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An approximation for zero-balanced Appell function F_1 near $(1,1)$

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Abstract

We suggest an approximation for the zero-balanced Appell hypergeometric function F_1 near the singular point $(1,1)$. Our approximation can be viewed as a generalization of Ramanujan's approximation for zero-balanced ${}_2F_1$ and is expressed in terms of ${}_3F_2$. We find an error bound and prove some basic properties of the suggested approximation which reproduce the similar properties of the Appell function. Our approximation reduces to the approximation of Carlson–Gustafson when the Appell function reduces to the first incomplete elliptic integral.

Keywords: Appell function; Hypergeometric function; Incomplete elliptic integral; Approximation near logarithmic singularity

1. Introduction

The generalized hypergeometric function is defined by [10, formula 4.1(1)]

$$ {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \tag{1} $$

where $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$, $k = 1,2,\dots,$ is shifted factorial. This function is called zero-balanced if $p = q+1$ and $\sum_{i=1}^{p} a_i = \sum_{i=1}^{q} b_i$.

Ramanujan (see [3–5]) suggested the following approximations for zero-balanced ${}_2F_1$ and ${}_3F_2$:

$$ B(a,b)\,{}_2F_1(a,b;a+b;x) = -\ln(1-x) + \gamma(a,b) + O\big((1-x)\ln(1-x)\big), \quad x \to 1-, \tag{2} $$

where

$$ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{3} $$

is Euler's beta function,

$$ \gamma(a,b) = 2\psi(1) - \psi(a) - \psi(b), \qquad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{4} $$

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and

$$\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(b_1)\Gamma(b_2)}{}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}\middle|\,x\right)=-\ln(1-x)+L+O\big((1-x)\ln(1-x)\big),\quad x\to 1-,$$

where $\Re(a_3)>0$ and

$$L=2\psi(1)-\psi(a_1)-\psi(a_2)+\sum_{k=1}^{\infty}\frac{(b_2-a_3)_k(b_1-a_3)_k}{k(a_1)_k(a_2)_k}.$$

These formulas have been generalized to ${}_{q+1}F_q$ by Nørlund [17], Saigo and Srivastava in [18], Marichev and Kalla in [15] and Bühring in [7], see details in the survey paper by Bühring and Srivastava [8].

The Appell function F_1 generalizes ${}_2F_1$ to two variables and is defined by [10]:

$$F_1(\alpha;\beta_1,\beta_2;\nu;z_1,z_2)=\sum_{k,n=0}^{\infty}\frac{(\alpha)_{k+n}(\beta_1)_k(\beta_2)_n}{(\nu)_{k+n}k!n!}z_1^kz_2^n,\tag{5}$$

for $|z_1|<1$, $|z_2|<1$ and by analytic continuation for other values of z_1, z_2. An asymptotic expansion for F_1 in the neighbourhood of infinity has been studied by Ferreira and López in [11]. Their expansion can be converted into an approximation around $(1,1)$ using the formula

$$F_1(a;b,c;d;1-xz,1-yz)=z^{-b-c}x^{-b}y^{-c}F_1\left(d-a;b,c;d;1-\frac{1}{xz},1-\frac{1}{yz}\right).$$

It has been noticed by B.C. Carlson in [6] that the incomplete elliptic integral of the first kind $F(\lambda,k)$ is a particular case of F_1:

$$F_1\big(1/2;1/2,1/2;3/2;\lambda^2,k^2\lambda^2\big)=\frac{1}{\lambda}F(\lambda,k).\tag{6}$$

Carlson and Gustafson studied the asymptotic approximation for $F(\lambda,k)$ in [9]. Their expansion can be shown to be a particular case of the expansion for F_1 given later in [11]. We will show below that both expansions (but not the error bounds!) can be obtained by simple rearrangement of (5) and use of known transformation formulas for F_1. More precise approximations for $F(\lambda,k)$ which cannot be reduced to expansions from [11] have been given recently by S.M. Sitnik and the author in [13].

The purpose of this paper is to give an analogue of (2) for the "zero-balanced" Appell function F_1 with $\nu=\alpha+\beta_1+\beta_2$. Important properties of F_1 are permutation symmetry

$$F_1(\alpha;\beta_1,\beta_2;\nu;z_1,z_2)=F_1(\alpha;\beta_2,\beta_1;\nu;z_2,z_1),\tag{7}$$

reduction formulas

$$F_1(\alpha;\beta_1,\beta_2;\nu;z,1)={}_2F_1(\alpha,\beta_2;\nu;1){}_2F_1(\alpha,\beta_1;\nu-\beta_2;z),\tag{8}$$

$$F_1(\alpha;\beta_1,\beta_2;\nu;z,z)={}_2F_1(\alpha,\beta_1+\beta_2;\nu;z),\tag{9}$$

and reduction formula (6). Our approximation reproduces the permutation symmetry (7), reduces to Ramanujan approximation given in (2) in cases given by (8) and (9) and reproduces Carlson–Gustafson approximation for the values of parameters given in (6).

