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Holomorphic horospherical duality "sphere-cone"

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ABSTRACT

We describe a construction of complex geometrical analysis which corresponds to the classical theory of spherical harmonics.

I believe that the connection of harmonic analysis and complex analysis has an universal character and is not restricted by the case of complex homogeneous manifolds. It looks as a surprise that such a connection exists and though it is quite natural for finite dimensional representations and compact Lie groups [2,4]. In this note we describe the complex picture which corresponds to harmonic analysis on the real sphere. The basic construction is a version of horospherical transform which in this case is a holomorphic integral transform between holomorphic functions on the complex sphere and the complex spherical cone. This situation looks quite unusual from the point of view of complex analysis and I believe presents a serious interest also in this setting. It can be considered as a version of the Penrose transform, but in a purely holomorphic situation when there is neither cohomology nor complex cycles.

1. GEOMETRICAL PICTURE

Let

$$\Delta(z) = z_1^2 + \cdots + z_{n+1}^2$$

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be the quadratic form in $\mathbb{C}_z^{n+1}$. Let $\mathbb{C}_z^{n+1}$ and $\mathbb{C}_\zeta^{n+1}$ are dual spaces relative to the form

$$\zeta \cdot z = \zeta_1 z_1 + \cdots + \zeta_{n+1} z_{n+1}.$$

Let the complex sphere $\mathbb{C}S = \mathbb{C}S^n \subset \mathbb{C}_z^{n+1}$ be defined by the equation

$$\Delta(z) = 1$$

and the cone $\mathbb{C}\Xi = \mathbb{C}\Xi^n \subset \mathbb{C}_\zeta^{n+1}$ be defined as

$$\Delta(\zeta) = 0, \quad \zeta \neq 0.$$

Both these complex manifolds are homogeneous relative to $SO(n+1, \mathbb{C})$, but we want to describe our objects in the language of complex geometry without an appeal to groups.

We consider on $\mathbb{C}S$ the family of sections $E(\zeta), \zeta \in \mathbb{C}\Xi$, by the isotropic hyperplanes

$$\zeta \cdot z = 1.$$

Let us call $E(\zeta)$ horospheres. They are paraboloids. Correspondingly on $\mathbb{C}\Xi$ we consider the hyperplane sections $L(z), z \in \mathbb{C}S$, by the hyperplanes with the same equations. They are hyperboloids. These dual families of complex submanifolds are the basic element of our geometrical picture.

There is one more essential element of the picture: we consider the family of real spheres $S(u)$, $u \in U$, which are real forms of the complex sphere $\mathbb{C}S$. The parametrical space U is the homogeneous space $SO(n+1, \mathbb{C})/SO(n+1)$, but for us it is essential that $S(u)$ are totally real cycles of the (maximal) dimension n. They are all mutually homotopic, but they are not homological to zero.

2. HOROSPHERICAL CAUCHY TRANSFORM

The complex sphere $\mathbb{C}S$ is the Stein manifold; the complex cone $\mathbb{C}\Xi$ is the holomorphicaly separated complex manifold. Thus there are many holomorphic functions on both of them. We will construct some remarkable integrable operators between the spaces of holomorphic functions on them.

We will denote through $[a_1, a_2, \ldots, a_{n+1}]$ the determinant of the matrix with the columns $a_1, a_2, \ldots, a_{n+1}$ some of which can be 1-forms. We expand such determinants from left to right using the exterior product for the multiplication of 1-forms. We will write $a^{\{k\}}$ if a column a repeats k times. Let

$$\omega = \left[z, dz^{\{n\}}\right] = n! \sum_{1 \leqslant j \leqslant (n+1)} (-1)^{j-1} z_j \bigwedge_{i \neq j} dz_i.$$

It is an invariant holomorphic n-form on $\mathbb{C}S$.

