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A Classification of 2-Simple Prehomogeneous Vector Spaces of Type I

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INTRODUCTION

Let $\rho: G \to GL(V)$ be a rational representation of a connected linear algebraic group G on a finite-dimensional vector space V, all defined over an algebraically closed field K of characteristic zero. If V has a Zariski-dense G-orbit, we call a triplet (G, ρ, V) *a prehomogeneous vector space* (abbrev. P.V.). When ρ is irreducible, such P.V.s have been classified in [1]. Since then, it has turned out gradually that the complete classification of reductive P.V.s (i.e., P.V.s with reductive groups G) is an extremely laborious task. Therefore it is natural to classify some restricted class of P.V.s (e.g., [2]) to get some insight into the general situation.

A P.V. (G, ρ, V) is called *a 2-simple P.V.* when (1) $G = GL(1)^l \times G_1 \times G_2$ with simple algebraic groups G_1 and G_2, (2) ρ is the composition of a rational representation ρ' of $G_1 \times G_2$ of the form $\rho' = \rho_1 \otimes \rho'_1 + \cdots + \rho_k \otimes \rho'_k + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ with $k + s + t = l$, where ρ_i, σ_i (resp. ρ'_j, τ_j) are nontrivial irreducible representations of G_1 (resp. G_2), and the scalar multiplications $GL(1)^l$ on each irreducible component V_i for $i = 1, ..., l$, where $V = V_1 \oplus \cdots \oplus V_l$. We say that a 2-simple P.V. (G, ρ, V) is *of type I* if $k \geqslant 1$ and at least one of $(GL(1) \times G_1 \times G_2, \rho_i \otimes \rho'_i)$ $(i = 1, ..., k)$ is a nontrivial P.V. (see Definition 5, p. 43 in [1]). On the other hand, if $k \geqslant 1$ and all $(GL(1) \times G_1 \times G_2, \rho_i \otimes \rho'_i)$ $(i = 1, ..., k)$ are trivial P.V.s, it is called a 2-simple P.V. *of type II*. In [3], all 2-simple P.V.s of type II has been already classified. In this paper, we shall classify all 2-simple P.V.s of type I. Thus, together with [3], we complete a classification of all 2-simple P.V.s. For example, the fact that all irreducible P.V.s are castling-equivalent to 2-simple P.V.s (or to $(SL(m) \times SL(m) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ with $m = 2, 3$) (see [1]) indicates the importance of 2-simple P.V.s. For simplicity, we write (G, ρ', V) or (G, ρ') instead of (G, ρ, V).

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This paper consists of the following four sections: Introduction. 1. Preliminaries. 2. A classification. 3. List.

In Section 1, we give also some correction of [2]. In Section 3, we shall give the list of 2-simple P.V.s of type I, which are not catling-equivalent to simple P.V.s. For regular P.V.s (see Section 4 in [1]), we also give the generic isotropy subgroups and the number of basic relative invariants.

1. Preliminaries

First we start from the correction of [2].

PROPOSITION 1.1. (1) *The triplet* $(GL(1)^3 \times SL(5),\ \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^*,\ V(10) \oplus V(10) \oplus V(5)^*)$ *is a nonregular P.V. with the generic isotropy subalgebra*

$$\mathfrak{h} = \left\{ (\varepsilon, \varepsilon, 3\varepsilon) \oplus \left(\begin{array}{c|c} -3\varepsilon I_2 & A \\ \hline 0 & 2\varepsilon I_3 \end{array} \right); A = \begin{pmatrix} \gamma & -\gamma & -\gamma \\ -\gamma & \gamma & \gamma \end{pmatrix} \right\}.$$

If we identify $V(10) \oplus V(10) \oplus V(5)^*$ with $\{(X, Y; Z) \mid X, Y \in M(5),\ {}^tX = -X,\ {}^tY = -Y,\ Z \in K^5\}$, the action ρ is given by $\rho(g)x = (\alpha A X {}^tA,\ \beta A Y {}^tA;\ \gamma {}^tA^{-1} \cdot Z)$ for $x = (X, Y; Z)$ and $g = (\alpha, \beta, \gamma; A) \in GL(1)^3 \times SL(5)$. The basic relative invariants are given by

$$f_1(x) = Pf \left(\begin{array}{c|c} X & YZ \\ \hline -{}^tZ {}^tY & 0 \end{array} \right) \quad \text{and} \quad f_2(x) = Pf \left(\begin{array}{c|c} Y & XZ \\ \hline -{}^tZ {}^tX & 0 \end{array} \right),$$

where Pf denotes the Pfaffian.

