A Classification of 2-Simple Prehomogeneous Vector Spaces of Type I

TATSUO KIMURA, SHIN-ICHI KASAI, MASAAKI INUZUKA,
AND OSAMI YASUKURA

The Institute of Mathematics,
University of Tsukuba, Ibaraki, 305, Japan

Communicated by Nagayoshi Iwahori

Received June 30, 1986

INTRODUCTION

Let \( \rho: G \to GL(V) \) be a rational representation of a connected linear algebraic group \( G \) on a finite-dimensional vector space \( V \), all defined over an algebraically closed field \( K \) of characteristic zero. If \( V \) has a Zariski-dense \( G \)-orbit, we call a triplet \((G, \rho, V)\) a prehomogeneous vector space (abbrev. P.V.). When \( \rho \) is irreducible, such P.V.s have been classified in [1]. Since then, it has turned out gradually that the complete classification of reductive P.V.s (i.e., P.V.s with reductive groups \( G \)) is an extremely laborious task. Therefore it is natural to classify some restricted class of P.V.s (e.g., [2]) to get some insight into the general situation.

A P.V. \((G, \rho, V)\) is called a 2-simple P.V. when (1) \( G = GL(1) \times G_1 \times G_2 \) with simple algebraic groups \( G_1 \) and \( G_2 \), (2) \( \rho \) is the composition of a rational representation \( \rho' \) of \( G_1 \times G_2 \) of the form \( \rho' = \rho_1 \otimes \rho'_1 + \cdots + \rho_k \otimes \rho'_k + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t) \) with \( k + s + t = l \), where \( \rho_i, \sigma_i, \tau_i \) (resp. \( \rho'_i, \tau_i \)) are nontrivial irreducible representations of \( G_i \) (resp. \( G_2 \)), and the scalar multiplications \( GL(1)' \) on each irreducible component \( V_i \) for \( i = 1, \ldots, l \), where \( V = V_1 \oplus \cdots \oplus V_l \). We say that a 2-simple P.V. \((G, \rho, V)\) is of type I if \( k \geq 1 \) and at least one of \((GL(1) \times G_1 \times G_2, \rho_i \otimes \rho'_i)\) \((i = 1, \ldots, k)\) is a nontrivial P.V. (see Definition 5, p. 43 in [1]). On the other hand, if \( k \geq 1 \) and all \((GL(1) \times G_1 \times G_2, \rho_i \otimes \rho'_i)\) \((i = 1, \ldots, k)\) are trivial P.V.s, it is called a 2-simple P.V. of type II. In [3], all 2-simple P.V.s of type II has been already classified. In this paper, we shall classify all 2-simple P.V.s of type I. Thus, together with [3], we complete a classification of all 2-simple P.V.s. For example, the fact that all irreducible P.V.s are castling-equivalent to 2-simple P.V.s (or to \((SL(m) \times SL(m) \times GL(2), A_1 \otimes A_1 \otimes A_1)\) with \( m = 2, 3 \) (see [1]) indicates the importance of 2-simple P.V.s. For simplicity, we write \((G, \rho', V)\) or \((G, \rho')\) instead of \((G, \rho, V)\).
This paper consists of the following four sections: Introduction, 1. Preliminaries, 2. A classification, 3. List.

In Section 1, we give also some correction of [2]. In Section 3, we shall give the list of 2-simple P.V's of type I, which are not catling-equivalent to simple P.V's. For regular P.V's (see Section 4 in [1]), we also give the generic isotropy subgroups and the number of basic relative invariants.

1. Preliminaries

First we start from the correction of [2].

**Proposition 1.1.** (1) The triplet \((GL(1)^3 \times SL(5), A_2 \oplus A_2 \oplus A_2^*, V(10) \oplus V(10) \oplus V(5)^*)\) is a nonregular P.V. with the generic isotropy subalgebra

\[ h = \left\{ (e, e, 3e) \oplus \begin{pmatrix} -3eI_2 & A \\ 0 & 2eI_3 \end{pmatrix}; A = \begin{pmatrix} i & -\bar{y} & \bar{y} \\ \bar{y} & y & \gamma \\ -\bar{y} & y & \gamma \end{pmatrix} \right\}. \]

If we identify \(V(10) \oplus V(10) \oplus V(5)^*\) with \(\{(X, Y; Z) | X, Y \in M(5), \bar{X} = -X, \bar{Y} = -Y, Z \in K^5\}\), the action \(\rho\) is given by \(\rho(g)x = (\alpha AX'A, \beta A Y'A; \gamma A^{-1} \cdot Z)\) for \(x = (X, Y; Z)\) and \(g = (\alpha, \beta, \gamma; A) \in GL(1)^3 \times SL(5)\).

The basic relative invariants are given by

\[ f_1(x) = Pf \left( \begin{array}{c|c} X & YZ \\ \hline -'Z'Y & 0 \end{array} \right) \quad \text{and} \quad f_2(x) = Pf \left( \begin{array}{c|c} Y & XZ \\ \hline -'Z'X & 0 \end{array} \right), \]

where Pf denotes the Pfaffian.

(2) The triplet \((GL(1)^3 \times SL(5), A_2 \oplus A_2 \oplus A_1, V(10) \oplus V(10) \oplus V(5))\) is not a P.V.

**Proof.** We may also identify \(V(10)\) with \(\Sigma K \cdot e_i \wedge e_j (1 \leq i < j \leq 5)\). Then the isotropy subalgebra at a generic point \(x_0 = (e_2 \wedge e_3 + e_1 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_5)\) is given by

\[ g_{x_0} = \left\{ (e_1, e_2; \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}); A_1 = \begin{pmatrix} -\varepsilon_1 - 2\varepsilon_2 \\ -2\varepsilon_1 - \varepsilon_2 \end{pmatrix}, \right. \]

\[ \left. A_2 = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_1 & \gamma_4 \end{pmatrix}, A_3 = \begin{pmatrix} \varepsilon_1 + \varepsilon_2 \\ 2\varepsilon_2 \\ 2\varepsilon_1 \end{pmatrix} \right\}. \]
The dual action of $g_{x_0}$ on $K^5$ is a P.V., since the isotropy subalgebra at $e_1 + e_5 \in K^5$ is given by $\mathfrak{h}$, and hence we have (1). The standard action of $g_{x_0}$ on $K^5$ is a non-P.V., since $f(Z) = z_4 z_5 z_3^{-2}$ for $Z = \Sigma z_i e_i \in K^5$ is a non-constant absolute invariant. Q.E.D.

Remark 1.2. There is a mistake in Proposition 2.2, p. 80 in [2]. It should be corrected to “For $n = 2m + 1$, the triplet (5) for $n = 5$ and the triplet (2) are P.V.s, and the triplets (3), (4), (5) with $n \not= 5$, (6) are not P.V.s.” Thus the triplet $(GL(1)^3 \times SL(5), A_2 \oplus A_2 \oplus A_1^*)$ should be added in the table of simple P.V.s, p. 100 in [2] as the nineteenth P.V. Thus we obtain the following theorem.

**Theorem 1.3 ([2] with the correction above).** All non-irreducible simple P.V.s with scalar multiplications are given as follows:

1. $(GL(1)^k+1 \times SL(n), A_1 \oplus \cdots \oplus A_1 \oplus A_1^{(*)}) \ (1 \leq k \leq n, \ n \geq 2)$.
2. $(GL(1)^{k+1} \times SL(n), A_2 \oplus A_1^{(*)} \oplus \cdots \oplus A_1^{(*)}) \ (1 \leq k \leq 3, \ n \geq 4)$ except $(GL(1)^4 \times SL(n), A_2 \oplus A_1 \oplus A_1 \oplus A_1^*)$ with $n = \text{odd}$.
3. $(GL(1)^2 \times SL(2m+1), A_2 \oplus A_2)$ for $m \geq 2$.
4. $(GL(1)^2 \times SL(n), 2A_1 \oplus A_1^{(*)})$.
5. $(GL(1)^3 \times SL(5), A_2 \oplus A_2 \oplus A_1^*)$.
6. $(GL(1)^2 \times SL(n), A_3 \oplus A_1^{(*)}) \ (n = 6, 7)$.
7. $(GL(1)^3 \times SL(6), A_3 \oplus A_1 \oplus A_1)$.
8. $(GL(1)^l \times Sp(n), A_1 \oplus \cdots \oplus A_1) \ (l = 2, 3)$.
9. $(GL(1)^2 \times Sp(2), A_2 \oplus A_1)$.
10. $(GL(1)^2 \times Sp(3), A_3 \oplus A_1)$.
11. $(GL(1)^2 \times Spin(n), \ (\text{half-)spin rep.} \oplus \text{vector rep.}) \ (n = 7, 8, 10, 12)$.
12. $(GL(1)^2 \times Spin(10), A \oplus A)$, where $A = \text{the even half-spin representation}$.

Here $A^{(*)}$ stands for $A$ or its dual $A^\ast$. Note that $(G, \rho, V) \simeq (G, \rho^\ast, V^\ast)$ as triplets if $G$ is reductive.

Now let us consider the triplet $(GL(1) \times SL(2m+1) \times SL(2), A_2 \oplus A_1, V(m(2m+1)) \otimes V(2))$. Let $g_{x_0}$ be the isotropy subalgebra of $gl(1) \oplus gl(2m+1) \oplus gl(2)$ at a generic point $X_0$ given in p. 94 in [1]. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in gl(2)$, let $nA(A) = (\alpha_i, \beta_i)$ be an $(n+1) \times (n+1)$ matrix with $\alpha_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $0 \leq k \leq n$, $\alpha_{k+1} = kb, \alpha_{k+1+k} = (n+1-k)c (1 \leq k \leq n)$, all other $\alpha_i = 0$. Put $nA_1^\ast(A) = -i(\alpha_i)$. By simple calculation, we have the following lemma.
LEMMA 1.4. The generic isotropy subalgebra \( g_{x_0} \) is given as follows:

\[
g_{x_0} = \left\{ (\delta) \oplus \begin{pmatrix} mA^*_0(A) + m \delta I_{m+1} & 0 \\ B & (m-1)A_1(A) - (m+1)\delta I_m \end{pmatrix} \right\} \oplus (A);
\]

\( A \in \mathfrak{sl}(2), \quad B = (b_u) \in M(m, m+1) \quad (1.1) \)

with \( b_u = a_{i+i_1-1} \)

\[
(\mathfrak{gl}(1) \oplus \mathfrak{so}(2)) \oplus \mathfrak{v}(2m).
\]

THEOREM 1.5 [1]. All nontrivial irreducible (reduced or nonreduced) 2-simple P.V.s are given as follows. Here \( H \sim H_1 \) implies that the generic isotropy subgroup \( H \) is locally isomorphic to a group \( H_1 \):

(I)

1. \((SL(2m+1) \times GL(2m^2 + m - 2), A_2 \otimes A_1) \quad (m \geq 5) \) with \( H \sim (GL(1) \times SL(2)) \cdot G^m_a \).

2. \((SL(5) \times GL(4), A_2 \otimes A_1) \) with \( H \sim \{1\} \).

3. \((SL(5) \times GL(6), A_2 \otimes A_1) \) with \( H \sim \{1\} \).

4. \((SL(5) \times GL(7), A_2 \otimes A_1) \) with \( H \sim SL(2) \).

5. \((SL(2) \times GL(3), 3A_1 \otimes A_1) \) with \( H \sim \{1\} \).

6. \((SL(3) \times GL(2), 2A_1 \otimes A_1) \) with \( H \sim \{1\} \).

7. \((SL(3) \times GL(4), 2A_1 \otimes A_1) \) with \( H \sim \{1\} \).

8. \((SL(8) \times GL(55), A_3 \otimes A_1) \) with \( H \sim SL(3) \).

9. \((Spin(7) \times GL(5), \text{spin rep.} \otimes A_1) \) with \( H \sim SL(2) \times SL(2) \).

10. \((Spin(9) \times GL(15), \text{spin rep.} \otimes A_1) \) with \( H \sim Spin(7) \).

11. \((Spin(10) \times GL(13), \text{half-spin rep.} \otimes A_1) \) with \( H \sim SL(2) \times O(3) \).

12. \((Spin(11) \times GL(31), \text{spin rep.} \otimes A_1) \) with \( H \sim SL(5) \).

13. \((Spin(14) \times GL(63), \text{half-spin rep.} \otimes A_1) \) with \( H \sim (G_2) \times (G_2) \).

14. \(((G_2) \times GL(5), A_2 \otimes A_1) \) with \( H \sim GL(2) \).

15. \((E_6 \times GL(26), A_1 \otimes A_1) \) with \( H \sim E_4 \).

16. \((E_6 \times GL(2), A_1 \otimes A_1) \) with \( H \sim Spin(8) \).

17. \((E_6 \times GL(25), A_1 \otimes A_1) \) with \( H \sim Spin(8) \).

18. \((E_7 \times GL(55), A_6 \otimes A_1) \) with \( H \sim E_6 \).

(II)

19. \((SL(6) \times GL(2), A_2 \otimes A_1) \) with \( H \sim SL(2) \times SL(2) \times SL(2) \).

20. \((SL(6) \times GL(13), A_2 \otimes A_1) \) with \( H \sim SL(2) \times SL(2) \times SL(2) \).
(21) \((SL(7)\times GL(19), A_2 \otimes A_1)\) with \(H \sim (GL(1) \times SL(2)) \cdot G_a^6\).

(22) \((SL(9)\times GL(34), A_2 \otimes A_1)\) with \(H \sim (GL(1) \times SL(2)) \cdot G_a^8\).

(23) \((SL(2m)\times GL(2m^2 - m - 1), A_2 \otimes A_1)\) \((m \geq 3)\) with \(H \sim Sp(m)\).

(24) \((SL(m)\times GL(\frac{1}{2}m(m + 1) - 1), 2A_1 \otimes A_1)\) \((m \geq 3)\) with \(H \sim O(m)\).

(25) \((SL(6)\times GL(19), A_3 \otimes A_1)\) with \(H \sim SL(3) \times SL(3)\).

(26) \((SL(7)\times GL(34), A_3 \otimes A_1)\) with \(H \sim (G_2)\).

(27) \((Sp(3)\times GL(13), A_1 \otimes A_1)\) with \(H \sim SL(3)\).

(28) \((Spin(12)\times GL(31), \text{half-spin rep.} \otimes A_1)\) with \(H \sim SL(6)\).

(III)

(29) \((SL(5)\times GL(3), A_2 \otimes A_1)\) with \(H \sim SL(2)\).

(30) \((SL(2m + 1)\times GL(2), A_2 \otimes A_1)\) \((m \geq 5)\) with \(H \sim (GL(1) \times SL(2)) \cdot G_a^{2m}\) (see (1.1)).

(31) \((Sp(n)\times GL(2), A_1 \otimes 2A_1)\) with \(H \sim (Sp(n - 2) \times SO(2)) \cdot U(2n - 3)\) \((n \geq 2)\).

(32) \((SO(n)\times GL(m), A_1 \otimes A_1)\) with \(H \sim SO(m) \times SO(n - m)\) for \(n = 9, 11,\) or \(n \geq 13,\) and \(n > m \geq 2\).

(33) \((Spin(7)\times GL(2), \text{spin rep.} \otimes A_1)\) with \(H \sim SL(3) \times O(2)\).

(34) \((Spin(7)\times GL(3), \text{spin rep.} \otimes A_1)\) with \(H \sim SL(2) \times O(3)\).

(35) \((Spin(7)\times GL(6), \text{spin rep.} \otimes A_1)\) with \(H \sim SL(3) \times O(2)\).

(36) \((Spin(10)\times GL(2), \text{half-spin rep.} \otimes A_1)\) with \(H \sim (G_2) \times SL(2)\).

(37) \((Spin(10)\times GL(3), \text{half-spin rep.} \otimes A_1)\) with \(H \sim SL(2) \times O(3)\).

(38) \((Spin(10)\times GL(14), \text{half-spin rep.} \otimes A_1)\) with \(H \sim (G_2) \times SL(2)\).