Some new reduction formulas for F_1 have been discovered in [12].

2. Main results

To save space let us introduce the notation

$$f_{a,b_1,b_2}(x,y)=B(a,b_1+b_2)F_1(a;b_1,b_2;a+b_1+b_2;x,y).\tag{10}$$

Our main approximation is given by

$$g_{a,b_1,b_2}(x,y)=\ln\frac{1}{1-x}+\gamma(a,b_1+b_2)+\frac{b_2(y-x)}{(b_1+b_2)(1-x)}{}_3F_2\left(\begin{matrix}1,1,b_2+1\\ 2,b_1+b_2+1\end{matrix}\middle|\,\frac{y-x}{1-x}\right),\tag{11}$$

where $\gamma(a, b_1+b_2)$ is defined in (4). The following theorem confirms that g_{a,b_1,b_2} is indeed a correct analogue of the right-hand side of (2).

Theorem 1. *For* $0 \leqslant x < 1$, $0 \leqslant y < 1$, $a, b_1, b_2 > 0$:

$$f_{a,b_1,b_2}(x,y) = g_{a,b_1,b_2}(x,y) + R_{a,b_1,b_2}(x,y), \tag{12}$$

with

$$0 < R_{a,b_1,b_2}(x,y) < r\bigl(1+a-a\ln(r)\bigr) = O\bigl(r\ln(r)\bigr), \tag{13}$$

where in the last formula $x, y \to 1$, $r = (1-x)b_1 + (1-y)b_2 \to 0$ *is the "rhombic" distance to* $x = y = 1$ *which is asymptotically equivalent to Euclidean distance, i.e.*

$$\frac{r}{\sqrt{(1-x)^2+(1-y)^2}} \to \frac{b_1+Ab_2}{\sqrt{1+A^2}} \quad \text{as } x, y \to 1,\ \text{where } A = \lim_{x,y\to 1}\frac{1-y}{1-x}.$$

Corollary 1.1. *Formulas* (12) *and* (13) *imply, in particular, the inequality*

$$f_{a,b_1,b_2}(x,y) > g_{a,b_1,b_2}(x,y) \tag{14}$$

for all $x, y \in (0,1)$.

Proof of Theorem 1. A simple rearrangement of (5) gives

$$F_1(\alpha;\beta_1,\beta_2;\nu;z_1,z_2) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta_1)_k}{(\nu)_k k!}\,{}_2F_1(\alpha+k,\beta_2;\nu+k;z_2)z_1^k. \tag{15}$$

Suppose $\nu = \alpha+\beta_2$, then ${}_2F_1$ in (15) is zero-balanced and we can apply [10, formula 2.10(12)]

$$\frac{\Gamma(\eta)\Gamma(\beta)}{\Gamma(\eta+\beta)}\,{}_2F_1(\eta,\beta;\eta+\beta;z) = \sum_{n=0}^{\infty}\frac{(\eta)_n(\beta)_n}{(n!)^2}\bigl[-\log(1-z)+2\psi(n+1)-\psi(\eta+n)-\psi(\beta+n)\bigr](1-z)^n. \tag{16}$$

It gives

$$\begin{aligned}&\frac{\Gamma(\alpha)\Gamma(\beta_2)}{\Gamma(\alpha+\beta_2)}F_1(\alpha;\beta_1,\beta_2;\alpha+\beta_2;z_1,z_2)\\ &\quad= \sum_{n,k=0}^{\infty}\frac{(\alpha+k)_n(\beta_2)_n(\beta_1)_k z_1^k}{(n!)^2k!}\bigl[-\ln(1-z_2)+2\psi(1+n)-\psi(\beta_2+n)-\psi(\alpha+k+n)\bigr](1-z_2)^n.\end{aligned} \tag{17}$$

