JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **101**, 64–77 (1984)

A Parametric Approach to Complementarity Theory

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Submitted by K. Fan

Received September 24, 1982

INTRODUCTION

Various problems arising in game theory, mathematical economics, and optimization theory may be formulated as complementarity problems. For example, the necessary Kuhn–Tucker conditions for the general optimization problem produce a problem of this type. The list of contributions to the theory of the linear complementarity problem is very extensive, comprising Lemke and Howson [6], Cottle and Dantzig [1], Eaves [2], Ingleton [4], Karamardian [5], Murty [8, 9], and Saigal [10], although many computational problems in mathematical economics and game theory, e.g., the computation of equilibria of the generalized von Neumann model or N-person games, are typical examples for nonlinear complementarity problems. For that reason the question arises of how far some aspects of the linear theory are appropriate for a generalization to the nonlinear case.

Some classes of linear complementarity problems produce the constant parity property, i.e., the property that the number of solutions is either odd within the whole class or even (see [9, 10]). Problems of this type depend on some real parameters and are connected by homotopies preserving the parity of the number of solutions. Moreover, these homotopies generate paths, which connect different solutions. Thus, there are some chances to find new solutions if some solutions were known. The investigation of classes of nonlinear problems will lead to similar results in some cases. Of course, as in the linear case these results will not be true without the nondegeneracy assumption. Therefore the question arises whether this assumption will be satisfied within a sufficiently large subclass. It turns out that for special types of classes the assumption holds at least for almost all problems with respect to the Lebesgue measure on the parameter space.

0022-247X/84 $3.00

1. Fundamental Definitions and Notations

A *linear complementarity problem* is given by a $n \times n$ matrix M and an n-vector q. One is interested in the nonnegative solutions (w, z) of the equations

$$LK(M, q): \quad \begin{aligned} w - Mz - q &= 0 \\ w^T z &= 0. \end{aligned}$$

The theory of these problems is presented in [7, 8], for instance. In [8] there are to be found expositions on *parametric linear complementarity problems*, i.e., problems of the type $LK(M, q)$, where q depends on a real parameter λ ($q = q(\lambda)$). This is a special case of a much more general situation. Given a mapping $f\colon W \to \mathbb{R}^n$, where $W \subset \mathbb{R}^m$ is an interval of the type $W = \mathbb{R}^k_+ \times \mathbb{R}^k_+ \times Q$ (where W takes the form $\mathbb{R}^k_+ \times \mathbb{R}^k_+$ for $2k = m$ and W is an arbitrary interval in $\mathbb{R}^m$ for $k = 0$, respectively) with nonempty interior. Do there exist solutions of the problem

$$\begin{aligned} f(x, y, z) &= 0 \\ \mathrm{PCP}_{k,m,n}(f, W): \qquad xy^T &= 0 \\ (x, y, z) \in \mathbb{R}^k_+ \times \mathbb{R}^k_+ \times Q &= W, \end{aligned}$$

and are there any chances to find at least one of them?

1.1 Definition. A *parametric complementarity problem* (PCP)[1] is a problem of the type $\mathrm{PCP}(f, W) = \mathrm{PCP}_{k,m,n}(f, W)$.

The integer

$$\mathrm{ord}(\mathrm{PCP}(f, W)) := m - k - n$$

is called the *order* of $\mathrm{PCP}(f, W)$.

Since W is an interval in $\mathbb{R}^m$ with nonempty interior, there are intervals $W_1, \ldots, W_m$ in $\mathbb{R}$ with nonempty interior such that

$$W = W_1 \times \cdots \times W_m.$$

Define a *space segment* H_i in $\mathbb{R}^m$ by

$$H_i = \{w \in \mathbb{R}^m \mid w_i \in W_i\} \qquad (i = 1, \ldots, m).$$

[1] The conception "complementarity problem," is derived from the requirement $x, y \geqslant 0$, $xy^T = 0$.

Then

$$W = \bigcap_{i=1}^{m} H_i$$

and

$$H_i = \mathbb{R} \times \cdots \times \mathbb{R} \times \underset{\substack{\uparrow \\ i}}{\mathbb{R}_+} \times \mathbb{R} \times \cdots \times \mathbb{R} \qquad (i = 1, \ldots, 2k).$$

Each space segment H_i is a manifold of dimension m with boundary. The boundary is a manifold of dimension $m-1$ without boundary and is denoted by ∂H_i. Further let us assume f to be arbitrarily often differentiable in an open neighborhood U of W. For a subset $L \subset \{1, \ldots, m\}$ let $\partial_L f$ be the restriction of f to the $m - |L|$-dimensional manifold $U \cap \bigcap_{i \in L} \partial H_i$.