(39) \(((G_2)\times GL(2), A_2 \otimes A_1)\) with \(H \sim GL(2)\).

(40) \(((G_2)\times GL(6), A_2 \otimes A_1)\) with \(H \sim SL(3)\).

(IV)

(41) \((SL(2)\times GL(2), 2A_1 \otimes A_1)\) with \(H \sim O(2)\).

(42) \((SL(5)\times GL(8), A_2 \otimes A_1)\) with \(H \sim (GL(1) \times SL(2)) \cdot G_a^4\).

(43) \((SL(9)\times GL(2), A_2 \otimes A_1)\) with \(H \sim (GL(1) \times SL(2)) \cdot G_a^8\).

(44) \((SL(2m + 1)\times GL(2m^2 + m - 1), A_2 \otimes A_1)\) \((m \geq 4)\) with \(H \sim (GL(1) \times Sp(m)) \cdot G_a^{2m}\).

(45) \((SO(10)\times GL(m), A_1 \otimes A_1)\) \((2 \leq m \leq 9)\) with \(H \sim SO(10 - m) \times SO(m)\).
(46) \((SO(12) \times GL(m), A_1 \otimes A_1) (2 \leq m \leq 11)\) with \(H \sim SO(12-m) \times SO(m)\).

(47) \((Spin(7) \times GL(7), \text{spin rep. } \otimes A_1)\) with \(H \sim (G_2)\).

(48) \((Spin(10) \times GL(15), \text{half-spin rep. } \otimes A_1)\) with \(H \sim (GL(1) \times Spin(7)) \cdot G^8_d\).

(V)

(49) \((SL(5) \times GL(2), A_2 \otimes A_1)\).

(50) \((SL(5) \times GL(9), A_2 \otimes A_1)\).

(51) \((SL(7) \times GL(2), A_2 \otimes A_1)\).

(52) \((SL(7) \times GL(20), A_2 \otimes A_1)\).

(53) \((SO(5) \times GL(m), A_1 \otimes A_1) \simeq (Sp(2) \times GL(m), A_2 \otimes A_1) (m = 2, 3, 4)\).

(54) \((SO(6) \times GL(m), A_1 \otimes A_1) \simeq (SL(4) \times GL(m), A_2 \otimes A_1) (2 \leq m \leq 5)\).

(55) \((SO(7) \times GL(m), A_1 \otimes A_1) \simeq (Spin(7) \times GL(m), \text{vector rep. } \otimes A_1) (2 \leq m \leq 6)\).

(56) \((SO(8) \times GL(m), A_1 \otimes A_1) (2 \leq m \leq 7)\).

(57) \((Sp(n) \times GL(2m), A_1 \otimes A_1) (n \geq m \geq 1)\).

(58) \((Sp(n) \times GL(2m + 1), A_1 \otimes A_1) (n > m \geq 1)\).

The following lemma is almost obvious.

**Lemma 1.6.** Let \(H\) be a generic isotropy subgroup of \((GL(1) \times G \times G', \rho_1 \otimes \rho'_1)\). Let \(d\) and \(d'\) be the minimum of degree of nontrivial representations of \(G\) and \(G'\), respectively:

1. If \(1 + \dim H \leq \min\{d, d'\}\), then there exists no non-irreducible 2-simple P.V. with an irreducible component \((GL(1) \times G \times G', \rho_1 \otimes \rho'_1)\).

2. If \(1 + \dim H \leq d\) (resp. \(d'\)), then \((GL(1)^2 \times G \times G', \rho_1 \otimes \rho_1' + \rho_2 \otimes \rho_2')\) with \(\rho_2 \neq 1\) (resp. \(\rho_2' \neq 1\)) is not a P.V.

2. A Classification

In this section, for each nontrivial 2-simple P.V. \((GL(1) \times G \times G', \rho_1 \otimes \rho'_1)\) in Theorem 1.5, we shall determine all nonirreducible 2-simple P.V.s which have \((GL(1) \times G \times G', \rho_1 \otimes \rho'_1)\) as one of their irreducible components. For this purpose, we shall investigate the prehomogeneity of \((GL(1)^2 \times G \times G', \rho_1 \otimes \rho_1' + \rho_2 \otimes \rho_2'),\) where we do not assume the non-triviality of \(\rho_2\) and \(\rho'_2\) in general.
**Theorem 2.1.** There exists no nonirreducible 2-simple P.V. which has one of (1) in Theorem 1.5 as an irreducible component.

**Proof.** For (1), we have $\rho'_2 = 1$ by (2) of Lemma 1.6 and Lemma 1.4, since $2m^2 + m - 2 \geq \dim(g(1) \oplus g_{m+1}) = 2m + 5$ for $m \geq 5$. If $\rho_2 \neq 1$, we have $\rho_2 = A_1$ or $A_1^\ast$ by dimension reason. Then the castling transform $(GL(1)^2 \times SL(2m + 1) \times SL(2), A_2^\ast \otimes A_1 + \rho_2 \otimes 1)$ is also a P.V., and by (1.1), $(GL(2), (m-1)A_1)$ (resp. $(GL(2), mA_1)$) must be a P.V. if $\rho_2 = A_1$ (resp. $\rho_2 = A_1^\ast$), which is a contradiction since $m \geq 5$. By (1) of Lemma 1.6, we have (2)–(7) and (11) in Theorem 1.5. For (8), by (2) of Lemma 1.6, we have $\rho'_2 = 1$. If $\rho_2 \neq 1$, then its castling transform $(GL(1)^2 \times SL(8), A_2^\ast + \rho_2) \simeq (GL(1)^2 \times SL(8), A_1 + \rho_2^\ast)$ is a P.V. which is a contradiction by Theorem 1.3. Similarly, we have (12), (13), (15), and (18). For (9), by dimension reason, if $\rho'_2 \neq 1$, then we have $\rho_2 = 1$ and $\rho'_2 = A_1$ or $A_1^\ast$.

If $\rho_2 = 1$ and $\rho'_2 = A_1$, its castling transform $(GL(1)^2 \times Spin(7) \times SL(4), \text{spin rep.} \otimes A_1 + 1 \otimes A_1)$ must be also a P.V. Since $(Spin(7) \times GL(4), \text{spin rep.} \otimes A_1)$ is a non-P.V. (see p. 118 in [1]), the case for $\rho_2 = 1$ and $\rho'_2 = A_1$ is a non-P.V. Since a generic isotropy subgroup of (9) is reductive, the case for $\rho_2 = 1$ and $\rho'_2 = A_1^\ast$ is also a non-P.V. Hence $\rho'_2 = 1$. If $\rho_2 \neq 1$, then $\deg \rho_2 \leq 7 = \dim(GL(1) \times SL(2) \times SL(2))$ and hence $\rho_2$ must be the vector representation. By (5.37), p. 118 in [1], it is a P.V. if and only if the triplet $(GL(1) \times SL(2) \times SL(2), A_1 \otimes A_1 \otimes A_1 + A_1 \otimes 1 \otimes 2A_1, V(4) \oplus V(3))$ is a P.V. However, it is clearly not a P.V. and we have finished the case (9).

For (10), if $\rho'_2 \neq 1$, then $\deg \rho_2 \leq 7 = \dim(GL(1) \times Spin(7)) = 22$, and hence $\rho'_2 = A_1$ or $A_1^\ast$ and $\rho_2 = 1$. In this case, it is a P.V. if and only if $(GL(1) \times Spin(7), A_1 \otimes (\text{spin rep.} + \text{vector rep.}))$ is a P.V. By p. 96 in [2], it is not a P.V. If $\rho'_2 = 1$, it reduces to the simple case by a castling transformation. By pp. 77, 89 in [2], it is not a P.V. for any $\rho_2 \neq 1$. For (14), if $\rho_2 \otimes \rho'_2 \neq 1$, then $\rho_2 = 1$ and $\rho'_2 = A_1$ (or $A_1^\ast$) by dimension reason. If $\rho'_2 = A_1$, we have its castling transform $(GL(1)^2 \times G_2 \times SL(3), A_2^\ast \otimes A_1 + 1 \otimes A_1)$. Since $(G_2 \times GL(3), A_2 \otimes A_1)$ is a non-P.V. by p. 136 in [1], the case for $\rho_2 = 1$ and $\rho'_2 = A_1$ (hence also the case for $\rho'_2 = A_1^\ast$) is a non-P.V. For (16) and (17), we have our desired result from the fact that the restriction of $(E_6, A_1)$ (resp. $(GL(2), A_1), (GL(25), A_1)$) to a generic isotropy subgroup $H \sim Spin(8)$ is given by $(Spin(8), 1 + 1 + 1 + A_1 + A_v + A_0, V(27))$ (resp. $(Spin(8), 1 + 1, V(2)), (Spin(8), 1 + A_1 + A_v + A_0, V(25))$, where $A_1$ (resp. $A_v, A_0$) denotes the vector (resp. even half-spin, odd half-spin) representation of Spin(8). One can check this fact by simple calculation of weights.

**Q.E.D.**

**Lemma 2.2.** Let $(GL(1) \times G \times G', \rho_1 \otimes \rho'_1)$ be one of (II) in Theorem 1.5. If $(GL(1)^2 \times G \times G', \rho_1 \otimes \rho'_1 + \rho_2 \otimes \rho'_2)$ is also a P.V., then we have $\rho'_2 = 1$.

**Proof.** By (5.10) in p. 93 in [1], we have (19). By (2) of Lemma 1.6, we
have (20)-(22) and (24)-(27). Since the restriction of \((GL(2m^2-m-1), A_1)\)
to \(H=Sp(m)\) is \((Sp(m), A_2)\), we have (23) by p. 106 in [1]. For (28), by
dimension reason, only the possibility for \(\rho_2 \neq 1\) is \(\rho_2 = A_1\) or \(A_1^*\). If
\(\rho_2 = A_1\), we have its castling transform \((GL(1)^2 \times Spin(12) \times SL(2), \text{half-spin rep. } \otimes A_1 + 1 \otimes A_1)\)
which is a non-P.V. by p. 130 in [1]. Since \(H\) is reductive, the case for \(\rho_2 = A_1^*\) is also a non-P.V.
Q.E.D.

**Theorem 2.3.** All non-irreducible 2-simple P.V.s which have one of (II)
in Theorem 1.5 as an irreducible component are given as follows:

\[
\begin{align*}
(GL(1)^2 \times SL(6) \times SL(9), A_2 \otimes A_1 + A_1^* \otimes 1) & \quad (2.1) \\
(GL(1)^2 \times SL(7) \times SL(19), A_3 \otimes A_1 + A_1^* \otimes 1) & \quad (2.2) \\
(GL(1)^2 \times SL(8) \times SL(34), A_3 \otimes A_1 + A_1 \otimes 1) & \quad (2.3) \\
(GL(1)^2 \times SL(9) \times SL(31), A_3 \otimes A_1 + A_1 \otimes 1) & \quad (2.4) \\
(GL(n)^2 \times SL(2m) \times SL(2m^2-m-1), A_2 \otimes A_1 + \Sigma \otimes 1), & \quad \text{where } m \geq 3, \quad s = 1, 2, 3; \quad \Sigma = A_2^* + \Sigma_1^*, \quad \Sigma_1 = A_2^* A_1 A_1^* + \Sigma_2 = A_1^* A_1 A_1^* + \Sigma_3^*, \\
\text{and } & \quad \Sigma_3 = A_1^* A_1^* A_1^*. \\
(GL(1)^2 \times SL(n) \times SL(2n(n+1)-1), 2A_1 \otimes A_1 + A_1^* \otimes 1) & \quad (n \geq 3) \\
(GL(1)^2 \times SL(6) \times SL(19), A_3 \otimes A_1 + A_1 \otimes 1) & \quad (2.7) \\
(GL(1)^2 \times SL(9) \times SL(34), A_3 \otimes A_1 + A_1^* \otimes 1) & \quad (2.8) \\
(GL(1)^2 \times Spin(3) \times SL(13), A_3 \otimes A_1 + A_1 \otimes 1) & \quad (2.9) \\
(GL(1)^2 \times Spin(12) \times SL(31), \text{half-spin rep. } \otimes A_1 + \text{vector rep. } \otimes 1) & \quad (2.10) \\
(GL(n)^2 \times SL(9) \times SL(34), A_2 \otimes A_1 + A_1^* \otimes 1) & \quad (2.11)
\end{align*}
\]

Note that \(A_2 \otimes A_1 + A_2^* \otimes 1\) for (22) is not a P.V., and \(A_3^* = A_3\) for
\(SL(6)\) in (2.7) and (2.8).

**Proof.** For (19) (resp. (21), (22)), we have \(\dim(GL(1)^{l-1} \times H) = 8 + l\)
(resp. \(9 + l, \quad 11 + l\) \(\geq\) \(\deg \rho_2 + \cdots + \deg \rho_l \geq (l-1) \deg A_1 = 6(l-1)\) (resp.
\(7(l-1), \quad 9(l-1)\)), and hence \(l = 2, \quad \rho_2 = A_1\) or \(A_1^*\). Since (19) and the
castling transform of (21) are F.P.s (see [4]), the case (19) and (21) are
actually P.V.s.

By Lemma 2.2 and a castling transformation, (20) reduces to (19). For
(22), first note that the castling transform of \((GL(1)^2 \times SL(9) \times SL(34),
A_2 \otimes A_1 + A_1 \otimes 1\) (resp. \(A_2 \otimes A_1 + A_1^* \otimes 1\)) is given by \((GL(1)^2 \times SL(9) \times
SL(2), \quad A_2 \otimes A_1 + A_1^* \otimes 1\) (resp. \(A_2 \otimes A_1 + A_1 \otimes 1\)). If the case for
\(A_2 \otimes A_1 + A_1 \otimes 1\) is a P.V., then by (1.1), the triplet \((GL(2), \quad 4A_1, \quad V(5))\)
must be also a P.V., which is a contradiction by dimension reason.

By (1.1), \((GL(1)^2 \times SL(9) \times SL(2), \quad A_2 \otimes A_1 + A_1^* \otimes 1)\) is a P.V. if and
only if \(g = \{ (z) \oplus (-C); \quad C \text{ is the second matrix in (1.1)} \} \) acts on \(K^o\)
prehomogeneously. Since \(x_0 = e_6 + e_9 \in K^o\) is a generic point, \((g, K^o)\) (and
hence (22)) is a P.V. By a castling transformation, (23)–(28) reduce to the simple P.V.s, and by Theorem 1.3, we have our results. Q.E.D.

**Lemma 2.4.** Let \((GL(1) \times G \times G', \rho_1 \otimes \rho'_1)\) be one of (III) in Theorem 1.5. If \((GL(1)^2 \times G \times G', \rho_1 \otimes \rho'_1 + \rho_2 \otimes \rho'_2)\) is also a P.V., then we have \(\rho_2 = 1\).

**Proof.** By (2) of Lemma 1.6, we have the cases (29), (37), and (39). For (30), if \(\rho_2 \neq 1\), then we have \(\rho'_2 = 1\) since otherwise \(\deg \rho_2 \otimes \rho'_2 \geq 2(2m + 1) > \dim H + 1 = 2m + 5\). Then, by the castling transformation and (1) in Theorem 2.1, we have our result. For (31), \(\rho_2 \otimes \rho'_2\) must be one of (a) \(A_1 \otimes A_1\), \(A_2 \otimes A_1\) for \(n = 2\), (b) \(A_3 \otimes 1\) for \(n = 3\), (c) \(A_1 \otimes 2A_1\), \(A_1 \otimes A_1\) for \(n \geq 3\), \(A_1 \otimes 1\). However, (a) and (b) are impossible by dimension reason. If \(\rho_2 \otimes \rho'_2 = A_1 \otimes 2A_1\) (resp. \(A_1 \otimes A_1\), \(A_1 \otimes 1\)), it is a P.V. if and only if \((GL(1)^2 \times SL(2), A_1^2(2A_1 + 2A_1))\) (resp. \(A^2(2A_1 + A_1), A^2(2A_1 + 1))\) is a P.V. by pp. 40–41 in [1], which is impossible by dimension reason. Now before going ahead, we shall prove several sublemmas.