(2) *The triplet* $(GL(1)^3 \times SL(5),\ \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1,\ V(10) \oplus V(10) \oplus V(5))$ *is not a P.V.*

Proof. We may also identify $V(10)$ with $\Sigma K \cdot e_i \wedge e_j$ $(1 \leqslant i < j \leqslant 5)$. Then the isotropy subalgebra at a generic point $x_0 = (e_2 \wedge e_3 + e_1 \wedge e_4,\ e_1 \wedge e_3 + e_2 \wedge e_5)$ is given by

$$\mathfrak{g}_{x_0} = \left\{ \left(\varepsilon_1, \varepsilon_2; \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array} \right) \right); A_1 = \begin{pmatrix} -\varepsilon_1 - 2\varepsilon_2 & \\ & -2\varepsilon_1 - \varepsilon_2 \end{pmatrix}, \right.$$

$$\left. A_2 = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_1 & \gamma_4 \end{pmatrix}, A_3 = \begin{pmatrix} \varepsilon_1 + \varepsilon_2 & & \\ & 2\varepsilon_2 & \\ & & 2\varepsilon_1 \end{pmatrix} \right\}.$$

The dual action of $\mathfrak{g}_{x_0}$ on K^5 is a P.V., since the isotropy subalgebra at $e_1+e_2\in K^5$ is given by $\mathfrak{h}$, and hence we have (1). The standard action of $\mathfrak{g}_{x_0}$ on K^5 is a non-P.V., since $f(Z)=z_4z_5z_3^{-2}$ for $Z=\Sigma z_ie_i\in K^5$ is a non-constant absolute invariant. Q.E.D.

Remark 1.2. There is a mistake in Proposition 2.2, p. 80 in [2]. It should be corrected to "For $n=2m+1$, the triplet (5) for $n=5$ and the triplet (2) are P.V.s, and the triplets (3), (4), (5) with $n\neq 5$, (6) are not P.V.s." Thus the triplet $(GL(1)^3\times SL(5), \Lambda_2\oplus\Lambda_2\oplus\Lambda_1^*)$ should be added in the table of simple P.V.s, p. 100 in [2] as the nineteenth P.V. Thus we obtain the following theorem.

THEOREM 1.3 ([2] with the correction above). *All non-irreducible simple P.V.s with scalar multiplications are given as follows*:

(1) $(GL(1)^{k+1}\times SL(n), \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{k}\oplus\Lambda_1^{(*)})$ $(1\leqslant k\leqslant n,\ n\geqslant 2)$.

(2) $(GL(1)^{k+1}\times SL(n), \Lambda_2\oplus\overbrace{\Lambda_1^{(*)}\oplus\cdots\oplus\Lambda_1^{(*)}}^{k})$ $(1\leqslant k\leqslant 3,\ n\geqslant 4)$ *except* $(GL(1)^4\times SL(n), \Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1^*)$ *with* $n=$ *odd*.

(3) $(GL(1)^2\times SL(2m+1), \Lambda_2\oplus\Lambda_2)$ *for* $m\geqslant 2$.

(4) $(GL(1)^2\times SL(n), 2\Lambda_1\oplus\Lambda_1^{(*)})$.

(5) $(GL(1)^3\times SL(5), \Lambda_2\oplus\Lambda_2\oplus\Lambda_1^*)$.

(6) $(GL(1)^2\times SL(n), \Lambda_3\oplus\Lambda_1^{(*)})$ $(n=6, 7)$.

(7) $(GL(1)^3\times SL(6), \Lambda_3\oplus\Lambda_1\oplus\Lambda_1)$.

(8) $(GL(1)^l\times \mathrm{Sp}(n), \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{l})$ $(l=2, 3)$.

(9) $(GL(1)^2\times \mathrm{Sp}(2), \Lambda_2\oplus\Lambda_1)$.

(10) $(GL(1)^2\times \mathrm{Sp}(3), \Lambda_3\oplus\Lambda_1)$.

(11) $(GL(1)^2\times \mathrm{Spin}(n)$, *(half-)spin rep.* $\oplus$ *vector rep.*) $(n=7, 8, 10, 12)$.