If $w \in \partial H_i$, let us say that w "*satisfies a boundary condition* (*w.r.t.* H_i)." If $w \in \partial H_i$ for some $i \in \{1, \ldots, 2k\}$ we say that w satisfies a *complementary* boundary condition; if $w \in \partial H_i$ for some $i \in \{2k+1, \ldots, m\}$ we say that w satisfies a *noncomplementary* boundary condition.

Hence, if (x, y, z) is a solution of PCP(f, W), as a consequence of $xy^T = 0$ it satisfies at least k boundary conditions, with respect to H_i and H_{i+k} at least one $(i = 1, \ldots, k)$. Let

$$\Lambda := \{L \subset \{1, \ldots, m\} \mid \{i, i+k\} \cap L \neq \varnothing \ (i = 1, \ldots, k)\}.$$

We call Λ the system of *complementary subsets* of $\{1, \ldots, m\}$.

The following definition of nondegeneracy is an extension of the same conception used in linear complementarity theory.

1.2. Definition. The parametric complementarity problem PCP(f, W) is called *nondegenerate*, iff 0 is a regular value for all mappings $\partial_L f$ $(L \in \Lambda)$.

The assumption of nondegeneracy ensures that the sets

$$\{\partial_L f = 0\} = \left\{ w \in U \cap \bigcap_{i \in L} \partial H_i \mid \partial_L f(w) = 0 \right\}$$

are geometrically tractable objects, if $L \in \Lambda$.

1.3. Proposition. *If* PCP(f, W) *is nondegenerate, then the sets* $\{\partial_L f = 0\}$ $(L \in \Lambda)$ *are manifolds of dimension* $m - |L| - n$.[2]

[2] A manifold of dimension <0 is always empty.

Proof. Since O is a regular value of the mapping $\partial_L f$ for $L \in \Lambda$ and $\partial_L f$ is a mapping from a manifold of dimension $m - |L|$ into $\mathbb{R}^n$, the theorem is a consequence of the preimage theorem in differential topology (cf. [3, p. 21]). ∎

Let $\mathscr{S}$ be the set of solutions of PCP(f, W). It can easily be verified that

$$\mathscr{S} = \bigcup_{L \in \Lambda} \left(\{\partial_L f = 0\} \cap \bigcap_{i \in L^c} H_i \right). \tag{1.4}$$

Hence, as $|L| \geqslant k$ ($L \in \Lambda$) and

$$\{\partial_L f = 0\} = \varnothing \qquad (L \in \Lambda, |L| > m - n)$$

in a nondegenerate case (otherwise 0 would never be a regular value of $\partial_L f$), $\mathscr{S}$ consists of manifolds with boundary, whose dimensions vary between 0 and $m - k - n = \operatorname{ord}(\text{PCP}(f, W))$. Most examples for PCPs yield problems of order 0. In this case the solutions will be singletons. But from a computational viewpoint problems of order 1 are more interesting, because in this case the solution set consists of curves; and by these means one would expect to have a chance to connect the solution sets of two problems of order 0. This is a reason for a further analysis of those problems.

2. PCPs of Order 1 with Compact Solution Sets

In this section let us assume that

(1) $\operatorname{ord}(\text{PCP}(f, W)) = m - k - n = 1$;

(2) the solution set $\mathscr{S}$ of PCP(f, W) is compact;

(3) PCP(f, W) is nondegenerate.

In this case $\mathscr{S}$ is included in the union of curves $\{\partial_L f = 0\}$ with $L \in \Lambda$, $|L| = k$, and the union of all singletons $\{\partial_L f = 0\}$ with $L \in \Lambda$, $|L| = k + 1$.

2.1. Proposition. *Let $L \in \Lambda$, $|L| = k + 1$. Then the set $L^* := \{i \in L \mid L - \{i\} \in \Lambda\}$ contains exactly one element if $L \not\subset \{1,..., 2k\}$, and exactly two elements if $L \subset \{1,..., 2k\}$.*

Proof. Since $L \in \Lambda$, we have

$$\{i, i + k\} \cap L \neq \varnothing \qquad (i = 1,..., k). \tag{i}$$

Thus, if $L \not\subset \{1,..., 2k\}$

$$|\{i, i + k\} \cap L| = 1 \qquad \text{for all} \quad i = 1,..., k,$$

and there is exactly one $j \in L - \{1,..., 2k\}$. This is the only element of L we may omit without violating (i).