Taking account of

$$(\alpha)_{k+n} = (\alpha)_k(\alpha+k)_n = (\alpha)_n(\alpha+n)_k,$$

formula (3) for Euler's beta function and the derivative formula

$${}_2F_1'(a,b;c;x) \equiv \frac{\partial}{\partial a}\,{}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{\psi(a+k)(a)_k(b)_k x^k}{(c)_k k!} - \psi(a){}_2F_1(a,b;c;x), \tag{18}$$

identity (17) can be rewritten as

$$\begin{aligned}&B(\alpha,\beta_2)F_1(\alpha;\beta_1,\beta_2;\alpha+\beta_2;z_1,z_2)\\ &\quad= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta_2)_n}{(n!)^2}(1-z_2)^n\\ &\qquad\times\bigl\{\bigl[-\ln(1-z_2)+2\psi(1+n)-\psi(\alpha+n)-\psi(\beta_2+n)\bigr]{}_2F_1(\alpha+n,\beta_1;\alpha;z_1) - {}_2F_1'(\alpha+n,\beta_1;\alpha;z_1)\bigr\}.\end{aligned} \tag{19}$$

Applying the transformation

$$F_1(a;b_1,b_2;a+b_1+b_2;x,y)=\left(\frac{1-y}{1-x}\right)^{b1}F_1\left(b_1+b_2;b_1,a;a+b_1+b_2;\frac{y-x}{1-x},y\right)$$

to (17) and (19) in view of (10) gives

$$\begin{aligned}
&f_{a,b_1,b_2}(x,y)\\
&\quad=\left(\frac{1-y}{1-x}\right)^{b_1}\sum_{k,n=0}^{\infty}\frac{(b_1)_k(b_1+b_2+k)_n(a)_n(-\ln(1-y)+2\psi(1+n)-\psi(a+n)-\psi(b_1+b_2+k+n))}{k!(n!)^2(1-y)^{-n}}\\
&\qquad\times\left(\frac{y-x}{1-x}\right)^k\\
&\quad=\left(\frac{1-y}{1-x}\right)^{b_1}\left\{\sum_{n=0}^{\infty}\frac{(a)_n(b_1+b_2)_n}{(n!)^2}\left[\ln\frac{1}{1-y}+2\psi(1+n)-\psi(a+n)-\psi(b_1+b_2+n)\right]\right.\\
&\qquad\times{}_2F_1\left(b_1+b_2+n,b_1;b_1+b_2;\frac{y-x}{1-x}\right)(1-y)^n\\
&\qquad\left.-\sum_{n=0}^{\infty}\frac{(a)_n(b_1+b_2)_n}{(n!)^2}\,{}_2F_1'\left(b_1+b_2+n,b_1;b_1+b_2;\frac{y-x}{1-x}\right)(1-y)^n\right\}.
\end{aligned}\tag{20}$$

Taking $n=0$ in the above formula and applying

$${}_2F_1\left(b_1+b_2,b_1;b_1+b_2;\frac{y-x}{1-x}\right)=\left(\frac{1-y}{1-x}\right)^{-b_1}$$

we get

$$\begin{aligned}
f_{a,b_1,b_2}(x,y)&=\ln\frac{1}{1-y}+2\psi(1)-\psi(a)-\psi(b_1+b_2)-\left[\frac{1-y}{1-x}\right]^{b_1}{}_2F_1'\left[\begin{matrix}b_1+b_2,b_1\\ b_1+b_2\end{matrix}\,\middle|\,\frac{y-x}{1-x}\right]\\
&\quad+R_{a,b_1,b_2}(x,y),
\end{aligned}\tag{21}$$

where it is clear from (20) that

$$R_{a,b_1,b_2}(x,y)=O\big((1-y)\ln(1-y)\big),$$

which is equivalent to the second formula in (13). Formula (21) can be easily put into a different form by differentiating the identity

$${}_2F_1(a,b;c;x)=(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x)$$

with respect to a:

$${}_2F_1'(a,b;c;x)=-\ln(1-x)(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x)-(1-x)^{c-a-b}\,{}_2F_1'(c-a,c-b;c;x).$$

Hence:

$${}_2F_1'\left[\begin{matrix}b_1+b_2,b_1\\ b_1+b_2\end{matrix}\,\middle|\,\frac{y-x}{1-x}\right]=\left(\frac{1-x}{1-y}\right)^{b_1}\ln\frac{1-x}{1-y}-\left(\frac{1-x}{1-y}\right)^{b_1}{}_2F_1'\left[\begin{matrix}0,b_2\\ b_1+b_2\end{matrix}\,\middle|\,\frac{y-x}{1-x}\right].$$