(12) $(GL(1)^2\times \mathrm{Spin}(10), \Lambda\oplus\Lambda)$, *where* Λ = *the even half-spin representation*.

Here $\Lambda^{(*)}$ *stands for* Λ *or its dual* Λ^*. *Note that* $(G, \rho, V)\simeq(G, \rho^*, V^*)$ *as triplets if* G *is reductive*.

Now let us consider the triplet $(GL(1)\times SL(2m+1)\times SL(2), \Lambda_2\otimes\Lambda_1, V(m(2m+1))\otimes V(2))$. Let $\mathfrak{g}_{x_0}$ be the isotropy subalgebra of $\mathfrak{gl}(1)\oplus\mathfrak{sl}(2m+1)\oplus\mathfrak{sl}(2)$ at a generic point X_0 given in p. 94 in [1]. For $A=\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\in\mathfrak{sl}(2)$, let $n\Lambda_1(A)=(\alpha_{ij})$ be an $(n+1)\times(n+1)$ matrix with $\alpha_{k+1,k+1}=(n-2k)a$ $(0\leqslant k\leqslant n)$, $\alpha_{k,k+1}=kb$, $\alpha_{k+1,k}=(n+1-k)c$ $(1\leqslant k\leqslant n)$, all other $\alpha_{ij}=0$. Put $n\Lambda_1^*(A)=-{}^t(\alpha_{ij})$. By simple calculation, we have the following lemma.

LEMMA 1.4. *The generic isotropy subalgebra* $\mathfrak{g}_{x_0}$ *is given as follows*:

$$\mathfrak{g}_{x_0} = \left\{ (\delta) \oplus \left(\begin{array}{c|c} mA_1^*(A) + m\,\delta I_{m+1} & 0 \\ \hline B & (m-1)\,A_1(A) - (m+1)\,\delta I_m \end{array} \right) \oplus (A); \right.$$

$$A \in \mathfrak{sl}(2),\ B = (b_{ij}) \in M(m, m+1) \tag{1.1}$$

$$\left. \text{with } b_{ij} = a_{i+j-1} \right\} \simeq (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus V(2m).$$

THEOREM 1.5 [1]. *All nontrivial irreducible* (*reduced or nonreduced*) 2-*simple P.V.s are given as follows. Here* $H \sim H_1$ *implies that the generic isotropy subgroup H is locally isomorphic to a group* H_1:

(I)

(1) $(SL(2m+1) \times GL(2m^2+m-2),\ \Lambda_2 \otimes \Lambda_1)$ $(m \geqslant 5)$ *with* $H \sim (GL(1) \times SL(2)) \cdot G_a^m$.

(2) $(SL(5) \times GL(4),\ \Lambda_2 \otimes \Lambda_1)$ *with* $H \sim \{1\}$.

(3) $(SL(5) \times GL(6),\ \Lambda_2 \otimes \Lambda_1)$ *with* $H \sim \{1\}$.

(4) $(SL(5) \times GL(7),\ \Lambda_2 \otimes \Lambda_1)$ *with* $H \sim SL(2)$.

(5) $(SL(2) \times GL(3),\ 3\Lambda_1 \otimes \Lambda_1)$ *with* $H \sim \{1\}$.

(6) $(SL(3) \times GL(2),\ 2\Lambda_1 \otimes \Lambda_1)$ *with* $H \sim \{1\}$.

(7) $(SL(3) \times GL(4),\ 2\Lambda_1 \otimes \Lambda_1)$ *with* $H \sim \{1\}$.

(8) $(SL(8) \times GL(55),\ \Lambda_3 \otimes \Lambda_1)$ *with* $H \sim SL(3)$.

(9) $(\mathrm{Spin}(7) \times GL(5)$, spin rep. $\otimes \Lambda_1)$ *with* $H \sim SL(2) \times SL(2)$.

(10) $(\mathrm{Spin}(9) \times GL(15)$, spin rep. $\otimes \Lambda_1)$ *with* $H \sim \mathrm{Spin}(7)$.

(11) $(\mathrm{Spin}(10) \times GL(13)$, half-spin rep. $\otimes \Lambda_1)$ *with* $H \sim SL(2) \times O(3)$.

(12) $(\mathrm{Spin}(11) \times GL(31)$, spin rep. $\otimes \Lambda_1)$ *with* $H \sim SL(5)$.