Now let $L \subset \{1,..., 2k\}$. Then

$$\{j, j+k\} \subset L \qquad \text{for one} \quad j \in \{1,..., k\},$$

and this is the only pair of this type contained in L, i.e.,

$$|\{i, i+k\} \cap L| = 1 \qquad \text{for} \quad i \in \{1,..., k\},\ i \neq j.$$

Thus j and $j+1$ are the only elements of L we may omit without violating (i). ∎

By this proposition, (1.4), and the notation

$$\mathscr{S}_L := \{\partial_L f = 0\} \cap \bigcap_{i \notin L} H_i \qquad (L \in \Lambda)$$

we have also proved that

$$\mathscr{S} = \bigcup_{|L| = k} \mathscr{S}_L. \tag{2.2}$$

The first step towards detecting the structure of $\mathscr{S}$ is the answer of the question for the boundary behaviour of the curves $\{\partial_L f = 0\}$ w.r.t. some halfspace H_j.

2.3. Proposition. *Let $L \in \Lambda$ and $w \in \{\partial_L f = 0\}$. If w satisfies a boundary condition w.r.t. some H_j, $j \notin L$, then this is the only boundary condition of this type w satisfies.*

Proof. Let $i, j \notin L$, $i \neq j$, and $w \in \partial H_i \cap \partial H_j$. Then

$$w \in \{\partial_{L \cup \{i,j\}} f = 0\}.$$

Since $L \in \Lambda$, we have $|L \cup \{i,j\}| \geqslant k+2$. From Proposition 1.3 we derive that the dimension of $\{\partial_{L \cup \{i,j\}} f = 0\}$ is less than $m - k - n - 1 = 0$. So our assumption must have been wrong. ∎

The following proposition yields a characterization of the boundary points of the curves $\{\partial_L f = 0\}$ w.r.t. some space segment H_j.

2.4. Proposition. *Let $L \in \Lambda$, $|L| = k$ and $j \notin L$. Then the intersection $\{\partial_L f = 0\} \cap H_j$ is a one-dimensional manifold with boundary. The boundary of this manifold equals the set $\{\partial_{L \cup \{j\}} f = 0\}$.*

Proof. Since PCP(f, W) is nondegenerate, 0 is a regular value for both mappings

$$\partial_L f\colon U \cap \bigcap_{i \in L} \partial H_i \cap H_j \to \mathbb{R}^n, \qquad \partial_{L \cup \{j\}} f\colon U \cap \bigcap_{i \in L} \partial H_i \cap \partial H_j \to \mathbb{R}^n.$$

Hence, a preimage theorem in differential topology (cf. [3, p. 60]) implies that $\{\partial_L f = 0\} \cap H_j$ is a manifold with boundary, and the boundary equals the set

$$\{\partial_L f = 0\} \cap \partial H_j = \{\partial_{L \cup \{j\}} f = 0\}.$$

Furthermore the codimension of $\{\partial_L f = 0\} \cap H_j$ equals n. Hence the dimension of this manifold is given by $m - |L| - n = m - k - n = 1$. ∎

Since ∂H_i is a closed subset of $\mathbb{R}^m$, Proposition 2.3 shows that for sufficiently small neighborhoods V of points $w \in \mathscr{S}_L = \{\partial_L f = 0\} \cap \bigcap_{i \notin L} H_i$ we have

$$\mathscr{S}_L \cap V = \{\partial_L f = 0\} \cap H_j \cap V. \tag{2.5}$$

for some $j \in L^c$. Thus by Proposition 2.4 we conclude that $\mathscr{S}_L$ is a one-dimensional manifold ($L \in \Lambda$, $|L| = k$). As a consequence of (2.2) $\mathscr{S}_L$ is a subset of $\mathscr{S}$. Since $\mathscr{S}_L$ is closed and $\mathscr{S}$ is compact, $\mathscr{S}_L$ must also be compact. By these means the number of connected components of $\mathscr{S}_L$ is finite, and every connected component is a compact smooth curve, hence either diffeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$ or the unit interval $[0, 1] \subset \mathbb{R}$. Hence, by (2.2) $\mathscr{S}$ is the union of a finite number of smooth curves. Let us denote the set of all these curves by Γ and the set of all endpoints of these curves by Γ^*. In conformity with a graph-theoretical terminology let us denote the elements of Γ by *arcs* and the elements of Γ^* by *nodes*. Then the following proposition is valid.