Since

$$F'(a,b;c;z)_{|a=0}=\sum_{k=1}^{\infty}\frac{d}{da}(a)_k\frac{(b)_kz^k}{(c)_kk!}\bigg|_{a=0}=\sum_{k=1}^{\infty}\frac{(b)_k(k-1)!}{(c)_kk!}z^k=\frac{bz}{c}\,{}_3F_2\left(\begin{matrix}1,1,b+1\\ 2,c+1\end{matrix}\,\middle|\,z\right),\tag{22}$$

we will have

$${}_2F_1'\left[\begin{matrix}0,b_2\\ b_1+b_2\end{matrix}\,\middle|\,\frac{y-x}{1-x}\right]=\frac{b_2(y-x)}{(b_1+b_2)(1-x)}\,{}_3F_2\left(\begin{matrix}1,1,b_2+1\\ 2,b_1+b_2+1\end{matrix}\,\middle|\,\frac{y-x}{1-x}\right).$$

In view of definition (11) of $g_{a,b_1,b_2}(x,y)$ formula (21) transforms into (12).

To estimate the remainder term we will use the ideas from [14]. First, define the Mellin transform $M[g;z]$ of g as the integral $\int_0^\infty t^{z-1}g(t)\,dt$ when it exists or its analytic continuation in z if it fails to exist. Particular cases of Theorems 2.10 and 2.12 from [14] combined and adopted for our situation read

Theorem A. *(See López [14].) Let the locally integrable functions f and h defined on $(0,\infty)$ have the following asymptotic expansions*:

$$\begin{aligned} &f(t)=\sum_{k=0}^{n-1}\frac{B_k}{t^{k+1}}+f_n(t), \quad n=1,2,3,\dots,\ f_n(t)=O\big(t^{-n-1}\big) \text{ as } t\to\infty;\\ &f(t)=O\big(t^{a-1}\big) \quad \text{as } t\to 0,\ a>0;\\ &h(t)=\sum_{k=0}^{n-1}A_kt^k+h_n(t), \quad n=1,2,3,\dots,\ h_n(t)=O\big(t^n\big) \text{ as } t\to 0;\\ &h(t)=O\big(t^{-b}\big) \quad \text{as } t\to\infty,\ b>0, \end{aligned} \tag{23}$$

with some complex constants A_k, B_k. Then

$$\int_0^\infty f(t)h(\varepsilon t)\,dt=\sum_{k=0}^{n-1}\varepsilon^k\Big\{-B_kA_k\log\varepsilon+\lim_{z\to 0}\big(B_kM[h;z-k]+A_kM[f;z+k+1]\big)\Big\}+\int_0^\infty f_n(t)h_n(\varepsilon t)\,dt \tag{24}$$

and

$$\int_0^\infty f_n(t)h_n(\varepsilon t)\,dt=O\big(\varepsilon^{n+1}\log\varepsilon\big) \quad \text{as } \varepsilon\to 0.$$

An application of the integral representation [10, formula 5.8(5)] and a change of variable give ($u=1-x$, $v=1-y$):

$$\begin{aligned} &F_1(a;b_1,b_2;a+b_1+b_2;1-u,1-v)\\ &\quad=u^{-b_1}v^{-b_2}F_1\left(b_1+b_2;b_1,b_2;a+b_1+b_2;1-\frac{1}{u},1-\frac{1}{v}\right)\\ &\quad=\frac{\Gamma(a+b_1+b_2)}{\Gamma(a)\Gamma(b_1+b_2)}\int_0^\infty\frac{t^{a-1}(1+t)^{-a}dt}{(1+ut)^{b_1}(1+vt)^{b_2}}=\frac{\Gamma(a+b_1+b_2)}{\Gamma(a)\Gamma(b_1+b_2)}\int_0^\infty f_a(t)h_{b_1,b_2}(u,v;t)\,dt, \end{aligned} \tag{25}$$

where

$$f_a(t)=t^{a-1}(1+t)^{-a}=\sum_{k=0}^{n-1}(-1)^k\frac{(a)_k}{k!t^{k+1}}+f_{a,n}(t) \quad \text{as } t\to\infty; \tag{26}$$

$$f_a(t)=O\big(t^{a-1}\big) \quad \text{as } t\to 0, \tag{27}$$

and

$$h_{b_1,b_2}(u,v;t)=\frac{1}{(1+tu)^{b_1}(1+tv)^{b_2}}=\sum_{k=0}^{n-1}(-1)^kt^k\sum_{m=0}^{k}\frac{(b_1)_m(b_2)_{k-m}}{m!(k-m)!}u^mv^{k-m}+h_{b_1,b_2,n}(t), \quad t\to 0; \tag{28}$$

