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Stability under Persistent Disturbances for Systems with Impulse Effect

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Problems concerning the stability of the solution of systems of ordinary differential equations with impulse effect under persistent disturbances are investigated for the first time. Definitions for stability and unstability of the system considered are introduced. The main results are formulated in two theorems. In the proofs of the theorems a new analogue of the Gronwall–Belmann inequality for piecewise continuous functions is applied. © 1985 Academic Press, Inc.

1. INTRODUCTION

The present paper considers the following system of differential equation with impulse effect:

$$\frac{dx}{dt} = f(t, x), \qquad t \neq \tau_i(x),$$

$$\Delta x|_{t=\tau_i(x)} = B_i(x).$$

Systems of such type can be found in many problems of physics, engineering and biology. Its investigation was initiated by the papers of Mil'man and Myshkis [1, 2] and Myshkis and Samoilenko [3]. The stability of the solutions is treated in papers by Samoilenko and Perestiuk [4, 5], where linear and quasi-linear systems are considered in detail.

This paper deals with the questions of stability of the solutions of systems with impulse effect under persistent disturbances. The definitions of stability are those of [4] and are given in the form used in [6].

2. PRELIMINARY REMARKS

Let R^n be n-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\alpha \in R$, $I = [\alpha, \infty)$ and let $\Omega \subset R^n$ be a region.

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Consider the following system with impulse effect consisting of n differential equations,

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \qquad t \neq \tau_i(x), \\ \Delta x|_{t=\tau_i(x)} &= B_i(x) \end{aligned} \tag{1}$$

and the corresponding perturbed system with impulse effect,

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) + g(t, x), \qquad t \neq \tau_i(x), \\ \Delta x|_{t=\tau_i(x)} &= B_i(x) + P_i(x), \end{aligned} \tag{2}$$

where $x\colon I \to R^n$; $f, g\colon I \times \Omega \to R^n$; $B_i, P_i\colon \Omega \to R^n$; $\tau_i\colon R^n \to R$ $(i = 1, 2, ...)$.

Such systems are subject to momentary forces when meeting the mapping point (t, x) of the extended phase space with the hypersurfaces, given by the equations

$$t = \tau_i(x), \qquad \alpha < \tau_1(x) < \tau_2(x) < \cdots < \tau_i(x) < \cdots, \quad i = 1, 2, ...$$

Under the momentary effect (hit, impulse) the mapping point "instantly" jumps from the position (t, x) in the position $(t, x + B_i(x))$. We shall suppose that the solutions of the systems (1) and (2) are continuous on the left, i.e., the following conditions hold when the integral curve $(t, x(t))$ meets the hypersurfaces $t = \tau_i(x)$:

$$\begin{aligned} x(t_i - 0) &= x(t_i), \\ \Delta x|_{t=t_i} &\equiv x(t_i + 0) - x(t_i - 0). \end{aligned}$$

We say that the conditions (A) hold if the following conditions are fulfilled:

A1. The function $f(t, x)$ is continuous in the domain $I \times \Omega$.

A2. The function $f(t, x)$ satisfies the Lipschitz condition with respect to x with a constant $a \geqslant 0$, i.e.,

$$\|f(t, x) - f(t, y)\| \leqslant a\, \|x - y\| \qquad \forall t \in I, \quad \forall x, y \in \Omega.$$

A3. $\|f(t, x)\| \leqslant M$ $\forall t \in I$, $\forall x \in \Omega$ $(M \geqslant 0)$.

We say that the conditions (B) hold if the following conditions are fulfilled:

B1. The function $g(t, x)$ is continuous in the domain $I \times \Omega$.

B2. The function $g(t, x)$ satisfies the Lipschitz condition with respect to x with a constant $\mathscr{C} \geqslant 0$, i.e.,

$$\|g(t, x) - g(t, y)\| \leqslant \mathscr{C} \, \|x - y\| \qquad \forall t \in I, \quad \forall x, y \in \Omega.$$

We say that the conditions (C) hold if the following conditions are fulfilled:

C1. $\|B_i(x) - B_i(y)\| \leqslant c \, \|x - y\| \; \forall x, y \in \Omega, \forall i = 1, 2, \ldots$.

C2. $\|P_i(x) - P_i(y)\| \leqslant c \, \|x - y\| \; \forall x, y \in \Omega, \forall i = 1, 2, \ldots \; (c \geqslant 0)$.

We say that the conditions (D) hold if the following conditions are fulfilled:

D1. The functions $\tau_i(x)$, $i = 1, 2, \ldots$, are continuously differentiable in the domain Ω.

D2. $\sup_{x \in \Omega} \|\partial \tau_i(x)/\partial x\| \leqslant N \; (N \geqslant 0)$.

D3. $\inf_{x \in \Omega} \tau_{i+1}(x) - \sup_{x \in \Omega} \tau_i(x) \geqslant \theta > 0$.

Remark 1. From the condition D2 it follows that $\forall x, y \in \Omega$ and $\forall i = 1, 2, \ldots$, the following inequality holds:

$$|\tau_i(x) - \tau_i(y)| \leqslant N \, \|x - y\|.$$

We say that the conditions (E) holds if the following condition is fulfilled:

E. There exists a number $h > 0$ such that

$$\sup_{\substack{x \in \Omega \\ 0 \leqslant s \leqslant 1 \\ \|z\| \leqslant h}} \left\langle \frac{\partial \tau_i(x + s(B_i(x) + z))}{\partial x}, B_i(x) + z \right\rangle \leqslant 0, \qquad i = 1, 2, \ldots .$$

Let $t_0 \in I$ and $x_0 \in \Omega$. Denote as $x(t; t_0, x_0)$ the solution of the system (1) (or (2)) for which $x(t_0 + 0; t_0, x_0) = x_0$. We denote as $J^+(t_0, x_0)$ the maximal interval of the type $(t_0, \bar{t})$ in which the solution $x(t; t_0, x_0)$ is continuable on the right.