**Sublemma 2.4.1.** The triplet \((GL(1)^2 \times SO(n) \times SL(m), A_1 \otimes A_1 + A_1 \otimes A_1^*, M(n, m) \oplus M(n, m))\) is a non-P.V. for \(n \geq m \geq 1\).

**Proof.** For \(x = (X, Y) \in M(n, m) \oplus M(n, m), g = (x, \beta; A, B) \in GL(1)^2 \times SO(n) \times SL(m)\) and \(\rho = A_1 \otimes A_1 + A_1 \otimes A_1^*\) (resp. \(A_1 \otimes A_1 + A_1 \otimes A_1^*, M(n, m) \oplus M(n, m))\), we have \(\rho(g)x = (xAX'B, \beta Y'B)\) (resp. \(xAX'B, \beta Y'B^{-1})\) and hence, \(f(x) = \det('XX) \cdot \det('YY) \cdot \det('XY)^{-2}\) is a nonconstant absolute invariant. Q.E.D.

**Sublemma 2.4.2.** For \(n \geq m \geq 1\), the triplet \((GL(1)^2 \times SO(n) \times SL(m), A_1 \otimes 1 + A_1 \otimes A_1, V(n) \oplus M(n, m))\) is a non-P.V.

**Proof.** By pp. 109–110 in [1], it is a P.V. if and only if \((GL(1) \times SO(n-m) \times SO(m), A_1 \otimes A_1 \otimes 1 + A_1 \otimes 1 \otimes A_1)\) is a P.V. In this case, a triplet \((SO(m), A_1, V(m))\) without scalar multiplication must be a P.V., which is a contradiction. Q.E.D.

**Sublemma 2.4.3.** For \(m_1, m_2 \geq n \geq 1\), the triplet \((SO(m_1) \times SO(m_2) \times GL(n), A_1 \otimes 1 \otimes A_1 + 1 \otimes A_1 \otimes A_1^{*}, M(m_1, n) \oplus M(m_2, n))\) is a non-P.V.

**Proof.** For \(x = (X, Y) \in M(m_1, n) \oplus M(m_2, n), g = (A, B, C) \in SO(m_1) \times SO(m_2) \times GL(n)\) and \(\rho^{(*)} = A_1 \otimes 1 \otimes A_1 + 1 \otimes A_1 \otimes A_1^{*}\), we have \(\rho(g)x = (AX'C, BY'C)\) (resp. \(\rho^{(*)}(g)x = (AX'C, BY'C^{-1})\), and hence \(f(x) = \det('XX) \cdot \det('YY)^{\cdot 2}\) (resp. \(f(x) = \det('XX) \cdot \det('YY)^{\cdot 1}\)) is a nonconstant absolute invariant. Q.E.D.
For $n \geq m \geq 1$, the triplet $(GL(n) \times GL(m), (1 + A_1 + A_1^*) \otimes A_1, V(m) \oplus M(n, m) \oplus M(n, m))$ is a non-P.V.

Proof. For $x = (y, X_1, X_2) \in V(m) \oplus M(n, m) \oplus M(n, m)$, $g = (A, B) \in GL(n) \times GL(m)$ and $\rho = (1 + A_1 + A_1^*) \otimes A_1$, we have $\rho(g)x = (By, AX_1', B'y, A'X_2')$. Hence, if $m \geq 2$, then $f(x) = \det((X'_2X_1 + X_1'X_2) \cdot \det((X'_2X_1)^{-1})$ is a nonconstant absolute invariant. If $m = 1$, then $f(x) = (X'_1X_2) \cdot y^{-2}$ is a nonconstant absolute invariant.

Q.E.D.

For $n \geq m \geq 1$, the triplet $(SO(n) \times GL(m), 1 \otimes A_1 + A_1 \otimes A_1^*, V(m) \oplus M(n, m))$ is a non-P.V.

Proof. By pp. 109–110 in [1], it is a P.V. if and only if $(SO(m), A_1, V(m))$ is a P.V. without scalar multiplication, which is a contradiction.

Q.E.D.

Now we start to prove the case (32). Note that if $\rho_2' = 1$, we may assume $n \geq 2m$ by a castling transformation. If $\rho_2 \neq 1$, then $\rho_2 \otimes \rho_2'$ must be one of $A_1 \otimes A_1^* \oplus A_1 \otimes 1$, or $A \otimes 1$ with $A = (\text{half-})$ spin representation by [1]. If $\rho_2 \otimes \rho_2' = A_1 \otimes A_1^*$ (resp. $A_1 \otimes 1$), then it is a non-P.V. by Sublemma 2.4.1 (resp. Sublemma 2.4.2). For $n = 9$ and $\rho_2 \otimes \rho_2' = A \otimes 1$, it is a non-P.V. by p. 127 in [1], $A(\text{Spin}(7)) \subset SO(8)$, and Sublemma 2.4.5. For $n = 11$ and $\rho_2 \otimes \rho_2' = A \otimes 1$, it is a non-P.V. by p. 130 in [1] and Sublemma 2.4.4. For $n = 14$ and $\rho_2 \otimes \rho_2' = A \otimes 1$, it is a non-P.V. by p. 133 in [1] and Sublemma 2.4.3. For (33), we have $\rho_2 \otimes \rho_2' = A \otimes 1$ ($A$ is the spin rep.) or $A_1 \otimes 1$ ($A_1$ is the vector rep.) by dimension reason. If $\rho_2 \otimes \rho_2' = A \otimes 1$, then it is a non-P.V. by $A(\text{Spin}(7)) \subset SO(8)$ and Sublemma 2.4.2. If $\rho_2 \otimes \rho_2' = A_1 \otimes 1$, then it is a non-P.V. by (5.35), p. 117 in [1], and Sublemma 2.4.4. For (34), we have $\rho_2 \otimes \rho_2' = \text{vector rep.} \otimes 1$, and it is a P.V. if and only if its castling transform $(\text{Spin}(7) \times GL(5), \text{spin rep.} \otimes A_1 + \text{vector rep.} \otimes 1)$ is a P.V. which is a contradiction by Theorem 2.1 for (9) in Theorem 1.5. For (35), if $\rho_2 \neq 1$, then we have $\rho_2' = 1$, since otherwise $\dim H + 1 = 10 \geq \deg \rho_2 \otimes \rho_2' \geq 7 \cdot 6 = 42$, which is a contradiction. Hence we can reduce (35) to (33) by a castling transformation. For (36), we have $\rho_2 \otimes \rho_2' = A \otimes 1$ ($A$ = half-spin rep.) or $A_1 \otimes 1$ ($A_1$ = the vector rep.) by dimension reason. If $\rho_2 \otimes \rho_2' = A \otimes 1$, it is a P.V. if and only if $(GL(1) \times (G_2) \times SL(2), A_1 \otimes A_2 \otimes A_1 + A_1 \otimes 1 \otimes A_1, V(14) \oplus V(2))$ is a P.V. by calculation of weights (cf. p. 123 in [1]). By (5.53), p. 136 in [1], it is a P.V. if and only if $(O(2), A_1, V(2))$ is a P.V. Since $\dim O(2) = 1 < \dim V(2) = 2$, it is a non-P.V. If $\rho_2 \otimes \rho_2' = A_1 \otimes 1$, it is a non-P.V. by (5.42), p. 123 in [1], $A_2(G_2) \subset SO(7)$, and Sublemma 2.4.3. For (38), if $\rho_2 \neq 1$, then $\rho_2' = 1$ by dimension reason, and hence (38) reduces to the case (36) by a castling transformation. For (40), if $\rho_2 \neq 1$, we have $\rho_2 \otimes \rho_2' = A_2 \otimes 1$ by dimension reason. It is a non-P.V. by $A_2(G_2) \subset SO(7)$ and Sublemma 2.4.2.
THEOREM 2.5. All non-irreducible 2-simple P.V.s which have one of (III) in Theorem 1.5 as an irreducible component are given as follows:

\[ (GL(1)^2 \times SL(5) \times SL(3), A_2 \otimes A_1 + 1 \otimes A_1^{(*)}) \]  
\[ (GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes \rho) \quad (m \geq 5), \]  
where \( \rho = A_1, 2A_1, \text{ or } 3A_1. \)  
\[ (GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1 + 1 \otimes \rho) \quad (m \geq 5), \]  
where \( \rho = A_1 \) or \( 2A_1. \)  
\[ (GL(1)^4 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes (A_1 + A_1 + A_1)) \quad (m \geq 5) \]  
\[ (GL(1)^2 \times Sp(n) \times SL(2), A_1 \otimes 2A_1 + 1 \otimes A_1) \quad (n \geq 2) \]  
\[ (GL(1)^2 \times SO(n) \times SL(m), A_1 \otimes A_1 + 1 \otimes A_1^{(*)}) \]  
\[ (GL(1)^3 \times Spin(7) \times SL(2), \text{ spin rep. } \otimes A_1 + 1 \otimes A_1) \]  
\[ (GL(1)^3 \times Spin(7) \times SL(3), \text{ spin rep. } \otimes A_1 + 1 \otimes A_1^{(*)}) \]  
\[ (GL(1)^2 \times Spin(10) \times SL(2), \text{ half-spin rep. } \otimes A_1 + 1 \otimes \rho), \]  
where \( \rho = A_1, 2A_1, \text{ or } 3A_1. \)  
\[ (GL(1)^2 \times Spin(10) \times SL(2), \text{ half-spin rep. } \otimes A_1 + 1 \otimes A_1 + 1 \otimes \rho), \]  
where \( \rho = A_1 \) or \( 2A_1. \)  
\[ (GL(1)^2 \times (G_2) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1) \]  
\[ (GL(1)^2 \times (G_2) \times SL(6), A_2 \otimes A_1 + 1 \otimes A_1^{(*)}). \]

Proof. First note that if \((GL(1)^k \times SO(n), \rho_1 \oplus \cdots \oplus \rho_k)\) is a P.V., then we have \( k = 1 \) and \( \rho_1 = A_1. \) The \( SL(m)\)-part of the generic isotropy subgroup of (29) (resp. (31), (32), (33), (34), (37), (39)) is \( SO(m) \) by p. 96 (resp. pp. 104, 109, 117, 118, 125, 136) in [1], and hence we have (2.12), (2.16)-(2.19), (2.24), and (2.26). Now, if \((GL(1)^k \times SL(2), \rho_1 \oplus \cdots \oplus \rho_k)\) is a P.V., then we have \( k \leq 3 \) and \( \rho_1 \oplus \cdots \oplus \rho_k = A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \) \( (k = 3); \) \( 2A_1 \oplus A_1, A_1 \oplus A_1 \) \( (k = 2); \) \( 3A_1, 2A_1, A_1 \) \( (k = 1). \) The \( SL(2)\)-part of the generic isotropy subgroup of (30) (resp. (36)) is \( SL(2) \) by (1.1) in Lemma 1.4 (resp. p. 112 in [1]) and hence, we have (2.13)-(2.15) and (2.21)-(2.23). For (35), (38), and (40), we have \( \rho_2 \otimes \rho_2^2 + \cdots + \rho_l \otimes \rho_l = 1 \otimes A_1^{(*)}, \) i.e., \( l = 2 \) and \( \rho_2 = A_1^{(*)} \) by dimension reason. Since the generic isotropy subgroups of (35), (38), (40) in Theorem 1.5 are reductive, we
may assume that $\rho_2 \otimes \rho'_3 = 1 \otimes A_1$ to see the prehomogeneity. Then, by a castling transformation, (35) (resp. (38), (40)) is reduced to (2.19) (resp. (2.24), (2.26)) and we have (2.20), (2.25), and (2.27). Q.E.D.

**Lemma 2.6.** Let $(GL(1) \times G \times G', \rho_1 \otimes \rho'_1)$ be one of (IV) in Theorem 1.5. Then, (i) $(GL(1)^2 \times G \times G', \rho_1 \otimes \rho'_1 + \rho_2 \otimes \rho'_2)$ is a non-P.V. for any $\rho_3 \neq 1$ and $\rho_2 \neq 1$; (ii) $(GL(1)^3 \times G \times G', \rho_1 \otimes \rho'_1 + \rho_2 \otimes 1 + 1 \otimes \rho'_3)$ is a non-P.V. for any $\rho_1 \neq 1$ and $\rho'_3 \neq 1$.

**Proof.** For (41), we have (i) by $\dim GL(1) \times H = 2 < 4 = 2 \times 2 \leq \deg \rho_2 \otimes \rho'_2$ and (ii) by $\dim GL(1)^2 \times H = 3 < 4 = 2 + 2 \leq \deg (\rho_1 \otimes 1 + 1 \otimes \rho'_3)$. For (42), we have (i) by $\dim GL(1) \times H = 9 < 40 = 5 \times 8 \leq \deg \rho_2 \otimes \rho'_2$ and (ii) by $\dim GL(1)^2 \times H = 10 < 13 = 8 + 5 \leq \deg (\rho_1 \otimes 1 + 1 \otimes \rho'_3)$. For (43), we have (i) by $\dim GL(1) \times H = 9 < 40 = 5 \times 8 \leq \deg \rho_2 \otimes \rho'_2$. Now if $(GL(1)^2 \times SL(9) \times SL(2), A_2 \otimes A_1 + \rho_2 \otimes 1)$ is a P.V., then $(GL(1)^2 \times SL(9) \times SL(34), \rho_2 \otimes A_1 + \rho'_2 \otimes 1)$ is also a P.V., and hence, by (2.4), we have $\rho_2 = A_1^*$. If $(GL(1)^3 \times SL(9) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes \rho'_3)$ is a P.V., then $(GL(1)^2 \times SL(2), 3A_1 + \rho'_3)$ must be also a P.V. by Lemma 1.4, and hence $\rho'_3 = 1$. For (44), we have (i) by $\dim GL(1) \times H = 2m^2 + 3m + 2 < 4m^2 + 4m^2 - m - 1 = (2m + 1) \times (2m^2 + m - 1) \leq \deg \rho_2 \otimes \rho'_2$ for $m \geq 4$. Now assume that $(GL(1)^2 \times SL(2m + 1) \times SL(2m^2 + m - 1), A_2 \otimes A_1 + 1 \otimes \rho'_3)$ is a P.V. We shall see that $\rho'_3 = A_1$ (and $\rho'_3 \neq A_1^*$). Since $\dim GL(1) \times H = 2m^2 + 3m + 2 \geq \deg \rho'_3 \geq 2m^2 + m - 1$, we have $\rho'_3 = A_1$ or $A_1^*$. By calculating the weights, the $SL(2m^2 + m - 1)$ part of the generic isotropy subgroup $H$ of $(GL(1) \times SL(2m + 1) \times SL(2m^2 + m - 1), A_2 \otimes A_1 + 1 \otimes A_1)$.

\[
\begin{pmatrix}
A_2(A) + \varepsilon_1 I & * \\
0 & A_1(A) - \varepsilon_2 I
\end{pmatrix}, \quad \varepsilon_1 = (2m^2 + m)e,
\]

\[
2\varepsilon_2 = (m - 1)(2m + 1)^2e, \quad A \in Sp(m)
\]

or

\[
\begin{pmatrix}
A_2(A) + \varepsilon_1 I & 0 \\
* & A_1(A) - \varepsilon_2 I
\end{pmatrix}.
\]

Now if $\rho'_3 = A_1$, its castling transform is $(GL(1)^2 \times SL(2m + 1) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1)$, which is a P.V. by (2.13). Note that it is a P.V. for
This implies that the $SL(2m^2 + m - 1)$-part of $H$ must be of the form

$$\left\{ \begin{array}{c|c} A_2(A) + v_1 I & * \\ \hline 0 & A_1(A) - v_2 I \end{array} \right\}; \ A \in \text{Sp}(m),$$

since $(GL(1) \times \text{Sp}(m), A_2)$ is a non-P.V. for $m \geq 3$. Therefore, if $\rho_3 = A_1^\star$, it is a non-P.V. Assume that $(GL(1)^3 \times SL(2m+1) \times SL(2m^2 + m - 1), A_2 \otimes A_1 + 1 \otimes A_1 + \rho_2 \otimes 1)$ is a P.V. Then its castling transform

