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Fine Topology, Šilov Boundary, and $(dd^c)^n$

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1. INTRODUCTION

In classical potential theory, the fine topology provides a useful and natural way to give sharp statements of many results. The same is true for the case of the potential theory associated with plurisubharmonic (psh) functions. In Section 2 we give a short discussion of the (pluri-) fine topology of psh functions. Almost all the results are the same as for the classical fine topology, even with the same proof, so we have omitted all proofs in this section. However, there is one crucial difference—the notions of "thin set" and "polar set" are not equivalent in psh potential theory.

The use of the fine topology allows us to give sharp statements of some convergence theorems proved in [BT]. For example, if u_j is a uniformly bounded sequence of psh functions which converges monotonically a.e. to a psh function u, then $(dd^cu_j)^n \to (dd^cu)^n$ weak* in the fine topology. That is, if ψ is a bounded, fine continuous function with compact support, then $\int \psi(dd^cu_j)^n \to \int \psi(dd^cu)^n$. A more refined version of this result is given in Section 3, where it is also shown that a version of "balayage" is possible for fine closed sets (Corollary 3.4).

In Section 4, we discuss the definition of $(dd^c)^n$. The operator $(dd^c)^n$ converges for monotone limits of bounded, psh functions, and this may be used to justify the extension of $(dd^c)^n$ from smooth, psh functions (cf. [BT]). (We note that $(dd^c)^n$ does not behave well under nonmonotone limits, as was shown by Cegrell [Ce] and Lelong [L].) For bounded, psh u, Oberguggenberger [O] has shown that if one computes the exterior product $dd^cu \wedge \cdots \wedge dd^cu$, using the algebra of distributions of Colombeau, one obtains the same $(dd^c)^n$ as before. Here we shown that this coin-

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cides also with the definition proposed by Kiselman [K]. For unbounded psh u, $(dd^c u)^n$ may put infinite mass on a single point. Let us note, however, that Sibony [Sb] has shown that under certain hypotheses on the set $\{u = -\infty\}$, $(dd^c u)^n$ will be a regular Borel measure. With the fine topology, it is natural to define the "nonpolar part" of $(dd^c)^n$, which extends the definition for bounded functions. For $\alpha = n - 1 + \varepsilon$ there is also the operator

$$\tilde{M}v = (-v)^{-\alpha}(dd^c v)^n,$$

which is shown to be well defined for all negative, psh functions. Finally, it is shown that the nonpolar part of $(dd^c v)^n$ is the same as $(-v)^\alpha \tilde{M}(v)$.

The fine topology also enters in the study of the Domination Principle for envelopes of psh functions. Let E be a subset of a bounded, strictly pseudoconvex domain Ω in $\mathbb{C}^n$, and define

$$U_E^*(z) = \limsup_{\zeta \to z} U_E(\zeta),$$

where

$$U_E(z) = \sup\{v(z): v \text{ psh}, v < 0 \text{ on } \Omega, v \leqslant -1 \text{ on } E\}.$$

We showed in [BT] that the nonnegative Borel measure $\lambda_E = (dd^c U_E^*)^n$ is supported on $\bar{E}$ and is, in many ways, an analogue of the equilibrium measure in classical potential theory. The associated relative (or condenser) capacity is given by

$$C(E, \Omega) = \int (dd^c U_E^*)^n$$

(see [BT]). In Section 5 we consider the question of when two sets $F \subset E \subset \Omega$ have the same capacity. The main result, Theorem 5.1, is that the capacities are the same if and only if

$$\lambda_E(E^f - F^f) = 0,$$

where E^f denotes the fine closure of E. This then expresses the principle that, in terms of the equilibrium measure "full measure implies full capacity."

In Section 6, we obtain corresponding results for the generalized Green function in $\mathbb{C}^n$, which is closely related to the study of polynomials in $\mathbb{C}^n$. We give a necessary and sufficient condition for a family of polynomials to satisfy the "Leja polynomial condition."

Section 7 concerns the problem of characterizing the support and the fine support of the measure λ_E for a subset $E \subset \Omega$. From Section 5, it follows that if E is compact, then the support of λ_E is the Shilov boundary of the function algebra $A_E(\Omega)$ (defined in Sect. 7). For fine closed E, the fine support of λ_E is shown to be the fine Shilov boundary of the base of E.

We wish to thank Professor Siciak for helpful comments on the material in Section 6.

2. The (Pluri-) Fine Topology

The pluri-fine topology on an open set Ω in $\mathbb{C}^n$ is the smallest topology on Ω for which all the psh functions are continuous. The fine closure of a set $A \subset \Omega$ will be denoted by A^f. We will omit the adjective "pluri-" unless some other fine topology is also being discussed. In this section we will outline a few basic facts, usually without proof. Doob has pointed out [D, p. 800] that for several years the fine topology in classical potential theory was "merely a tool for phrasing results elegantly."

Almost all the facts we give about the fine topology can be proved exactly as in the case of classical potential theory. We shall omit all such proofs. An excellent discussion of the classical case is given in [D, Chap. 1 XI], and also in [B].

We first discuss the notion of *thinness* and give the analogues of Cartan's theorems relating thinness and the fine topology. It is with the notion of thinness where the difference between the pluri-fine and fine topologies arise. Namely, in the classical case the thin sets coincide with the polar sets. But, that is not the case here. Thin sets are polar, but not conversely. The study of thin sets appears to be very delicate. Sadullaev [SA] has given a discussion of thinness and provided many very interesting examples and open questions concerning this notion.

Definition 2.1. A subset E of Ω is *thin at* $p \in \Omega$ if and only if either $p \notin \bar{E}$ or there is an open set $\sigma \in p$ and a psh function u on σ such that

$$\limsup_{z \to p, z \in E} u(z) < u(p). \tag{2.1}$$

A subset E is *thin* if it is thin at every point $p \in \Omega$.

As in the classical case, a basic fact is that the "jump" in (2.1) can be infinite if E is thin at p.

Proposition 2.2. *The set E is thin at p if and only if there exists a psh function v on $\mathbb{C}^n$, locally bounded outside a neighborhood of p, such that $v(p) > -\infty$ and*

$$\limsup_{z \to p, z \in E} v(z) = -\infty.$$

This is Proposition 10.2 of [BT], except for the observation that v can be a global psh function. However, the proof given there easily shows that v can be a global function, even with $v(z)=O(\log|z|)$ as $z\to\infty$.

Since polar sets are not necessarily thin, it is not possible to give a necessary and sufficient criterion for thinness in terms of capacity. The unregularized functions U_K, however, may be used to give something analogous to the Wiener Criterion (cf. [D, p. 249]), where the specific values $r_j=\alpha^j$ are used, for some $\alpha\in(0,1)$.

THINNESS CRITERION. *Let $A\subset\Omega\subset\subset\mathbb{C}^n$ be given, and let z_0 be a point of $\bar{A}\cap\Omega$. Then A is thin at z_0 if and only if there exist $r_1>r_2>r_3>\cdots$, such that*

$$\sum_{j=1}^{\infty} U_{A_j}(z_0)>-\infty, \tag{2.2}$$

where $A_j=A\cap\{r_j>|z|\geqslant r_{j+1}\}$, and $U_{A_j}=U^{\Omega}_{A_j}$ is the relative (unregularized) extremal function.

Proof. If (2.2) holds, then there exist $v_j\in P(\Omega)$, $v_j<0$, with $v_j\leqslant -1$ on A_j and

$$v_j(z_0)>U_{A_j}(z_0)-2^{-j-1}.$$

It follows, then, that $V_J=\sum_{j=J}^{\infty}V_j\in P(\Omega)$, and $2^{-J}+\sum_{j=J}^{\infty}U_{A_j}(z_0)<V_J(z_0)$. If J is chosen large enough, then $V_J(z_0)>-1$ but

$$\limsup_{\substack{\zeta\to z_0\\ \zeta\in A}} V_J(\zeta)\leqslant -1,$$

and so A is thin at z_0.

Conversely, if A is thin at z_0, we let $v\in P(\Omega)$ be the function given by Proposition 2.2. Without loss of generality, we may assume that $v<0$. Now let us choose $r_1>r_2>r_3>\cdots$, such that

$$\sup_{\{|\zeta|<r_j\}\cap A} v(\zeta)<-2^j.$$

It follows, then, that

$$U_{A_j}(z_0)\geqslant 2^{-j}v(z_0)$$

and so (2.2) holds.