$$h_{b_1,b_2}(u,v;t)=O\big(t^{-b_1-b_2}\big), \quad t\to\infty. \tag{29}$$

Representation (25) is not precisely a Mellin convolution. However, if we approach the point $u = v = 0$ (i.e. $x = y = 1$) along straight lines we can put $u = \gamma_1\varepsilon$, $v = \gamma_2\varepsilon$, where γ_1 and γ_2 are non-negative constants at least one of them is strictly positive and $\varepsilon \to 0$. It this case

$$h_{b_1,b_2}(u,v;t) = h_{b_1,b_2,\gamma_1,\gamma_2}(\varepsilon t), \quad \varepsilon = \frac{1}{\gamma_1}(1-x) = \frac{1}{\gamma_2}(1-y),$$

and (25) takes the form of the Mellin convolution on the left-hand side of (24). Since every point (u, v) lies on some straight line segment with endpoint $(1, 1)$ and all our further speculations assume sufficiently small but fixed u, v there are always γ_1, γ_2 and ε (of course non-unique) which are implied. Hence, Theorem A is applicable to the integral representation (25). It is not difficult to compute the Mellin transforms of $f_a(t)$ and $h_{b_1,b_2}(t)$ and the limit in (24). Then, some manipulations with hypergeometric functions similar to those from Remark 4 below show that the expansion (24) applied to (25) and (truncated) expansion (20) are, in fact, the same. However, we do not need these computations since it is sufficient to note that both expansion (24) and (truncated) expansion (20) use the same asymptotic sequences $(1-y)^k$, $(1-y)^k\log(1-y)$ (or $(1-x)^k$, $(1-x)^k\log(1-x)$ if $y = 1$) and hence are identical. It remains to take $n = 1$ in (24) to see that the remainder term defined by (21) can be expressed by the formula

$$R_{a,b_1,b_2}(u,v) = \int_0^\infty f_{a,1}(t)h_{b_1,b_2,1}(u,v;t)\,dt = \int_0^\infty \left[\frac{t^{a-1}}{(1+t)^a} - \frac{1}{t}\right]\left[\frac{1}{(1+ut)^{b_1}(1+vt)^{b_2}} - 1\right]dt. \tag{30}$$

The bound for $R_{a,b_1,b_2}(u,v)$ is based on the following lemma whose proof we postpone until the end of the proof of the theorem.

Lemma 1. *For all $t \in (0,\infty)$ the inequalities*

$$-a/t^2 < f_{a,1}(t) < 0, \tag{31}$$
$$-1/t < f_{a,1}(t) < 0, \tag{32}$$
$$-1 < h_{b_1,b_2,1}(u,v;t) < 0, \tag{33}$$
$$-t(ub_1 + vb_2) < h_{b_1,b_2,1}(u,v;t) < 0 \tag{34}$$

hold true.

The integral in (30) may be decomposed as follows

$$R_{a,b_1,b_2}(u,v) = \int_0^1 f_{a,1}(t)h_{b_1,b_2,1}(u,v;t)\,dt + \int_1^{1/r} f_{a,1}(t)h_{b_1,b_2,1}(u,v;t)\,dt + \int_{1/r}^\infty f_{a,1}(t)h_{b_1,b_2,1}(u,v;t)\,dt,$$

where r can be any positive number (it is not needed that $r < 1$!). Set $r = ub_1 + vb_2$ and use estimates (32) and (34) in the first integral, (31) and (34) in the second and (31) and (33) in the third. This gives the estimate (13). □

Remark 1. We could use Proposition 3.1 from [14] to give an estimate for the error term. However, in our specific situation we are able to derive a much better bound based on Lemma 1 using the method of proof of this proposition but not its statement.

Proof of Lemma 1. (a) Inequality (31). Write $f_{a,1}(t) = g_a(t)/t^2$, where

$$g_a(t) = \frac{t^{a+1}}{(1+t)^a} - t.$$

Then (31) is equivalent to $-a < g_a(t) < 0$. Clearly, $g_a(0) = 0$. It is an easy exercise to check that $g_a(\infty) = -a$. If we prove that $g_a'(t) < 0$ we are done. Differentiating and multiplying both sides by $(1+t)^{a+1}$ we see that the required inequality takes the form

$$(1+a)(1+t)t^a < (1+t)^{a+1} + at^{a+1} \quad \Leftrightarrow \quad \frac{(1+t)^{a+1}}{t^a(1+a+t)} > 1 \quad \Leftrightarrow \quad (1+x)^{a+1} > 1 + (1+a)x,$$

where $x = 1/t$ and the last inequality is the classical Bernoulli inequality valid for $a > 0$ and $x > -1$ [16, formula III(1.2)].

(b) Inequality (32) is proved similarly but simpler.

(c) Inequality (33) is obvious from the definition (28) of $h_{b_1,b_2}(u, v; t)$.