$(GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1 + \rho_2^\star \otimes 1)$

must be a P.V. If $m \geq 5$, then we have $\rho_2^\star = 1$ by (30) of Lemma 2.4. If $m = 4$, by (43) of our Lemma 2.6, we have $\rho_2^\star = 1$ for (45). Therefore, if $\rho_3 = A_1^\star$, it is a non-P.V. Assume that $(GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1 + \rho_2^\star \otimes 1)$ is a P.V. Then its castling transform

$(GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1 + \rho_2 \otimes 1)$

must be a P.V. If $m \geq 5$, then we have $\rho_2^\star = 1$ by (30) of Lemma 2.4. If $m = 4$, by (43) of our Lemma 2.6, we have $\rho_2^\star = 1$. For (45), assume that $(GL(1)^2 \times SO(10) \times SL(m), A_1 \otimes A_1 + \rho_2 \otimes \rho_1^\star)$ is a P.V. for $\rho_2 \neq 1$ and $\rho_3^\star \neq 1$. Then we have $\text{dim } G = m^2 + 46 \geq \text{dim } V \geq 20m$, i.e., $(m - 10)^2 \geq 54 (2 \leq m \leq 9)$, and hence $m = 2$. Thus $\rho_2 \otimes \rho_2^\star$ must be $A_1 \otimes A_1^\star$ or half-spin rep. $\otimes A_1$. By Sublemma 2.4.1, $\rho_2 \otimes \rho_2^\star \neq A_1 \otimes A_1^\star$. If $\rho_2 \otimes \rho_2^\star$ is a half-spin rep. $\otimes A_1$, then $\text{dim } G = 50 < \text{dim } V = 52$, which is a contradiction. Thus we have (i) for (45). Now assume that $(GL(1)^3 \times SO(10) \times SL(m), A_1 \otimes A_1 + \rho_2 \otimes 1 + 1 \otimes \rho_1^\star)$ is a P.V. with $\rho_2 \neq 1$ and $\rho_3^\star \neq 1$. By Sublemma 2.4.2, $\rho_2$ must be a half-spin representation of $\text{Spin}(10)$. Since $SL(m)$-part of the generic isotropy subgroup of $(SO(10) \times GL(m), A_1 \otimes A_1)$ is $O(m)$, $\rho_3^\star$ must be $A_1$ or $A_1^\star$. The generic isotropy subgroup of $(GL(1)^2 \times \text{Spin}(10) \times SL(m), \text{vector rep. } \otimes A_1 + 1 \otimes A_1^\star)$ is locally isomorphic to $O(10 - m) \times O(m - 1)$ (p. 110 in [1]) and by calculation of weights, we see that the restriction of a half-spin representation of $\text{Spin}(10)$ to $O(10 - m) \times O(m - 1)$ is given by $(\text{Spin}(8), \text{even half-spin rep. } \oplus \text{odd half-spin rep.})$ for $m = 2, 9$; $(GL(1) \times \text{Spin}(7), \ A_1 \oplus A_1^\star \otimes \text{spin rep.})$ for $m = 3, 8$; $(SL(2) \times SL(4), A_1 \otimes (A_1 + A_1^\star))$ for $m = 4, 7$; $(SL(2) \times SL(2) \times Spin(2), A_1 \otimes (A_1 + A_1^\star \otimes A_1))$ for $m = 5, 6, 7$. Since they are not P.V.s even with a scalar multiplication (see the proof of (2.9) in [4] for $m = 5, 6$), we have (ii) for (45). For (46), assume that $(GL(1)^2 \times \text{Spin}(12) \times SL(m), \text{vector rep. } \otimes A_1 + \rho_2 \otimes \rho_2^\star)$ is a P.V. with $\rho_2 \neq 1$ and $\rho_2^\star \neq 1$. By Theorem 1.5, $\rho_2 \otimes \rho_2^\star$ must be vector rep. $\otimes A_1$. By Sublemma 2.4.1, it is a non-P.V. and hence we have (i) for (46). Now assume that $(GL(1)^3 \times \text{Spin}(12) \times SL(m), \text{vector rep. } \otimes A_1 + \rho_2 \otimes 1 + 1 \otimes \rho_1^\star)$ is a P.V. Then $\rho_2$ must be a half-spin representation by Sublemma 2.4.2, and $\rho_3^\star = A_1$ or $A_1^\star$ (see the proof for (45)). Since the generic isotropy subgroup of $(GL(1) \times \text{Spin}(12), \rho_2)$ is $SL(6)$ (p. 129 in [1]), $(GL(1)^2 \times \text{Spin}(12) \times SL(m), \text{vector rep. } \otimes A_1 + \rho_2 \otimes 1)$ is a P.V. if and only if $(SL(6) \times GL(m), (A_1 + A_1^\star) \otimes A_1)$ is a P.V. By the proof of Sublemma 2.4.4 (and by a castling transformation if necessary), it is not a P.V. for $2 \leq m \leq 10$. Since the generic isotropy subgroup of $(GL(1)^2 \times \text{Spin}(12) \times SL(11), \text{vector rep. } \otimes A_1 + \rho_2 \otimes 1)$ is reductive, we may assume $\rho_3^\star = A_1$ as far as we consider the prehomogeneity. Then
its castling transform is $(GL(1)^3 \times \text{Spin}(12) \times SL(2))$, vector rep. $\otimes A_1 + \rho_2 \otimes 1 + 1 \otimes A_1$, which is not a P.V. as we have seen above. Thus we have (ii) for (46). For (47), if $(GL(1)^3 \times \text{Spin}(7) \times SL(7))$, spin rep. $\otimes A_1 + \rho_2 \otimes \rho_2'$ is a P.V. for $\rho_2' \neq 1$ and $\rho_2' \neq 1$, then dim $GL(1)^3 \times H = 15 \geq \text{deg} \rho_2 \otimes \rho_2' \geq 49$, which is a contradiction, and hence we have (i) for (7). If $(GL(1)^3 \times \text{Spin}(7) \times SL(7))$, spin rep. $\otimes A_1 + \rho_2 \otimes 1 + 1 \otimes \rho_2')$ is a P.V., then $\rho_2'$ must be the vector representation by Theorem 1.3, since a castling transform $(GL(1)^3 \times \text{Spin}(7) \times SL(7))$, spin rep. $\otimes A_1 + \rho_2 \otimes 1 + 1 \otimes A_2$ must be a P.V. By dimension reason, we have $\rho_3 = A_1$ or $A_1^\ast$. Since the generic isotropy subgroup of $(GL(1)^3 \times \text{Spin}(7) \times SL(7))$, spin rep $\otimes A_1 + \rho_2 \otimes 1$ is reductive, we may assume $\rho_3 = A_1$. Then, by a castling transformation, we have $(GL(1)^3 \times \text{Spin}(7) \times SL(2))$, spin rep. $\otimes A_1 + \rho_2 \otimes 1 + 1 \otimes A_1)$, which is not a P.V. by (33) of Lemma 2.4. Thus we have (ii) for (47). For (48), if $(GL(1)^3 \times \text{Spin}(10) \times SL(15))$, half-spin rep. $\otimes A_1 + \rho_2 \otimes \rho_2'$ is a P.V. for $\rho_2' \neq 1$ and $\rho_2' \neq 1$, then dim $GL(1)^3 \times H = 31 \geq \text{deg} \rho_2 \otimes \rho_2' \geq 150$, which is a contradiction, and hence we have (i) for (48). If $(GL(1)^3 \times \text{Spin}(10) \times SL(15))$, half-spin rep. $\otimes A_1 + \rho_2 \otimes 1 + 1 \otimes \rho_2')$ is a P.V., then $\rho_2'$ must be the half-spin representation or the vector representation by [2], and $\rho_3 = A_1$ or $A_1^\ast$ by dimension reason. Since the generic isotropy subgroup of $(GL(1)^3 \times \text{Spin}(10) \times SL(15))$, half-spin rep. $\otimes A_1 + \rho_2 \otimes 1$ is reductive (see pp. 96, 97 in [2]), we may assume that $\rho_3 = A_1$. Then, by a castling transformation, we have $(GL(1)^3 \times \text{Spin}(10) \times SL(2))$, half-spin rep. $\otimes A_1 + \rho_2 \otimes 1 + 1 \otimes A_1)$, which is not a P.V. by (36) of Lemma 2.4. Thus we have (ii) for (48). Q.E.D.

**Theorem 2.7.** All non-irreducible 2-simple P.V.s which have one of (IV) in Theorem 1.5 as an irreducible component are given as follows:

$$(GL(1)^3 \times SL(2) \times SL(2), 2A_1 \otimes A_1 + A_1 \otimes 1)$$  \hspace{1cm} (2.28)

$$(GL(1)^3 \times SL(2) \times SL(2), 2A_1 \otimes A_1 + 1 \otimes A_1)$$  \hspace{1cm} (2.29)

$$(GL(1)^3 \times SL(5) \times SL(8), A_2 \otimes A_1 + \Sigma_1 \otimes 1)$$  \hspace{1cm} (s = 1, 2),

where $\Sigma_1 = A_1^{\ast \ast}$ and $\Sigma_2 = A_1 + A_1^{\ast \ast}$.  \hspace{1cm} (2.30)

$$(GL(1)^3 \times SL(5) \times SL(8), A_2 \otimes A_1 + 1 \otimes A_1^{\ast \ast})$$  \hspace{1cm} (2.31)

$$(GL(1)^3 \times SL(9) \times SL(2), A_2 \otimes A_1 + A_1^{\ast \ast} \otimes 1)$$  \hspace{1cm} (2.32)

$$(GL(1)^3 \times SL(9) \times SL(2), A_2 \otimes A_1 + 1 \otimes T_1, \hspace{1cm} (t = 1, 2, 3),$$

where $T_1 = A_1, 2A_1, 3A_1; T_2 = A_1 + A_1, A_1 + 2A_1; \hspace{1cm} (2.33)$

$T_3 = A_1 + A_1 \otimes A_1$.

$$(GL(1)^3 \times SL(2m + 1) \times SL(2m^2 + m - 1),$$

$A_2 \otimes A_1 + \Sigma_3 \otimes 1) \hspace{1cm} (s = 1, 2, 3), \hspace{1cm} (2.34)$

where $\Sigma_1 = A_1^{\ast \ast}$, $A_2^{\ast \ast}$;

$A_1^{\ast \ast} \otimes A_1^{\ast \ast}$; $\Sigma_3 = A_1^{\ast \ast} \otimes A_1^{\ast \ast} \otimes A_1^{\ast \ast}$ except for $\Sigma_3 \otimes A_1^{\ast \ast} \otimes A_1^{\ast \ast}$, \hspace{1cm} (m \geq 4).
(\text{GL}(1)^2 \times SL(2m+1) \times SL(2m^2 + m - 1), \\
A_2 \otimes A_1 + 1 \otimes A_1) \quad (m \geq 4) \quad (2.35)

(\text{GL}(1)^2 \times \text{Spin}(10) \times SL(m), \text{vector rep.} \otimes A_1 + \text{half-spin rep.} \otimes 1) \quad (2 \leq m \leq 9, \ m \neq 5) \quad (2.36)

(\text{GL}(1)^2 \times SO(10) \times SL(m), A_1 \otimes A_1 + 1 \otimes A_1^{(*)}) \quad (2 \leq m \leq 9) \quad (2.37)

(\text{GL}(1)^2 \times \text{Spin}(12) \times SL(11), \text{vector rep.} \otimes A_1 + \text{half-spin rep.} \otimes 1) \quad (2.38)

(\text{GL}(1)^2 \times SO(12) \times SL(m), A_1 \otimes A_1 + 1 \otimes A_1^{(*)}) \quad (2 \leq m \leq 11) \quad (2.39)

(\text{GL}(1)^2 \times \text{Spin}(7) \times SL(7), \text{spin rep.} \otimes A_1 + \text{vector rep.} \otimes 1) \quad (2.40)

(\text{GL}(1)^2 \times \text{Spin}(7) \times SL(7), \text{spin rep.} \otimes A_1 + 1 \otimes A_1^{(*)}) \quad (2.41)

(\text{GL}(1)^2 \times \text{Spin}(10) \times SL(15), A_0 \otimes A_1 + \rho \otimes 1) \quad (2.42)

Note that A_2 \otimes A_1 + A_1 \otimes 1 for (2.32) and A_2 \otimes A_1 + 1 \otimes A_1^* for (2.35) are non-P.V.s.

Proof. For (2.28) (resp. (2.29)), we have \( \rho_2 = A_1 \) (resp. \( \rho'_2 = A_1 \)), since \( \deg \rho_2 \) (resp. \( \deg \rho'_2 \)) \( \leq \dim G - \deg 2A_1 \otimes A_1 = 2 \). Since the SL(2) part of the generic isotropy subgroup is O(2), (2.28) and (2.29) are actually P.V.s. For (2.30), \( (\text{GL}(1)^2 \times SL(10) \times SL(m), A_2 \otimes A_1 + \Sigma, \otimes 1) \) with \( \Sigma = \sigma_1 + \cdots + \sigma_s \), is a P.V. if and only if \( (\text{GL}(1)^2 \times SL(10) \times SL(2), A_2 \otimes A_1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1) \) is a P.V. Since \( \dim G \geq \dim V \), we have \( 5s \leq \deg \sigma_1^* + \cdots + \deg \sigma_s^* \leq s + 8 \), and hence \( s = 1 \) or \( 2 \). Thus we have \( \sigma_i^* = A_1^{(*)} \) for \( s = 1 \) and \( \sigma_1^* \oplus \sigma_s^* = A_1^{(*)} \oplus A_1^{(*)} \). However, \( \sigma_1^* \oplus \sigma_s^* \neq A_1 \oplus A_1 \) since otherwise \( (\text{GL}(1)^2 \times SL(2), A_2 \otimes A_1 + (A_1^* + A_1^*) \otimes 1) \) becomes a P.V. by (1.1), which is a contradiction by dimension reason. By calculating the isotropy subalgebra at \( (X_0, e_5, e_1 + e_3 + e_4 + e_5) \) (resp. \( (X_0, e_5, e_1 + e_3 + e_2) \)) of \( (\text{GL}(1)^2 \times SL(10) \times SL(2), A_2 \otimes A_1 + (A_1^* + A_1^*) \otimes 1) \) (resp. \( A_2 \otimes A_1 + (A_1^* + A_1^*) \otimes 1) \) (see Lemma 1.4), we see that they are actually P.V.s. For (2.31), if \( A_2 \otimes A_1 + 1 \otimes (\tau_1 + \cdots + \tau_i) \) is a P.V., then we have \( 8t \leq \deg \tau_1 + \cdots + \deg \tau_i \leq 8 + t \) and hence \( t = 1 \), \( \tau_1 = A_1^* \). If \( \tau_1 = A_1^* \), then it is castling-equivalent to (2.12), and hence it is a P.V. If \( \tau_1 = A_1^* \), we identify the representation space of \( A_2 \otimes A_1 + 1 \otimes A_1^* \) with

\[ V = V_2 \otimes \bigoplus_{i=1}^{8} V_2 \otimes K^8, \]