An important consequence of Proposition 2.2 is the following description of a basis for the fine topology on Ω. Thus a fine open set must have positive Lebesgue measure, although it need not have Euclidean interior.

THEOREM 2.3. *If $p \in \Omega$, a neighborhood base for the fine topology at p is given by those sets containing p whose complement is thin at p, i.e., by sets of the form* $\{z\colon |z-p|<\varepsilon,\ u(z)>0\}$.

See, e.g., [D, p. 166–169] or [B, p. 3] for the proof of similar facts. Also, one can find there proofs of Cartan's theorem relating fine limits and ordinary limits (Theorem 2.4).

One consequence of Theorem 2.3 is that the definition of the fine-open sets is independent of Ω. That is, if $\mathscr{T}_f(\Omega)$ is the fine topology on Ω, then a subset S of Ω belongs to $\mathscr{T}_f(\Omega)$ if and only if $S = T \cap \Omega$ for some $T \in \mathscr{T}_f(\mathbb{C}^n)$. Further, S is a fine-open subset of Ω if and only if it is locally fine-open.

THEOREM 2.4. *If a real valued function f on a subset E of Ω has a fine limit λ at p, then there is a fine neighborhood V of p such that the ordinary limit*

$$\lim_{\substack{z \to p \\ z \in V \cap E}} f(z) = \lambda$$

exists at p.

A slight variation on Theorem 2.3 is that we can also use a neighborhood base of the fine topology of the form

$$\{z\colon |z-p| \leqslant \varepsilon,\ u(z) \geqslant 0\}.$$

Since these sets are compact, we may argue as on p. 167 of [D] to obtain the *Baire property* for the fine topology: *if $\mathcal{O}_1, \mathcal{O}_2, \ldots,$ are fine-open and fine-dense, then $\bigcap \mathcal{O}_j$ is fine-dense.*

We now consider more global properties of the fine topology. The resuls depend, for the most part, on the equivalence of negligible sets and pluripolar sets.

DEFINITION 2.5. A subset E of Ω is said to be (pluri-) *polar* if for each $p \in E$, there is a neighborhood $\mathcal{O}$ of p and a psh function u on $\mathcal{O}$, not identically $-\infty$, such that $\mathcal{O} \cap E \subset \{u = -\infty\}$. The set E is (pluri-) *negligible* if there is a locally bounded family $\{u_\alpha : \alpha \in A\}$ of psh functions such that $E \subset \{u^* > u\}$, where $u(z) = \sup\{u_\alpha(z) \colon \alpha \in A\}$ is the upper envelope of the family and $u^*(z) = \limsup_{\zeta \to z} u(\zeta)$ is the upper semicontinous regularization of u.

It is a basic theorem, due to Josefson [J], that polar sets can be given by global functions. That is, if E is polar, then there exists u psh on $\mathbb{C}^n$, not

$\equiv -\infty$, such that $E \subset \{u = -\infty\}$. Another basic fact for the study of the fine topology is given by Theorem 7.1 and Proposition 6.3 of [BT].

THEOREM 2.6. *A set $E \subset \Omega$ is negligible if and only if it is polar.*

A direct consequence of this Theorem is the following property of the fine topology (see [D, p. 181] for the proof).

THEOREM 2.7 (Quasi–Lindelöf property). *An arbitrary union of fine-open subsets differs from a countable subunion by at most a polar set.*

For any $E \subset \mathbb{C}^n$, we may define the *base* of E to be

$$b(E) = \{z \in E: E - P \text{ is not thin at } z \text{ for any polar } P\}.$$

It may also be seen that

$$b(E) = \bigcap \{E': E' \text{ is } f\text{-closed and } E \setminus E' \text{ is polar}\}. \tag{2.3}$$

It follows easily that $b(E)$ is f-closed, and from Theorems 2.6 and 2.7 it follows that $E \setminus b(E)$ is polar. Thus $\supseteq$ holds in (2.3). The inequality $\subseteq$ follows from Theorem 2.4.

Let $C(E)$ denote the capacity used in Section 3.

DEFINITION 2.8. A function f on Ω is called *quasi-continuous* if for each $\varepsilon > 0$, there is an open subset $\mathcal{O}$ of Ω such that $C(\mathcal{O}) < \varepsilon$ and f is continuous on $\Omega \setminus \mathcal{O}$ (for the Euclidean topology).

An important property of the capacity is the following analogue of Cartan's theorem for subharmonic functions (Theorem 3.5 of [BT]).

THEOREM 4.9. *Plurisubharmonic functions are quasi-continuous.*

3. FINE CONVERGENCE OF $(dd^c)^n$

We want to discuss integrals such as $\int_E (dd^c u)^n$ where E is a fine open or closed set, but not all fine-open or closed sets are Borel sets. However, it is clear that the Borel measure $(dd^c u)^n$ associated to a bounded psh function u has a natural extension to the σ-algebra of "quasi-Borel" sets

$$\text{QB} = \sigma\text{-algebra generated by the Borel sets and the pluripolar sets}$$

since the measure $(dd^c u)^n$ puts zero mass on each pluripolar set. Thus, QB is contained in the σ-algebra associated with the completion of the measure. It can be verified that QB consists exactly of sets of the form $B \setminus E$ where B is a Borel set and E is a polar set.

PROPOSITION 3.1. *The σ-algebra* QB *contains the fine-Borel sets; i.e., the σ-algebra generated by the fine open sets.*

Proof. By the quasi-Lindelöf property, each open set $\mathcal{O}$ can be written, up to a polar set, as a countable subunion of such sets. $B \cap \{u > 0\}$. However, each of the basic open sets is a Borel set. Thus, all fine-open sets are in QB.

THEOREM 3.2. *Let $\{T_j\}$ denote a sequence of positive currents of bidegree (k, k) such that $T_j \to T$ on $\Omega \subset \mathbb{C}^n$, in the weak topology on the space of currents. Then the following are equivalent (where convergence is in the sense of currents of order* 0).

(1) *T has zero mass on any polar set and $uT_j \to uT$ for every locally bounded psh function u on Ω;*

(2) *T has zero mass on any polar set and $\psi T_j \to \psi T$ for every bounded, quasi-continuous function ψ on Ω;*

(3) *$\psi_j T_j \to \psi T$ for each uniformly bounded sequence of quasi-continuous functions $\{\psi_j\}$ which converge monotonically, either increasing or decreasing, to ψ quasi-everywhere;*

(4) *the sequence T_j puts uniformly small mass on sets of small capacity; i.e., if $\omega \Subset \Omega$, then for any sequence $\mathcal{O}_j$ of open subsets of ω, with* $\lim_{j \to \infty} C(\mathcal{O}_j) = 0$, *we have*

$$\lim_{j \to \infty} \sup\{|T|(\mathcal{O}_j), |T_k|(\mathcal{O}_j): k = 1, 2, ...\} = 0.$$

Proof. That $(3) \Rightarrow (2) \Rightarrow (1)$ is trivial. We will next show that $(1) \Rightarrow (4)$. Assume not. Then we can find an open set $\Subset \Omega$, open sets $\mathcal{O}_j$, and a number $\delta > 0$ such that $|T_j|(\mathcal{O}_j) = \int_{\mathcal{O}_j} \beta_{n-k} \wedge T_j \geq \delta > 0$ and $C(\mathcal{O}_j) \leq 2^{-j}$. Here $\beta_j = (4^{-j}/j!)\ (dd^c|z|^2)^j$. The assertions (1) and (4) are local, so it is no loss of generality to assume that $\omega = \{|z| \leq r\} \subset \{|z| < 1\} = \Omega$. Consider the extremal functions $u_j = U^*_{\mathcal{O}_j}$ for $\mathcal{O}_j$ relative to the unit ball Ω. Since $C(\mathcal{O}_j) \to 0$, the bounded functions $u_j \to 0$ almost everywhere in Ω. Replacing $\mathcal{O}_j$ by $\bigcup_{l \geq j} \mathcal{O}_l$ if necessary, we can assume that $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots$, so that $u_2 \leq u_2 \leq \cdots$. Further, $u_j \to 0$ almost everywhere, hence also pointwise outside a set E of capacity zero; i.e., a polar set. It then also follows that $u_j \to 0$ locally in L^1. Hence, we can further assume that $\sum |u_j|$ converges locally in L^1.