In the sequel we give definitions for stability of systems with impulse effect which shall be further used.

For $t \geqslant \alpha$ let the solution $x = \varphi(t)$ of the system (1) be defined as the integral curve which meets the hypersurfaces $t = \tau_i(x)$ at the moments t_i $(\alpha < t_1 < \cdots t_i < \cdots)$ and $\lim_{i \to \infty} t_i = \infty$.

DEFINITION 1. The solution $x=\varphi(t)$ of the system (1) is called uniformly stable if

$$(\forall\varepsilon>0)(\forall\eta>0)(\exists\delta>0)\colon(\forall t_0\in I)(\forall x_0\in\Omega,\ \|x_0-\varphi(t_0+0)\|<\delta)$$
$$(\forall t\in J^+(t_0,x_0),\ |t-t_i|>\eta)$$
$$\|x(t;t_0,x_0)-\varphi(t)\|<\varepsilon.$$

DEFINITION 2. The solution $x=\varphi(t)$ of the system (1) is called uniformly attractive if

$$(\exists\lambda>0)(\forall\varepsilon>0)(\forall\eta>0)(\exists\sigma>0)$$
$$(\forall t_0\in I)(\forall x_0\in\Omega,\ \|x_0-\varphi(t_0+0)\|<\lambda)\ t_0\in\sigma\in J^+(t_0,x_0)$$

and

$$(\forall t\geqslant t_0+\sigma,\ t\in J^+(t_0,x_0),\ |t-t_i|>\eta)$$
$$\|x(t;t_0,x_0)-\varphi(t)\|<\varepsilon.$$

DEFINITION 3. The solution $x=\varphi(t)$ of the system (1) is called uniformly asymptotically stable, if it is uniformly stable and uniformly attractive.

DEFINITION 4. The solution $x=\varphi(t)$ of the system (1) is called unstable if

$$(\exists\varepsilon>0)(\exists\eta>0)(\exists t_0\in I)(\forall\delta>0)$$
$$(\exists x_0\in\Omega,\ \|x_0-\varphi(t_0+0)\|<\delta)(\exists t^*\in J^+(t_0,x_0),\ |t^*-t_i|>\eta)$$
$$\|x(t^*;t_0,x_0)-\varphi(t^*)\|\geqslant\varepsilon.$$

DEFINITION 5. The solution $x=\varphi(t)$ of the system (1) is called uniformly stable under persistent disturbances if

$$(\forall\varepsilon>0)(\forall\eta>0)(\exists r>0)(\exists\rho>0)(\forall x_0\in\Omega,\ \|x_0-\varphi(t_0+0)\|<r)$$

the solution $x(t;t_0,x_0)$ of the system (2) satisfies

$$(\forall g\colon\forall(t,x)\in I\times\Omega\ \|g(t,x)\|<\rho)$$
$$(\forall i=1,2,\ldots\ \forall P_i\colon\forall x\in\Omega\ \|P_i(x)\|<\rho)$$
$$(\forall t\in J^+(t_0,x_0),\ |t-t_i|>\eta)$$
$$\|x(t;t_0,x_0)-\varphi(t)\|<\varepsilon.$$

DEFINITION 6. The solution $x=\varphi(t)$ of the system (1) is called strongly unstable under persistent disturbances if

$$(\exists\varepsilon>0)(\exists\eta>0)(\exists t_0\in I)(\forall r>0)(\exists\rho>0)$$
$$(\exists x_0\in\Omega,\ \|x_0-\varphi(t_0+0)\|<r)$$

the solution $x(t;t_0,x_0)$ of the system (2) satisfies

$$(\forall g\colon \forall(t,x)\in I\times\Omega\ \|g(t,x)\|<\rho)$$
$$(\forall i=1,2,...\ \forall P_i\colon \forall x\in\Omega\ \|P_i(x)\|<\rho)$$
$$(\exists t^*\in J^+(t_0,x_0),\ |t^*-t_i|>\eta)$$
$$\|x(t^*;t_0,x_0)-\varphi(t^*)\|\geqslant\varepsilon.$$

We say that the solution $x=\varphi(t)$ of the system (1) fulfills the condition (F) if the following condition holds:

F. For $t\in I$ the solution $x=\varphi(t)$ has values in Ω and it has no limit points on the boundary of Ω, i.e., there exists a number $d>0$ such that $\{x\in R^n\colon \|x-\varphi(t)\|\leqslant d$ for some $t\in I\}\subset\Omega$.

3. MAIN RESULTS

For proving the main results we shall use the following lemmas.

LEMMA 1 (Absence of beating). *Let the conditions* (A), C1, (D) *and* (E) *hold and let* $\varphi(t)$ *be a solution of the system* (1) *with values in* Ω *for* $t\in[t_0,t_0+T]$.

Then if $MN<1$ *the integral curve* $(t,\varphi(t))$ *for* $t\in[t_0,t_0+T]$ *meets every hypersurface* $t=\tau_i(x)$ *only once.*

The Lemma 1 gives a sufficient condition excluding "beating" of the solution on the hypersurfaces $t=\tau_i(x)$, i.e., the effect when the integral curve meets the hypersurface $t=\tau_i(x)$ several or infinite many times. The proof of Lemma 1 is given in [4].