For a holomorphic function $f(z) \in \mathcal{O}(\mathbb{C}S)$ let

$$\hat{f}(\zeta) = \int_S f(z) \frac{\omega}{1 - \zeta \cdot z}.$$

Here S is any real sphere $S(u)$ which does not intersect the horosphere $E(\zeta)$. Such cycles exist and they are homological. It is true since the group $SO(n+1, \mathbb{C})$ transitively acts on both sets of horospheres $E(\zeta)$ and real spheres $S(u)$ and for the sphere S there is an open set of horospheres which do not intersect it (see below the exact description of this set). The integral is independent of the choice $S(u)$ since the integrand is closed as a holomorphic form of maximal degree. Let us remark that this form has the singularity on the horosphere $E(\zeta)$ and that we integrate it along the compact manifold. The result is a holomorphic function on $\mathbb{C}\Xi$. This operator $f \to \hat{f}$ acting from $\mathcal{O}(\mathbb{C}S)$ to $\mathcal{O}(\mathbb{C}\Xi)$ we will call *the horospherical Cauchy transform*.

For a real sphere $S(u)$ let $\Xi(u)$ be the set of such $\zeta \in \Xi$ that $S(u) \cap E(\zeta) = \emptyset$. It is simple to check [5] that, if $S(0) = S$ is the sphere $\{y = \Im z = 0\}$, then $\Xi(0) = \Xi_S$ is the domain described by the conditions

$$\Delta(\xi) = \Delta(\eta) < 1.$$

These domains $\Xi(u)$ for different u can be obtained by the action of elements of the group $G_{\mathbb{C}} = SO(n+1, \mathbb{C})$. Dual connected sets $U(\zeta)$ are described similarly.

3. DUAL HOROSPHERICAL CAUCHY TRANSFORM

Let us consider on $L(z)$ the normalized holomorphic forms of the maximal degree $n-1$:

$$\nu_z(d\zeta) = \frac{[\lambda, \zeta, d\zeta^{\{n-1\}}]}{\lambda \cdot z},$$

where λ is any vector such that $\lambda \cdot z \neq 0$. The restriction of the form ν_z on L_z is independent of a choice of λ: up to a constant factor it is the residue of the form

$$\frac{\omega(d\zeta)}{1 - \zeta \cdot z}$$

on L_z. We can take, in particular, $\lambda = z$ and then

$$\nu_z(d\zeta) = \left[z, \zeta, d\zeta^{\{n-1\}}\right].$$

Of course we could be directly start from this form but we want explain its nature and to connect integrations for different z. Let us put

$$\check{F}(z) = \int_{L_{\mathbb{R}}(z)} F(\zeta)\nu_z(d\zeta), \quad F \in \mathcal{O}(\mathbb{C}\Xi),$$

Here $L_{\mathbb{R}}(z)$ is any cycle in $L(z)$ which is its real form (similar to $S(u)$). We have an operator from $\mathcal{O}(\mathbb{C}\Xi)$ to $\mathcal{O}(\mathbb{C}S)$ which we will call *the dual horospherical Cauchy transform*.

Let us compare the two transforms which we defined. They are very similar indeed. To see this we remark that in the definition of the dual transform we can replace the integral by the integral of the n-form

$$\frac{F(\zeta)[\zeta, d\zeta^{\{n\}}]}{1 - z \cdot \zeta}$$

along a n-dimensional cycle contractible to $L_{\mathbb{R}}(z)$. The integral in the definition is the residue of this integral. This version shows that $\check{F}$ is holomorphic on z since we can fix the cycle for small changes of z. So the definitions are similar but there is an essential difference originating in a difference in the geometry of $\mathbb{C}S$ and $\mathbb{C}\Xi$: we can take the residue of ω on the horosphere $E(\zeta)$ but we can not contract the cycle $S(u)$ on $E(\zeta)$ and on the horosphere there are no appropriate cycles. In a sense $\hat{f}$ characterizes f on "infinity".