For each $l = 1, 2, ...,$ let $\tilde{v}_l = \sum_{j \geq l} u_j$ and $v_l = \max(\tilde{v}_l, -1)$. Then $\tilde{v}_l, v_l$ are psh on Ω, $v_1 \leq v_2 \leq \cdots$, and $v_l \to 0$ locally in L^1. It therefore follows that

$V_l(z) \to 0$, except possibly on a set E of capacity zero (see, e.g., Theorem 7.2 of [BT]). Thus, if $V = \sup v_l$ then $V\beta_{n-k} \wedge T = 0$. But, from the bounded convergence theorem, $v_l \beta_{n-k} \wedge T = 0$. Thus, since $v_l \cdot \beta_{n-k} \wedge T$ is a negative Borel measure, we can choose l so large that

$$-\frac{\delta}{2} < \int_{\omega} v_l \beta_{n-k} \wedge T \leqslant 0.$$

For this fixed value of l, we apply the hypothesis (1) to the psh function v_l. Then

$$-\frac{\delta}{2} < \lim_{j \to \infty} \int_{W} v_l \beta_{n-k} \wedge T_j \leqslant \lim_{j \to \infty} \int_{\mathcal{O}_j} u_j \beta_{n-k} \wedge T_j$$

$$\leqslant -\lim_{j \to \infty} \int_{\mathcal{O}_j} \beta_{n-k} \wedge T_j \leqslant -\delta,$$

which is a contradiction. Thus, (1) ⇒ (4) is proved.

It remains to show that (4) ⇒ (3). The capacity is subadditive, $C(UE_j) \leqslant \Sigma C(E_j)$. Therefore, if the ψ_j, ψ are quasicontinuous and $\psi \to \psi$ monotonically, quasi-everywhere, then for each $\varepsilon > 0$, there exists an open set $\mathcal{O}$ with $C(\mathcal{O}) < \varepsilon$, ψ_j, ψ continuous on $\Omega \backslash \mathcal{O}$, and $\psi_j \to \psi$ monotonically on $\Omega \backslash \mathcal{O}$. It then follows from Dini's theorem that $\psi_j \to \psi$ uniformly on compact subsets of $\Omega \backslash \mathcal{O}$. Thus we can write $\psi_j = \tilde{\psi}_j + \eta_j$, $\psi = \tilde{\psi} + \eta$, where $\tilde{\psi}_j$, $\tilde{\psi}$ are continuous on Ω, $|\tilde{\psi}_j| \leqslant \sup|\tilde{\psi}_j| \leqslant M$, η_j, $\eta = 0$ on $\Omega \backslash \mathcal{O}$, and $\tilde{\psi}_j \to \tilde{\psi}$ uniformly on $\Omega \backslash \mathcal{O}$. Thus, if φ is any continuous $(n-k, n-k)$ form with compact support in Ω, then

$$\limsup_{j \to \infty} \left| \int \varphi \wedge (\psi_j T_j - \psi T) \right|$$

$$\leqslant \limsup_{j \to \infty} \left| \int_{\Omega} \varphi \wedge ((\tilde{\psi}_j - \tilde{\psi})\, T_j \right|$$

$$+ \left| \int_{\Omega} \varphi \wedge (\tilde{\psi}(T_j - T)) \right| + \left| \int_{\Omega} \varphi \wedge (\eta_j T_j - \eta T) \right|.$$

Because $\tilde{\psi}_j \to \tilde{\psi}$ uniformly on $[\Omega \backslash \mathcal{O}] \cap [\text{support } \varphi]$, the first term in the last expression does not exceed

$$\limsup_{j \to \infty} \left| \int_{\mathcal{O}} \varphi \wedge (\tilde{\psi}_j - \tilde{\psi})\, T_j \right| \leqslant C \cdot \limsup_{j} |T_j|(\mathcal{O}),$$

where C is a constant depending on a maximum of ψ_j, ψ, and φ. This term thus tends to zero as $C(\mathcal{O}) \to 0$, by hypothesis 4). The second term $\to 0$, because $T_j \to T$ weakly as currents on Ω and φ, $\tilde{\psi}$ are continuous on Ω. The integrand in the third term is uniformly bounded and equal to zero off of $\mathcal{O}$. Hence, the third term does not exceed $\text{const}(|T_j|(\mathcal{O}) + |T|(\mathcal{O}))$, which also tends to zero as $C(\mathcal{O}) \to 0$. Thus we have $\psi_j T_j \to \psi T$, as asserted. This completes the proof.

Let $\{u_j^0\}, \ldots, \{u_j^n\}$, $j = 1, 2, \ldots$, denote $n+1$ sequences of psh functions on Ω. In [BT], the following types of currents were considered.

$$\begin{aligned} S_j &= dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n, \\ T_j &= u_j^0 \, dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n = u_j^0 s_j, \\ U_j &= du_j^0 \wedge d^c u_j^1 \wedge \cdots \wedge dd^c u_j^n. \end{aligned} \tag{3.1}$$

Suppose that the sequences u_j^k are uniformly bounded and converge monotonically almost everywhere, either increasing or decreasing, to psh functions u^k, $k = 0, 1, \ldots,$. Let S, T, U denote the corresponding currents with u_j^k replaced by u^k. We then have the following corollary of Theorem 3.2.

COROLLARY 3.3. *Let $\{X_j\}$, X denote any of the 3 sequences of currents S_j, S, etc. of (3.1). If $u_j^0 \geq 0$, then we have:*

(a) *for any fine-open subset $\mathcal{O}$ of Ω,*

$$\int_{\mathcal{O}} X \leq \liminf_{j \to \infty} \int_{\mathcal{O}} x_j,$$

(b) *for any fine-closed subset F of Ω,*

$$\int_F X \geq \limsup_{j \to \infty} \int_F X_j.$$

Proof. This follows directly from Theorem 3.2, since the characteristic function of a fine-open (fine-closed) set is finely lower semicontinuous (upper semicontinuous), the currents are all nonnegative, and satisfy (4) of Theorem 3.2.

Remark. The convergence theorems for the complex Monge–Ampère operator proved in [BT] showed that the currents involved converged in a stronger sense than the usual weak topology. Theorem 3.2 shows that the convergence yields "weak convergence in fine-topology," and it results entirely from the fact that the currents involved satisfy condition (4); they put small mass on sets of small capacity. This also allows us to perform a "balayage" with respect to fine-closed sets.

COROLLARY 3.4. *Let $\mathcal{O} \subset \Omega$ be a fine open set, and let $u \in P(\Omega)$ be locally bounded. If*

$$\tilde{u}_{\Omega - \mathcal{O}} = (\sup\{v \in P(\Omega): v \leqslant u \text{ on } \Omega - \mathcal{O}\})^*$$

then $(dd^c \tilde{u}_{\Omega - \mathcal{O}})^n = 0$ on $\mathcal{O}$.

Proof. By the quasi-Lindelöf property, we may assume that $\mathcal{O}$ is a Borel set (since no mass is put on a polar set). Thus we may take open sets $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots$ such that $\tilde{u}_{\Omega - \mathcal{O}_j}$ increases to $\tilde{u}_{\Omega - \mathcal{O}}$ q.e. Since $(dd^c \tilde{u}_{\Omega - \mathcal{O}_j})^n$ puts no mass on $\mathcal{O}$, the result follows from Corollary 3.3.

COROLLARY 3.5. *If $E \subset \Omega$, then $(dd^c U_E^*)^n$ puts zero mass on the fine interior of $\Omega - E$, i.e., on ${}^c(E^f)$.*

4. DEFINITION OF $(dd^c)^n$

Recall that the extension of dd^c from smooth psh functions to the general case is made by continuity: if u_j is any sequence of smooth psh functions which converge to a psh function u in some weak topology, say locally in L^1, then $dd^c u_j \to dd^c u$ as currents. However for the case of the higher exterior powers, $(dd^c u)^k$, $k > 1$ the situation becomes more complicated for two reasons. First, the mass of the current $(dd^c u)^k$ need not be locally bounded; the paper of Kiselman [K] gives an excellent discussion of this. In particular, a function can put infinite mass on a single point, e.g., if we set

$$u(z, w) = \sum_{n=1}^{\infty} 2^{-n} \log(|z|^2 + |w|^{2^{2n}}),$$

then $(dd^c u)^2$ puts infinite mass at the origin. Second, the operator $u \to (dd^c u)^k$ is badly discontinuous for the usual topologies on the space of psh functions. This was proved by Cegrell [Ce]. And, by modifying Cegrell's technique, Lelong [L] showed the following remarkable fact: Given u psh on $|z| < 1$, $0 < u < 1$, there exists a sequence of psh functions $\{u_j\}$ such that $0 \leqslant u_j < 1$, $u_j \to u$ in L^1, but $(dd^c u_j)^2 = 0$.