2.6. PROPOSITION. (1) $\Gamma^* = \bigcup_{L \in \Lambda, |L| = k+1} \mathscr{S}_L$.

(2) *For each node* $w \in \mathscr{S}_L$ *with* $L \in \Lambda$, $|L| = k + 1$, *the number of arcs* $\gamma \in \Gamma$, *for which* w *is an endpoint, equals* 1, *if* $L \not\subset \{1, \ldots, 2k\}$ *and* 2, *if* $L \subset \{1, \ldots, 2k\}$.

Proof. (1) By (2.5) we have for a sufficiently small neighborhood V of a point $w \in \mathscr{S}_L$ with $L \in \Lambda$, $|L| = k$,

$$\mathscr{S}_L \cap V = \{\partial_L f = 0\} \cap H_j \cap V \tag{i}$$

for some $j \in L^c$. So $w \in \Gamma^*$ if and only if it belongs to the boundary of the manifold $\{\partial_L f = 0\} \cap H_j$. By Proposition 2.4 this is true if and only if $w \in \{\partial_{L \cup \{j\}} f = 0\}$. But

$$\mathscr{S}_L \cap \{\partial_{L \cup \{j\}} f = 0\} = \mathscr{S}_{L \cup \{j\}}.$$

(2) Let $w \in \Gamma^*$. Then there is some $L \in \Lambda$, $|L| = k+1$ with $w \in \mathscr{S}_L = \{\partial_L f = 0\} \cap \bigcap_{i \notin L} H_i$. By Proposition 2.4 w is on the boundary of each set $\{\partial_K f = 0\} \cap H_j$ with $K \in \Lambda$ and $K \cup \{j\} = L$. By (i) w is a boundary point of each set $\mathscr{S}_K$ with $K \in \Lambda$, $|K| = k$, $K \subset L$, and obviously no other set $\mathscr{S}_K$. But by Proposition 2.1 the number of sets $K \subset L$ with $K \in \Lambda$, $|K| = k$, equals 1, if $L \not\subset \{1,..., 2k\}$, and 2, if $L \subset \{1,..., 2k\}$. Now we have only to verify that two different sets K_1, K_2 yield two different arcs. But this is obvious, because

$$\mathscr{S}_{K_1} \cap \mathscr{S}_{K_2} = \mathscr{S}_L$$

and $\mathscr{S}_L$ is finite; but $\mathscr{S}_{K_1}$ and $\mathscr{S}_{K_2}$ are one-dimensional manifolds. ∎

By Proposition 2.6 we have a subdivision of Γ^* into two different types of nodes: those being an endpoint w.r.t. exactly one arc, which will be denoted by *nodes of degree* 1, and those being an endpoint w.r.t. exactly two arcs, which will be denoted by *nodes of degree* 2. The corresponding subsets of Γ^* will be denoted by $\Gamma^*_{\deg 1}$ and $\Gamma^*_{\deg 2}$, respectively. Then by Proposition 2.6 we obtain a simple characterization of nodes of degree 1 and 2, respectively.

2.7. COROLLARY.

$$\Gamma^*_{\deg 1} = \bigcup_{\substack{L \in \Lambda \\ L \not\subset \{1,\ldots,2k\}}} \mathscr{S}_L, \qquad \Gamma^*_{\deg 2} = \bigcup_{\substack{L \in \Lambda \\ \exists i \in \{1,\ldots,k\}: \\ \{i, i+k\} \subset L}} \mathscr{S}_L.$$

This is now the incentive for some graph-theoretical reflections leading to a description of the solution set $\mathscr{S}$. A subset $\mathscr{P} \subset \mathscr{S}$ is called a *path*, if it can be arranged in the following way: There is a sequence

$$w^{(0)},\ \alpha^{(0,1)},\ w^{(1)},\ldots,\ w^{(r-1)},\ \alpha^{(r-1,r)},\ w^{(r)}$$

of pairwise different arcs $\alpha^{(i-1,i)} \in \Gamma$ $(i = 1,..., r)$ and nodes $w^{(i)} \in \Gamma^*$ $(i = 0,..., r)$ such that $w^{(i-1)}$ and $w^{(i)}$ are the endpoints of $\alpha^{(i-1,i)}$, and $\mathscr{P}$ is the union of all these arcs and nodes. If $w^{(0)} \neq w^{(r)}$ let us call $\mathscr{P}$ a *loop.* Each arc diffeomorphic to S^1 let also be a loop. If $w^{(0)} \neq w^{(r)}$ let us call $\mathscr{P}$ a *channel* and $w^{(0)}$, $w^{(r)}$ the endpoints (of the channel).[3]

Then the following theorem holds.