(13) $(\mathrm{Spin}(14) \times GL(63)$, half-spin rep. $\otimes \Lambda_1)$ *with* $H \sim (G_2) \times (G_2)$.

(14) $((G_2) \times GL(5),\ \Lambda_2 \otimes \Lambda_1)$ *with* $H \sim GL(2)$.

(15) $(E_6 \times GL(26),\ \Lambda_1 \otimes \Lambda_1)$ *with* $H \sim F_4$.

(16) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1)$ *with* $H \sim \mathrm{Spin}(8)$.

(17) $(E_6 \times GL(25),\ \Lambda_1 \otimes \Lambda_1)$ *with* $H \sim \mathrm{Spin}(8)$.

(18) $(E_7 \times GL(55),\ \Lambda_6 \otimes \Lambda_1)$ *with* $H \sim E_6$.

(II)

(19) $(SL(6) \times GL(2),\ \Lambda_2 \otimes \Lambda_1)$ *with* $H \sim SL(2) \times SL(2) \times SL(2)$.

(20) $(SL(6) \times GL(13),\ \Lambda_2 \otimes \Lambda_1)$ *with* $H \sim SL(2) \times SL(2) \times SL(2)$.

(21) $(SL(7)\times GL(19),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim(GL(1)\times SL(2))\cdot G_a^6$.

(22) $(SL(9)\times GL(34),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim(GL(1)\times SL(2))\cdot G_a^8$.

(23) $(SL(2m)\times GL(2m^2-m-1),\ \Lambda_2\otimes\Lambda_1)$ $(m\geqslant 3)$ *with* $H\sim \mathrm{Sp}(m)$.

(24) $(SL(m)\times GL(\frac{1}{2}m(m+1)-1),\ 2\Lambda_1\otimes\Lambda_1)$ $(m\geqslant 3)$ *with* $H\sim O(m)$.

(25) $(SL(6)\times GL(19),\ \Lambda_3\otimes\Lambda_1)$ *with* $H\sim SL(3)\times SL(3)$.

(26) $(SL(7)\times GL(34),\ \Lambda_3\otimes\Lambda_1)$ *with* $H\sim(G_2)$.

(27) $(\mathrm{Sp}(3)\times GL(13),\ \Lambda_3\otimes\Lambda_1)$ *with* $H\sim SL(3)$.

(28) $(\mathrm{Spin}(12)\times GL(31)$, half-spin rep. $\otimes\Lambda_1)$ *with* $H\sim SL(6)$.

(III)

(29) $(SL(5)\times GL(3),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim SL(2)$.

(30) $(SL(2m+1)\times GL(2),\ \Lambda_2\otimes\Lambda_1)$ $(m\geqslant 5)$ *with* $H\sim(GL(1)\times SL(2))\cdot G_a^{2m}$ (see (1.1)).

(31) $(\mathrm{Sp}(n)\times GL(2),\ \Lambda_1\otimes 2\Lambda_1)$ *with* $H\sim(\mathrm{Sp}(n-2)\times SO(2))\cdot U(2n-3)$ $(n\geqslant 2)$.

(32) $(SO(n)\times GL(m),\ \Lambda_1\otimes\Lambda_1)$ *with* $H\sim SO(m)\times SO(n-m)$ *for* $n=9$, 11, *or* $n\geqslant 13$, *and* $n>m\geqslant 2$.

(33) $(\mathrm{Spin}(7)\times GL(2)$, spin rep. $\otimes\Lambda_1)$ *with* $H\sim SL(3)\times O(2)$.

(34) $(\mathrm{Spin}(7)\times GL(3)$, spin rep. $\otimes\Lambda_1)$ *with* $H\sim SL(2)\times O(3)$.

(35) $(\mathrm{Spin}(7)\times GL(6)$, spin rep. $\otimes\Lambda_1)$ *with* $H\sim SL(3)\times O(2)$.

(36) $(\mathrm{Spin}(10)\times GL(2)$, half-spin rep. $\otimes\Lambda_1)$ *with* $H\sim(G_2)\times SL(2)$.

(37) $(\mathrm{Spin}(10)\times GL(3)$, half-spin rep. $\otimes\Lambda_1)$ *with* $H\sim SL(2)\times O(3)$.

(38) $(\mathrm{Spin}(10)\times GL(14)$, half-spin rep. $\otimes\Lambda_1)$ *with* $H\sim(G_2)\times SL(2)$.