LEMMA 2. *Let the conditions* (A), C1, (D) *and* (E) *hold and* $MN<1$. *Let* $\varphi(t)$ *be a solution of the system* (1) *which fulfills the condition* (F) *and the integral curve* $(t,\varphi(t))$ *meets the hypersurfaces* $t=\tau_i(x)$ *at the moments* t_i, $i=1,2,...$.

Let $x(t)=x(t;\alpha,x_0)$ *be a solution of the system* (1).

If for $\varepsilon > 0$, $\eta > 0$: $(1+c)(\varepsilon + 5M\eta) < d$, $\eta < \theta/2$ *and* $t \in J^+(\alpha, x_0)$, $|t - t_i| > \eta$ *the following condition holds,*

$$\|x(t) - \varphi(t)\| < \varepsilon \tag{3}$$

then $J^+(t_0, x_0) = (\alpha, \infty)$.

Proof. According to Lemma 1 the integral curve $(t, \varphi(t))$ meets each of the hypersurfaces $t = \tau_i(x)$ only once at the moments $t = t_i$ for which, according to the condition D3, we have

$$\inf_{x \in \Omega} \tau_i(x) \leqslant t_i \leqslant \sup_{x \in \Omega} \tau_i(x), \qquad i = 1, 2, \ldots,$$

$$t_{i+1} - t_i \geqslant \theta, \qquad i = 1, 2, \ldots .$$

Let

$$\alpha_i = \tfrac{1}{2}[\sup_{x \in \Omega} \tau_i(x) + \inf_{x \in \Omega} \tau_{i+1}(x)], \qquad i = 1, 2, \ldots .$$

At first we show that the solution $x(t)$ is continuable in the interval $(\alpha, \alpha_1]$. From the condition D3 it follows that for $t \in (\alpha, \alpha_1]$ the integral curve $(t, x(t))$ can meet the hypersurface $t = \tau_1(x)$ only and t_1' is the first moment of such meeting.

Let $t_1' < t_1 - \eta$. Then according to the inequality (3) and the conditions (A) and (F) the solution $x(t)$ is continuable to $t = t_1'$ and $\|x(t_1' + 0) - \varphi(t_1')\| < \varepsilon < d$; i.e., $x(t_1 + 0) \in \Omega$ and $x(t)$ is continuable to $t = t_1 - \eta$. Moreover

$$\|x(t_1 - \eta) - \varphi(t_1 - \eta)\| \leqslant \varepsilon.$$

It follows from the inequality (3) and Lemma 1 that for $t \in (\alpha, t_1 - \eta]$ $x(t)$ does not leave Ω and the integral curve $(t, x(t))$ no longer meets the hypersurface $t = \tau_1(x)$.

For $t \in J^+(\alpha, x_0) \cap [t_1 - \eta, \alpha_1]$ the following estimates hold:

$$\|x(t) - \varphi(t)\| < \varepsilon \qquad \text{for} \quad t \in (t_1 + \eta, \alpha_1]$$

and

$$\begin{aligned} \|x(t) - \varphi(t_1 - \eta)\| &\leqslant \|x(t) - x(t_1 - \eta)\| + \|x(t_1 - \eta) - \varphi(t_1 - \eta)\| \\ &\leqslant 2M\eta + \varepsilon \qquad \text{for} \quad |t - t_1| \leqslant \eta. \end{aligned} \tag{4}$$

Hence, for $t \in J^+(\alpha, x_0) \cap (\alpha, \alpha_1]$ the solution $x(t)$ does not leave Ω, it is continuable to $t = \alpha_1$ and the integral curve $(t, x(t))$ meets the hypersurface $t = \tau_1(x)$ only once. Moreover, for $|t - t_1| \leqslant \eta$ the estimate (4) holds.

If $t_1' > t_1 + \eta$ by analogical argument we come to the same conclusions.

Let $t_1' \in [t_1 - \eta, t_1 + \eta]$. Then the solution $x(t)$ is continuable to t_1' and the following estimates hold:

$$\|x(t_1 - \eta) - \varphi(t_1 - \eta)\| \leqslant \varepsilon,$$
$$\|x(t) - \varphi(t_1 - \eta)\| \leqslant 2M\eta + \varepsilon \qquad \text{for} \quad t \in [t_1 - \eta, t_1'].$$

Particularly,

$$\|x(t_1') - \varphi(t_1 - \eta)\| \leqslant 2M\eta + \varepsilon.$$

Then

$$\begin{aligned} &\|x(t_1' + 0) - \varphi(t_1 + 0)\| \\ &\quad \leqslant \|x(t_1') + B(x(t_1')) - \varphi(t_1) - B_1(\varphi(t_1))\| \\ &\quad \leqslant (1 + c)\, \|x(t_1') - \varphi(t_1)\| \\ &\quad \leqslant (1 + c)(\|x(t_1') - \varphi(t_1 - \eta)\| + \|\varphi(t_1 - \eta) - \varphi(t_1)\|) \\ &\quad \leqslant (1 + c)(2M\eta + \varepsilon + M\eta) = (1 + c)(3M\eta + \varepsilon) < d. \end{aligned}$$

Hence, $x(t_1' + 0) \in \Omega$ and the solution $x(t)$ is continuable on the right of t_1'.