2.8. THEOREM. *The number of connected components of* $\mathscr{S}$ *is finite. Each connected component* $\mathscr{P}$ *of* $\mathscr{S}$ *is a path.* $\mathscr{P}$ *is a loop, if it does not contain any node of degree* 1; *otherwise it is a channel. In the last case* $\mathscr{P}$ *contains exactly two different nodes of degree* 1 *being the endpoints of the channel* $\mathscr{P}$. *The whole number of nodes of degree* 1 *is even.*

[3] They are not identical with the endpoints of the arcs.

As the proof of this theorem is obvious by the preceding expositions, we are satisfied with some remarks: Since $\mathscr{S}$ consists of a finite number of arcs and nodes, the first statement of the theorem is obvious. The other statements are simple graph-theoretical results for the case where all nodes have degree 1 or 2. But this is true in our case.

Theorem 2.8 is less of theoretical but more of practical interest. By Corollary 2.7 the endpoints of channels can easily be recognized. If one of these nodes of degree 1 is known, we may start a path-following procedure, which ends up in a new node of degree 1. This procedure may be very helpful for many computational problems. The same procedure is used in the famous Lemke–Howson–Algorithm for computing solutions of linear complementarity problems.

3. A.E. Nondegenerate Classes of PCPs

Firstly, it seems to be a very strong requirement for a PCP to satisfy the nondegeneracy assumption. The primary difficulty consists in the fact that in many cases there is no possibility to prove nondegeneracy without knowing all solutions of the PCP. But, since we should like to determine solutions, this trouble cannot be overcome. However, some theorems of the type of Sard's lemma let us place our confidence in the assumption to be satisfied, whenever we are given a class of PCPs depending on real parameters. Fortunately, a lot of examples produce such classes.

In this section let $\mathrm{PCP}_{k,m,n}(f, W \times W^*)$ be a PCP for which $W \subset \mathbb{R}^r$ and $2k \leqslant r$, i.e., complementarity touches only W not W^*. Thus W^* has to be a "parametric part" of the interval.

Defining

$$f(\cdot, w^*)(w) := f(w, w^*) \qquad (w \in W, w^* \in W^*)$$

we obtain a class of PCPs in a very natural way by

$$\Theta := \{\mathrm{PCP}_{k,r,n}(f(\cdot, w^*), W) \mid w^* \in W^*\}.$$

3.1. Definition. The class Θ is said to be almost everywhere (a.e.) *nondegenerate*, if $\mathrm{PCP}(f(\cdot, w^*), W)$ is nondegenerate for almost every $w^* \in W^*$, w.r.t. the Lebesgue measure on W^*.

Then the following theorem holds.

3.2. Theorem. *If* $\mathrm{PCP}(f, W \times W^*)$ *is nondegenerate, then* Θ *is an a.e. nondegenerate class.*

Proof. Since the boundary of W^* has Lebesgue measure zero, we may restrict ourselves to the interior of W^*. Hence let us assume W^* to be an open interval.

Now let L be a complementary subset of $\{1,..., r\}$. It suffices to prove that 0 is a regular value of $\partial_L f(\cdot, w^*)$ for a.e. $w^* \in W^*$. Since there are only finitely many complementary subsets of $\{1,..., r\}$, we conclude that $\mathrm{PCP}(f(\cdot, w^*), W)$ is nondegenerate for a.e. $w^* \in W^*$.