where \( V_2 = \Sigma Ke_i \wedge e_j \) (\( 1 \leq i < j \leq 5 \)). Then the action is given by \( x \mapsto \alpha \cdot A_3(A) \cdot (x_1, ..., x_8)B + \beta'B^{-1}y \) for \( x = (x_1, ..., x_8; y) \in V \) and \( g = (x, \beta; A, B) \in GL(1)^2 \times SL(5) \times SL(8) \). By calculating the isotropy sub-
algebra at $x = (\omega_1, 2\omega_2, 2\omega_3, \omega_{10}, \omega_4 - \omega_8, \omega_4 - \omega_9, \omega_6, \omega_7; e_2 + e_8)$ (see p. 95 in [1]), we see that it is a P.V. For $(GL(1)^{1+s} \times SL(9) \times SL(2), A_2 \otimes A_1 + \Sigma_1 \otimes 1)$, we have (2.32) from (2.4) by a castling transformation. For (2.33), since the $SL(2)$ part of the generic isotropy subgroup of $(GL(1) \times SL(9) \times SL(2), A_2 \otimes A_1)$ is $SL(2)$ by Lemma 1.4, we have our result by [1]. For (2.34), it is castling-equivalent to a simple P.V. $(GL(1)^{1+s} \times SL(2m + 1), A_2 + \Sigma_s^*)$, and hence we obtain our result by [2]. For (2.35), if $A_2 \otimes A_1 + 1 \otimes (\tau_1 + \cdots + \tau_i)$ is a P.V., then we have $(2m^2 + m - 1)t \leq \deg \tau_1 + \cdots + \deg \tau_i \leq t + (2m^2 + 3m + 1)$ and hence $t = 1$ and $\tau_1 = A_1^{(*)}$. By the proof of (44) of Lemma 2.6, we have our result. For (2.36), if vector rep. $\otimes A_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ $(2 \leq m \leq 9)$ is a P.V., then $\sigma_1, \ldots, \sigma_s \neq$ the vector representation by Sublemma 2.4.2 and $\sigma_1 = A_c$ or $\sigma_1 + \sigma_2 = A_c + A_c$, $s \leq 2$ by [2]. If $\sigma_1 + \sigma_2 = A_c + A_c$, then $\dim G \geq \dim V$ implies $(m - 5)^2 \geq 10$ $(2 \leq m \leq 9)$ and hence $m = 9$. Then, it is castling-equivalent to $(GL(1)^3 \times Sp(10), \text{vector rep.} \oplus A_c \oplus A_c)$, which is a non-P.V. by [2], and hence we have $\sigma_1 = A_c$. In this case, it is a P.V. for $m = 1, 2, 3$ (and hence $m = 9, 8, 7$) by Theorems 3.3 and 5.7 in Kimura et al. [4]. For $m = 4$ (resp. $m = 5$), the restriction of $(GL(1) \times Sp(10) \times SL(m), A_c \otimes 1)$ to the generic isotropy subgroup $SO(10 - m) \times SL(m)$ is equivalent to $(GL(1) \times SL(2) \times SL(2) \times SL(4), A_1 \otimes A_1 \otimes 1 \otimes A_1 + A_1 \otimes 1 \otimes A_1 \otimes A_1^*)$ for $m = 4$ (resp. $(GL(1) \times Sp(2) \times Sp(2), A_1 \otimes A_1 \otimes A_1)$ for $m = 5$). Thus the case for $m = 4$ (and hence $m = 6$) is a P.V. (see the corollary of Theorem 1.16 in [3]), and the case $m = 5$ is a non-P.V. For (2.37), since the $SL(m)$ part of the generic isotropy subgroup of $(GL(1) \times SO(10) \times SL(m), A_1 \otimes 1)$ is $SO(m)$, we have our result by [2]. If $(GL(1)^{1+s} \times Spin(12) \times SL(m), \text{vector rep.} \otimes A_1 + \Sigma_1 \otimes 1)$ is a P.V. with $2 \leq m \leq 11$, then we have $m = 11$ by the proof of Lemma 2.6. Hence it is castling-equivalent to a simple P.V. $(GL(1)^{1+s} \times Spin(12), \text{vector rep.} \oplus \Sigma_s^*)$. Thus we obtain (2.38) by Theorem 1.3. For (2.39), we have our result similarly as (2.37). For (2.40), it is castling-equivalent to a simple P.V. (2.41). Since the $SL(7)$ part of the generic isotropy subgroup of $(GL(1) \times Spin(7) \times SL(7), \text{spin rep.} \otimes A_1)$ is $((G_2), A_2)$, we have our result by [2]. (2.42), it is castling-equivalent to a simple P.V. and we have our result by Theorem 1.3. Now assume that $(GL(1)^{1+t} \times Spin(10) \times SL(15), A_0 \otimes A_1 + 1 \otimes (\tau_1 + \cdots + \tau_i))$ is a P.V. Then we have $15t \leq \deg \tau_1 + \cdots + \deg \tau_i \leq 30 + t$ and hence $t = 1$ or 2. By dimension reason, we have $\tau_1 = A_1^{(*)}$ for $t = 1$ and $\tau_1 + \tau_2 = A_1^{(*)} + A_1^{(*)}$ for $t = 2$. If $t = 1$ and $\tau_1 = A_1$, it is castling-equivalent to $(GL(1)^2 \times Spin(10) \times SL(2), A_c \otimes A_1 + 1 \otimes A_1)$ which is a P.V. by (2.21). If $t = 2$ and $\tau_1 + \tau_2 = A_1 + A_1$, it is castling-equivalent to $(GL(1)^3 \times Spin(10) \times SL(3), A_c \otimes A_1 + 1 \otimes A_1 + 1 \otimes A_1)$ which is a non-P.V. by Theorem 2.5 for (37). Let $V_c$ be the vector space spanned by $1, e, e_i, (1 \leq i \leq 5)$, $e_je_ie_i, (1 \leq k < l < s < t \leq 5)$ over $K$. Let $\rho_1$ by the even half-spin representation $A_c$ on $V_c$. Then, the odd half-spin representation
PREHOMOGENEOUS VECTOR SPACES

$A_0$ is the dual $\rho_1^*$ of $\rho_1$. Now the representation space $V$ of $(GL(1)^2 \times \text{Spin}(10) \times SL(15), A_0 \otimes A_1 + 1 \otimes A_1^*)$ is identified with

$$V = V_e \oplus \cdots \oplus V_e \otimes K^{15}. $$

The action is given by $x \rightarrow (\alpha^* (A)(X_1, \ldots, X_n)^B; \beta B^{-1} y)$ for $x = (X_1, \ldots, X_n; y) \in V$, $g = (\alpha, \beta; A, B) \in GL(1)^2 \times \text{Spin}(10) \times SL(15)$. Put $x_0 = (e_1 e_5, e_2 e_5, e_3 e_5, e_4 e_5, e_2 e_3 e_4 e_5, -e_1 e_3 e_4 e_5, e_1 e_2 e_4 e_5, -e_1 e_2 e_3 e_5, -1 + e_1 e_2 e_3 e_4, e_1 e_2, e_1 e_4, -e_3 e_4, e_2 e_4, -e_2 e_3; e_0) \in V$. The isotropy subalgebra of $g_0(1) \oplus g_0(1) \oplus o(10) \oplus o(15)$ at $x_0$ is given by $\{(16e), \{e\}, \{A \oplus (30e) \oplus (A - 14\epsilon A_4) \oplus (16e) \oplus (30e)\} \mid A \in o(4), e \in g_0(1) \} \approx g_0(1) \oplus o(4)$. Hence it is a P.V. Since $(GL(1)^3 \times SL(4), A_1 \otimes 1 + A_1 \otimes A_1^*)$ is a non-P.V., $(GL(1)^2 \times \text{Spin}(10) \times SL(15), A_0 \otimes A_1 + 1 \otimes A_1^* + 1 \otimes A_1^*)$ is a non-P.V. Q.E.D.

**Theorem 2.8.** All non-irreducible 2-simple P.V.s which have $(SL(5) \times GL(2), A_2 \otimes A_1)$ ((49) in Theorem 1.5) as an irreducible component are given as follows:

$$(GL(1)^{s-1} \times SL(5) \times SL(2), A_2 \otimes A_1 + \Sigma_i \otimes 1) \quad (s = 1, 2),$$

where $\Sigma_1 = A_1^* \otimes A_1$ and $\Sigma_2 = A_1^* + A_1^*$ ($\Sigma_2 \neq A_1 + A_1$).

$$(GL(1)^{i-1} \times SL(5) \times SL(2), A_2 \otimes A_1 + 1 \otimes T_i) \quad (i = 1, 2, 3),$$

where $T_1 = A_1, 2A_1, 3A_1; T_2 = A_1 + A_1, 2A_1 + A_1$;

$T_3 = A_1 + A_1 + A_1$.

$$(GL(1)^3 \times SL(5) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes A_1)$$

$$(GL(1)^{2-i} \times SL(5) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes T_i)$$

$(i = 1, 2)$, where $T_1 = 2A_1; T_2 = A_1 + A_1$.

**Proof.** By dimension reason, $A_2 \otimes A_1 + \rho_2 \otimes \rho_2' (\rho_2 \neq 1, \rho_2' \neq 1)$ is a non-

P.V. If $A_2 \otimes A_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V., then its castling transform

$(GL(1)^{s-1} \times SL(5) \times SL(8), A_2 \otimes A_1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1)$ is also a P.V., and hence, by (2.30), we have $\sigma_1^* = A_1^* \otimes A_1$ and $\sigma_1^* + \sigma_2^* = A_1 + A_1^* \otimes 1$, i.e., (2.44). We have (2.45) similarly as (2.13)–(2.15). By dimension reason, $A_2 \otimes A_1 + A_1^* \otimes 1 + 1 - A_1$ is a non-P.V. for any $\rho_4 \otimes \rho_4' \neq 1$. Assume that $A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$, where $T_1 = 2A_1; T_2 = A_1 + A_1$.

Thus it is enough to prove that (2.46) and (2.47) are actually P.V.s. (2.46) is a F.P. (see (5.19) in [4]) and hence a P.V. For (2.47), the generic isotropy subgroup of $(GL(1)^2 \times SL(2), A_1 + A_1)$ or
(GL(1) × SL(2), 2A₁) is O(2), and hence (2.47) is a P.V. if and only if
(GL(1)³ × SL(5), A₂ ⊕ A₂ ⊕ A*₁) is a P.V. By (5) of Theorem 1.3, we have
our results. Q.E.D.

**Theorem 2.9.** All non-irreducible 2-simple P.V.s which have (SL(5) ×
GL(9), A₂ ⊗ A₁) ((50) in Theorem 1.5) as an irreducible component are given
as follows:

\[
(GL(1)^{1+s} × SL(5) × SL(9), A₂ ⊗ A₁ + S₁ ⊗ 1) \quad (s = 1, 2, 3),
\]
where \(S₁ = A_{1}^{*₁}, A_{2}^{*₂}, S₂ = A_{1}^{*₁} + A_{1}^{*₂}, A_{2}^{*₃} + A₁;\)
\[S₃ = A_{1}^{*₁} + A_{1}^{*₂} + A_{1}^{*₃} \text{ except for } S₃ = A_{1}^{*₁} + A_{1}^{*₃} + A₁. \quad (2.48)\]

\[
(GL(1)^{2} × SL(5) × SL(9), A₂ ⊗ A₁ + 1 ⊗ A_{1}^{*₁}) \quad (2.49)
\]

\[
(GL(1)^{3} × SL(5) × SL(9), A₂ ⊗ A₁ + A_{1}^{*₁} ⊗ 1 + 1 ⊗ A_{1}^{*₁}). \quad (2.50)
\]

**Proof.** By dimension reason, \(A₂ ⊗ A₁ + p₁ ⊗ p₂ (p₁ ≠ 1, p₂ ≠ 1)\) is a non-P.V. Since \(A₂ ⊗ A₁ + S₁ ⊗ 1\) is castling-equivalent to
(GL(1)^{1+s} × SL(5), A₂ + S₁ *), we have our result by Theorem 1.3. If
\(A₂ ⊗ A₁ + 1 ⊗ Tₜ (Tₜ = τ₁ + ⋯ + τᵣ)\) is a P.V., then \(t = 1\) and \(τ₁ = A_{1}^{*₁}\) by
dimension reason. The prehomogeneity of (2.49) comes from that of (2.50).
If \(A₂ ⊗ A₁ + 1 ⊗ A_{1}^{*₁} + S₁ ⊗ 1\) is a P.V., then we have \(s = 1\) and
\(A₁ = A_{1}^{*₁}\) by dimension reason. Now \(A₂ ⊗ A₁ + A_{1}^{*₁} ⊗ 1 + 1 ⊗ A₁\) is
castling-equivalent to (2.46) and hence it is a P.V. Since \(A₂ ⊗ A₁ + A_{1}^{*₁} ⊗ 1\)
is castling-equivalent to a regular P.V. (GL(1)^{2} × SL(5), A₂ ⊗ A₁), its
generic isotropy subgroup is reductive. Since \(A₂ ⊗ A₁ + A_{1}^{*₁} ⊗ 1 + 1 ⊗ A₁\)
is a P.V., \(A₂ ⊗ A₁ + A_{1}^{*₁} ⊗ 1 + 1 ⊗ A_{1}^{*₁}\) is also a P.V. By Lemma 2.10, it is
castling-equivalent to \(A₂ ⊗ A₁ + A₁ ⊗ 1 + 1 ⊗ A_{1}^{*₁}\), and hence (2.50) is
actually a P.V. Q.E.D.

**Lemma 2.10.** Assume that G is reductive and \(\text{deg } p₁ = m ≥ 3\). Then
(GL(1)^{3} × G × SL(m - 1), \(p₁ ⊗ A₁ + p₂ ⊗ 1 + 1 ⊗ A_{j}^{*₁}\) is castling-equivalent
to (GL(1)^{3} × G × SL(m - 1), \(p₁ ⊗ A₁ + p₂ ⊗ 1 + 1 ⊗ A_{j}^{*₁}\)).

**Proof.** It is equivalent to ((GL(1) × G) × GL(m - 1) × GL(1), \(p₁ ⊗ A₁ ⊗ 1 + p₂ ⊗ 1 ⊗ 1 + 1 ⊗ A_{j}^{*₁} ⊗ A₁\)) \(\sim\)
((GL(1) × G) × GL(m - 1) × GL(m - 2), \(p₁ ⊗ A₁ ⊗ 1 + p₁ ⊗ A₁ ⊗ 1 + p₂ ⊗ 1 ⊗ 1 + 1 ⊗ A₁ ⊗ A_{j}^{*₁}\)) \(\sim\)
\(p₁ ⊗ A₁ ⊗ 1 + p₂ ⊗ 1 ⊗ 1 + 1 ⊗ A₁ ⊗ A_{j}^{*₁} \sim (GL(1)^{3} × G × SL(m - 1), \(p₁ ⊗ A₁ + p₂ ⊗ 1 + 1 ⊗ A_{j}^{*₁}\)), \(\sim\) denotes
the castling- (resp. reductive-)equivalence. Q.E.D.

**Theorem 2.11.** All non-irreducible 2-simple P.V.s which have (SL(7) ×
GL(2), A₂ ⊗ A₁) ((51) in Theorem 1.5) as an irreducible component are given
as follows:
(GL(1)^2 x SL(7) x SL(2), A_2 \otimes A_1 + A_i^{(*1) \otimes 1}) \tag{2.51}

(GL(1)^1+1 x SL(7) x SL(2), A_2 \otimes A_1 + 1 \otimes T_t) \quad (t = 1, 2, 3),

where \( T_1 = A_1, \ 2A_1, \ 3A_1; T_2 = A_1 + A_1, A_1 + 2A_1; T_3 = A_1 + A_1 + A_1. \) \tag{2.52}

(GL(1)^3 x SL(7) x SL(2), A_2 \otimes A_1 + A^*_i \otimes 1 + 1 \otimes A_1). \tag{2.53}

Note that \( A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1 \) is a non-P.V. for (2.53).

Proof. \( A_2 \otimes A_1 + \rho_2 \otimes \rho'_2 (\rho_2 \neq 1, \rho'_2 \neq 1) \) is a non-P.V. by dimension reason. \( A_2 \otimes A_1 + \Sigma_i \otimes 1 \) is castling-equivalent to \((GL(1)^1+1 x SL(7) x SL(19), A_i^{*} \otimes A_1 + \Sigma_i \otimes 1)\) and hence we obtain (2.51) from (2.3). If \( A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes T_t \) is a P.V., then \((GL(1)^1+1 x SL(2), 3A_1 + T_t)\) must be a P.V. by Lemma 1.4. Hence we have \( t = 0, T_t = 1. \) If \( A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes T_t (T_t \neq 1) \) is a P.V., then \((GL(1)^1+1 x SL(2), 2A_1 + T_t)\) must be a P.V. and hence \( t = 1, T_t = A_1, \) i.e., (2.53). It is actually a P.V. For example, \( X_0; \ (0000010), (1, 1) \) (see Lemma 1.4 for \( X_0 \)) is a generic point. Q.E.D.