4. THE INVERSION FORMULA

We want to find a Radon's type inversion formula for the horospherical transform. This formula will combine the dual transform and a remarkable differential operator $\mathcal{L}$ on $\mathbb{C}\Xi$. Let

$$D = \zeta \cdot \frac{\partial}{\partial \zeta} = \zeta_1 \frac{\partial}{\partial \zeta_1} + \cdots + \zeta_{n+1} \frac{\partial}{\partial \zeta_{n+1}}$$

be the operator along generators of the cone $\mathbb{C}\Xi$. The operator $\mathcal{L}$ will be a polynomial $\mathcal{L}(D)$ of D. We extend functions $F(\zeta)$ as homogeneous functions $F(\zeta, p)$, $p \in \mathbb{C}$, of degree -1. It corresponds to the homogeneous coordinates of the sections $(\zeta \cdot z = p)$. Then

$$\mathcal{L} = c\left(\frac{n-1}{2}\frac{\partial^{(n-2)}}{\partial p^{(n-2)}} - 2\frac{\partial^{(n-1)}}{\partial p^{(n-1)}}\right)\Bigg|_{p=1}, \qquad c = \frac{n}{(-2\pi i)^n}.$$

This operator is the polynomial $\mathcal{L}(D)$ of D.

Theorem. *There is an inversion formula*

$$f = (\mathcal{L}\hat{f})^{\vee}, \quad f \in \mathcal{O}(\mathbb{C}S).$$

It is sufficient (using the closeness of forms) to check the formula in one point $z \in \mathbb{C}S$ and for a specific choice of the cycles $S(u)$ and $L_{\mathbb{R}}(z)$. Let $z = x$ be a point of the real sphere $S = S(0)$ and

$$L_{\mathbb{R}}(x) = \{\zeta = x + i\eta,\ x \cdot \eta = \eta \cdot \eta = 0\}.$$

The inversion formula in this specification was proved in [5] for arbitrary continuous functions on the real sphere S.

We can consider this inversion formula as the integral formula of Cauchy–Fantappie type on $\mathbb{C}S$ (cf. [8]).

5. THE CONNECTION WITH SPHERICAL HARMONICS

All our constructions so far can be described only in the language of the geometry of two families of complex submanifolds. They all are invariant relative to $SO(n+1;\mathbb{C})$ but we can avoid the group language. Now in our considerations will appear a group but this group will be Abelian: the Abelian group $\mathbb{C}^\times$ acts on $\mathbb{C}\Xi$ by the multiplications $\zeta \mapsto c\zeta$, $c \in \mathbb{C}\setminus 0$. Let F be the flag manifold—the factorization of $\mathbb{C}\Xi$ relative to this action. It is the projectivization of this cone—the quadric in $\mathbb{C}P^n$.

The action of $\mathbb{C}^\times$ commutates with the action of $SO(n+1;\mathbb{C})$. Let us decompose the space of holomorphic functions $\mathcal{O}(\mathbb{C}\Xi)$ on invariant subspaces

$$\mathcal{O}(\mathbb{C}\Xi) = \bigoplus_{n\geqslant 0} \mathcal{O}_n(\mathbb{C}\Xi).$$

Functions in $\mathcal{O}_n(\mathbb{C}\Xi)$ are homogeneous polynomials of degree n and they can be interpreted as sections of line bundles on the quadric F. In $\mathcal{O}_n(\mathbb{C}\Xi)$ there are realized the irreducible (finite-dimensional) representations of $SO(n+1;\mathbb{C})$.