(39) $((G_2)\times GL(2),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim GL(2)$.

(40) $((G_2)\times GL(6),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim SL(3)$.

(IV)

(41) $(SL(2)\times GL(2),\ 2\Lambda_1\otimes\Lambda_1)$ *with* $H\sim O(2)$.

(42) $(SL(5)\times GL(8),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim(GL(1)\times SL(2))\cdot G_a^4$.

(43) $(SL(9)\times GL(2),\ \Lambda_2\otimes\Lambda_1)$ *with* $H\sim(GL(1)\times SL(2))\cdot G_a^8$.

(44) $(SL(2m+1)\times GL(2m^2+m-1),\ \Lambda_2\otimes\Lambda_1)$ $(m\geqslant 4)$ *with* $H\sim(GL(1)\times \mathrm{Sp}(m))\cdot G_a^{2m}$.

(45) $(SO(10)\times GL(m),\ \Lambda_1\otimes\Lambda_1)$ $(2\leqslant m\leqslant 9)$ *with* $H\sim SO(10-m)\times SO(m)$.

(46) $(SO(12)\times GL(m),\ \Lambda_1\otimes\Lambda_1)$ $(2\leqslant m\leqslant 11)$ *with* $H\sim SO(12-m)\times SO(m)$.

(47) $(\mathrm{Spin}(7)\times GL(7),$ spin rep. $\otimes\Lambda_1)$ *with* $H\sim (G_2)$.

(48) $(\mathrm{Spin}(10)\times GL(15),$ half-spin rep. $\otimes\Lambda_1)$ *with* $H\sim(GL(1)\times \mathrm{Spin}(7))\cdot G_a^8$.

(V)

(49) $(SL(5)\times GL(2),\ \Lambda_2\otimes\Lambda_1)$.

(50) $(SL(5)\times GL(9),\ \Lambda_2\otimes\Lambda_1)$.

(51) $(SL(7)\times GL(2),\ \Lambda_2\otimes\Lambda_1)$.

(52) $(SL(7)\times GL(20),\ \Lambda_2\otimes\Lambda_1)$.

(53) $(SO(5)\times GL(m),\ \Lambda_1\otimes\Lambda_1)\simeq(Sp(2)\times GL(m),\ \Lambda_2\otimes\Lambda_1)$ $(m=2,3,4)$.

(54) $(SO(6)\times GL(m),\ \Lambda_1\otimes\Lambda_1)\simeq(SL(4)\times GL(m),\ \Lambda_2\otimes\Lambda_1)$ $(2\leqslant m\leqslant 5)$.

(55) $(SO(7)\times GL(m),\ \Lambda_1\otimes\Lambda_1)\simeq(\mathrm{Spin}(7)\times GL(m),$ vector rep. $\otimes\Lambda_1)$ $(2\leqslant m\leqslant 6)$.

(56) $(SO(8)\times GL(m),\ \Lambda_1\otimes\Lambda_1)$ $(2\leqslant m\leqslant 7)$.

(57) $(Sp(n)\times GL(2m),\ \Lambda_1\otimes\Lambda_1)$ $(n>m\geqslant 1)$.

(58) $(Sp(n)\times GL(2m+1),\ \Lambda_1\otimes\Lambda_1)$ $(n>m\geqslant 1)$.

The following lemma is almost obvious.

LEMMA 1.6. *Let H be a generic isotropy subgroup of $(GL(1)\times G\times G',\ \rho_1\otimes\rho_1')$. Let d and d' be the minimum of degree of nontrivial representations of G and G', respectively*:

(1) *If* $1+\dim H\lneqq\min\{d,d'\}$, *then there exists no non-irreducible 2-simple P.V. with an irreducible component* $(GL(1)\times G\times G',\ \rho_1\otimes\rho_1')$.

(2) *If* $1+\dim H\lneqq d$ (*resp.* d'), *then* $(GL(1)^2\times G\times G',\ \rho_1\otimes\rho_1'+\rho_2\otimes\rho_2')$ *with* $\rho_2\neq 1$ (*resp.* $\rho_2'\neq 1$) *is not a P.V.*

2. A CLASSIFICATION

In this section, for each nontrivial 2-simple P.V. $(GL(1)\times G\times G',\ \rho_1\otimes\rho_1')$ in Theorem 1.5, we shall determine all nonirreducible 2-simple P.V.s which have $(GL(1)\times G\times G',\ \rho_1\otimes\rho_1')$ as one of their irreducible components. For this purpose, we shall investigate the prehomogeneity of $(GL(1)^2\times G\times G',\ \rho_1\otimes\rho_1'+\rho_2\otimes\rho_2')$, where we do not assume the nontriviality of ρ_2 and ρ_2' in general.