THEOREM 2.12. All non-irreducible 2-simple P.V.s which have \((SL(7) \times GL(20), A_2 \otimes A_1) \) ((52) in Theorem 1.5) as an irreducible component are given as follows:

\((GL(1)^1+1 x SL(7) x SL(20), A_2 \otimes A_1 + \Sigma_i \otimes 1)\) (s = 1, 2, 3),

where \( \Sigma_1 = A_i^{*}, A_i^{*}; \ \Sigma_2 = A_i^{*} + A_i^{*}; \ \Sigma_3 = A_i^{*} + A^*_i + A_i^{*} \) except for \( \Sigma_3 = A_i + A_i^* + A_i^*. \) \tag{2.54}

(GL(1)^2 x SL(7) x SL(20), A_2 \otimes A_1 + 1 \otimes A_1) \tag{2.55}

(GL(1)^3 x SL(7) x SL(20), A_2 \otimes A_1 + A_i \otimes 1 + 1 \otimes A_1). \tag{2.56}

Note that \( A_2 \otimes A_1 + 1 \otimes A_i^* \) for (2.55), and \( A_2 \otimes A_1 + A_i^* \otimes 1 + 1 \otimes A_1 \) for (2.56), are not P.V.s.

Proof. If \( A_2 \otimes A_1 + \rho_2 \otimes \rho'_2 (\rho_2 \neq 1, \rho'_2 \neq 1) \) is a P.V., then we have \( \dim G = 449 > \dim V > 420 + 7 \times 20 = 560, \) which is a contradiction. Similarly as (2.34) and (2.35), we have (2.54) and (2.55). Since \( A_2 \otimes A_1 + 1 \otimes A_1 + \Sigma_i \otimes 1 \) is castling-equivalent to \((GL(1)^2+1 x SL(7) x SL(2), A_2 \otimes A_1 + 1 \otimes A_1 + \Sigma_i \otimes 1)\), we have \( s = 1 \) and \( \Sigma_1 = A_1 \) by (2.53). Q.E.D.

THEOREM 2.13. All non-irreducible 2-simple P.V.s which have \((SO(5) \times GL(m), A_1 \otimes A_1) \simeq (Sp(2) \times GL(m), A_2 \otimes A_1) \) (m = 2, 3, 4) ((53) in Theorem 1.5) as an irreducible component are given as follows:

(GL(1)^2 x Sp(2) x SL(m), A_2 \otimes A_1 + A_1 \otimes 1) \tag{2.57}
\[(GL(1)^3 \times Sp(2) \times SL(m), A_2 \otimes A_1 + 1 \otimes A_1^{(*)}) \quad (2.58)\]
\[(GL(1)^3 \times Sp(2) \times SL(m), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^{(*)}), \quad \text{for } m = 2, 4 \ (m \neq 3). \quad (2.59)\]

**Proof.** If \(A_2 \otimes A_1 + \rho_2 \otimes \rho_2' \ (\rho_2 \neq 1, \rho_2' \neq 1)\) is a P.V., then we have \(\dim G = m^2 + 11 \geq \dim V \geq 5m + 4m \ (m = 2, 3, 4)\), which is a contradiction. First note that \(A_2 \otimes A_1 + A_2 \otimes 1\) is a non-P.V. by Sublemma 2.4.2. Hence if \(A_2 \otimes A_1 + \Sigma_3 \otimes 1\) is a P.V., then \(s = 1, \Sigma_3 = A_1\) or \(s \leq 2, \Sigma_3 = A_1 + A_1\), for \(m = 4\), by dimension reason. However, a castling transform \((GL(1)^3 \times Sp(2), A_2 + A_1 + A_1)\) of \((GL(1)^3 \times Sp(2) \times SL(4), A_2 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1)\) is a non-P.V., we have (2.57). Actually (2.57) is a P.V. by (5.10) in [4]. Since the \(SL(m)\) part of the generic isotropy subgroup of \((GL(1)^3 \times Sp(2) \times SL(m), A_2 \otimes A_1) \ (m = 2, 3, 4)\) is \(O(m)\), we have (2.58). For (2.59), we have \(m \neq 3\) by dimension reason. Since the generic isotropy subalgebra of
\[
(\text{Sp}(2) \times GL(2), A_2 \otimes A_1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & -'A \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} -\text{Tr} A & 0 \\ 0 & \text{Tr} A \end{pmatrix} \right\}; A \in gl(2)\]
(see p. 455 in Kimura and Kasai [5]), \((GL(1)^3 \times Sp(2) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1)\) is a P.V., and hence \((GL(1)^3 \times Sp(2) \times SL(4), A_2 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1)\) is a P.V. by a castling transformation. Since the \(SL(4)\) part of the generic isotropy subgroup of \((GL(1)^3 \times Sp(2) \times SL(4), A_2 \otimes A_1)\) is \(O(4)\), \((GL(1)^2 \times Sp(2) \times SL(4), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^{(*)})\) is also a P.V. Q.E.D.

**Theorem 2.14.** All non-irreducible 2-simple P.V.s which have \((SO(6) \times GL(m), A_1 \otimes A_1) \simeq (SL(4) \times GL(m), A_2 \otimes A_1) \ (2 \leq m \leq 5)\) \((54) \text{ in Theorem 1.5)}\) as an irreducible component are given as follows:

\[(GL(1)^2 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes A_1) \quad (2.60)\]
\[(GL(1)^3 + SL(4) \times SL(m), A_2 \otimes A_1 + \Sigma_2 \otimes 1) \quad (s = 1, 2, 3), \quad (m = 5); \quad \Sigma_3 = A_1 + A_1^{(*)} + A_1^{(*)} \quad (m = 5) \quad (2.61)\]
\[(GL(1)^2 \times SL(4) \times SL(m), A_2 \otimes A_1 + 1 \otimes A_1^{(*)}) \quad (2.62)\]
\[(GL(1)^3 \times SL(4) \times SL(m), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^{(*)}). \quad (2.63)\]

Note that \(A_2 \otimes A_1^{(*)}\) for \(SL(4)\).

**Proof.** If \(A_2 \otimes A_1 + \rho_2 \otimes \rho_2' \ (\rho_2 \neq 1, \rho_2' \neq 1)\) is a P.V., we have \(\dim G = 16 + m^2 \geq \dim V \geq 6m + 4m \ (2 \leq m \leq 5)\) and hence \(m = 2, \rho_2 \otimes \rho_2' =\)
PREHOMOGENEOUS VECTOR SPACES

Then it is acturally a P.V., since \((e_1 + e_2, e_3 + e_4, e_1 + e_3, e_2 + e_4)\) is a generic point. Note that \(A_2 \otimes A_1 + A_1 \otimes A_1 + \rho_3 \otimes \rho_3\) is not a P.V. for any \(\rho_3 \otimes \rho_3 \neq 1\) by dimension reason. Thus we have (2.60). Let us consider \(A_2 \otimes A_1 + \Sigma_s \otimes 1\) with \(\Sigma_s = \sigma_1 + \cdots + \sigma_s\). If \(m = 5\), it is castleing-equivalent to a simple P.V. \((GL(1)^1 \times SL(4), A_2 + \sigma_1 + \cdots + \sigma_s)\) and hence we have \(s = 1, 2, 3\); \(\sigma_1 = A_1\); \(\sigma_1 + \sigma_2 = A_1 + A_1^*\); \(\sigma_1 + \sigma_2 + \sigma_3 = A_1 + A_1^* + A_i^*\). For \(m = 2, 3, 4\), \(A_2 \otimes A_1 + \sigma_1 \otimes 1\) with \(\sigma_1 = A_2\) (resp. \(\sigma_1 = 2A_1\)) is not a P.V. by Sublemma 2.4.2 (resp. by dimension reason), and hence

\[\Sigma_s = A_1^* + \cdots + A_i^*.\]

Since \(\dim G = 15 + s + m \geqslant \dim V = 6m + \deg \Sigma_s \geqslant 6m + 4s\), i.e., \((m - 3)^2 + 6 \geqslant 3s\) \((2 \leqslant m \leqslant 4)\), we have \(s = 1\) or \(2\). Since the \(SL(4)\) part of the generic isotropy subgroup of \((SL(4) \times GL(3), A_2 \otimes A_1, V(6) \oplus V(6) \oplus V(6))\) at \((e_1 \wedge e_2, e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4)\) is \(SO(4),(GL(1)^2 \times SL(4) \times SL(3), A_2 \otimes A_1 + A_1 \otimes 1)\) is a P.V. and \((GL(1)^3 \times SL(4) \times SL(3), A_2 \otimes A_1 + A_1 \otimes 1 + A_i^* \otimes 1)\) is a non-P.V. Since the generic isotropy subalgebra of \((SL(4) \times GL(2), A_2 \otimes A_1)\) at \((e_1 \wedge e_2, e_3 \wedge e_4)\) is given by

\[
\begin{pmatrix}
A + \alpha I_2 & 0 \\
0 & B - \alpha I_2
\end{pmatrix}
\begin{pmatrix}
-2\alpha & 0 \\
0 & 2\alpha
\end{pmatrix}; A, B \in \mathcal{O}(2), \alpha \in \mathfrak{g}(1)
\]

one can check easily that \((GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1)\) (resp. \(A_2 \otimes A_1 + A_1 \otimes 1 + A_i^* \otimes 1\)) is a P.V. (resp. is a non-P.V.), and so is the case for \(m = 4\) by a castling transformation. Thus we have (2.61). Since the \(SL(m)\)-part of the generic isotropy subgroup of \((GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1)\) is \(SO(m)\), we have (2.62). Assume that \(A_2 \otimes A_1 + \Sigma_s \otimes 1 + 1 \otimes A_i^*\) is a P.V. Then we have \(\dim G = s + 16 + m^2 \geqslant 7m + \deg \Sigma_s \geqslant 7m + 4s\), and hence \(s = 1; s = 2\) \((m = 2, 5)\). We shall see that \(s \neq 2\). Since the \(SL(2)\) part of the generic isotropy subalgebra of \((GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + A_i \otimes 1)\) is zero, we have \(s \neq 2\) for \(m = 2\). By p. 94 in [2], the generic isotropy subgroup of \((GL(1)^3 \times SL(4) \times SL(5), A_2 \otimes A_1 + A_1 \otimes 1 + A_i^* \otimes 1)\) is reductive, \(A_2 \otimes A_i + A_1 \otimes 1 + A_i^* \otimes 1 + 1 \otimes A_i^*\) and \(A_2 \otimes A_i + A_1 \otimes 1 + A_i^* \otimes 1 + 1 \otimes A_i^*\) are P.V.-equivalent. However, its castling transform \((GL(1)^4 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + A_i^* \otimes 1 + 1 \otimes A_i^*)\) is a non-P.V. as above, we have \(s \neq 2\) for \(m = 5\). If \(s = 1\), it is a F.P. by Theorem 5.17 in [4], and hence it is a P.V. Thus we have (2.63).

Q.E.D.

**Theorem 2.15.** Let \(A\) (resp. \(A_1\)) be the spin (resp. the vector) representation of Spin(7). All non-irreducible 2-simple P.V.s which have
(SO(7) × GL(m), A₁ ⊗ A₁) ∼ (Spin(7) × GL(m), A₁ ⊗ A₁) (2 ≤ m ≤ 6) ((55) in Theorem 1.5) as an irreducible component are given as follows:

\[(GL(1)^2 \times Spin(7) \times SL(m), A₁ \otimes A₁ + A \otimes 1) \quad (m = 2, 5, 6)\]  \quad (2.64)

\[(GL(1)^2 \times Spin(7) \times SL(m), A₁ \otimes A₁ + 1 \otimes A₁^{*1}) \quad (2 ≤ m ≤ 6)\]  \quad (2.65)

\[(GL(1)^3 \times Spin(7) \times SL(m), A₁ \otimes A₁ + A \otimes 1 + 1 \otimes A₁^{*1}) \quad (m = 2, 6).\]  \quad (2.66)

Proof. If \(A₁ \otimes A₁ + \rho₂ \otimes \rho₂' \ (\rho₂ ≠ 1, \rho₂' ≠ 1)\) is a P.V., then we have \(\dim G = 22 + m^2 \geq \dim V \geq 7m + 7m, \) i.e., \((m - 7)^2 \geq 27 \ (2 ≤ m ≤ 6), \) which is a contradiction. If \(A₁ \otimes A₁ + (\sigma₁ + \cdots + \sigmaₖ) \otimes 1\) is a P.V., then we have \(s = 1\) and \(\sigma₁ = A₁\) by Sublemma 2.4.2 and [2]. Since the restriction of \((GL(1)^2 \times Spin(7), A₁)\) to a generic isotropy subgroup of \((GL(1)^2 \times Spin(7), A₁)\) is equivalent to \(((G₂), A₂, V(7))\) (see p. 116 in [1]), we have (2.64). Since the SL(m) part of a generic isotropy subgroup of \((GL(1)^2 \times Spin(7) \times SL(m), A₁ \otimes A₁)\) is SO(m), we have (2.65). Now \(A₁ \otimes A₁ + A \otimes 1 + 1 \otimes A₁^{*1}\) is a P.V. if and only if \((GL(1)^2 \times (G₂) \times SL(m), A₂ \otimes A₁ + 1 \otimes A₁^{*1}) \ (m = 2, 5, 6)\) is a P.V. Thus we have \(m ≠ 5\) by (14) of Theorem 2.1. It is a P.V. for \(m = 2, 6\) by (2.26) and (2.27). Thus we have our result. Q.E.D.

Let \(A_e\) (resp. \(A_o, A_i\)) be the even half-spin (resp. the odd half-spin, the vector) representation of Spin(8). Then it is well known that \((Spin(8), A_e) ≅ (Spin(8), A_o) \sim (Spin(8), A_i) \sim (SO(8), A₁)\) as triplets (see p. 36 in [1]).

Theorem 2.16. All non-irreducible 2-simple P.V.s which have \((SO(8) \times GL(m), A₁ \otimes A₁, 2 ≤ m ≤ 7)\) ((56) in Theorem 1.5) as an irreducible component are given as follows:

\[(GL(1)^2 \times Spin(8) \times SL(m), A_e \otimes A₁ + A₁ \otimes 1) \quad (2 ≤ m ≤ 7, m ≠ 4)\]  \quad (2.67)

\[(GL(1)^2 \times Spin(8) \times SL(m), A_e \otimes A₁ + 1 \otimes A₁^{*1}) \quad (2 ≤ m ≤ 7)\]  \quad (2.68)

\[(GL(1)^3 \times Spin(8) \times SL(m), A_e \otimes A₁ + A₁ \otimes 1 + 1 \otimes A₁^{*1}) \quad \text{for } m = 2, 3, 6, 7.\]  \quad (2.69)

Proof. If \(A_e \otimes A₁ + \rho₂ \otimes \rho₂' \ (\rho₂ ≠ 1, \rho₂' ≠ 1)\) is a P.V., then we have \(\dim G = 29 + m^2 \geq \dim V \geq 8m + 8m,\) i.e., \((m - 8)^2 \geq 35 \ (2 ≤ m ≤ 7),\) and hence \(m = 2.\) Note that \((Spin(8) \times SL(2), \rho₂ \otimes \rho₂') \sim (SO(8) \times SL(2), A₁ \otimes A₁, V(8) \otimes V(m))\) as triplets if \(\rho₂ ≠ 1\) and \(\rho₂' ≠ 1.\) Hence the SL(2) part of a generic isotropy subgroup of \((GL(1)^2 \times Spin(8) \times SL(2), \rho₂ \otimes \rho₂')\) is \(O(2) = \{(xₙ); \alpha β = ±1\}.\) Thus \((GL(1)^2 \times Spin(8), A_e + A_i)\) must be a P.V., which is a contradiction by Theorem 1.3. Assume that \(A_e \otimes A₁ +
(σ₁ + ⋯ + σ₁) ⊗ 1 is a P.V. Then, by Sublemma 2.4.2 and Theorem 1.3, we have s = 1, 2; σ₁ = A₁; σ₂ = A₁ + A₀. Since the restriction of A₁ and A₀ of Spin(8) to a generic isotropy subgroup of (GL(1) × Spin(8), A₁) gives both the spin representation A of Spin(7) and A(Spin(7)) ⊂ SO(8), we have s ≠ 2, i.e., s = 1 by Sublemma 2.4.2. Since A₁ ⊗ A₁ + A₂ ⊗ 1 is P.V.-equivalent to (GL(1) × Spin(7) × SL(m), A ⊗ A₁), we have (2.67) by Theorem 1.5. Since the SL(m) part of a generic isotropy subgroup of (GL(1) × Spin(8) × SL(m), A₁ ⊗ A₁) is O(m), we have (2.68). For (2.69), A₁ ⊗ A₁ + A₁ ⊗ 1 + 1 ⊗ A₁(∗) (2 ≤ m ≤ 7, m ≠ 4) is P.V.-equivalent to (GL(1) × Spin(7) × SL(m), A ⊗ A₁ + 1 ⊗ A₁(∗)). Hence we have m ≠ 5 by (9) of Theorem 2.1. By (2.18)–(2.20) and (2.41), we have (2.69). Q.E.D.