(d) To prove (34) we again apply Bernoulli's inequality [16, formula III(1.2)] in the form ($b_1, b_2 > 0$):

$$(1+tu)^{-b_1} > 1 - b_1 tu, \qquad (1+tu)^{-b_2} > 1 - b_2 tu.$$

Multiplying these two inequalities we get the estimate

$$1 - \frac{1}{(1+tu)^{b_1}(1+tv)^{b_2}} < t(ub_1 + vb_2) - t^2 uvb_1 b_2 \tag{35}$$

which is even stronger than (34). □

Remark 2. An application of (35) instead of (34) in the proof of Theorem 1 leads to a bound for the remainder term which is better than (13). However, numerically it is only a very minor improvement, so we decided to keep the simpler estimate (13) in the theorem.

Remark 3. Expansion [11, formula (53)] can be cast into the form

$$\begin{aligned}
&\frac{\Gamma(b_1+b_2)\Gamma(a)}{\Gamma(a+b_1+b_2)} F_1\left(a; b_1, b_2; a+b_1+b_2; 1-\frac{\gamma_1}{z}, 1-\frac{\gamma_2}{z}\right) \\
&\quad = \sum_{k=0}^{n-1}\left[\frac{D_k(a,b_1,b_2;\gamma_1,\gamma_2)}{z^k} + \log(z)\frac{E_k(a,b_1,b_2;\gamma_1,\gamma_2)}{z^k}\right] + R_n(a,b_1,b_2,\gamma_1,\gamma_2;z).
\end{aligned} \tag{36}$$

Substituting $x = 1 - \gamma_1/z$, $y = 1 - \gamma_2/z$ into (20) we see that both (36) and (20) are asymptotic expansions for $|z| \to \infty$ in the same asymptotic sequences z^{-k}, $z^{-k}\log(z)$ and so their coefficients are the same. Hence, (20) can be viewed as a simpler form of [11, formula (53)]. The appearance of the coefficients D_k and E_k is very different from that of the coefficients of (20) and direct reduction is non-trivial. For instance, the first term of [11, formula (53)] reads (after some simple manipulations) ($F = {}_2F_1$, $M = (1-y)/(1-x)$):

$$\begin{aligned}
&B(a, b_1+b_2) F_1(a; b_1, b_2; a+b_1+b_2; x, y) \\
&\quad = \psi(1) - \psi(a) + \frac{-\ln(1-v) + \ln(M) + \psi(1) - \psi(b_1+b_2)}{b_1+b_2} \\
&\qquad \times \left(Mb_2 F\left[\begin{matrix}1, b_2+1\\ b_1+b_2+1\end{matrix}\middle|1-M\right] + b_1 F\left[\begin{matrix}1, b_2\\ b_1+b_2+1\end{matrix}\middle|1-M\right]\right) \\
&\qquad + \frac{1}{b_1+b_2}\left(Mb_2 F'\left[\begin{matrix}1, b_2+1\\ b_1+b_2+1\end{matrix}\middle|1-M\right] + b_1 F'\left[\begin{matrix}1, b_2\\ b_1+b_2+1\end{matrix}\middle|1-M\right]\right) + R_1.
\end{aligned} \tag{37}$$

Now using the relation [10, formula 2.8(36)]

$$(c-a-b)F(a,b;c;z) - (c-a)F(a-1,b;c;z) + b(1-z)F(a,b+1;c;z) = 0 \tag{38}$$

we immediately get

$$Mb_2 F\left[\begin{matrix}1, b_2+1\\ b_1+b_2+1\end{matrix}\middle|1-M\right] + b_1 F\left[\begin{matrix}1, b_2\\ b_1+b_2+1\end{matrix}\middle|1-M\right] = b_1 + b_2.$$

Differentiating (38) with respect to a and putting $a = 0$ we obtain:

$$(c-b-1)F'(1,b;c;z) + b(1-z)F'(1,b+1;c;z) = F(1,b;c;z) + (c-1)F'(0,b;c;z) - 1.$$