On the other hand, for bounded psh functions, there is a good definition of $(dd^c u)^k$ as a positive current of bidegree (k, k). The estimate of Chern, Levine, and Nirenberg [CLN] shows that the mass in $(dd^c u)^k$ is locally bounded by const.$(\sup|u|)^k$. And it was proved in [BT] that the operator $(dd^c)^k$ is continuous under bounded, monotone limits. Sibony [Sb] has given conditions for $(dd^c u)^n$ to have finite mass, in which case $(dd^c u)^n$ may be defined by decreasing limits.

In this section, we show how we may define the *nonpolar part of* $(dd^c)^n$, i.e., what the measure $(dd^c u)^n$ must be when restricted to the set $\{u > -\infty\}$. However, we do not mean for this terminology to imply that it is always possible to define $(dd^c u)^n$ on the set $\{u = -\infty\}$.

LEMMA 4.1. *Let $\Omega \subset \mathbb{C}^n$ be open, and let $\mathcal{O} \subset \Omega$ be fine open. Let $\{u_j\}$ (resp. $\{v_j\}$) be a sequence of bounded, psh functions converging monotonically to $u \in P(\Omega) \wedge L^\infty(\Omega, \text{loc})$ (resp. v). If*

$$(dd^c u_j)^n|_{\mathcal{O}} = (dd^c v_j)^n|_{\mathcal{O}}$$

then

$$(dd^c u)^n)|_{\mathcal{O}} = (dd^c v)^n|_{\mathcal{O}},$$

i.e., the measures agree on measurable subsets of $\mathcal{O}$.

Proof. By the quasi-Lindenlöf property, we may write $\mathcal{O}$ (modulo a polar set) as a countable union of sets of the form

$$B \cap \{\psi > 0\},$$

where B is an open ball, $B \Subset \Omega$, and $\psi \in P(B)$. Since polar sets have measure zero, it suffices to prove the result for $\mathcal{O} = B \cap \{\psi > 0\}$. Obviously there is a fine continuous function $\tilde{\psi}$ with compact support in Ω such that

$$\{z \in \Omega: \tilde{\psi}(z) > 0\} = \mathcal{O}.$$

By Theorem 3.1,

$$\lim_{j \to \infty} \int \tilde{\psi}(dd^c u_j)^n = \int \tilde{\psi}(dd^c u)^n.$$

This holds also if $\tilde{\psi}$ is replaced by $f\tilde{\psi}$ for any continuous f. We conclude that

$$\int f\tilde{\psi}(dd^c u)^n = \int f\tilde{\psi}(dd^c v)^n$$

holds for all $f \in C(\Omega)$. Thus $(dd^c u)^n - (dd^c v)^n$ vanishes on $\{\tilde{\psi} > 0\}$, which proves the lemma.

PROPOSITION 4.2. *Let u, $v \in P(\Omega) \cap L^\infty(\Omega, \text{loc})$ be given, and let $\mathcal{O} = \{u > v\}$. Then*

$$(dd^c \max(u, v))^n|_{\mathcal{O}} = (dd^c u)^n|_{\mathcal{O}},$$

i.e., the measures coincide on subsets of $\mathcal{O}$.

Proof. If $\mathcal{O}$ is open, then the Proposition is obvious. Let $\{u_k\}$ be a sequence of smooth, psh functions decreasing to u, and set

$$\mathcal{O}_k = \{u_k > v\}.$$

Since $\mathcal{O} = \cap \mathcal{O}_k$ is fine open, and since $(dd^c \max(u_k, v))^n|_{\mathcal{O}} = (dd^c u_k)^n|_{\mathcal{O}}$ the corollary follows from Lemma 4.1.

COROLLARY 4.3. *Let $\Omega \subset \mathbb{C}^n$ be open, and let $u, v \in P(\Omega)$ be locally bounded. If $\mathcal{O} \subset \Omega$ is fine open, and if $u = v$ on $\mathcal{O}$, then*

$$(dd^c u)^n|_{\mathcal{O}} = (dd^c v)^n|_{\mathcal{O}}.$$

Proof. Since $u = \max(u, v - \varepsilon)$ holds on $\mathcal{O}$, we have

$$(dd^c u)^n|_{\mathcal{O}} = (dd^c \max(u, v - \varepsilon))^n|_{\mathcal{O}}.$$

Thus the Corollary follows from Lemma 4.1.

DEFINITION. If $u \in P(\Omega)$, the *nonpolar part* of $(dd^c u)^n$, $NP(dd^c u)^n$, is the measure which is zero on $\{u = -\infty\}$, and for a Borel set $E \subset \{u > -\infty\}$,

$$\int_E NP(dd^c u)^n = \lim_{j \to \infty} \int_{E \cap \{u > -j\}} (dd^c \max(u, -j))^n.$$

Note that if $E \subset \{u > -k\}$, then by Lemma 4.2

$$\int_E (dd^c \max(u, -j))^n = \int_E (dd^c \max(u, -k))^n$$

holds for $j \geq k$.

In general $NP(dd^c u)^n$ is not locally finite. However, the following convergence property is a consequence of Lemma 4.1.

PROPOSITION 4.4. *Let $u \in P(\Omega)$ and a compact subset $K \subset \{u > \infty\}$ be given. If $\{u_j\} \subset P(\Omega) \cap L^\infty(\Omega, \text{loc})$ is monotone decreasing to u, then*

$$\lim_{j \to \infty} \int_K (dd^c u_j)^n = \int_K NP(dd^c u)^n.$$

Next we consider the extended definition of $(dd^c)^n$ in terms of the Monge–Ampère measure carried by the finite graph. Let us summarize the approach of Kiselman [K]. For a domain $\Omega \subset \mathbb{C}^n$, we set

$$\tilde{\Omega} = \{(z, \zeta) \in \Omega \times \mathbb{C} : |\text{Im}\, \zeta| < 1\},$$

and for a function $u \in P(\Omega)$, we set

$$\tilde{u}(z, \zeta) = (u(z) - \operatorname{Re} \zeta)^+.$$

If $u \in P(\Omega)$ is locally bounded, then

$$(dd^c u)^n = p_\#(dd^c \tilde{u})^{n+1},$$

where $p: \tilde{\Omega} \to \Omega$ is the projection $p(z, \zeta) = z$. Further, if u is continuous, then $(dd^c \tilde{u})^{n+1}$ is supported on the finite graph $\Gamma_u = \{(z, \zeta) \in \tilde{\Omega}: \operatorname{Re} \zeta = u(z) > -\infty\}$ of u (Lemma 5.1 of [K]).

Now by Lemma 4.1, we see that for locally bounded $u \in P(\Omega)$, the fine support of $(dd^c \tilde{u})^{n+1}$ is in the (fine closed) graph Γ_u. For general $u \in P(\Omega)$ we may compute $(dd^c \tilde{u})^{n+1}$ in the following heuristic manner.

First we claim that $(dd^c \tilde{u})^{n+1}$ *is supported on* Γ_u. For $(z_0, \zeta_0) \notin \Gamma_u$, we let $\mathcal{O}$ be a fine open set $\mathcal{O} \cap \Gamma_u = \emptyset$, $(z_0, \zeta_0) \in \mathcal{O}$. Now $dd^c \tilde{u} = dd^c u$ holds on $\mathcal{O}$, and by reason of dimension we have $(dd^c u)^{n+1} = 0$. (We obtain the same result also if we consider any sequence $u_j \in P(\Omega) \cap C^\infty(\Omega)$ decreasing to u; $\tilde{u}_j$ decreases to u but $(dd^c \tilde{u}_j)^{n+1} = 0$ on $\mathcal{O}_j = \mathcal{O} \cap \{\tilde{u}_j \neq 0\}$ and $\mathcal{O}_j$ increases to $\mathcal{O}$.)