**Lemma 2.17.** For 2n > m ≥ 2, (GL(1)² × Sp(n) × SL(m), A₁ ⊗ A₁ + A₁ ⊗ A₁(∗)) is a non-P.V.

**Proof.** The representation space of A₁ ⊗ A₁ + A₁ ⊗ A₁ (resp. A₁ ⊗ A₁ + A₁ ⊗ A₁(∗)) is identified with V = M(2n, m) ⊕ M(2n, m), where the action is given by \((X, Y) → (\alpha AX'B, \beta AY'B)\) (resp. \((\alpha AX'B, \beta AY'B)\)) for \(g = (\alpha, \beta; A, B) \in GL(1)² \times Sp(n) \times SL(m)\) and \(v = (X, Y) \in V\). Then a rational function \(f(x) = det('XJY - 'YJX) · det('XJY)' · det('YJX)'⁻¹\) is a nonconstant absolute invariant for \(m ≥ 2\), where

\[
J = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
\]

Q.E.D.

**Lemma 2.18.** All 2-simple P.V.s which contain (GL(1)² × Sp(2) × SL(m), A₁ ⊗ A₁ + A₂ ⊗ 1) (m = 2, 3) as a component, are given as follows:

(1) (GL(1)² × Sp(2) × SL(3), A₁ ⊗ A₁ + A₂ ⊗ 1) (2.70)

(2) (GL(1)³ × Sp(2) × SL(3), A₁ ⊗ A₁ + A₂ ⊗ 1 + 1 ⊗ A₁(∗)). (2.71)

**Proof.** By Lemma 4.6 in [4], (GL(1)² × Sp(2) × SL(2), A₁ ⊗ A₁ + A₂ ⊗ 1) is a non-P.V. Now (2.70) is actually a P.V., since it is castling-equivalent to (9) in Theorem 1.3. If \(A₁ ⊗ A₁ + A₂ ⊗ 1 + \sigma₁ ⊗ τ₁ + ⋯ + \sigmaₖ ⊗ τₖ\) is a P.V., then we have dim G = k + 20 ≥ dim V > 17 + 3k, we have k = 1. In this case, we have deg(σ₁ ⊗ τ₁) ≤ 4, and hence σ₁ ⊗ τ₁ = A₁ ⊗ 1 or 1 ⊗ A₁(∗). If σ₁ ⊗ τ₁ = A₁ ⊗ 1, it is castling-equivalent to (GL(1)³ × Sp(2), A₂ + A₁ + A₁), which is a non-P.V. by Theorem 1.3. If σ₁ ⊗ τ₁ = 1 ⊗ A₁, then it is castling-equivalent to (GL(1)³ × Sp(2) × SL(2), A₁ ⊗ A₁ + A₂ ⊗ 1 + 1 ⊗ A₁) which is a non-P.V. as we have seen above. If σ₁ ⊗ τ₁ = 1 ⊗ A₁, then it is a P.V., since
is a generic point. Q.E.D.

**Lemma 2.19.** All 2-simple P.V.s which contain \((GL(1)^2 \times Sp(3) \times SL(m), A_1 \otimes A_1 + A_3 \otimes 1)(2 \leq m \leq 5)\) as a component, are given as follows:

\[
(GL(1)^2 \times Sp(3) \times SL(5), A_1 \otimes A_1 + A_3 \otimes 1).
\] (2.72)

**Proof:** Since the generic isotropy subalgebra of \((GL(1) \times Sp(3), A_3)\) at 
\(e_1 \land e_2 \land e_4 \land e_5 \land e_6\) is given by \(\{(0) \oplus \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \}; A \in \mathcal{M}(3)\} \) (see [1]), 
\(A_1 \otimes A_1 + A_3 \otimes 1\) is P.V.-equivalent to \((SL(3) \times GL(m), (A_1 + A_1^*) \otimes A_1)\) which is a P.V. (resp. a non-P.V.) for \(m = 5\) (resp. \(m = 2, 3, 4\)) by the proof of Sublemma 2.4.4 (and a castling transformation for \(m = 4\)). If 
\((GL(1)^3 \times Sp(3) \times SL(5), A_1 \otimes A_1 + A_3 \otimes 1 + \rho_3 \otimes \rho_3')\) is a P.V., then we have \(\dim G = 48 \geq \dim V = 44 + \deg \rho_3 \otimes \rho_3' \geq 49\), which is a contradiction.

Q.E.D.

**Lemma 2.20.** For \(n > m \geq 1\), a triplet \((GL(1)^3 \times Sp(n) \times SL(2m), A_1 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1, M(2n, 2m) \oplus K^{2n} \oplus K^{2n})\) is a non-P.V.

**Proof:** The action is given by \(x \rightarrow \left( \begin{array}{c} AAX'B; BAY; CAZ \end{array} \right) \) for \(g = (x, \beta, \gamma; A, B) \in GL(1)^3 \times Sp(n) \times SL(2m)\) and \(x = (X, y, z) \in M(2n, 2m) \oplus K^{2n} \oplus K^{2n}\). Then a rational function \(f(x) = \left( \begin{array}{c} X' \end{array} \right) \cdot Pf(XJX)^{-1} Pf(X'JX')^{-1}\) is a nonconstant absolute invariant, where \(X' = (X, y, z) \in M(2n, 2n + 2)\) and \(Pf\) denotes the Pfaffian.

Q.E.D.

**Theorem 2.21.** All non-irreducible 2-simple P.V.s which have \((Sp(n) \times GL(2m), A_1 \otimes A_1, V(2n) \otimes V(2m))\) (\(n > m \geq 1\)) ((57) in Theorem 1.5) as an irreducible component are given as follows:

\[
(GL(1)^2 \times Sp(n) \times SL(2m), A_1 \otimes A_1 + A_1 \otimes 1)
\] (2.73)

\[
(GL(1)^{t+1} \times Sp(n) \times SL(2m), A_1 \otimes A_1 + 1 \otimes T_i) \quad (t = 1, 2, 3,)
\]

where \(T_1 = A_1^{(i)}, T_2 = 2A_1 (m = 1), T_3 = 3A_1 (m = 1), T_2 = A_1^{(i)} + A_1^{(i)}\)

\[
(T_1 = A_1^{(i)} + A_1^{(i)} + A_1^{(i)}). \quad (2.74)
\]

\[
(GL(1)^{2+t} \times Sp(n) \times SL(2m), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes T_i),
\]

where \(t = 1, 2; T_1 = A_1^{(i)}; T_1 = 2A_1 (m = 1); T_2 = A_1^{(i)} + A_1^{(i)}\).

\[
(T_1 = A_1^{(i)} + A_1^{(i)}). \quad (2.75)
\]
Proof. If \( A_1 \otimes A_1 + \rho_2 \otimes \rho'_2 (\rho_2 \neq 1, \rho'_2 \neq 1) \) is a P.V., then we have \( \rho_2 \otimes \rho'_2 = A_1 \otimes A_1^{(*)} \) by (2.16), Theorem 2.13, and Theorem 1.5. By Lemma 2.17, it is a contradiction. Now assume that \( A_1 \otimes A_1 + (\sigma_1 + \cdots + \sigma_4) \otimes 1 \) is a P.V. By Lemmas 2.18–2.20, we have \( s = 1 \) and \( \sigma_1 = A_1 \), i.e., (2.73). Now \( A_1 \otimes A_1 + 1 \otimes (\tau_1 + \cdots + \tau_s) \) is P.V.-equivalent to \( (GL(1)^t \times Sp(m), \tau_1 + \cdots + \tau_s) \), and hence we have (2.74) by Theorem 1.3. By p. 40 in [1], \( A_1 \otimes A_1 + 1 \otimes (\tau_1 + \cdots + \tau_s) \) is P.V.-equivalent to \( (GL(1)^t \times SL(2m), A_2 + \tau_1 + \cdots + \tau_s) \). By a careful check for scalar multiplications, we see that the latter is also P.V.-equivalent to \( (GL(1)^t \times SL(2m), A_2 + \tau_1 + \cdots + \tau_s) \), and hence we have (2.75). Note that \( (GL(1)^t \times SL(2), A_1 \otimes A_2) \sim (GL(1), A_1) \), and that the prehomogeneity of (2.73) has been also proved. Q.E.D.

LEMMA 2.22. For \( n > m + 1 \), a triplet \((GL(1)^t \times Sp(n) \times SL(2m + 1), A_1 \otimes A_1 + 1 \otimes 1 + 1 \otimes A_1^{(*)}, M(2n, 2m + 1) \otimes K^{2n} \otimes K^{2n} \otimes K^{2n})\) is a non-P.V.

Proof. The action is given by \( x \rightarrow (aAX'B; \beta_1 Ay_1, \beta_2 Ay_2, \beta_3 Ay_3) \) for \( g = (a, \beta_1, \beta_2, \beta_3; A, B) \in GL(1)^t \times Sp(n) \times SL(2n + 1) \) and \( x = (X, y_1, y_2, y_3) \in M(2n, 2m + 1) \otimes K^{2n} \otimes K^{2n} \otimes K^{2n} \). Then the polynomials \( f_i(x) = Pf('X'iX'i) \) with \( i = 1, 2, 3 \) are relative invariants corresponding to the characters \( \chi_i(g) = \alpha^m \beta_i (i = 1, 2, 3) \) and \( \chi_0(g) = \beta_1 \beta_2 (1 \leq i < j \leq 3) \), respectively. Now assume that \( n \geq m + 2 \). Then we have \( 2n \geq (2m + 1) + 3 \) and hence \( h(x) = Pf('X'X') \) with \( X' = (X, y_1, y_2, y_3) \in M(2n, 2m + 4) \) is a nonzero relative invariant corresponding to the character \( \chi(g) = \alpha^{2n} \beta_1 \beta_2 \beta_3 \). Hence, \( f(x) = f_1 f_2 f_3 g_{12} g_{23} g_{13} h^{-1}(x) \) is a nonconstant absolute invariant. Thus our triplet is a non-P.V. for \( n \geq m + 2 \). If \( n = m + 1 \), then we have \( 2m + 1 = 2n - 1 \), and it is castling-equivalent to \((GL(1)^t \times Sp(n), A_1 + A_1 + A_1 + A_1)\), which is a non-P.V. by Theorem 1.3. Q.E.D.

LEMMA 2.23. For \( n > m + 1 \), a triplet \((GL(1)^t \times Sp(n) \times SL(2m + 1), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^{(*)}, M(2n, 2m + 1) \otimes K^{2n} \otimes K^{2n} \otimes K^{2n} \otimes K^{2n + 1})\) is a non-P.V.

Proof. Since \( A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1 \) is castling-equivalent to \((GL(1)^t \times Sp(n) \times SL(2n - 2m), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^{(*)})\), it is a non-P.V. by Lemma 2.20. For \( A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^{(*)} \), the action is given by \( x \rightarrow (aAX'B; \beta_1 Ay_1, \beta_2 Ay_2, \gamma'B^{-1}z) \) for \( g = (x, \beta_1, \beta_2, \gamma; A, B) \in GL(1)^t \times Sp(n) \times SL(2m + 1) \) and \( x = (X, y_1, y_2, z) \in M(2n, 2m + 1) \otimes K^{2n} \otimes K^{2n} \otimes K^{2n + 1} \). Then the polynomials \( f_i(x) = Pf('X'iX'i) \) with \( X_i = (X, y_i) \in M(2n, 2m + 2) \) (\( i = 1, 2 \)) and \( g_j(x) = 'y_jXz \) (\( j = 1, 2 \)) are
relative invariants corresponding to the characters \( \chi_i(g) = \alpha^{2m+1} \beta_i (i = 1, 2) \) and \( \chi'_i(g) = \alpha \beta_i (j = 1, 2) \), respectively, where

\[
J = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}.
\]

Then a rational function \( f(x) = (g_1 f_2) \cdot (g_2 f_1)^{-1}(x) \) is a nonconstant absolute invariant, and hence it is a non-P.V. Q.E.D.

**THEOREM 2.24.** All non-irreducible 2-simple P.V.s which have \( (\text{Sp}(n) \times \text{GL}(2m+1), A_1 \otimes A_1) \) \((n > m \geq 1) ((58) \text{ in Theorem } 1.5)\) as an irreducible component are given by (2.70)-(2.72) and the following (2.76)-(2.78):

\[
(GL(1)^{s+1} \times \text{Sp}(n) \times \text{SL}(2m+1), A_1 \otimes A_1 + \Sigma_1 \otimes 1),
\]

where \( s = 1, 2; \Sigma_1 = A_1, \Sigma_2 = A_1 + A_1 \). (2.76)

\[
(GL(1)^{t+1} \times \text{Sp}(n) \times \text{SL}(2m+1), A_1 \otimes A_1 + 1 \otimes T_{t}),
\]

where \( t = 1, 2, 3; T_1 = A_1^{*1}, A_2, T_1 = 2A_1 (m = 1); T_2 = A_1^{*1} + A_1^{*1}; T_3 = A_2 + A_1^* (m = 2); T_5 = A_1^{*1} + A_1^{*1} + A_1^{*1}, \) except for \( T_3 \approx A_1 + A_1 + A_1^* \). (2.77)

\[
(GL(1)^{2+s} \times \text{Sp}(n) \times \text{SL}(2m+1), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes T_{t}),
\]

where \( t = 1, 2; T_1 = A_1^{*1}; T_1 = A_1 + A_1, A_1^* + A_1^* \). (2.78)

**Proof.** If \( A_1 \otimes A_1 + \rho_2 \otimes \rho_2^* (\rho_2 \neq 1, \rho_2^* \neq 1) \) is a P.V., then we have \( \rho_2 \otimes \rho_2^* = A_1 \otimes A_1^{*1} \) by Theorem 1.5, which is a contradiction by Lemma 2.17. Now assume that \( A_1 \otimes A_1 + (\sigma_1 + \ldots + \sigma_s) \otimes 1 \) is a P.V. Then, by Lemmas 2.18-2.20, we have \( s = 1, 2; \sigma_1 = A_1, \sigma_1 = A_2 (n = 2, m = 1), \sigma_1 = A_3 (n = 3, m = 2) \) and \( \sigma_1 + \sigma_2 = A_1 + A_1 \). We shall show that \( A_1 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1 \) is actually a P.V. If \( n - m + 1 \), then it is castling-equivalent to a simple P.V. \( (GL(1)^3 \times \text{Sp}(n), A_1 + A_1 + A_1) \). If \( n \geq m + 2 \), we can use Proposition 13 in p. 40 in [1], and it is P.V.-equivalent to \( (GL(1) \times GL(1) \times SL(2m+1), A_1 \otimes A_1 \otimes A_2 + A_1 \otimes 1 \otimes A_1 + 1 \otimes A_1 \otimes A_1) \) which is a P.V. with a generic point

\[
\begin{pmatrix}
J & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 \cdots 0 \\
1 \cdots 0
\end{pmatrix}.
\]