Using (22) we see that

$$\begin{aligned}
&Mb_2 F'\left[\begin{matrix}1, b_2+1\\ b_1+b_2+1\end{matrix}\middle|1-M\right] + b_1 F'\left[\begin{matrix}1, b_2\\ b_1+b_2+1\end{matrix}\middle|1-M\right] \\
&\quad = F\left[\begin{matrix}1, b_2\\ b_1+b_2+1\end{matrix}\middle|1-M\right] + \frac{b_2(b_1+b_2)(1-M)}{(b_1+b_2+1)}\, {}_3F_2\left[\begin{matrix}1, 1, b_2+1\\ 2, b_1+b_2+2\end{matrix}\middle|1-M\right] - 1
\end{aligned}$$

and

$$B(a,b_1+b_2)F_1(a;b_1,b_2;a+b_1+b_2;x,y)$$
$$=\ln\frac{1}{1-x}+2\psi(1)-\psi(a)-\psi(b_1+b_2)+\frac{1}{b_1+b_2}F\left[\begin{matrix}1,b_2\\ b_1+b_2+1\end{matrix}\middle|1-M\right]$$
$$+\frac{b_2(1-M)}{(b_1+b_2+1)}\,{}_3F_2\left[\begin{matrix}1,1,b_2+1\\ 2,b_1+b_2+2\end{matrix}\middle|1-M\right]-\frac{1}{b_1+b_2}+R_1. \tag{39}$$

Finally, (39) is reduced to (11) with the help of the following formula found at http://functions.wolfram.com/07.27.17.0029.01:

$${}_3F_2(a,b,c;a+1,e;z)=\frac{1}{a-e+1}\big[a\,{}_2F_1(b,c;e;z)-(e-1)\,{}_3F_2(a,b,c;a+1,e-1;z)\big].$$

Recalling that $M=(1-y)/(1-x)$ we get (11). The direct reduction for further terms is even more complicated.

Theorem 2. *The following properties are true*:

1. *The function g is permutation symmetric*:
$$g_{a,b_1,b_2}(x,y)=g_{a,b_2,b_1}(y,x). \tag{40}$$
2. *For $y=1$ (and $x=1$ due to (40)) the function $g_{a,b_1,b_2}(x,y)$ reduces to Ramanujan's approximation*:
$$g_{a,b_1,b_2}(x,1)=\ln\frac{1}{1-x}+2\psi(1)-\psi(a)-\psi(b_1), \tag{41a}$$
$$g_{a,b_1,b_2}(1,y)=\ln\frac{1}{1-y}+2\psi(1)-\psi(a)-\psi(b_2). \tag{41b}$$
3. *For $x=y$ the function $g_{a,b_1,b_2}(x,y)$ again becomes Ramanujan's approximation*:
$$g_{a,b_1,b_2}(x,x)=\ln\frac{1}{1-x}+2\psi(1)-\psi(a)-\psi(b_1+b_2). \tag{42}$$
4. *For the values of parameters $a=b_1=b_2=1/2$ we have*
$$g_{1/2,1/2,1/2}(x,y)=2\ln\frac{4}{\sqrt{1-x}+\sqrt{1-y}}, \tag{43}$$
which is the approximation of Carlson–Gustafson [9].

Proof. To prove the first statement we need the following elementary lemma:

Lemma 2. *For $b\neq 1$ the following relation holds true*:

$${}_3F_2\left(\begin{matrix}1,b,c\\ 2,e\end{matrix}\middle|\frac{z}{z-1}\right)=\frac{(1-z)^b(c-e)}{c-1}\,{}_3F_2\left(\begin{matrix}1,b,e-c+1\\ 2,e\end{matrix}\middle|z\right)+\frac{(e-1)(1-z)(1-(1-z)^{b-1})}{(c-1)(b-1)z}. \tag{44}$$

For $b=1$ it reduces to

$${}_3F_2\left(\begin{matrix}1,1,c\\ 2,e\end{matrix}\middle|\frac{z}{z-1}\right)=\frac{(z-1)(e-c)}{c-1}\,{}_3F_2\left(\begin{matrix}1,1,e-c+1\\ 2,e\end{matrix}\middle|z\right)+\frac{(e-1)(1-z)}{(c-1)z}\ln\frac{1}{1-z}. \tag{45}$$

Proof. The proof is based on the following easily verifiable relation (which can be also found at http://functions.wolfram.com/07.27.03.0120.01):

$${}_3F_2\left(\begin{matrix}1,b,c\\ 2,e\end{matrix}\middle|z\right)=\frac{e-1}{(b-1)(c-1)z}\left[{}_2F_1\left(\begin{matrix}b-1,c-1\\ e-1\end{matrix}\middle|z\right)-1\right]. \tag{46}$$

To prove (44) write this relation for $z/(z-1)$ in place of z, apply Pfaff's transformation

$${}_2F_1\left(\begin{matrix}b-1,c-1\\ e-1\end{matrix}\middle|\frac{z}{z-1}\right)=(1-z)^{b-1}\,{}_2F_1\left(\begin{matrix}b-1,e-c\\ e-1\end{matrix}\middle|z\right)$$

and substitute ${}_2F_1$ from the right-hand side by ${}_2F_1$ expressed from (46). To prove (45) let b tend to 1 and apply the L'Hopital rule. □

Combining (45) with the definition (11) of $g_{a,b_1,b_2}(x,y)$ we immediately obtain (40).