For $t \in J^+(\alpha, x_0) \cap (t_1', \alpha_1]$ the following estimates hold:

$$\|x(t) - \varphi(t)\| < \varepsilon \qquad \text{for} \quad t_1 + \eta < t \leqslant \alpha_1$$

and

$$\begin{aligned} \|x(t) - \varphi(t_1 + 0)\| &\leqslant \|x(t) - x(t_1' + 0)\| + \|x(t_1' + 0) - \varphi(t_1 + 0)\| \\ &\leqslant 2M\eta + (1 + c)(3M\eta + \varepsilon) \\ &< (1 + c)(5M\eta + \varepsilon) < d \qquad \text{for} \quad t_1' < t < t_1 + \eta. \end{aligned}$$

Hence, for $t \in J^+(\alpha, x_0) \cap (\alpha, \alpha_1]$ the solution $x(t)$ does not leave Ω, it is continuable to $t = \alpha_1$ and the integral curve $(t, x(t))$ meets the hypersurface $t = \tau_1(x)$ only once.

By mathematical induction and analogical arguments one can easily check that the solution $x(t)$ is continuable for each $t = \alpha_i$, i.e., $J^+(t_0, x_0) = (\alpha, \infty)$.

LEMMA 3. *Suppose that the following conditions hold:*

1. *For each $t \geqslant t_0$ the function $u(t)$ is nonnegative and piecewise continuous with discontinuities from the first type, at which $u(t)$ is continuous from the left.*

2. *For each $t \geqslant t_0$ the function $a(t)$ is nonnegative and continuous.*
3. *The sequence $\{t_i\}_{i=1}^{\infty}$ fulfills the condition*

$$t_0 < t_1 < t_2 < \cdots < t_i < \cdots, \qquad \lim_{i \to \infty} t_i = \infty.$$

4. *For $t \geqslant t_0$ the following inequality holds,*

$$u(t) \leqslant u_0 + \int_{t_0}^{t} a(s)\, u(s)\, ds + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0) \tag{5}$$

where $u_0 \geqslant 0$, $\beta_i \geqslant 0$ are constants.

Then for $t \geqslant t_0$

$$u(t) \leqslant u_0 \prod_{t_0 < t_i < t} (1 + \beta_i) \exp\left[\int_{t_0}^{t} a(s)\, ds\right]. \tag{6}$$

Proof. Let $t \in [t_0, t_1]$. Then the inequality (5) will take the form

$$u(t) \leqslant u_0 + \int_{t_0}^{t} a(s)\, u(s)\, ds \tag{7}$$

and the inequality (6) becomes

$$u(t) \leqslant u_0 \exp\left[\int_{t_0}^{t} a(s)\, ds\right].$$

Denote the right hand side of (7) as $z(t)$. Then we get the inequalities

$$u(t) \leqslant z(t)$$

and

$$z(t) \leqslant u_0 + \int_{t_0}^{t} a(s)\, z(s)\, ds. \tag{8}$$

Since for $t \geqslant t_0$ the function $z(t)$ is continuous, applying the Gronwall–Belmann lemma to the integral inequality (8) we get

$$u(t) \leqslant z(t) \leqslant u_0 \exp\left[\int_{t_0}^{t} a(s)\, ds\right].$$

This proves Lemma 3 for $t \in [t_0, t_1]$.

Let $t \in (t_1, t_2]$. Then

$$\begin{aligned} u(t) &\leqslant u_0 + \int_{t_0}^{t} a(s)\, u(s)\, ds + \beta_1 u(t_1 - 0) \\ &\leqslant u_0 + \int_{t_0}^{t_1} a(s)\, u_0 \exp\left[\int_{t_0}^{s} a(\tau)\, d\tau\right] ds \\ &\quad + \int_{t_1}^{t} a(s)\, u(s)\, ds + \beta_1 u_0 \exp\left[\int_{t_0}^{t_1} a(s)\, ds\right] \\ &= u_0(1+\beta_1) \exp\left[\int_{t_0}^{t_1} a(s)\, ds\right] + \int_{t_1}^{t} a(s)\, u(s)\, ds. \end{aligned}$$

Hence

$$\begin{aligned} u(t) &\leqslant u_0(1+\beta_1) \exp\left[\int_{t_0}^{t_1} a(s)\, ds\right] \exp\left[\int_{t_1}^{t} a(s)\, ds\right] \\ &= u_0(1+\beta_1) \exp\left[\int_{t_0}^{t} a(s)\, ds\right]. \end{aligned}$$

Using the mathematical induction method and analogical arguments one can easily check that the inequality (6) holds for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots$, i.e., for each $t \geqslant t_0$.

THEOREM 1. *Suppose that the following conditions hold*:

1. *The conditions* (A), (B), (C), (D) *and* (E) *hold and* $MN < 1$.

2. *The function* $x = \varphi(t)$ *is an uniformly asymptotically stable solution of the system* (1) *which fulfills the condition* (F).

Then the solution $x = \varphi(t)$ *is uniformly stable under persistent disturbances.*

Proof. According to Lemma 1 the integral curve $(t, \varphi(t))$ meets each hypersurface $t = \tau_i(x)$ only once and this occurs at the moments t_i, $i = 1, 2, \ldots$.

Let $\varepsilon > 0$, $\eta > 0$ be given and $(1+c)(5M\eta + \varepsilon) < d$. Let

$$\bar{\varepsilon} = \min\left(\varepsilon, \frac{1-MN}{3N}\eta\right), \qquad \bar{\eta} = \min\left(\eta, \frac{1-MN}{8MN}\eta\right). \tag{9}$$

Let us denote as $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; t_0, x_0)$ the solutions of the system (1) and the system (2), and let their integral curves meet the hypersurfaces $t = \tau_i(x)$ at the moments $\{t_i'\}$ and $\{t_i''\}$, respectively. Let

$J^+(x)$ and $J^+(y)$ be the maximal intervals on which the solutions $x(t)$ and $y(t)$ are continuable on the right.