Since L is a complementary subset of $\{1,..., r\}$ and $2k \leqslant r$, L is a complementary subset of $\{1,..., m\}$, too. But $\mathrm{PCP}(f, W \times W^*)$ is nondegenerate. Hence 0 is a regular value of $\partial_L f$, where $\partial_L f$ is defined on a boundaryless manifold of the type $H \times W^*$. Now, by a parametric transversality theorem (cf. [3, p. 68]) we conclude that 0 is a regular value of $\partial_L f(\cdot, w^*)$ for a.e. $w^* \in W^*$, which is what was to be shown. ∎

It is not very surprising that Theorem 3.1 also yields a possibility to connect the solution sets of two nondegenerate problems $\mathrm{PCP}(f(\cdot, w_0^*), W)$ and $\mathrm{PCP}(f(\cdot, w_1^*), W)$ out of Θ by a homotopy between $f(\cdot, w_0^*)$ and $f(\cdot, w_1^*)$ in such a way that this homotopy produces a nondegenerate PCP. Since Θ is a parametric class, we may do this by means of a connection of the parameter values w_0^* and w_1^*. In general it will not suffice to connect them by a line segment, because this may yield a degenerate problem. The next attempt would be to do it by means of an additional parameter $s \in W^*$, defining $w_s^*(t)$ as a convex combination of w_0^*, w_1^* and s of the following type

$$w_s^*(t) := \frac{1}{1 + t(1-t)} \left((1-t)\, w_0^* + t w_1^* + t(1-t)s\right) \qquad (t \in \mathbb{R}). \quad (3.2)$$

By

$$g_s(w, t) := f(w, w_s^*(t)) \qquad ((w, t) \in U), \quad (3.3)$$

where U is a neighborhood of $W \times [0, 1]$, we obtain a class

$$\tilde{\Theta} := \{\mathrm{PCP}_{k, r+1, n}(g_s, W \times [0, 1]) \mid s \in W^*\} \quad (3.4)$$

of PCPs. This way seems to be successful.

3.5. Corollary. *If* $\mathrm{PCP}(f, W \times W^*)$ *is nondegenerate, then* $\tilde{\Theta}$ *is an a.e. nondegenerate class.*

Proof. Without loss of generality let W^* be an open interval. Define

$$G(w, s, t) := g_s(w, t) = f(w, w_s^*(t))$$

in a suitable neighbourhood U of $W \times W^* \times [0, 1]$.

In order to prove nondegeneracy for PCP$(G, W \times W^* \times [0, 1])$ we have to verify that 0 is a regular value for each of the mappings $\partial_L G$ (where L is a complementary subset of $\{1,..., m+1\}$). It suffices to prove that 0 is a regular value for each of the mappings

$$\partial_L G_t := \partial_L G(\cdot, \cdot, t),$$

where t is fixed in a neighborhood of $[0, 1]$ and L is a complementary subset of $\{1,..., m\}$. Since W^* is open, $s \in W^*$ may never satisfy a boundary condition. This is why we may assume L to be a complementary subset of $\{1,..., r\}$. First let $t \in \{0, 1\}$. Then by the definition of $w_s^*(t)$ we have

$$\partial_L G_t = \partial_L f(\cdot, w_0^*) \qquad \text{or} \qquad \partial_L G_t = \partial_L f(\cdot, w_1^*).$$

But PCP$(f(\cdot, w_0^*), W)$ and PCP$(f(\cdot, w_1^*), W)$ are nondegenerate problems. Hence 0 is a regular value for both $\partial_L f(\cdot, w_0^*)$ and $\partial_L f(\cdot, w_1^*)$, and thus for $\partial_L G_t$.

Now let $t \notin \{0, 1\}$ and define a mapping h_t on a suitable neighbourhood V of $W \times W^*$ by

$$h_t(w, s) = (w, w_s^*(t)).$$

Let

$$X_L := \bigcap_{i \in L} \partial H_i \cap V,$$

where H_i is the space segment corresponding to $i \in L$. Then the derivative of $\partial_L h_t : X_L \to X_L$ is given by

$$D\partial_L h_t(w) = \begin{pmatrix} I_{r-|L|} & 0 \\ 0 & \dfrac{t(1-t)}{1+t(1-t)} I_{m-r} \end{pmatrix} \qquad (w \in X_L)$$

where I_j is the identity matrix in $\mathbb{R}^j$. Since $t \notin \{0, 1\}$, $\partial_L h_t$ is a local diffeomorphism. Furthermore

$$\partial_L G_t = \partial_L f \circ \partial_L h_t.$$

Since PCP$(f, W \times W^*)$ is nondegenerate, 0 is a regular value for $\partial_L f$, hence for $\partial_L G_t$, too. Thus, we have proved nondegeneracy for PCP$(G, W \times W^* \times [0, 1])$. Now Theorem 3.1 completes the proof. ∎

4. Classes of Equivalent Standardized Problems

By the results of Sections 2 and 3 we are in a position to connect solution sets of PCPs of order 0 by PCPs of order 1. If we are given two PCPs of order 0 out of a class of PCPs, then the question arises whether there is a PCP of order 1 such that the endpoints of the paths generated by this PCP coincide with the solutions of the PCPs of order 0. In the case where these solution sets are finite, we shall obtain some results on the parity of the number of solutions (and thus, in the case of oddness, existence of solutions). We shall present some sufficient requirements for classes of PCPs of order 0, which often can easily be checked, implying the constant parity property and the existence of paths connecting the solution sets.