Let us call *the holomorphic spherical Fourier transform* $\tilde{f}(n;\zeta)$ the composition of the horospherical Cauchy transform and the projection on $\mathcal{O}_n$. By the direct decomposition of the kernel $1/(1-\zeta\cdot z)$ in the geometrical series we obtain

$$\tilde{f}(n;\zeta) = \int\limits_{S(u)} f(z)(\zeta\cdot z)^n\omega.$$

The differential operator $\mathcal{L}(D)$ on $\mathcal{O}_n$ is the multiplication on the $\mathcal{L}(n)$. As a result we have the inversion formula for the spherical Fourier transform

$$f(z) = \sum_{n\geqslant 0} \mathcal{L}(n) f_n(z), \qquad f_n(z) = \int\limits_{L_{\mathbb{R}}(z)} \tilde{f}(n;\zeta)\nu_z(d\zeta).$$

The preimages of $\mathcal{O}_n(\mathbb{C}\Xi)$ relative to the horospherical transform are subspaces of spherical harmonics $\mathcal{O}_n(\mathbb{C}S)$; $f \mapsto f_n$ are the projectors on them. The integral operator $f(n;\zeta) \mapsto f_n(z)$ can be considered as an analogue of the Poisson integral. The quadric F in a sense plays the role of "the complex boundary" of $\mathbb{C}S$. The connection of spherical harmonics and homogeneous polynomials on the cone $\mathbb{C}\Xi$ goes back to Maxwell.

6. HOLOMORPHIC EXTENSIONS OF HOROSPHERICAL TRANSFORM

We can remove in the definition of the horospherical transform $\hat{f}(\zeta)$ the condition $\Delta(\zeta) = 0$. We need only the existence of a cycle $S(u)$ which does not intersect the section by $\zeta\cdot z = 1$. Correspondingly, we can extend the definition of the dual horospherical transform for such z that the hyperplane $z \cdot \zeta = 1$ intersects the cone $\mathbb{C}\Xi$ on a hyperboloid. The extended holomorphic functions will satisfy to some differential equation. The direct differentiation of the integral representation of $\check{F}(z)$ shows that it is harmonic:

$$\Delta\left(\frac{\partial}{\partial z}\right)\check{F}(z) = 0.$$

The combination with the inversion formula gives the harmonic extension of holomorphic functions $f(z)$ from $\mathbb{C}S$. The elements of $\mathcal{O}_n(\mathbb{C}S)$ extend as harmonic polynomials.

To describe the extension of $\hat{f}(\zeta)$ it is convenient to extend them as homogeneous functions $\check{f}(\zeta, p)$. Then we have

$$\left[\Delta\left(\frac{\partial}{\partial \zeta}\right) - \frac{\partial^2}{\partial p^2}\right]\hat{f}(\zeta, p) = 0.$$

Of course using the homogeneity of $\hat{f}$ we can eliminate p.

7. UNITARY RESTRICTION

The principle of the unitary restriction means that irreducible finite-dimensional representations of complex semisimple Lie groups give the complete system of such representations for compact forms of these groups. In our geometrical picture it corresponds to the possibility to restrict the construction of the horospherical Cauchy transform on the real sphere S (indeed we already used this possibility in the opposite direction). More precisely, let us consider for the sphere $S \subset \mathbb{C}S$ the dual domain $\Xi_S \subset \mathbb{C}\Xi$ of such ζ that the horosphere $E(\zeta)$ does not intersect S. As we remarked (with the reference [5]) this domain is described by the condition

$$\Delta(\xi) = \Delta(\eta) < 1.$$

Correspondingly, we can construct the duality between the real spheres $S(u)$ and the domains $\Xi(u)$ in $\mathbb{C}\Xi$. I believe that it is remarkable that the natural dual object to the real sphere is a complex domain.

For functions $f \in C(S)$ we can define using our definition the horospherical Cauchy transform $\hat{f}(\zeta), \zeta \in \Xi_S$. We can take the boundary values of $\hat{f}$ and apply the inversion formula. The crucial circumstance is that submanifold $L(x), x \in S$, does not intersect the domain Ξ_S but intersect the boundary on the cycle

$$L_{\mathbb{R}}(x) = \{\zeta = x + i\eta,\ x \cdot \eta = \eta \cdot \eta = 0\}.$$

In such a way the boundary $\partial(\mathbb{C}\Xi)$ fibers over S with the fibers $L_{\mathbb{R}}(x)$. The inversion formula holds in this situation [5].