THEOREM 2.1. *There exists no nonirreducible 2-simple P.V. which has one of* (I) *in Theorem* 1.5 *as an irreducible component.*

Proof. For (1), we have $\rho'_2 = 1$ by (2) of Lemma 1.6 and Lemma 1.4, since $2m^2 + m - 2 \gneqq \dim(\mathfrak{gl}(1) \oplus \mathfrak{g}_{x_0}) = 2m + 5$ for $m \geqslant 5$. If $\rho_2 \neq 1$, we have $\rho_2 = \Lambda_1$ or Λ_1^* by dimension reason. Then the castling transform $(GL(1)^2 \times SL(2m+1) \times SL(2),\ \Lambda_2^* \otimes \Lambda_1 + \rho_2 \otimes 1)$ is also a P.V., and by (1.1), $(GL(2),\ (m-1)\Lambda_1)$ (resp. $(GL(2), m\Lambda_1)$) must be a P.V. if $\rho_2 = \Lambda_1$ (resp. $\rho_2 = \Lambda_1^*$), which is a contradiction since $m \geqslant 5$. By (1) of Lemma 1.6, we have (2)–(7) and (11) in Theorem 1.5. For (8), by (2) of Lemma 1.6, we have $\rho'_2 = 1$. If $\rho_2 \neq 1$, then its castling transform $(GL(1)^2 \times SL(8),\ \Lambda_3^* + \rho_2) \simeq (GL(1)^2 \times SL(8),\ \Lambda_3 + \rho_2^*)$ is a P.V. which is a contradiction by Theorem 1.3. Similarly, we have (12), (13), (15), and (18). For (9), by dimension reason, if $\rho'_2 \neq 1$, then we have $\rho_2 = 1$ and $\rho'_2 = \Lambda_1$ or Λ_1^*. If $\rho_2 = 1$ and $\rho'_2 = \Lambda_1$, its castling transform $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(4),$ spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1)$ must be also a P.V. Since $(\mathrm{Spin}(7) \times GL(4),$ spin rep. $\otimes \Lambda_1)$ is a non-P.V. (see p. 118 in [1]), the case for $\rho_2 = 1$ and $\rho'_2 = \Lambda_1$ is a non-P.V. Since a generic isotropy subgroup of (9) is reductive, the case for $\rho_2 = 1$ and $\rho'_2 = \Lambda_1^*$ is also a non-P.V. Hence $\rho'_2 = 1$. If $\rho_2 \neq 1$, then $\deg \rho_2 \leqslant 7 = \dim(GL(1) \times SL(2) \times SL(2))$ and hence ρ_2 must be the vector representation. By (5.37), p. 118 in [1], it is a P.V. if and only if the triplet $(GL(1) \times SL(2) \times SL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes 2\Lambda_1,\ V(4) \oplus V(3))$ is a P.V. However, it is clearly not a P.V. and we have finished the case (9). For (10), if $\rho'_2 \neq 1$, then $\deg \rho_2 \otimes \rho'_2 \leqslant \dim(GL(1) \times \mathrm{Spin}(7)) = 22$, and hence $\rho'_2 = \Lambda_1$ or Λ_1^* and $\rho_2 = 1$. In this case, it is a P.V. if and only if $(GL(1) \times \mathrm{Spin}(7),\ \Lambda_1 \otimes$ (spin rep. + vector rep.)) is a P.V. By p. 96 in [2], it is not a P.V. If $\rho'_2 = 1$, it reduces to the simple case by a castling transformation. By pp. 77, 89 in [2], it is not a P.V. for any $\rho_2 \neq 1$. For (14), if $\rho_2 \otimes \rho'_2 \neq 1$, then $\rho_2 = 1$ and $\rho'_2 = \Lambda_1$ (or Λ_1^*) by dimension reason. If $\rho'_2 = \Lambda_1$, we have its castling transform $(GL(1)^2 \times G_2 \times SL(3),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$. Since $(G_2 \times GL(3),\ \Lambda_2 \otimes \Lambda_1)$ is a non-P.V. by p. 136 in [1], the case for $\rho_2 = 1$ and $\rho'_2 = \Lambda_1$ (hence also the case for $\rho'_2 = \Lambda_1^*$) is a non-P.V. For (16) and (17), we have our desired result from the fact that the restriction of (E_6, Λ_1) (resp. $(GL(2), \Lambda_1)$, $(GL(25), \Lambda_1)$) to a generic isotropy subgroup $H \sim \mathrm{Spin}(8)$ is given by $(\mathrm{Spin}(8),\ 1 + 1 + 1 + \Lambda_1 + \Lambda_e + \Lambda_0,\ V(27))$ (resp. $(\mathrm{Spin}(8),\ 1 + 1,\ V(2))$, $(\mathrm{Spin}(8),\ 1 + \Lambda_1 + \Lambda_e + \Lambda_0,\ V(25))$), where Λ_1 (resp. Λ_e, Λ_0) denotes the vector (resp. even half-spin, odd half-spin) representation of Spin(8). One can check this fact by simple calculation of weights. Q.E.D.