One can also show the prehomogeneity of \( A_1 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1 \) by the calculation of the isotropy subalgebra at

\[
\begin{pmatrix}
I_m & O_{m,n} \\
O_{m+1,n} & I_{m+1}
\end{pmatrix}, e_{m+1}, e_1 + e_{m+1} + e_{m+2} + e_{n+m+1}.
\]
Since \((GL(1)^{1'+}\times Sp(n)\times SL(2m+1), A_1\otimes A_1 + 1 \otimes (\tau_1 + \cdots + \tau_i))\) is P.V.-equivalent to \((GL(1)^{1'+}\times SL(2m+1), A_2 + \tau_1 + \cdots + \tau_i))\) by p. 40 in [1], we have (2.77) by Theorem 1.3. Similarly \((GL(1)^{2'+}\times Sp(n)\times SL(2m+1), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_i))\) is P.V.-equivalent to \((GL(1)^{2'+}\times SL(2m+1), A_2 + A_1 + \tau_1 + \cdots + \tau_i))\) and hence we have (2.78) by Theorem 1.3. Now assume that \(n > m + 2\). Then, by p. 40 in [1], \(A_1 \otimes A_1 + A_1 \otimes 1 + A_1 \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_i)\) is a P.V., then \((GL(1)^{3'+}\times SL(2m+1), A_2 + A_1 + \tau_1 + \cdots + \tau_i)\) must be a P.V., and hence \(t = 1, \tau_1 = A_1\). However, in this case, it is a non-P.V. by Lemma 2.23. Finally, assume that \(n = m + 1\), i.e., \(2m + 1 = 2n - 1\), and \(A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes 1 + \tau_1 + \cdots + \tau_i\) is a P.V. Then \(\tau\) must be one of \(A_1^*, A_2^*, (2A_1)^*, A_3^*(n = 4)\). However, we have \(\tau \neq A_1^*\) by Lemma 2.23 and \(\tau \neq A_2^*\), \((2A_1)^*, A_3^*(n = 4)\) by dimension reason.

Thus we obtain the following theorem.

**Theorem 2.25.** All non-irreducible 2-simple P.V.s of type I are given by (2.1)–(2.78).

3. **List of 2-Simple P.V.s of Type I**

By Theorem 2.25, any 2-simple P.V.s of type I is castling-equivalent (cf. [1]) to a simple P.V. in Theorem 1.3 or to one of the 2-simple P.V.s in the following list. For example, a 2-simple P.V. \((GL(1)^3 \times SL(4) \times SL(4), A_3 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1)\) is castling-equivalent to (4) in (1) with \(A_1^* = A_1\) in the list. Here \(H\) denotes the generic isotropy subgroup and \(H \sim H_1\) implies that \(H\) is locally isomorphic to \(H_1\). The number of the basic relative invariants is denoted by \(N\) and \(A_1^*\) stands for \(A_1\) or its dual \(A_1^*\).

**Notation.**

- \(A = \) the spin representation of Spin\((2n + 1)\).
- \(A' = \) a half-spin representation of Spin\((2n)\).
- \(\chi = \) the vector representation of Spin\((n)\), so that \((\text{Spin}(n), \chi) = (\text{SO}(n), A_1)\).

**List**

(I) **Regular 2-Simple P.V.s of Type I**

\((1) \ (GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes A_1), H \sim \{1\}, N = 2.\)
(2) \((GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + (A_1 + A_1) \otimes 1), H \sim GL(1), N = 2.\)

(3) \((GL(1)^2 \times SL(4) \times SL(3), A_2 \otimes A_1 + A_1 \otimes 1), H \sim SO(3), N = 2.\)

(4) \((GL(1)^3 \times SL(4) \times SL(3), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^\ast), H \sim SO(2), N = 3.\)

(5) \((GL(1)^3 \times SL(4) \times SL(4), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^\ast), H \sim SO(2), N = 3.\)

(6) \((GL(1)^3 \times SL(5) \times SL(2), A_2 \otimes A_1 + (A_1^\ast + A_1^{\ast \ast}) \otimes 1), H \sim \{1\}, N = 3.\)

(7) \((GL(1)^2 \times SL(5) \times SL(3), A_2 \otimes A_1 + 1 \otimes A_1^\ast), H \sim SO(2), N = 2.\)

(8) \((GL(1)^2 \times SL(5) \times SL(8), A_2 \otimes A_1 + 1 \otimes A_1^\ast), H \sim SO(2), N = 2.\)

(9) \((GL(1)^2 \times SL(5) \times SL(9), A_2 \otimes A_1 + 1 \otimes A_1^\ast), H \sim GL(1) \times SL(2) \times SL(2), N = 1.\)

(10) \((GL(1)^3 \times Sp(n) \times SL(2m), A_1 \otimes A_1 + 1 \otimes (A_1^\ast + A_1^{\ast \ast})), H \sim GL(1) \times Sp(n-m) \times Sp(m-1), N = 2.\)

(11) \((GL(1)^2 \times Sp(n) \times SL(2), A_1 \otimes A_1 + 1 \otimes 2A_1), H \sim Sp(n-1) \times SO(2), N = 2.\)

(12) \((GL(1)^2 \times Sp(n) \times SL(2), A_1 \otimes A_1 + 1 \otimes 3A_1), H \sim Sp(n-1), N = 2.\)

(13) \((GL(1)^3 \times Sp(n) \times SL(2), A_1 \otimes A_1 + 1 \otimes (2A_1 + A_1)), H \sim Sp(n-1), N = 3.\)

(14) \((GL(1)^2 \times Sp(n) \times SL(2m+1), A_1 \otimes A_1 + A_1 \otimes 1), H \sim GL(1) \times Sp(m) \times Sp(n-m-1), N = 1.\)

(15) \((GL(1)^4 \times Sp(n) \times SL(2m+1), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes (A_1 + A_1^{\ast \ast})), H \sim Sp(m-1) \times Sp(n-m-1), N = 4.\)

(16) \((GL(1)^3 \times Sp(2) \times SL(3), A_1 \otimes A_1 + A_2 \otimes 1 + 1 \otimes A_1^\ast), H \sim GL(1), N = 2.\)

(17) \((GL(1)^2 \times Sp(2) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1), H \sim SO(2), N = 2.\)

(18) \((GL(1)^3 \times Sp(2) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1), H \sim \{1\}, N = 3.\)

(19) \((GL(1)^3 \times Sp(2) \times SL(4), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^\ast), H \sim \{1\}, N = 3.\)

(20) \((GL(1)^2 \times SO(n) \times SL(m), A_1 \otimes A_1 + 1 \otimes A_1^{\ast \ast}), H \sim SO(m-1) \times SO(n-m), N = 2.\)

(21) \((GL(1)^2 \times Spin(7) \times SL(2), A \otimes A_1 + 1 \otimes A_1), H \sim SL(3), N = 2.\)

(22) \((GL(1)^2 \times Spin(7) \times SL(3), A \otimes A_1 + 1 \otimes A_1^{\ast \ast}), H \sim SL(2) \times SO(2), N = 2.\)
(23) \( (GL(1)^2 \times Spin(7) \times SL(6), \ A \otimes A_1 + 1 \otimes A^*_1), \ H \sim SL(2) \times SO(2), \ N = 2. \)

(24) \( (GL(1)^2 \times Spin(7) \times SL(7), \ A \otimes A_1 + 1 \otimes A^*_1), \ H \sim SL(3), \ N = 2. \)

(25) \( (GL(1)^2 \times Spin(7) \times SL(2), \ \chi \otimes A_1 + \chi \otimes 1), \ H \sim GL(2), \ N = 2. \)

(26) \( (GL(1)^2 \times Spin(7) \times SL(2), \ \chi \otimes A_1 + \chi \otimes 1 + 1 \otimes A_1), \ H \sim SL(2), \ N = 3. \)

(27) \( (GL(1)^3 \times Spin(7) \times SL(6), \ \chi \otimes A_1 + A \otimes 1 + 1 \otimes A^*_1), \ H \sim SL(2), \ N = 3. \)

(28) \( (GL(1)^2 \times Spin(8) \times SL(2), \ \chi \otimes A_1 + A' \otimes 1), \ H \sim SL(3) \times SO(2), \ N = 2. \)

(29) \( (GL(1)^2 \times Spin(8) \times SL(3), \ \chi \otimes A_1 + A' \otimes 1), \ H \sim SL(2) \times SO(3), \ N = 2. \)

(30) \( (GL(1)^3 \times Spin(8) \times SL(2), \ \chi \otimes A_1 + A' \otimes 1 + 1 \otimes A_1), \ H \sim SL(3), \ N = 3. \)

(31) \( (GL(1)^3 \times Spin(8) \times SL(3), \ \chi \otimes A_1 + A' \otimes 1 + 1 \otimes A^*_1), \ H \sim SL(2) \times SO(2), \ N = 3. \)

(32) \( (GL(1)^3 \times Spin(8) \times SL(6), \ \chi \otimes A_1 + A' \otimes 1 + 1 \otimes A^*_1), \ H \sim SL(2) \times SO(2), \ N = 3. \)

(33) \( (GL(1)^3 \times Spin(8) \times SL(7), \ \chi \otimes A_1 + A' \otimes 1 + 1 \otimes A^*_1), \ H \sim SL(3), \ N = 3. \)

(34) \( (GL(1)^2 \times Spin(10) \times SL(2), \ A' \otimes A_1 + 1 \otimes 2A_1), \ H \sim (G_2) \times SO(2), \ N = 2. \)

(35) \( (GL(1)^2 \times Spin(10) \times SL(2), \ A' \otimes A_1 + 1 \otimes 3A_1), \ H \sim (G_2), \ N = 2. \)

(36) \( (GL(1)^3 \times Spin(10) \times SL(2), \ A' \otimes A_1 + 1 \otimes (A_1 + A_1)), \ H \sim GL(1) \times (G_2), \ N = 2. \)

(37) \( (GL(1)^3 \times Spin(10) \times SL(2), \ A' \otimes A_1 + 1 \otimes (2A_1 + A_1)), \ H \sim (G_2), \ N = 3. \)

(38) \( (GL(1)^4 \times Spin(10) \times SL(2), \ A' \otimes A_1 + 1 \otimes (A_1 + A_1 + A_1)), \ H \sim (G_2), \ N = 4. \)

(39) \( (GL(1)^2 \times Spin(10) \times SL(3), \ A' \otimes A_1 + 1 \otimes A^*_1), \ H \sim SL(2) \times SO(2), \ N = 2. \)

(40) \( (GL(1)^2 \times Spin(10) \times SL(14), \ A' \otimes A_1 + 1 \otimes A^*_1), \ H \sim SL(2) \times SO(2), \ N = 2. \)

(41) \( (GL(1)^2 \times Spin(10) \times SL(15), \ A' \otimes A_1 + 1 \otimes A^*_1), \ H \sim GL(1) \times SL(4), \ N = 1. \)

(42) \( (GL(1)^2 \times Spin(10) \times SL(2), \ \chi \otimes A_1 + A' \otimes 1), \ H \sim (G_2), \ N = 2. \)
(43) \((GL(1)^2 \times \text{Spin}(10) \times SL(3), \chi \otimes A_1 + A' \otimes 1), H \sim SL(3) \times SO(2), N=2\).

(44) \((GL(1)^2 \times \text{Spin}(10) \times SL(4), \chi \otimes A_1 + A' \otimes 1), H \sim SL(2) \times SL(2), N=2\).

(45) \((GL(1)^2 \times (G_2) \times SL(2), A_2 \otimes A_1 + 1 \otimes A_1), H \sim SL(2), N=2\).

(46) \((GL(1)^2 \times (G_2) \times SL(6), A_2 \otimes A_1 + 1 \otimes A_1^*), H \sim SL(2), N=2\).

(II) Nonregular 2-Simple P.V.s of Type I

1. \((GL(1)^2 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes tA_1) (t = 1, 2, 3)\).

2. \((GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes (A_1 + tA_1)) (t = 1, 2)\).

3. \((GL(1)^3 \times SL(2m+1) \times SL(2), A_2 \otimes A_1 + 1 \otimes (A_1 + A_1 + A_1))\).

4. \((GL(1)^2 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1)\).

5. \((GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1)\).

6. \((GL(1)^3 \times SL(4) \times SL(2), A_2 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^*)\).

7. \((GL(1)^2 \times SL(5) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1)\).

8. \((GL(1)^3 \times SL(5) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes A_1)\).

9. \((GL(1)^3 \times SL(5) \times SL(9), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes A_1^*)\).

10. \((GL(1)^3 \times SL(5) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes 2A_1)\).

11. \((GL(1)^4 \times SL(5) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes (A_1 + A_1))\).

12. \((GL(1)^2 \times SL(6) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1)\).

13. \((GL(1)^2 \times SL(6) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1)\).

14. \((GL(1)^3 \times SL(6) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1 + 1 \otimes A_1)\).

15. \((GL(1)^2 \times SL(9) \times SL(2), A_2 \otimes A_1 + A_1^* \otimes 1)\).

16. \((GL(1)^2 \times \text{Sp}(n) \times SL(2m), A_1 \otimes A_1 + T) \text{ with } T = A_1 \otimes 1, 1 \otimes A_1, 1 \otimes A_1^*\).

17. \((GL(1)^3 \times \text{Sp}(n) \times SL(2m), A_1 \otimes A_1 + A_1 \otimes 1 + 1 \otimes A_1^*\).

18. \((GL(1)^2 \times \text{Sp}(n) \times SL(2m), A_1 \otimes A_1 + 1 \otimes (A_1^* + A_1^*) + T) \text{ with } T = A_1 \otimes 1, 1 \otimes A_1, 1 \otimes A_1^*\).

19. \((GL(1)^3 \times \text{Sp}(n) \times SL(2m), A_1 \otimes A_1 + 1 \otimes (A_1 + 2A_1)\).

20. \((GL(1)^2 \times \text{Sp}(n) \times SL(2m+1), A_1 \otimes A_1 + 1 \otimes T) \text{ with } T = A_1, A_1^*, A_2\).

21. \((GL(1)^3 \times \text{Sp}(n) \times SL(2m+1), A_1 \otimes A_1 + S + T) \text{ with } S, T = A_1 \otimes 1, 1 \otimes A_1, 1 \otimes A_1^*\).

22. \((GL(1)^3 \times \text{Sp}(n) \times SL(2m+1), A_1 \otimes A_1 + T) \text{ with } T = 1 \otimes A_1 + 1 \otimes (A_1 + A_1)^* + 1 \otimes (A_1^* + A_1^* + A_1^*)\).
PREHOMOGENEOUS VECTOR SPACES

(23) \((GL(1)^2 \times Sp(n) \times SL(3), A_1 \otimes A_1 + 1 \otimes 2A_1)\).
(24) \((GL(1)^3 \times Sp(n) \times SL(5), A_1 \otimes A_1 + 1 \otimes (A_2 + A_1^\perp))\).
(25) \((GL(1)^2 \times Sp(n) \times SL(2), A_1 \otimes 2A_1 + 1 \otimes A_1)\).
(26) \((GL(1)^2 \times Spin(10) \times SL(2), A' \otimes A_1 + 1 \otimes A_1)\).

REFERENCES

6. Z. Chen, A new prehomogeneous vector space of characteristic \(p\), preprint, East China Normal University.
13. Y. Teranishi, Relative invariants and \(b\)-functions of prehomogeneous vector spaces \((G \times GL(d_1, \ldots, d_r), \hat{p}_1, M(n, C))\), Nagoya Math. J. 98 (1985), 139-156.