From the asymptotic stability of $\varphi(t)$ it follows that there exists δ, $\delta < \bar{\varepsilon}/2$ such that for each $t_0 \in I$ for each $x_0 \in \Omega$, $\|x_0 - \varphi(t_0+0)\| < \delta$ and for each $t \in J^+(x)$, $|t - t_i| > \bar{\eta}$

$$\|x(t; t_0, x_0) - \varphi(t)\| < \frac{\bar{\varepsilon}}{2}. \tag{10}$$

Then $(1+c)(\bar{\varepsilon}/2 + 5M\bar{\eta}) < (1+c)(\varepsilon + 5M\eta) < d$, and according to Lemma 2, $J^+(x) = (t_0, \infty)$. Moreover

$$\begin{aligned} |t_i' - t_i| &= |\tau_i(x(t_i')) - \tau_i(\varphi(t_i))| \leqslant N\,\|x(t_i') - \varphi(t_i)\| \\ &\leqslant N\,\|x(t_i') - x(t_i - \bar{\eta})\| + N\,\|x(t_i - \bar{\eta}) - \varphi(t_i - \bar{\eta})\| \\ &\quad + N\,\|\varphi(t_i - \bar{\eta}) - \varphi(t_i)\| \leqslant MN\,|t_i' - t_i + \bar{\eta}| + N\frac{\bar{\varepsilon}}{2} + NM\bar{\eta} \\ &\leqslant MN\,|t_i' - t_i| + 2MN\bar{\eta} + N\frac{\bar{\varepsilon}}{2}. \end{aligned}$$

Hence the following estimate holds:

$$|t_i' - t_i| \leqslant \frac{4MN\bar{\eta} + N\bar{\varepsilon}}{2(1 - MN)}. \tag{11}$$

Since $\varphi(t)$ is an uniformly attractive solution of the system (1) it follows that there exist $\lambda > 0$ and $\sigma = \sigma(\varepsilon, \eta) > 0$ such that for each $t_0 \in I$ and for every $x_0 \in \Omega$, $\|x_0 - \varphi(t_0+0)\| < \lambda$, $t_0 + \sigma \in J^+(x)$ and

$$\|x(t; t_0, x_0) - \varphi(t)\| < \frac{\delta}{2} \tag{12}$$

for every $t \geqslant t_0 + \sigma$, $t \in J^+(x)$, $|t - t_i| > \eta$.

Without loss of generality we suppose that $t_0 + \sigma$ belongs to an interval of the type $[\sup_{x \in \Omega} \tau_m(x), \inf_{x \in \Omega} \tau_{m+1}(x)]$, since otherwise we can choose such a number $\sigma_1 > \sigma$ so that $t_0 + \sigma_1$ belongs to an interval of the given type.

Let the number ρ satisfy the conditions:

$$0 < \rho < \min\left(h, \frac{1 - MN}{2N}\right), \tag{13}$$

$$\rho\sigma\frac{\theta + 1}{\theta}\,[1 + c(1 - MN - \rho N)^{-1}]^{\sigma/\theta}\,e^{a\sigma} < \frac{\delta}{2}. \tag{14}$$

Suppose that the perturbations g and $\{P_i\}$ are such that

$$\|g(t, x)\| < \rho, \ \|P_i(x)\| < \rho \qquad \text{for} \quad t \in I \quad \text{and} \quad x \in \Omega.$$

Let in the interval $(t_0, t]$ the integral curves $(t, \varphi(t))$, $(t, x(t))$ and $(t, y(t))$ meet the hypersurfaces $t = \tau_i(x)$ $n(t)$, $n'(t)$ and $n''(t)$ times, respectively. We prove that for

$$t \in J^+(y), \qquad t_0 < t \leqslant t_0 + \sigma, \qquad |t - t_i| > \eta$$

the following relations hold:

$$\|x(t) - y(t)\| < \frac{\delta}{2}, \tag{15}$$

$$|t_i' - t_i| \leqslant \eta, \qquad |t_i'' - t_i| \leqslant \eta, \tag{16}$$

$$n(t) = n'(t) = n''(t). \tag{17}$$

For $t \in J^+(x)$, $t \in J^+(y)$ the solutions $x(t)$ and $y(t)$ satisfies the relations

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds + \sum_{t_0 < t_i' < t} B_i(x(t_i')), \tag{18}$$

$$y(t) = y_0 + \int_{t_0}^{t} [f(s, y(s)) + g(s, y(s))]\, ds$$
$$+ \sum_{t_0 < t_i'' < t} B_i(y(t_i'')) + \sum_{t_0 < t_i'' < t} P_i(y(t_i'')). \tag{19}$$

Then for $t_0 < t \leqslant \min(t_1', t_1'')$ the following estimate holds:

$$\|y(t) - x(t)\| \leqslant \int_{t_0}^{t} a\, \|y(s) - x(s)\|\, ds + \rho(t - t_0).$$

Hence, according to the Gronwall–Belmann lemma and (14), we get

$$\|y(t) - x(t)\| \leqslant \rho\sigma e^{a\sigma} < \frac{\delta}{2} < \frac{\bar{\varepsilon}}{4}.$$

Taking into account Remark 1 we have

$$|t_i'' - t_1'| = |\tau_1(y(t_1'')) - \tau_1(x(t_1'))| \leqslant N\, \|y(t_1'') - x(t_1')\|. \tag{20}$$

