, are assigned different colours $a$ and $b$. As above, we can check that the Hamilton cycles in $\{H_{1, i} \mid i \in A\} \cup \{H_{1, i} \mid i \in D\}$ contain all the edges of $W_{2r} \times (K_m \ast \overline{K_n})$ joining the vertices of $X_{2i-2}$ and $X_{2i-1}$.

Thus $\{H_{1, i} \mid i \in A\} \cup \{H_{1, i} \mid i \in D\} \cup F$ is a Hamilton cycle decomposition of $W_{2r} \times (K_m \ast \overline{K_n})$. This completes the proof.

We use the following results in the proof of Theorem 1.1.

Lemma 2.10 ([6]). If both $G_1$ and $G_2$ have Hamilton cycle decompositions and at least one of $G_1$ and $G_2$ is of odd order, then $G_1 \ast G_2$ is Hamilton cycle decomposable.

Lemma 2.11 ([10]). If $r \geq 3$, then $C_r \ast \overline{K_3}$ is Hamilton cycle decomposable.

Lemma 2.12 ([11]). Let $r \geq 3$ be odd. Then $K_{r, r}$ can be decomposed into Hamilton cycles and a $W_{2r}(\cong X(2r; \{1, r\}))$, that is, $K_{r, r} = C_r \ast C_r \ast \cdots \ast C_r \ast W_{2r}$.

Lemma 2.13 ([11]). If $m, n \geq 2$, then $C_{2n} \ast K_{2m}$ is Hamilton cycle decomposable.

Theorem 2.14 ([11]). If $m \geq 3$, then $K_{r, r} \ast K_m$ is Hamilton cycle decomposable.

Lemma 2.15. If $m \geq 4$ is even and $n, r \geq 3$ are odd, then $W_{2r} \times (K_m \ast \overline{K_n})$ is Hamilton cycle decomposable.

Proof. If $r = 3$, then the conclusion follows from Lemmas 2.4–2.6. If $r \geq 5$, then it follows from Lemma 2.9.

Proof of Theorem 1.1. If $n = 1$, then the result follows from Theorem 2.14. Hence we may assume that $n \geq 2$. We prove this theorem in two cases.
Case 1. \( r \) is even.
As \( K_{r,r} \) is Hamilton cycle decomposable, \( K_{r,r} = C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r} \) and hence
\[
K_{r,r} \times (K_m \ast \overline{K}_n) = (C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r}) \times (K_m \ast \overline{K}_n)
\]
\[
= C_{2r} \times (K_m \ast \overline{K}_n) \oplus C_{2r} \times (K_m \ast \overline{K}_n) \oplus \cdots \oplus C_{2r} \times (K_m \ast \overline{K}_n),
\]

since the tensor product is distributive over edge-disjoint subgraphs. It is enough to prove that \( C_{2r} \times (K_m \ast \overline{K}_n) \) is Hamilton cycle decomposable.

Clearly,
\[
C_{2r} \times (K_m \ast \overline{K}_n) \cong (C_{2r} \times K_m) \ast \overline{K}_n
\]
\[
= (C_{2rm} \oplus C_{2rm} \oplus \cdots \oplus C_{2rm}) \ast \overline{K}_n, \quad \text{by Lemma 2.10 or Lemma 2.13.}
\]
\[
= C_{2rm} \ast \overline{K}_n \oplus C_{2rm} \ast \overline{K}_n \oplus \cdots \oplus C_{2rm} \ast \overline{K}_n.
\]

But \( C_{2rm} \ast \overline{K}_n \) is Hamilton cycle decomposable, by Lemma 2.11. Thus we have a Hamilton cycle decomposition of \( K_{r,r} \times (K_m \ast \overline{K}_n) \).

Case 2. \( r \) is odd.
We complete the proof in two subcases.

Subcase 2.1. \( m \) is odd.
\[
K_{r,r} \times (K_m \ast \overline{K}_n) \cong (K_{r,r} \times K_m) \ast \overline{K}_n
\]
\[
= (C_{2rm} \oplus C_{2rm} \oplus \cdots \oplus C_{2rm}) \ast \overline{K}_n, \quad \text{by Theorem 2.14}
\]
\[
= C_{2rm} \ast \overline{K}_n \oplus C_{2rm} \ast \overline{K}_n \oplus \cdots \oplus C_{2rm} \ast \overline{K}_n.
\]

As \( C_{2rm} \ast \overline{K}_n \) is Hamilton cycle decomposable, by Lemma 2.11, we have a Hamilton cycle decomposition of \( K_{r,r} \times (K_m \ast \overline{K}_n) \).

Subcase 2.2. \( m \) is even.
\[
K_{r,r} = C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r} \oplus W_{2r}, \quad \text{by Lemma 12.}
\]

Consequently,
\[
K_{r,r} \times (K_m \ast \overline{K}_n) = (C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r} \oplus W_{2r}) \times (K_m \ast \overline{K}_n)
\]
\[
= C_{2r} \times (K_m \ast \overline{K}_n) \oplus C_{2r} \times (K_m \ast \overline{K}_n) \oplus \cdots \oplus C_{2r} \times (K_m \ast \overline{K}_n) \oplus W_{2r} \times (K_m \ast \overline{K}_n).
\]

It is enough to prove that \( C_{2r} \times (K_m \ast \overline{K}_n) \) and \( W_{2r} \times (K_m \ast \overline{K}_n) \) are Hamilton cycle decomposable.

Clearly,
\[
C_{2r} \times (K_m \ast \overline{K}_n) \cong (C_{2r} \times K_m) \ast \overline{K}_n
\]
\[
= (C_{2rm} \oplus C_{2rm} \oplus \cdots \oplus C_{2rm}) \ast \overline{K}_n, \quad \text{by Lemma 13.}
\]
\[
= C_{2rm} \ast \overline{K}_n \oplus C_{2rm} \ast \overline{K}_n \oplus \cdots \oplus C_{2rm} \ast \overline{K}_n.
\]

But \( C_{2rm} \ast \overline{K}_n \) is Hamilton cycle decomposable, by Lemma 2.11. Hence \( C_{2r} \times (K_m \ast \overline{K}_n) \) is Hamilton cycle decomposable. As \( m \geq 4 \) is even and \( r \geq 3 \) is odd, the existence of a Hamilton cycle decomposition of \( W_{2r} \times (K_m \ast \overline{K}_n) \) follows from Lemma 2.3 or Lemma 2.15 according to whether \( n \) is even or odd, respectively. This completes the proof. \( \square \)

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