4.1. Definition. A problem $\mathrm{PCP}_{k,m,n}(f, W)$ is called *a standardized problem*, if

(1) $\mathrm{ord}(\mathrm{PCP}(f, W)) = 0$.

(2) $\mathrm{PCP}(f, W)$ is nondegenerate.

(3) The solution set $\mathscr{S}(f, W)$ of $\mathrm{PCP}(f, W)$ is finite.

4.2. Definition. Let $\mathrm{PCP}_{k,m,n}(f_0, W)$ and $\mathrm{PCP}_{k,m,n}(f_1, W)$ be standardized problems. Then these problems are called *equivalent*, iff there is some $\mathrm{PCP}_{k,m+1,n}(f, W \times [0, 1])$ (of order 1) such that

(1) $f(\cdot, 0) = f_0, f(\cdot, 1) = f_1$ in a neighbourhood of W;

(2) $\mathrm{PCP}(f, W \times [0, 1])$ is nondegenerate;

(3) the solution set of $\mathrm{PCP}(f, W \times [0, 1])$ is compact;

(4) the set of all endpoints of channels generated by $\mathrm{PCP}(f, W \times [0, 1])$ (cf. Theorem 2.8) is given by

$$\{(w, 0) \mid w \in \mathscr{S}(f_0, W)\} \cup \{(w, 1) \mid w \in \mathscr{S}(f_1, W)\}.[4]$$

A $\mathrm{PCP}(f, W \times [0, 1])$ satisfying (1)–(4) is called a *bridge* from $\mathrm{PCP}(f_0, W)$ to $\mathrm{PCP}(f_1, W)$.

4.3. Lemma. *If* $\mathrm{PCP}_{k,m,n}(f_0, W)$ *and* $\mathrm{PCP}_{k,m,n}(f_1, W)$ *are equivalent standardized problems and* $\mathrm{PCP}_{k,m+1,n}(f, W \times [0, 1])$ *is a bridge from one to the other, then the number of channels in* $\mathscr{S}(f, W \times [0, 1])$ *connecting a solution* $(w_0, 0)$ *with a solution* $(w_1, 1)$ *has the same parity as* $|\mathscr{S}(f_0, W)|$ *and* $|\mathscr{S}(f_1, W)|$.

[4] This is really an equivalence relation, but, since it is not important for the following, the proof is omitted.

Proof. By property (4) of a bridge the set Γ^* of all endpoints of channels is given by

$$\Gamma^* = \Gamma_0^* \cup \Gamma_1^*,$$

where

$$\Gamma_0^* = \{(w, 0) \mid w \in \mathscr{S}(f_0, W)\}, \quad \Gamma_1^* = \{[w, 1) \mid w \in \mathscr{S}(f_1, W)\}.$$

Denote the set of all channels by Γ, the set of all channels connecting a point of Γ_0^* with another point of Γ_0^* by Γ_{00}, the set of all channels connecting a point of Γ_1^* with another point of Γ_1^* by Γ_{11} and the set of all channels connecting a point of Γ_0^* with a point of Γ_1^* by Γ_{01}. Then

$$\Gamma = \Gamma_{00} \cup \Gamma_{11} \cup \Gamma_{01}.$$

By Theorem 2.8 each channel in Γ has two different endpoints, and no two channels have common points. So, by deleting the endpoints of all channels in Γ_{00} from Γ_0^*, this set is reduced by an even number of points. All residual points are endpoints of channels from Γ_{01}. Hence, Γ_{01} may be counted by these points; whose number has the same parity as $|\Gamma_0^*|$. A similar consideration on Γ_1^* completes the proof. ∎

The following theorem presents sufficient conditions for the existence and constructability of bridges within a class of PCPs of order 0.