If $t_1'' \leqslant t_1'$ then

$$\|y(t_1'') - x(t_1')\| \leqslant \|y(t_1'') - x(t_1'')\| + \|x(t_1'') - x(t_1')\| \leqslant \frac{\bar{\varepsilon}}{4} + M\,|t_1'' - t_1'|. \tag{21}$$

If $t_1' \leqslant t_1''$ then

$$\|y(t_1'') - x(t_1')\| \leqslant \|y(t_1'') - y(t_1')\| + \|y(t_1') - x(t_1')\| \leqslant (M+\rho)\,|t_1'' - t_1'| + \frac{\bar{\varepsilon}}{4}. \tag{22}$$

From (20), (21) and (22) it follows that for $t_0 < t \leqslant \min(t_1', t_1'')$

$$|t_1'' - t_1'| \leqslant \frac{N\bar{\varepsilon}}{4(1 - MN - N\rho)} \tag{23}$$

and from (9), (11), (23) and (13) we get finally

$$|t_1' - t_1| \leqslant \frac{4MN\bar{\eta} + N\bar{\varepsilon}}{2(1 - MN)} < \eta,$$

$$\begin{aligned} |t_1'' - t_1| &\leqslant |t_1'' - t_1'| + |t_1' - t_1| \\ &\leqslant \frac{N\bar{\varepsilon}}{4(1 - MN - N\rho)} + \frac{4MN\bar{\eta} + N\bar{\varepsilon}}{2(1 - MN)} \\ &\leqslant \frac{8MN\bar{\eta} + 3N\bar{\varepsilon}}{4(1 - MN - N\rho)} < \frac{8MN\bar{\eta} + 3N\bar{\varepsilon}}{2(1 - MN)} \leqslant \eta. \end{aligned}$$

Hence for $t \in (t_0, t_1 - \eta]$ the inequalities (15), (16) hold and $n(t) = n'(t) = n''(t)$.

In the case when $t \in [t_i + \eta, t_{i+1} - \eta]$, $t \in J^+(y)$, $t_0 < t \leqslant t_0 + \sigma$ the inequalities (15), (16) and the equality (17) can be proved by induction with respect to i. For completeness we prove only that from the condition $n(t) = n'(t) = n''(t) = i$ the inequality (15) follows.

Apparently, from (18) and (19) for the function $u(t) = \|y(t) - x(t)\|$ we get

$$u(t) \leqslant \int_{t_0}^{t} au(s)\,ds + \rho(t - t_0) + \rho n(t) + \sum_{t_0 < t_i' < t} c\,\|y(t_i'') - x(t_i')\|. \tag{24}$$

If we denote $\bar{t}_i = \min(t_i', t_i'')$ then, as was done for the estimates (20), (21) and (22), we get the estimate

$$\|y(t_i'') - x(t_i')\| \leqslant (1 - MN - N\rho)^{-1} \|y(\bar{t}_i) - x(\bar{t}_i)\|. \tag{25}$$

From (24) and (25) and taking account that

$$t - t_0 \leqslant \sigma \qquad \text{and} \qquad n(t) \leqslant \frac{t - t_0}{\theta} \leqslant \frac{\sigma}{\theta}$$

we obtain

$$u(t) \leqslant \rho\sigma \frac{\theta + 1}{\theta} + \int_{t_0}^{t} au(s)\, ds + \sum_{t_0 < \bar{t}_i < t} c(1 - MN - N\rho)^{-1} u(\bar{t}_i). \tag{26}$$

Applying Lemma 3 to the integral inequality (26) we get the estimate

$$\|y(t) - x(t)\| \leqslant \rho\sigma \frac{\theta + 1}{\theta} [1 + c(1 - MN - N\rho)^{-1}]^{\sigma/\theta} e^{a\sigma} < \frac{\delta}{2}. \tag{27}$$

We show that for this choice of $\delta > 0$, $\rho > 0$ the solution $y(t) = y(t; t_0, x_0)$ of the system (2) for $t \in J^+(y)$, $|t - t_i| > \eta$ satisfies the inequality

$$\|y(t) - \varphi(t)\| < \varepsilon. \tag{28}$$

Clearly, from (10) and (27) it follows that for $t \in J^+(y)$, $t_0 < t \leqslant t_0 + \sigma$, $|t - t_i| > \eta$,

$$\|y(t) - \varphi(t)\| \leqslant \|y(t) - x(t)\| + \|x(t) - \varphi(t)\| < \frac{\delta}{2} + \frac{\bar{\varepsilon}}{2} < \frac{\bar{\varepsilon}}{4} + \frac{\bar{\varepsilon}}{2} < \varepsilon.$$

Since $(1 + c)(5M\eta + \varepsilon) < d$ from Lemma 2 it follows that for $t \in J^+(y)$, $t \in (t_0, t_0 + \sigma]$ the solution $y(t)$ of the system (2) does not leave Ω. Hence $(t_0, t_0 + \sigma] \subset J^+(y)$ and the integral curve $(t, y(t))$ for $t \in (t_0, t_0 + \sigma]$ meets the hypersurfaces only once.

From (12) and (27) for $t = t_0 + \sigma$ we get

$$\begin{aligned} \|y(t_0 + \sigma) - \varphi(t_0 + \sigma)\| &\leqslant \|y(t_0 + \sigma) - x(t_0 + \sigma)\| \\ &\quad + \|x(t_0 + \sigma) - \varphi(t_0 + \sigma)\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

By repeating the arguments for the intervals

$$(t_0 + \sigma, t_0 + 2\sigma], (t_0 + 2\sigma, t_0 + 3\sigma], \ldots, (t_0 + m\sigma, t_0 + m\sigma + \sigma], \ldots$$

we conclude that the inequality (28) holds for $t > t_0$, $|t - t_i| > \eta$. This completes the proof of Theorem 1.