Since the domain Ξ_S is invariant relative to the multiplication on the circle ($\zeta \mapsto \exp(i\theta)\zeta$), the decomposition in the Fourier series survives in this domain. Functions in invariant subspaces are automatically holomorphic in $\mathbb{C}\Xi$ and

$\mathcal{O}_n(\Xi_S) = \mathcal{O}_n(\mathbb{C}\Xi)$. Functions in dual subspaces on S also will be extended holomorphically on $\mathbb{C}S$. It reflects the elliptic nature of the problem (these functions are eigenfunctions of invariant differential operators on S which are elliptic). The holomorphic Fourier transform and its inversion are preserving for $f \in C(S)$.

It is natural to discuss functional spaces on S for which these constructions make sense. We can use our definition of the horospherical Cauchy transform for distributions on S. Moreover, we can consider the horospherical transform of hyperfunctions. So we consider the space of hyperfunctions

$$\mathrm{Hyp}(S) = H^{(n-1)}(\mathbb{C}S \backslash S, \mathcal{O}),$$

which is isomorphic to space of functionals on the space of holomorphic functions in neighborhoods of S on $\mathbb{C}S$. Then we can define the horospherical Cauchy transform of a functional $f \in \mathrm{Hyp}(S)$ as its value on the function $1/(1 - \zeta \cdot z)$, $\zeta \in \Xi_S$, which is holomorphic in a neighborhood of S. It is easy to reformulate this definition in the language of cohomology.

The horospherical Cauchy transform of cohomology $H^{(n-1)}(\mathbb{C}S \backslash S, \mathcal{O})$ is connected with the holomorphic cohomological language for this cohomology [1]. Let us define

$$\mathcal{E} = \{(z, \zeta);\ z \in \mathbb{C}S \backslash S,\ \zeta \in \Xi_S, z \in E(\zeta)\}.$$

It is the Stein manifold. The fibering $\mathcal{E} \to (\mathbb{C}S \backslash S)$ (with the fibers—the intersections $L(z) \cap \Xi_S$) satisfies the conditions in [1] and we can compute the analytic cohomology of $\mathbb{C}S \backslash S$ using the complex of holomorphic forms $\phi(z, \zeta; d\zeta)$ on $\mathcal{E}$ with the differentials only along the fibers. If we restrict these forms on a section of the fibering and take the $(0, q)$-part we obtain the operator on Dolbeault cohomology. We are interested in cohomology in the dimension $n - 1$ equal to the dimension of fibers. So the closeness of forms ϕ is trivial. Let us consider the operator

$$\mathcal{O}(\Xi_S) \to H^{(n-1)}\big((\mathbb{C}S \backslash S), \mathcal{O}\big) : F(\zeta) \mapsto \phi_F(z, \zeta; d\zeta) = F(\zeta) \nu_z(d\zeta).$$

The basic result is that this operator is an isomorphism. More exact, if $f \in H^{(n-1)}(\mathbb{C}S \backslash S, \mathcal{O})$ we take $F = \mathcal{L}\hat{f}$. We can interpret this construction as a holomorphic Hodge theorem: we pick up canonical representatives in cohomology classes but instead of Riemannian geometry we use the complex one. We already used such constructions in the context of the Penrose transform [7].

In conclusion let us remark that the boundary of $\mathcal{E}$ is the fibering over S with cycles $L_{\mathbb{R}}(x)$ as fibers. So if the holomorphic form $\phi(z, \zeta; d\zeta)$ has boundary values in some sense, then we can integrate the boundary form on the fibers and receive a function on S. The inversion formula realizes this correspondence between regular functions and hyperfunctions.