Next we clam that

$$(dd^c \tilde{u})^{n+1} = NP(dd^c \tilde{u})^{n+1}.$$

Since $(dd^c \tilde{u})^{n+1}$ is supported on Γ_u, it suffices to check how much mass is put on a compact set $K \subset \Gamma_u$. But since $K \subset \{\tilde{u} > -\infty\}$ our claim follows from Proposition 4.4.

To give a third approach to the nonpolar part of $(dd^c)^n$, we recall that Chern, Levine, and Nirenberg [CLN] showed that for negative $u \in C^2(\Omega) \cap P(\Omega)$ the mass of

$$\frac{(dd^c u)^n}{(-u)^n} + \frac{du \wedge d^c u \wedge (dd^c u)^{n-1}}{(-u)^{n+1}} \tag{4.1}$$

is locally bounded by a constant independent of u. Although we could use (4.1) to extend the definition of $(dd^c)^n$, we prefer to use the following sharper estimate.

THEOREM 4.5. *If u is a negative, C^2 psh function on the unit ball $\{|z| < 1\}$, then for $r < 1$ there is a constant C such that*

$$\int_{|z| \leqslant r} \frac{(dd^c u)^n}{(-u)^{n-1}} + (n-1) \int_{|z| \leqslant r} \frac{du \wedge d^c u \wedge (dd^c u)^{n-1}}{(-u)^n} \leqslant C|u(0)|. \tag{4.2}$$

Proof. Let β denote the Kähler form on C^n,

$$\beta = (i/2)(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n).$$

Fix r, $0 < r < 1$. Let χ denote a C^∞ function with compact support in $|z| < 1$ such that $0 \leqslant \chi \leqslant 1$ and $\chi(z) = 1$ for $|z| \leqslant r$. For $1 \leqslant k \leqslant n$, set

$$I_k = \int \frac{\chi^2 (dd^c u)^k \wedge \beta^{n-k}}{(-u)^{k-1}}$$

and

$$J_k = (k-1) \int \frac{\chi^2\, du \wedge d^c u \wedge (dd^c u)^{k-1} \wedge \beta^{n-k}}{(-u)^k}.$$

We will prove by induction on k that

$$I_k + J_k \leqslant C[-u(0)]. \tag{4.3}$$

When $k = n$, the left hand side of (4.1) is less than or equal to $I_k + J_k$ because $\chi = 1$ on $|z| \leqslant r$ and $\chi \geqslant 0$.

When $k = 1$, the integrand in I_k is χ^2 times the Laplacian of u and the estimate follows from Jensen's formula.

$$c_n \int_0^R \frac{n(r)}{r^{2n-1}}\, dr = -u(0) + \int_{|\alpha| = 1} u(R\alpha)\, d\sigma(\alpha), \tag{4.4}$$

where

$$n(r) = \int_{|z| \leqslant r} \Delta u(z)$$

is the mass of the Laplacian in the ball $|z| \leqslant r$, $d\sigma$ is normalized surface area measure on the ball, and c_n is a constant which depends only on n. To see this, note that because u is negative, the last term on the right-hand side of (4.4) is negative. Further, $n(r)$ is increasing because u is subharmonic. Thus, we have for $0 < \rho < R$,

$$-u(0) \geqslant \int_\rho^R \frac{n(r)}{r^{2n-1}}\, dr \geqslant \left\{\frac{1}{\rho^{2n}} - \frac{1}{R^{2n}}\right\} n(\rho)/2n.$$

If ρ and $R < 1$ are chosen so that the support of χ is contained in the ball $|z| \leqslant \rho$, then the estimate (4.3) follows in the case $k = 1$.

Suppose now that $k > 1$. In the integrand for I_k, write

$$\frac{\chi^2(dd^cu)^k}{(-u)^{k-1}} = d\left[\frac{\chi^2\, d^cu \wedge (dd^cu)^{k-1}}{(-u)^{k-1}}\right] - \frac{d\chi^2 \wedge d^cu \wedge (dd^cu)^{k-1}}{(-u)^{k-1}} - \frac{(k-1)\,\chi^2\, du \wedge d^cu \wedge (dd^cu)^{k-1}}{(-u)^k}$$

so that

$$I_k + J_k = -\int \frac{d\chi^2 \wedge d^cu \wedge (dd^cu)^{k-1} \wedge \beta^{n-k}}{(-u)^{k-1}}. \tag{4.5}$$

Write

$$\frac{d\chi^2 \wedge d^cu}{(-u)^{k-1}} = \left[\frac{2d\chi}{\sqrt{k-1}\,(-u)^{(k/2)-1}}\right] \wedge \left[\frac{\sqrt{k-1}\,\chi\, d^cu}{(-u)^{k/2}}\right]$$

and apply the Cauchy–Schwarz inequality (recall that u is psh so $(dd^cu)^{k-1} \wedge \beta^{n-k}$ is positive) to obtain from (4.5),

$$I_k + J_k \leqslant \left[\frac{4}{k-1}\int \frac{d\chi \wedge d^c\chi \wedge (dd^cu)^{k-1} \wedge \beta^{n-k}}{(-u)^{k-2}}\right]^{1/2} J_k^{1/2}.$$

But $AB \leqslant (A^2 + B^2)/2$, so we conclude

$$I_k + J_k/2 \leqslant \frac{4}{k-1}\int \frac{d\chi \wedge d^c\chi \wedge (dd^cu)^{k-1} \wedge \beta^{n-k}}{(-u)^{k-2}}.$$

If χ has support in $|z| \leqslant \rho < 1$, then because $(dd^cu)^{k-1}$ is positive, we have that the last integral is dominated by a constant times

$$\int_{|z| \leqslant \rho} \frac{(dd^cu)^{k-1} \wedge \beta^{n-k+1}}{(-u)^{k-2}}.$$

The integrand in this last expression is exactly of the same form as in the one for I_k, except that k has been replaced by $(k-1)$. This completes the inductive step of the proof.

Remark. The estimate (4.2) has the best possible power of $-u$ in the denominator. Kiselman [K] showed that the exponent $n-1$ in the left-hand side of (4.2) cannot be sharpened to $n-1-\varepsilon$.

Let us define, for C^2, negative, psh u and $\alpha = n-1+\varepsilon$

$$\tilde{M}(u) = (-u)^{-\alpha}(dd^cu)^n.$$

It follows that $\tilde{M}$ extends to all negative, psh functions.

COROLLARY 4.6. *If* $u \in P(\Omega)$, $u<0$ *and if the sequence* $\{u_j\} \subset P(\Omega) \cap C^2(\Omega)$ *decreases to* u, *then* $\tilde{M}u_j$ *converges weakly to a measure, called* $\tilde{M}u$.

The proof of Corollary 4.6 follows from Theorems 3.2 and 4.5. As a consequence, we find that for negative psh u, $\int |u|^{\varepsilon} \tilde{M}u < \infty$, and so $\tilde{M}u$ puts no mass on $\{u = -\infty\}$.

Our final characterization of $NP(dd^c u)^n$, is thus

COROLLARY 4.7. *If* $u \in P(\Omega)$, $u<0$, *then*

$$NP(dd^c u)^n = (-u)^{\alpha} \tilde{M}(u).$$

Remark. The function $\phi(x) = -(-x)^{1/n}$ is increasing and convex for $x<0$. Consequently, when u is a negative psh function, so is $\phi(u)$. It is interesting to note that

$$n^n[dd^c\phi(u)]^n = \frac{(dd^c u)^n}{(-u)^{n-1}} + (n-1)\frac{du \wedge d^c u \wedge (dd^c u)^{n-1}}{(-u)^n}$$

is the integrand on the left-hand side of (4.2). It seems to be an open problem to determine whether $NP(dd^c v)^n$ has locally finite mass when v is a psh function such that (locally) $v \geq \phi(u)$ for some negative psh function u.

5. COMPARISON OF CAPACITIES

Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary; the capacity of a Borel subset E of Ω is given by

$$C(E) = C(E, \Omega) = \sup\left\{\int_E (dd^c u)^n \colon u \text{ psh on } \Omega,\ 0 < u < 1\right\},$$

and the outer capacity of an arbitrary set by

$$C^*(E) = \inf\{C(\mathcal{O}) \colon \mathcal{O} \supset E,\ \mathcal{O} \text{ open}\}.$$

The capacity C is a Choquet capacity. It follows from Choquet's capacitability theorem that

$$C^*(E) = C_*(E) := \sup\{C(K) \colon K \subset E,\ K \text{ compact}\}$$

holds for all $\mathcal{K}$-analytic sets and thus for all Borel sets. See [BT] for proofs of these results.