THEOREM 2. *Suppose that the following conditions hold:*

1. *The conditions* (A), (B), (C), (D) *and* (E) *hold and* $MN < 1$.
2. *The function* $x = \varphi(t)$ *is an unstable solution of the system* (1) *and satisfies the condition* (F).

Then the solution $\varphi(t)$ *is strongly unstable under persistent disturbances.*

Proof. Since the solution $\varphi(t)$ is unstable there then exist $\varepsilon > 0$, $\eta > 0$, $(1 + c)(5M\eta + \varepsilon) < d$ and $t_0 \in I$ such that for every $\delta > 0$ there exist $x_0 \in \Omega$, $\|x_0 - \varphi(t_0 + 0)\| < \delta$ and $t^* \in J^+(t_0, x_0)$, $|t^* - t_i| > \eta$ such that

$$\|x(t; t_0, x_0) - \varphi(t)\| < \varepsilon \qquad \text{for} \quad t \in (t_0, t^*)$$

and

$$\|x(t^*, t_0; x_0) - \varphi(t^*)\| = \varepsilon. \tag{29}$$

Let $t^* \in (t_m + \eta, t_{m+1} - \eta)$. We choose the number μ such that

$$0 < \mu < \min\left(\frac{1}{2}, \frac{1 - MN}{2Nd}, (t^* - t_m - \eta)\right).$$

Let the number ρ satisfy

$$0 < \rho < \min\left(h, \frac{1 - MN}{2N}\right),$$

$$\rho \frac{\theta + 1}{\theta}(t^* - t_0)[1 + c(1 - MN - N\rho)^{-1}]^{(t^* - t_0)/\theta} e^{a(t^* - t_0)} < \mu\varepsilon$$

and the perturbations g, $\{P_i\}$ fulfills the inequalities

$$\|g(t, x)\| < \rho, \ \|P_i(x)\| < \rho \qquad \text{for} \quad t \in I, x \in \Omega.$$

We prove that for the solution $y(t; t_0, x_0)$ of the system (2) there exists $t \in (t_0, t^*]$, $t \in J^+(y)$, $|t - t_i| > \eta$ such that

$$\|y(t; t_0, x_0) - \varphi(t)\| \geqslant \frac{\varepsilon}{2}.$$

Assume the opposite, i.e., for each $t \in (t_0, t^*]$, $t \in J^+(y)$, $|t - t_i| > \eta$

$$\|y(t; t_0, x_0) - \varphi(t)\| < \frac{\varepsilon}{2}. \tag{30}$$

Then, since $(1+c)(5M\eta+\varepsilon)<d$ form Lemma 2, it follows that $y(t; t_0, x_0) \in \Omega$ for $t \in J^+(y)$, $t \in (t_0, t^*]$ and the solution $y(t; t_0, x_0)$ is continuable on the entire interval $(t_0, t^*]$. Moreover, according to Lemma 1 the integral curve meets the hypersurfaces $t=\tau_i(x)$ only once.

But from the choice of ρ it follows that for $t \in (t_0, t^*]$, $t \bar{\in} [t_i', t_i'']$ the following inequality holds:

$$\|y(t; t_0, x_0)-x(t; t_0, x_0)\| < \mu\varepsilon. \tag{31}$$

Furthermore, from the fact that $|t_m'-t_m|>\eta$ and from the estimation

$$|t_m''-t_m'| \leqslant \frac{N\mu\varepsilon}{1-MN-N\rho} \leqslant \frac{2N\mu\varepsilon}{1-MN}$$
$$\leqslant \frac{\varepsilon}{d}(t^*-t_m-\eta) < t^*-t_m-\eta$$

it follows that $t^*>t_m''$ and the estimate (31) holds for $t=t^*$, that is,

$$\|y(t^*; t_0, x_0)-x(t^*; t_0, x_0)\| < \mu\varepsilon < \frac{\varepsilon}{2}. \tag{32}$$

Comparing (29), (32) and the estimate (30) for $t=t^*$ we come to the contradiction

$$\begin{aligned}\varepsilon = \|x(t^*; t_0, x_0)-\varphi(t^*)\| &\leqslant \|x(t^*; t_0, x_0)-y(t^*; t_0, x_0)\| \\ &+ \|y(t^*; t_0, x_0)-\varphi(t^*)\| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus Theorem 2 is proved.

Remark 2. Let the impulse effects appear at fixed moments, i.e.,

$$\tau_i(x) \equiv t_i.$$

Then in Lemma 1, Lemma 2, Theorem 1, Theorem 2 and the conditions D1, D2, (E), $MN<1$ can be dropped and the condition D3 can be replaced by the condition

D3°. $t_{i+1}-t_i \geqslant \theta > 0$, $i=1, 2, \ldots$.

Remark 3. Let $\{x \in R^n: \|x\| \leqslant d\} \subset \Omega$, $f(t, 0)=0$ for $t \in I$ and $B_i(0)=0$, $i=1, 2, \ldots$. Then $\varphi(t) \equiv 0$ is a solution of the system (1). Using Lemmas 1 and 2 one can easily check that the Definitions 1–6 concerning the trivial solution are equivalent to the following definitions, respectively.