Next we check the behavior of the function $g_{a,b_1,b_2}(x,y)$ on the sides of the square $|x|<1$, $|y|<1$. Writing (46) for $z=1$ and using the Gauss formula for ${}_2F_1(1)$ we get

$$\begin{aligned}{}_3F_2\left(\begin{matrix}1,b,c\\2,e\end{matrix}\,\middle|\,1\right)&=\frac{e-1}{(b-1)(c-1)}\left[{}_2F_1\left(\begin{matrix}b-1,c-1\\e-1\end{matrix}\,\middle|\,1\right)-1\right]\\&=\frac{e-1}{(b-1)(c-1)}\left[\frac{\Gamma(e-1)\Gamma(e-b-c+1)}{\Gamma(e-b)\Gamma(e-c)}-1\right].\end{aligned}$$

Now let $b\to1$ and use the L'Hopital rule:

$$\begin{aligned}{}_3F_2\left(\begin{matrix}1,1,c\\2,e\end{matrix}\,\middle|\,1\right)&=\frac{(e-1)\Gamma(e-1)}{(c-1)\Gamma(e-c)}\frac{d}{db}\frac{\Gamma(e-b-c+1)}{\Gamma(e-b)}\bigg|_{b=1}\\&=\frac{\Gamma(e)}{(c-1)\Gamma(e-c)}\frac{-\Gamma(e-c)\psi(e-c)\Gamma(e-1)+\Gamma(e-1)\psi(e-1)\Gamma(e-c)}{[\Gamma(e-1)]^2}\\&=\frac{(e-1)}{(c-1)}\big(\psi(e-1)-\psi(e-c)\big).\end{aligned}$$

Substituting $e=b_1+b_2+1$, $c=b_2+1$ gives (41).

Identity (42) is obvious from the definition (11) of $g_{a,b_1,b_2}(x,y)$.

Finally, formula (43) follows from the reduction formula

$${}_3F_2\left(\begin{matrix}1,1,3/2\\2,2\end{matrix}\,\middle|\,z\right)=-\frac{4}{z}\ln\left(\frac{1}{2}+\frac{\sqrt{1-z}}{2}\right).$$

This completes the proof of the theorem. □

Corollary 2.1. *For $x,y\to1$*

$$f_{a,b_1,b_2}(x,y)=\ln\frac{1}{1-xy}+O(1). \tag{47}$$

Proof. Assume first that x and y approach $(1,1)$ in a way such that $(1-y)/(1-x)$ remains bounded. We have

$$\ln\frac{1}{1-xy}=\ln\frac{1}{1-x+x-xy}=\ln\frac{1}{(1-x)(1+x\frac{1-y}{1-x})}=\ln\frac{1}{1-x}+\ln\frac{1}{1+x\frac{1-y}{1-x}}.$$

Hence,

$$\begin{aligned}&\ln\frac{1}{1-xy}-g_{a,b_1,b_2}(x,y)\\&\quad=\ln\frac{1}{1+x\frac{1-y}{1-x}}-\gamma(a,b_1+b_2)-\frac{b_2(y-x)}{(b_1+b_2)(1-x)}\,{}_3F_2\left(\begin{matrix}1,1,b_2+1\\2,b_1+b_2+1\end{matrix}\,\middle|\,\frac{y-x}{1-x}\right)=O(1).\end{aligned}$$

If $(1-y)/(1-x)$ is unbounded, then exchange the roles of x and y and use (40). □

Remark 4. Finally, we remark that the authors of [1,2] consider monotonicity and ranges of the functions

$$\frac{1-{}_2F_1(a,b;a+b;x)}{\ln(1-x)},\qquad\frac{x\,{}_2F_1(a,b;a+b;x)}{\ln(1/(1-x))}$$

and

$$B(a,b)\,{}_2F_1(a,b;a+b;x)+\ln(1-x)$$

for $x \in (0,1)$. Our Corollary 2.1 shows that similar problems can be considered for the combinations

$$\frac{1 - F_1(\alpha; \beta_1, \beta_2; \alpha + \beta_1 + \beta_2; x, y)}{\ln(1 - xy)},$$

and

$$f_{\alpha,\beta_1,\beta_2}(x, y) - \ln \frac{1}{1 - xy}$$

for $x, y \in (0,1)$.

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