We found that the holomorphic inversion formula can be restricted on the compact real form of $\mathbb{C}S$—the real sphere S—and gives a possibility to reconstruct the harmonic analysis on it. It turns out that there is a possibility to restrict this formula also on noncompact real forms. We can observe here a principal difference with the theory of representations where we can not find representations of real forms in such a way.

Let us start from the hyperbolic space $H^n = H$ which we will realize as the one sheet of two-sheeted hyperboloid

$$\Box(x) = x_1^2 - x_2^2 - \cdots - x_{n+1}^2 = 1, \quad x_1 > 0.$$

It is convenient to replace the coordinates and to use here the bilinear form corresponding to this quadratic form:

$$\zeta \cdot z = \zeta_1 \cdot z_1 - \zeta_2 \cdot z_2 - \cdots - \zeta_{n+1} z_{n+1}.$$

Since H is noncompact we need to put some decreasing conditions on the class of functions. For simplicity, let us take the space $D(H)$ of finite continuous functions. The set of complex horospheres $E(\zeta)$ which do not intersect H is parameterized by the points of CR-submanifold

$$\Xi_H = \{\zeta;\ \zeta = \lambda\xi,\ \lambda \in \mathbb{C}\backslash\mathbb{R},\ \xi \in \mathbb{R}^{n+1},\ \Box(\xi) = 0\}.$$

We define the horospherical Cauchy transform $\hat{f}(\zeta)$, $\zeta \in \Xi_H$, $f \in D(H)$, by the same formula as earlier but we replace the integration on S by the integration on H. The boundary is

$$\{\partial\Xi_H = \xi \in \mathbb{R}^{n+1},\ \Box(\xi) = 0\}.$$

Let us take the boundary values $\hat{f}(\xi)$, $\xi \in \partial\Xi_H$, $\lambda \to 1 + i0$. In the definition of the dual horospherical transform $\check{F}$ for CR-functions F we take the cycle $\{\xi = x + \mu,\ \mu \cdot x = 0,\ \mu \cdot \mu = -1\}$ as the cycle $L_{\mathbb{R}}(x)$, $x \in H$, in the intersection-cone $L(x) \cap \partial\Xi_H$. So if $x = (1, 0, \ldots, 0)$, then $\mu_1 = 0$, $\mu_2^2 + \cdots + \mu_{n+1}^2 = 1$. Of course, we can replace this cycle by any other cycle which intersects once all generators of the cone. As it was shown in [3] then the inversion formula of this paper holds and it gives the possibility to reproduce the harmonic analysis in the hyperbolic space H.

The difference starts when we build the analogue of the spherical Fourier transform. The submanifold Ξ_H is invariant relative to the action noncompact Abelian subgroup $\mathbb{R}_+ \subset \mathbb{C}^\times$. The decomposition in the Mellin integral gives the spherical irreducible representations of $SO(1, n)$ and the composition with the horospherical Cauchy transform gives the spherical Fourier transform. So on Ξ_S the Abelian group was compact and the spectrum was discrete; on Ξ_H the Abelian group is noncompact and the spectrum is continuous.

We investigated the restrictions of the horospherical Cauchy transform on the symmetric Stein space $\mathbb{C}S$ for 2 Riemannian symmetric spaces S, H, compact and noncompact correspondingly. In the conclusion we will discuss the restriction on one pseudo Riemannian form—the hyperboloid X of the signature $(2, n-1)$:

$$\Box(x) = x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+1}^2 = 1$$

and we will replace the basic bilinear form by the form corresponding to this quadric form:

$$\zeta \cdot z = \zeta_1 \cdot z_1 + \zeta_2 \cdot z_2 - \zeta_3 \cdot z_3 - \cdots - \zeta_{n+1} z_{n+1}.$$

For $n = 3$ we have $X = SL(2; \mathbb{R})$. The corresponding results on the horospherical Cauchy transform were obtained in [2,4]. We will see that the picture in this case is a very interesting combination of the pictures for S and H.