THEOREM 5.1. *Let K be a compact subset of Ω of positive capacity. For a capacitable subset F of K (i.e., $C_*(F) = C^*(F)$) the following are equivalent.*

(α) $C(F) = C(K)$

(β) $U_F^*(z) = U_K^*(z)$ *for all* $z \in \Omega$,

(γ) $\int_{K \setminus F'} (dd^c U_K^*)^n = 0$.

For the proof, we will need the following technical result (Lemma 5.2) and an inequality (Lemma 5.3).

LEMMA 5.2. *Let Ω be a bounded strongly pseudoconvex domain in* $\mathbb{C}^n$. *Let* $u_0, u_1, \ldots, u_n \in P(\Omega_1) \cap L^\infty(\Omega_1, \text{loc})$, *where* $\Omega_1 \supset \bar{\Omega}$, *and suppose also that*

$$u_0(z) = \lim_{\zeta \to z} u_0(\zeta) = 0 \qquad \text{for all } z \in \partial\Omega.$$

Then

(a) $-\int_\Omega u_0 \, dd^c u_1 \wedge \cdots \wedge dd^c u_n = \int_\Omega du_0 \wedge d^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_n$ *and*

(b) $\int_\Omega du_0 \wedge d^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_n = \int_\Omega du_0 \wedge d^c u_2 \wedge dd^c u_1 \wedge dd^c u_3 \wedge \cdots \wedge dd^c u_n$.

The lemma is, formally, an integration by parts, the boundary term vanishing because $u_0 = 0$ and $du_0 = 0$ when restricted to $\partial\Omega$. The only problem is to justify the integration by parts. This may be done by using the standard smoothings

$$U_j^\varepsilon = U_j * \chi_\varepsilon$$

and making an approximation argument (cf. [BT]).

LEMMA 5.3. *Let F be a compact subset of K. Then*

$$\int_\Omega -U_K^*(dd^c U_F^*)^n \leqslant \int_\Omega -U_F^*(dd^c U_K^*)^n$$

and for $j = 0, 1, \ldots, n-1$,

$$\int dU_F^* \wedge d^c U_F^* \wedge (dd^c U_K^*)^j \wedge (dd^c U_F^*)^{n-1-j} \leqslant C(K).$$

Proof. For convenience of notation, let $u = U_K^*$ and $v = U_F^*$. Both u, v have extensions to be plurisubharmonic and bounded on a neighborhood of $\bar{\Omega}$. For, if ρ is a strictly plurisubharmonic defining function for $\Omega = \{\rho < 0\}$ on a neighborhood of $\bar{\Omega}$, then $u, v \geqslant A\rho$ for some constant $A > 0$. Hence, we can set $u = v = A\rho$ outside of Ω. In particular, u and v satisfy the hypotheses of Lemma 5.2 which justifies the integrations by parts we will make in the following arguments.

Define, for $j=0, 1,..., n$,

$$A_j=\int_\Omega (-v)(dd^c u)^j \wedge (dd^c v)^{n-j},$$

$$B_j=\int_\Omega (-u)(dd^c u)^j \wedge (dd^c v)^{n-j}$$

and note that we have the following relationship:

$$0 \leqslant A_j \leqslant B_j, \qquad j=0, 1,..., n; \tag{5.1}$$

$$B_j = A_{j+1}, \qquad j=0, 1,..., n-1; \tag{5.2}$$

$$B_j \leqslant \sqrt{A_j}\sqrt{B_{j+1}}, \qquad j=0, 1,..., n-1; \tag{5.3}$$

$$B_n = C(K); \tag{5.4}$$

$$B_0 = A_0 = C(F). \tag{5.5}$$

The proof of (5.1) is obvious, since $-u \geqslant -v$. To prove (5.2), one integrates by parts twice:

$$\begin{aligned} B_j &= \int_\Omega du \wedge d^c v \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &= \int_\Omega dv \wedge d^c u \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &= \int_\Omega (-v)(dd^c u)^{j+1} \wedge (dd^c v)^{n-j-1} = A_{j+1}. \end{aligned}$$

The proof of (5.3) is also easy, since

$$\begin{aligned} B_j &= \int_\Omega du \wedge d^c v \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ &\leqslant \left[\int_\Omega du \wedge d^c u \wedge (dd^c u)^j \wedge (dd^c v)^{n-j-1}\right]^{1/2} \\ &\quad \times \left[\int_\Omega dv \wedge d^c v \wedge (dd^c u)^j \wedge (dd^c v)^{n-1-j}\right]^{1/2} \end{aligned}$$

where the inequality is the Cauchy–Schwarz inequality (see, e.g., [BT, Sect. 3]). Integrating by parts one more time in each integral shows the last term is $\sqrt{B_{j+1}}\sqrt{A_j}$, which proves (5.3). Finally, the fact that $B_n = C(K)$ is because $u = U_K^* = -1$ on K, except on a set of capacity zero. Hence, because $(dd^c u)^n$ is supported on K, $B_n = \int_K (dd^c u)^n = C(K)$, by Proposition 5.3 of [BT]. Similarly, $A_0 = C(F) = B_0$.

Now, the first assertion of the lemma is that $B_0 \leqslant A_n$, which follows trivially from repeated application of (5.1) and (5.2). For we have $B_0 = A_1 \leqslant B_1 = A_2 \leqslant B_2 \leqslant \cdots \leqslant B_{n-1} = A_n$. But note also that

$$\log B_j \text{ is a convex, increasing sequence in } j \text{ for } j = 0, 1, ..., n. \tag{5.6}$$

Because, by (5.3) and (5.2), $B_j \leqslant \sqrt{A_j}\ \sqrt{B_{j+1}} = \sqrt{B_{j-1}}\ \sqrt{B_{j+1}}$, $j = 1, 2, ..., n-1$. Since $B_0 = C(F)$ and $B_n = C(K)$, this actually gives a stronger estimate than that asserted in the lemma, namely,

$$B_j \leqslant C(F)^{(1-j/n)}\, C(K)^{j/n}. \tag{5.7}$$

We now prove the Main Theorem.

Proof that $(\alpha) \Rightarrow (\beta)$. Since $F \subset K$, we have that $U_F^* \geqslant U_K^*$. We claim that $\{z \in K: U_F^*(z) > U_K^*(z)\}$ has $(dd^c U_K^*)^n$ measure zero. If this is true, then $U_F^*(z) \equiv U_K^*(z)$ by the Domination Principle, Corollary 4.5, of [BT]. Choose a sequence F_j of compact subsets of F such that $F_1 \subset F_2 \subset \cdots$, and $\sup_j C(F_j) = C(K)$. Such a sequence exists because $C_*(F) = C(K)$. If $u_j = U_{F_j}^*$, then $u_1 \geqslant u_2 \geqslant \cdots$. So, we have by the first inequality of Lemma 5.3 and the fact that $U_K^* = -1$ on F_j, except on a set of capacity zero, that

$$C(F_j) = \int_{F_j} (-U_K^*)(dd^c u_j)^n \leqslant \int_K (-u_j)(dd^c U_K^*)^n$$

$$\leqslant \int_K -U_F^*(z)(dd^c U_K^*)^n \leqslant C(K).$$

Letting $j \to \infty$, we see that we must have

$$\int_K -U_F^*(dd^c U_K^*)^n = C(K).$$

Because $0 \geqslant U_F^* \geqslant -1$, we must therefore have $U_F^* = -1$ except on a set of $(dd^c U_K^*)^n$ measure zero. It follows that $U_F^* \leqslant U_K^*$ holds $(dd^c U_K)^n$-almost everywhere, and so by the Domination Principle we have $U_F^* \leqslant U_K^*$. Thus $U_F^* = U_K^*$, as asserted.

Proof that $(\beta) \Rightarrow (\alpha)$. From Proposition 6.5 of [BT], we have for an arbitrary subset $F \Subset \Omega$,

$$C^*(F) = \int_\Omega (dd^c U_F^*)^n.$$

But, if $U_F^* = U_K^*$, then, clearly, $C^*(F) = C(K)$.