LEMMA 2.2. *Let* $(GL(1) \times G \times G',\ \rho_1 \otimes \rho'_1)$ *be one of* (II) *in Theorem* 1.5. *If* $(GL(1)^2 \times G \times G',\ \rho_1 \otimes \rho'_1 + \rho_2 \otimes \rho'_2)$ *is also a P.V., then we have* $\rho'_2 = 1$.

Proof. By (5.10) in p. 93 in [1], we have (19). By (2) of Lemma 1.6, we have (20)–(22) and (24)–(27). Since the restriction of $(GL(2m^2-m-1), \Lambda_1)$ to $H=\mathrm{Sp}(m)$ is $(\mathrm{Sp}(m), \Lambda_2)$, we have (23) by p. 106 in [1]. For (28), by dimension reason, only the possibility for $\rho_2' \neq 1$ is $\rho_2' = \Lambda_1$ or Λ_1^*. If $\rho_2' = \Lambda_1$, we have its castling transform $(GL(1)^2 \times \mathrm{Spin}(12) \times SL(2)$, half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1)$ which is a non-P.V. by p. 130 in [1]. Since H is reductive, the case for $\rho_2' = \Lambda_1^*$ is also a non-P.V. Q.E.D.

THEOREM 2.3. *All non-irreducible 2-simple P.V.s which have one of* (II) *in Theorem* 1.5 *as an irreducible component are given as follows*:

$$(GL(1)^2 \times SL(6) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1) \tag{2.1}$$

$$(GL(1)^2 \times SL(6) \times SL(13), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1) \tag{2.2}$$

$$(GL(1)^2 \times SL(7) \times SL(19), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1) \tag{2.3}$$

$$(GL(1)^2 \times SL(9) \times SL(34), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1) \tag{2.4}$$

$$(GL(1)^{s+1} \times SL(2m) \times SL(2m^2-m-1), \Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1),$$
where $m \geq 3$; $s = 1, 2, 3$; $\Sigma_1 = \Lambda_1^{(*)}$, $\Sigma_2 = \Lambda_1^{(*)} + \Lambda_1^{(*)}$, *and*
$$\Sigma_3 = \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)}. \tag{2.5}$$

$$(GL(1)^2 \times SL(n) \times SL(\tfrac{1}{2}n(n+1)-1), 2\Lambda_1 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1) \ (n \geq 3) \tag{2.6}$$

$$(GL(1)^2 \times SL(6) \times SL(19), \Lambda_3 \otimes \Lambda_1 + \Lambda_1 \otimes 1) \tag{2.7}$$

$$(GL(1)^3 \times SL(6) \times SL(19), \Lambda_3 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1) \tag{2.8}$$

$$(GL(1)^2 \times SL(7) \times SL(34), \Lambda_3 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1) \tag{2.9}$$

$$(GL(1)^2 \times Sp(3) \times SL(13), \Lambda_3 \otimes \Lambda_1 + \Lambda_1 \otimes 1) \tag{2.10}$$

$$(GL(1)^2 \times \mathrm{Spin}(12) \times SL(31), \text{half-spin rep.} \otimes \Lambda_1 + \text{vector rep.} \otimes 1). \tag{2.11}$$

Note that $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1$ for (22) is not a P.V., and $\Lambda_3^* = \Lambda_3$ for $SL(6)$ in (2.7) and (2.8).