4.4. THEOREM. *Let* $\mathrm{PCP}_{k,m,n}(f, W \times W^*)$, *where* $W \subset \mathbb{R}^r$ *with* $2k \leqslant r$, *be a problem with the following properties*:

(1) $\mathrm{PCP}(f, W \times W^*)$ *is nondegenerate.*

(2) *The set of all solutions of* $\mathrm{PCP}(f, W \times W^*)$ *in* $W \times C$ *is compact, if* C *is a compact subset of* W^*.

(3) *For each solution* (w, w^*) *of* $\mathrm{PCP}(f, W \times W^*)$ *the vector* $w \in W$ *satisfies only complementary boundary conditions.*

Furthermore, let $r - k - n = 0$; *thus*

$$\Theta := \{\mathrm{PCP}_{k,r,n}(f(\cdot, w^*), W) \mid w^* \in W^*\}$$

is a class of PCPs *of order* 0. *Then*

$$\Theta_{nd} := \{P \in \Theta \mid P \text{ is nondegenerate}\}$$

is a class of equivalent standardized problems. For two problems of Θ_{nd} *a class* $\tilde{\Theta}$ *of appropriate bridges may be constructed by* (3.2)–(3.4).

Proof. Let $\mathrm{PCP}(f(\cdot, w^*), W) \in \Theta_{nd}$. Then, by (2), the solution set of $\mathrm{PCP}(f(\cdot, w^*), W)$ is compact. Since the solution set of a nondegenerate PCP of order 0 consists of isolated points, this set must be finite. Hence Θ_{nd} is a class of standardized problems. Now let $\mathrm{PCP}(f(\cdot, w_0^*), W)$, $\mathrm{PCP}(f(\cdot, w_1^*), W) \in \Theta_{nd}$ and construct the class

$$\tilde{\Theta} := \{\mathrm{PCP}(g_s, W \times [0, 1]) \mid s \in W^*\}$$

for these two problems by (3.2)–(3.4). As a consequence of (1) and Corollary 3.5 $\tilde{\Theta}$ is an a.e. nondegenerate class. Thus there exists a nondegenerate problem $P := \mathrm{PCP}(g_s, W \times [0, 1])$.

We have to verify that P is a bridge.

By (3.2) and (3.3) we have

$$g_s(\cdot, 0) = f(\cdot, w_0^*),\ g_s(\cdot, 1) = f(\cdot, w_1^*).$$

Hence, P satisfies (1) in Definition 4.2.

Since the curve

$$C := \{w_s^*(t) \mid t \in [0, 1]\}$$

is compact, by (2), the set $\mathscr{S}_c$ of solutions of $\mathrm{PCP}(f, W \times W^*)$ in $W \times C$ is compact. But the solution set $\tilde{\mathscr{S}}$ of P is the preimage of $\mathscr{S}_c$ by the mapping $(w, t) \to (w, w_s^*(t))$. Hence, $\tilde{\mathscr{S}}$ is compact. Thus P satisfies (3) in Definition 4.2. Let $(w, t) \in \tilde{\mathscr{S}}$. Then $(w, w_s^*(t)) \in \mathscr{S}(f, W \times W^*)$. By (3), w satisfies only complementary boundary conditions. By Corollary 2.7 (w, t) is a node of degree 1 for P, iff it satisfies a noncomplementary boundary condition. Hence (w, t) is a node of degree 1, iff $t \in \{0, 1\}$. By Theorem 2.8 (w, t) is a node of degree 1, iff it is an endpoint of a channel. By these means and the Definition (3.3) of g_s we have proved property (4) of Definition 4.2 for P. Hence P is a bridge. ∎

Assumptions (1), (2), and (3) of Theorem 4.4 seem to be very strong. But it is not very difficult to find examples of classes satisfying them. One example is the noncooperative N-person game and the joined class of PCPs. In this case W^* is the set of all payoff vectors for the players when using pure strategies. Another example is the generalized von Neumann model and the joined class of PCPs. In this case W^* is the set of all fourtuples of matrices determining the model. In both cases the validity of (1) crucially depends on the components of f associated with W^*. Assumption (2) is always satisfied if W is closed and the solution sets of the problems $\mathrm{PCP}(f(\cdot, w^*), W)$ are uniformly bounded for bounded subsets of W^*. This can be proved for both examples. Of course, assumption (3) can be dropped if w contains only complementary components and no other parameters. This is not true for the second example, but from the type of equations it is easily verified.

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