Proof that $(\beta) \Rightarrow (\gamma)$. We can assume that F is fine-closed, because the extremal function of a set is clearly the same as the extremal function of the fine closure of the set. From the fact that $(\beta) \Rightarrow (\alpha)$ and F is capacitable, we can choose a sequence F_j of compact subsets of F such that $F_1 \subset F_2 \subset \cdots$, and $C(F_j) \to C(F) = C(K)$. If $u_j = U^*_{F_j}$, then $u_1 \geqslant u_2 \geqslant \cdots$, $u_j \to U^*_F = U^*_K$ on Ω, so also $(dd^c u_j) \to (dd^c U^*_K)^n$ in the sense of Theorem 3.2. Therefore, by Corollary 3.3,

$$C(K) = \int_F (dd^c U^*_F)^n \geqslant \limsup_j \int_F (dd^c u_j)^n$$

$$= \limsup_j \int_{F_j} (dd^c u_j)^n = \lim_j C(F_j) = C(K).$$

Since $(dd^c U^*_K)^n$ is concentrated on $K \supset F$ and $\int_K (dd^c U^*_K)^n = C(K)$, it follows that $K \backslash F$ has $(dd^c U^*_K)^n$-measure zero.

Proof that $(\gamma) \Rightarrow (\beta)$. If u is a plurisubharmonic function on Ω, $u < 0$ on Ω, and $u \leqslant -1$ on F, then also $u \leqslant -1$ on the fine closure F^f, because u is fine-continuous. Then, because $K \backslash F^f$ has $(dd^c U^*_K)^n$ measure zero, we have $u \leqslant -1$ for $(dd^c U^*_K)^n$-almost all $z \in K$. It follows from the Domination Principle (Corollary 4.5) of [BT]) that $u \leqslant U^*_K$. Thus, $U^*_F \leqslant U^*_K$. But, $F \subset K$, so $U^*_F = U^*_K$.

6. Application to Polynomial Envelopes.

We consider the family of psh functions on $\mathbb{C}^n$ with minimal growth

$$\mathscr{L} = \{u \text{ psh on } \mathbb{C}^n \colon u(z) \leqslant \log(1 + |z|) + C_u\}.$$

For $E \Subset \mathbb{C}^n$, we define

$$L^*_E(z) = \limsup_{\zeta \to z} L_E(\zeta),$$

where

$$L_E(z) = \sup\{u(z) \colon u \in \mathscr{L},\ u \leqslant 0 \text{ on } E\},$$

which is the generalized Green function on $\mathbb{C}^n$ with logarithmic pole at infinity. We will define $\lambda_E = (dd^c L^*_E)^n$ to be the *complex equilibrium measure* of E.

We will be able to apply the results of Section 5 to L^*_E and λ_E because of the following two facts relating the relative extremal function U_K^* global function L_K^*. The first of these (see Siciak [Si1]) is that

if $F \subset K \subset B^n$, then $U_F^*(z, B^n) = U_K^*(z, B^n)$ holds for all z if and only if $L_F^* = L_K^*$. (*)

The second (due to Levenberg [L]) is that

if K is a compact subset of the ball B^n, then λ_K is bounded above and below by constant multiples of $(dd^c U_K^*(B^n))^n$. (**)

The connection with polynomials arises since p is any polynomial, then

$$\frac{1}{\deg(p)} \log\left(\frac{|p(z)|}{\|p\|_E}\right) \leqslant L_E(z). \tag{6.1}$$

A compact set K is *regular* if L_K is continuous. For K regular, (6.1) yields a Bernstein–Markov-type inequality

for each $\lambda > 1$, there is an open set $\mathscr{U} \supset K$ such that for any polynomial p $|p(z)| \leqslant \|p\|_K \lambda^{\deg(p)}$ for all $z \in \mathscr{U}$. (6.2)

(Note that in (6.2), we take $U = \{z: L_K^*(z) < \log \lambda\}$, which is an open set containing K by the regularity assumption.) Siciak [Si2] has shown that (6.2) also holds with $\|p\|_K$ replaced by a constant times the integral of $|p|$ with respect to λ_K.

Now let $\mathscr{P}$ be an arbitrary family of polynomials. Given a compact set K in $\mathbb{C}^n$, we let

$$F = F(\mathscr{P}, K) = \{z \in K: \sup\{|p(z)|: p \in \mathscr{P}\} < \infty\} \tag{6.3}$$

be the set where the supremum is finite. We will consider the possibility of an estimate of the form

for each $\lambda > 1$, there exists an open set $\mathscr{U} \supset K$ and a constant $M > \infty$ such that $|p(z)| \leqslant M\lambda^{\deg(p)}$ for all $p \in \mathscr{P}$ and $z \in \mathscr{U}$. (6.4)

This is essentially the so-called Leja polynomial condition. Although (6.4) is related also to (6.2), it is possible for (6.4) to hold without $\{\|p\|_K: p \in \mathscr{P}\}$ being bounded.

If we set

$$F_j = \{z \in K: \sup\{|p(z)|: p \in \mathscr{P}\} \leqslant j\},$$

then F_j is compact, $F_1 \subset F_2 \subset \cdots$, and $\bigcup F_j = F$. It follows that $\lim_{j\to\infty} L^*_{F_j} = L^*_F$ (see, e.g., [BT, Proposition 8.1]). Now we define

$$\lambda_0 = \sup_{z\in K} \exp(L^*_F(z)),$$

so that λ_0 is a constant depending on $\mathcal{P}$. It is evident that we always have something weaker than (6.4), namely,

$$\text{for each } \lambda > \lambda_0, \text{ there exists an open set } \mathcal{U} \supset K \text{ and a constant } M < \infty \text{ such that } |p(z)| \leqslant M\lambda^{\deg(p)} \text{ for all } p \in \mathcal{P} \text{ and } z \in \mathcal{U}. \qquad (6.5)$$

(Note that this follows from (6.2), since we may take j large enough that $\mathcal{U} = \{z: L^*_{F_j}(z) < \log \lambda\}$ contains K, and then we set $M = j$.)

The main result of this section is a necessary and sufficient condition for (6.4) to hold.

THEOREM 6.1. *Let K be regular, let P be a family of polynomials, and let F be as in* (6.3). *Then*

(1) *if the fine closure of F, F^f, satisfies $\int_{K\setminus F^f} d\lambda_K = 0$, then $\mathcal{P}$ has the property* (6.4)

(2) *if F is a capacitable subset of K whose fine closure satisfies $\int_{K\setminus F^f} d\lambda_K > 0$, then there exists a family of polynomials $\mathcal{P}$ with $F \subset F(\mathcal{P}, K)$, and $\mathcal{P}$ fails to have the property* (6.4).

(3) *Condition* (1) *holds when $C(F) = C(K)$, and Condition* (2) *holds when $C(F) < C(K)$.*

(4) *Condition* (1) *holds when $U^*_F = U^*_K$ and Condition* (2) *holds when $U^*_F > U^*_K$ at some point.*

Part (1) of this theorem was essentially observed by Nguyen and Zeriahi in [NZ]. The converse, part (2), gives an extension to $\mathbb{C}^n$ of Ullman's theorem on "determining sets" [U]. Levenberg [Le] has shown that (1) and (2) hold with the conditions on F replaced by $U^*_F = U^*_K$. However, by Theorem 5.1, this is equivalent to the stated conditions.

For the proof of Theorem 6.1, it will suffice to prove just (1) and (2), for then (3) and (4) will follow by Theorem 5.1. We will give the proof of (2) only for the case when F is an F_σ – set. The proof when F is not F_σ requires some additional work to handle an exceptional polar set of points. Siciak [private communication] has given a good treatment of this case.

Proof of Theorem 6.1. (1) Taking $\Omega = B^n$ and using (*) and (**), we see that Theorem 5.1 yields $L^*_F = L^*_K$. Since K is regular, we have (6.4).