DEFINITION 1°. The solution $x \equiv 0$ of the system (1) is called uniformly stable if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t_0 \in I)(\forall x_0 \in \Omega, \|x_0\| < \delta)(\forall t \in J^+(t_0, x_0))$$
$$\|x(t; t_0, x_0)\| < \varepsilon.$$

DEFINITION 2°. The solution $x \equiv 0$ of the system (1) is called uniformly attractive if

$$(\exists \lambda > 0)(\forall \varepsilon > 0)(\exists \sigma > 0)$$
$$(\forall t_0 \in I)(\forall x_0 \in \Omega, \|x_0\| < \lambda)\ t_0 + \sigma \in J^+(t_0, x_0)$$

and

$$(\forall t \geqslant t_0 + \sigma, t \in J^+(t_0, x_0))$$
$$\|x(t; t_0, x_0)\| < \varepsilon.$$

DEFINITION 3°. The solution $x \equiv 0$ of the system (1) is called uniformly asymptotically stable if it is uniformly stable and uniformly attractive in the sense of the Definitions 1° and 2°.

DEFINITION 4°. The solution $x \equiv 0$ of the system (1) is called unstable if

$$(\exists \varepsilon > 0)(\exists t_0 \in I)(\forall \delta > 0)(\exists x_0 \in \Omega, \|x_0\| < \delta)(\exists t^* \in J^+(t_0, x_0))$$
$$\|x(t^*; t_0, x_0)\| \geqslant \varepsilon.$$

DEFINITION 5°. The solution $x \equiv 0$ of the system (1) is called uniformly stable under persistent disturbances if

$$(\forall \varepsilon > 0)(\exists r > 0)(\exists \rho > 0)(\forall t_0 \in I)(\forall x_0 \in \Omega, \|x_0\| < r)$$

the solution $x(t; t_0, x_0)$ of the system (2) satisfies

$$(\forall g: \forall (t, x) \in I \times \Omega\ \|g(t, x)\| < \rho)(\forall i = 1, 2, \ldots\ \forall P_i: \forall x \in \Omega\ \|P_i(x)\| < \rho)$$
$$(\forall t \in J^+(t_0, x_0))$$
$$\|x(t; t_0, x_0)\| < \varepsilon.$$

DEFINITION 6°. The solution $x \equiv 0$ of the system (1) is called strongly unstable under persistent disturbances if

$$(\exists \varepsilon > 0)(\exists t_0 \in I)(\forall r > 0)(\exists \rho > 0)(\exists x_0 \in \Omega, \|x_0\| < r)$$

the solution $x(t; t_0, x_0)$ of the system (2) satisfies

$$(\forall g\colon \forall (t, x) \in I \times \Omega\ \|g(t, x)\| < \rho)(\forall i = 1, 2, \ldots\ \forall P_i\colon \forall x \in \Omega\ \|P_i(x)\| < \rho)$$
$$(\exists t^* \in J^+(t_0, x_0))$$
$$\|x(t^*; t_0, x_0)\| \geqslant \varepsilon.$$

Furthermore, if we formulate Lemma 1, Lemma 2, Theorem 1, Theorem 2 for the trivial solution $\varphi(t) \equiv 0$ then the conditions A3 and $MN < 1$ can be omitted and the condition D3 can be replaced by the following condition:

D3*. $\tau_{i+1}(0) - \tau_i(0) \geqslant \theta > 0,\ i = 1, 2, \ldots$.

EXAMPLE. Consider the following linear system of differential equations with impulse effect at fixed moments of the time

$$\frac{dx}{dt} = Ax, \qquad t \neq t_i,$$
$$\Delta x|_{t = t_i} = Bx, \tag{33}$$

where the matrices A, B commute and the matrix $E + B$ is nonsingular.

Let the sequence of the moments $\{t_i\}$ be such that there exists a finite limit

$$\lim_{T \to \infty} \frac{i(t, t+T)}{T} = p,$$

uniformly in $t \in I$, where $i(t, t+T)$ is the number of the points of the sequence $\{t_i\}$ lying in the interval $(t, t+T)$.

Then, if the eigenvalues of the matrix $\Lambda = A + p \ln(E + B)$ have negative real parts, then all the solutions of (33) according to Theorem 2 in [5] are uniformly asymptotically stable and hence are uniformly stable under persistent disturbances. If at least one eigenvalue of the matrix Λ has positive real part, then according to Theorem 2 in [5] the solutions of the system (33) are unstable and then they are strongly unstable under persistent disturbances.

REFERENCES

1. V. D. MIL'MAN AND A. D. MYSHKIS, On the stability of motion in presence of impulses, *Siberian Math. J.* 1 (1960) 2, 233–237 [Russian].
2. V. D. MIL'MAN AND A. D. MYSHKIS, "Random Impulses in Linear Dynamic Systems. Asymptotic Methods for Solving Differential Equations," pp. 64–81, *Ed. AN UkSSR*, Kiev, 1963 [Russian].

3. A. D. MYSHKIS AND A. M. SAMOILENKO, Systems with impulses in prescribed moments of the time, *Mat. Sb.* **74** (1967) 2 [Russian].
4. A. M. SAMOILENKO AND N. A. PERESTIUK, Stability of the solutions of differential equations with impulse effect, *Differential Equations* **11** (1977), 1981–1992 [Russian].
5. A. M. SAMOILENKO AND N. A. PERESTIUK, On the stability of the solutions of systems with impulse effect, *Differential Equations* **11** (1981), 1995–2001 [Russian].
6. N. ROUCHE, P. HABETS, AND M. LALOY, "Stability Theory by Liapunov's Direct Method," Springer-Verlag, New York/Heidelberg/Berlin, 1977.