The parametric set of complex horospheres $E(\zeta)$ which do not intersect X has 3 components: the component Ξ_X^0 is exactly the same as the set Ξ_H in the last example (but for another bilinear form); 2 other the components are the connected components of the set

$$\Xi_X^{\pm} = \{\zeta - \xi + i\eta \colon \Box(\xi) = \Box(\eta) > 1\}.$$

Correspondingly, we can for $f \in D(X)$ separate 3 components $\hat{f}_0(\zeta), \hat{f}_{\pm}(\zeta)$ which have these 3 sets as the domains. They will be correspondingly CR-functions and holomorphic functions. We take their boundary values. Let F be a function with such 3 components. We define the dual horospherical transform $\check{F}$ of the same structure as above, but it will have 3 components. We need to describe the sets $L_{\mathbb{R}}^0(x), L_{\mathbb{R}}^{\pm}(x)$ along which we integrate in the dual Cauchy transform. In all cases they lie in intersection of the boundaries $\partial \Xi_X^0$, $\partial \Xi_X^{\pm}$ with $L(x)$, $x \in X$. The explicit descriptions are exactly as above: $L_{\mathbb{R}}^0(x)$ is the same as for H and $L_{\mathbb{R}}^{\pm}(x)$ is the same as for S. The difference is only in the real quadratic form. So the hyperboloid $\{\eta \colon \eta \cdot x = 0, \Box(\eta) = 1\}$ has 2 sheets. The important point is that all these 3 sets for X are not compact and we integrate along manifolds which are not cycles. Nevertheless, they all can be compactified by the parameters of degenerate horospheres

$$\xi \cdot z = 0, \quad \xi \in \mathbb{R}^{n+1}, \quad \Box(\xi) = 0, \quad \xi \cdot x = 0.$$

In such a way we construct the cycle which has 3 components which intersect on this set. We can define the boundary values of $\hat{f}$ on this set and the inversion formula can be written down in this case. It was proved in [2]. Let us emphasize that this proof as well as proofs in the previous cases, uses an universal inversion formula for hyperplane sections of a quadric and the inversion formula for horospheres is homotopic to the Radon inversion formula.

10. FINAL REMARKS

(1) There is one more version of the horospherical transform on the symmetric space $\mathbb{C}S^n$ which is connected with the Plancherel formula for $L^2(\mathbb{C}S^n)$. So we consider finite functions $f \in D(\mathbb{C}S)$ (instead of holomorphic functions) and define the (real) horospherical transform

$$\hat{f}(\zeta) = \int_{E(\zeta)} f(z)\omega \wedge \bar{\omega}, \quad \zeta \in \mathbb{C}\Xi.$$

The dual horospherical transform is defined as

$$\check{F}(z) = \int_{L(z)} F(\zeta)\nu_z(d\zeta) \wedge \overline{\nu_z(d\zeta)}, \quad z \in \mathbb{C}S.$$

There is an inversion formula

$$f = (\mathcal{L}\bar{\mathcal{L}}\hat{f})^\vee$$

which can be extended on $L^2(\mathbb{C}S)$. We consider the Fourier transform relative to the action of $\mathbb{C}^\times$ on $\mathbb{C}\Xi$ and its composition with the horospherical transform gives the spherical transform (the decomposition of the representation of $SO(n+1;\mathbb{C})$ on irreducible ones). Here it will be only one series (depending on one continuous and one discrete parameter): the polynomial symbol of the operator $\mathcal{L}\bar{\mathcal{L}}$ is the Plancherel density.

(2) This holomorphic horospherical duality can be generalized following the results of [6] on arbitrary symmetric Stein manifolds G/H where the semisimple Lie group G and its involutive subgroup H are complex.

(3) In the construction of the holomorphic horospherical duality we work only with two dual families of submanifolds and do not appeal to the groups (only the Abelian group appears if we want to have a Fourier type transform). It would be very interesting to find some geometrical conditions on dual families which admit similar explicit inversion formula.

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