(2) *In the case that F is a F_σ-set.* We write $F = \cup F_j$, where $F_1 \subset F_2 \subset \cdots$, are compact. By a result of Siciak [Si1] and Zaharjuta [Z],

$$L_{F_j}(z) = \sup \left\{ \frac{1}{d} \log|P_d(z)| : P_d \in \mathscr{P}^j_d \right\},$$

where

$$\mathscr{P}^j_d = \{\text{polynomials } P_d \text{ of degree } \leqslant d \text{ with } |P_d|_{F_j} \leqslant 1\}.$$

By Theorem 5.1, there is a point $z_0 \in K$ with $L^*_F(z_0) = \eta > 0$. Now we choose polynomials $P_j \in \mathscr{P}^j_{d_j}$ with $d_j \to \infty$ and

$$|P_j(z_0)| > e^{d_j \eta/2},$$

and it follows that $\mathscr{P} = \{P_1, P_2, ...\}$ fails to have (6.4).

7. Supports and Boundaries

In this section we show, as an application of Theorem 5.1 and Corollary 3.5, how the support of the relative extremal measure $(dd^c u^*_K)^n$ is related to the Silov, Jensen, and fine boundaries of K. For K compact in Ω, let $A(K) = A(K, \Omega)$ denote the subalgebra of the Banach algebra of continuous functions on K which is the closure of the functions holomorphic on Ω. We denote by $\partial_s K$ the Silov boundary of $A(K)$ and by spt λ_K the (closed) support of the measure λ_K. We define

$$K_0 = \overline{\{z \in K: U^*_K(z) = -1\}}.$$

THEOREM 7.1. *Let K be compact in Ω, a strictly pseudoconvex set in $\mathbb{C}^n$. Then* spt λ_K *and the Shilov boundary of $A(K)$ are related as follows.*

(1) spt $\lambda_K = \partial_S K_0 \subset \partial_S K$, *where K_0 is as above.*

(2) $\partial_S K \subseteq (K \backslash K_0) \cup$ spt λ_K *and thus $\partial_S K$ differs from* spt λ_K *by at most a polar set.*

(3) *If K is regular, then $\partial_S K =$* spt λ_K.

Proof of Theorem 7.1. (1) From the Domination Principle, it follows that if $f \in A(K)$, $\log|f| < 0$ on Ω, and $\log|f| \leqslant -1$ a.e. λ_K, then $\log|f(z)| \leqslant U^*_K(z)$. Hence, by continuity, $\log|f| \leqslant -1$ on K_0. It follows that the supremum of f on K_0 is exactly the same as the supremum of f on spt λ_K when f is a bounded analytic function on Ω. But, functions holomorphic on a neighborhood of $\bar{\Omega}$ are dense in $A(K)$. Thus, spt λ_K is a boundary for $A(K_0)$, so spt $\lambda_K \supset \partial_S K_0$.

To prove the other inclusion we note that by [GS], $U_K^* = U_{\partial_S K}^*$, and thus spt $\lambda_K =$ spt $\lambda_{\partial_S K} \subset \partial_S K$.

Statement (2) follows from (1) since $K \setminus K_0$ is polar. Case (3), follows from case (1) since $K \setminus K_0$ is empty.

Now we discuss the *fine support* of λ_K, $f-$spt λ_K, which is the intersection of all fine-closed sets whose complements have λ_K-measure zero. Since λ_K puts no mass on any polar set, it is clear that the fine support of λ_K lies inside the base of the support

$$f-\text{spt } \lambda_K \subset b(\text{spt } \lambda_K)$$

(see (2.3) for the definition of the base).

We will call a fine-closed subset F of a fine-closed set E a *fine boundary* for E if and only if

$$\sup_F \psi = \sup_E \psi$$

for all psh functions ψ on Ω. If there is a smallest fine boundary for E, i.e., if the intersection of all fine boundaries $\cap F$ is again a fine boundary for E, we will call it the *fine Silov boundary of E* and denote it by $\partial_S^f E$.

It is possible to pose a generalized Dirichlet problem for a lower semicontinuous (lsc) function φ on a compact set K in Ω. Namely, given φ lsc on K we let

$$U_\varphi(z) = \sup\{v(z) \colon v \in P(\Omega),\, v \leqslant \varphi \text{ on } K\}.$$

In general, U_φ will be lsc since the supremum may be taken in $P(\Omega) \cap C(\Omega)$. When Ω is strongly pseudoconvex, the supremum may be taken over functions v which are psh and continuous on a neighborhood of $\bar{\Omega}$.

By the boundary properties of $\partial_S K$, it follows that $U_\varphi = \varphi$ on a dense subset of $\partial_S K$. The *Jensen boundary* of K, denoted $\partial_J K$, consists of the points $z_0 \in K$ such that $U_\varphi(z_0) = \varphi(z_0)$ for all $\varphi \in C(K)$. This is a special case of the Choquet boundary (see the discussions in [G, GS, DG]). The set of peak points for $P(\Omega)$

$$\partial'_J K = \{z_0 \in K \colon \text{there exists } \psi \in P(\Omega) \text{ with } \psi(z_0) = 0 \\ \text{and } \psi(z) < 0 \text{ for } z \in K \setminus \{z_0\}\}.$$

is a subset of $\partial_J K$.

The Jensen and Silov boundaries are related as follows.

THEOREM 7.2. *Let K be a compact subset of Ω (strongly pseudoconvex). Then*

(1) $\partial_S K = \overline{\partial'_J K}$,

(2) $\partial^f_S K = (\partial'_J K)^f$.

Part (1) of Theorem 7.2 is well known, and the same proof may be applied to part (2). Here we will need the analogous result for the base of a compact set.

THEOREM 7.3. *Let K be a compact subset of $\Omega \Subset \mathbb{C}^n$. Then*

$$\partial_J b(K) \subset \partial_S K,$$

and

$$\partial^f_S(b(K)) = (\partial'_J K \cap b(K))^f.$$

The proof of Theorem 7.3 is similar to that of Theorem 7.2 and will be omitted.

THEOREM 7.4. *Let K be compact in Ω (strictly pseudoconvex). Then $f-\operatorname{spt} \lambda_K$ has the following boundary properties:*

(1) *The fine support of λ_K is the fine Silov boundary of $b(K)$, i.e.,*

$$f-\operatorname{spt} \lambda_K = \partial^f_S b(K).$$

(2) $f-\operatorname{spt} \lambda_K = \cap \{F\colon F \text{ is fine-closed and } U^*_F = U^*_K\}$

(3) $f-\operatorname{spt} \lambda_K = ((f-\operatorname{spt} \lambda_K) \cap \partial'_J K)^f$

Proof. (1) Let us set

$$E = \partial^f_S (b(K)).$$

Since E is the fine boundary, we have $U_E = U_{b(K)}$. Thus

$$U^*_E = U^*_{b(K)} = U^*_K.$$

By Corollary 3.4, we have

$$E \supset f-\operatorname{spt}(dd^c U^*_E)^n = f-\operatorname{spt} \lambda_K.$$

For the reverse inequality, we show that $f-\operatorname{spt} \lambda_K$ is a fine boundary for $b(K)$. Let $\psi \in P(\Omega)$, $\psi < 0$ be given, and suppose that $\psi \leqslant -1$ on $f-\operatorname{spt} \lambda_K$. Then by the Domination Principle, it follows that $\psi \leqslant U^*_K$. On the other hand, $U^*_K = -1$ on $b(K)$, so $\psi \leqslant -1$ on $b(K)$. Thus $f-\operatorname{spt} \lambda_K$ is a fine boundary.

(2) Let us set

$$\tilde{F} = \cap \{F\colon F \text{ fine closed, } U_F^* = U_K^*\}.$$

It is evident that $\tilde{F} \subset E$, defined in (1), and so by (1) we must show that $f-\operatorname{spt} \lambda_K \subset \tilde{F}$. Now suppose F is fine closed and $U_F^* = U_K^*$. By Corollary 3.4, the fine support of $(dd^c U_F^*)^n$ is contained in F. If $U_F^* = U_K^*$, then $f-\operatorname{spt} \lambda_K = f-\operatorname{spt} \lambda_F \subset F$.

(3) The inclusion $\supseteq$ is trivial. For the reverse inclusion, we have, by (1) and Theorem 7.3,

$$\begin{aligned} f-\operatorname{spt} \lambda_K &= \partial_S^f(b(K)) \\ &= (\partial_J K \cap b(K))^f \\ &= (\partial_J K \cap \partial_S^f(b(K)))^f \\ &= (\partial_J K \cap (f-\operatorname{spt} \lambda_K))^f. \end{aligned}$$

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