Proof. For (19) (resp. (21), (22)), we have $\dim(GL(1)^{l-1} \times H) = 8 + l$ (resp. $9 + l$, $11 + l) \geq \deg \rho_2 + \cdots + \deg \rho_l \geq (l-1) \deg \Lambda_1 = 6(l-1)$ (resp. $7(l-1)$, $9(l-1)$), and hence $l = 2$, $\rho_2 = \Lambda_1$ or Λ_1^*. Since (19) and the castling transform of (21) are F.P.s (see [4]), the case (19) and (21) are actually P.V.s.

By Lemma 2.2 and a castling transformation, (20) reduces to (19). For (22), first note that the castling transform of $(GL(1)^2 \times SL(9) \times SL(34)$, $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1$ (resp. $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1$)) is given by $(GL(1)^2 \times SL(9) \times SL(2)$, $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1$ (resp. $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1$)). If the case for $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1$ is a P.V., then by (1.1), the triplet $(GL(2), 4\Lambda_1, V(5))$ must be also a P.V., which is a contradiction by dimension reason. By (1.1), $(GL(1)^2 \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1)$ is a P.V. if and only if $\mathfrak{g} = \{(\alpha) \oplus (-{}^tC)$; C is the second matrix in (1.1)$\}$ acts on K^9 prehomogeneously. Since $x_0 = e_6 + e_9 \in K^9$ is a generic point, $(\mathfrak{g}, K^9)$ (and hence (22)) is a P.V. By a castling transformation, (23)–(28) reduce to the simple P.V.s, and by Theorem 1.3, we have our results. Q.E.D.

LEMMA 2.4. *Let* $(GL(1)\times G\times G', \rho_1\otimes\rho_1')$ *be one of* (III) *in Theorem* 1.5. *If* $(GL(1)^2\times G\times G', \rho_1\otimes\rho_1'+\rho_2\otimes\rho_2')$ *is also a P.V., then we have* $\rho_2=1$.

Proof. By (2) of Lemma 1.6, we have the cases (29), (37), and (39). For (30), if $\rho_2\neq 1$, then we have $\rho_2'=1$ since otherwise $\deg\rho_2\otimes\rho_2'\geqslant 2(2m+1)>\dim H+1=2m+5$. Then, by the castling transformation and (1) in Theorem 2.1, we have our result. For (31), $\rho_2\otimes\rho_2'$ must be one of (a) $\Lambda_1\otimes\Lambda_1$, $\Lambda_2\otimes 1$, $\Lambda_2\otimes\Lambda_1$ for $n=2$, (b) $\Lambda_3\otimes 1$ for $n=3$, (c) $\Lambda_1\otimes 2\Lambda_1$, $\Lambda_1\otimes\Lambda_1$ for $n\geqslant 3$, $\Lambda_1\otimes 1$. However, (a) and (b) are impossible by dimension reason. If $\rho_2\otimes\rho_2'=\Lambda_1\otimes 2\Lambda_1$ (resp. $\Lambda_1\otimes\Lambda_1$, $\Lambda_1\otimes 1$), it is a P.V. if and only if $(GL(1)^2\times SL(2), \Lambda^2(2\Lambda_1+2\Lambda_1)$ (resp. $\Lambda^2(2\Lambda_1+\Lambda_1)$, $\Lambda^2(2\Lambda_1+1)$)) is a P.V. by pp. 40–41 in [1], which is impossible by dimension reason. Now before going ahead, we shall prove several sublemmas.

SUBLEMMA 2.4.1. *The triplet* $(GL(1)^2\times SO(n)\times SL(m), \Lambda_1\otimes\Lambda_1+\Lambda_1\otimes\Lambda_1^*, M(n,m)\oplus M(n,m))$ *is a non-P.V. for* $n\geqslant m\geqslant 1$.

Proof. For $x=(X,Y)\in M(n,m)\oplus M(n,m)$, $g=(\alpha,\beta;A,B)\in GL(1)^2\times SO(n)\times SL(m)$ and $\rho=\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes\Lambda_1$ (resp. $\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes\Lambda_1^*$), we have $\rho(g)x=(\alpha AX{}^tB, \beta AY{}^tB)$ (resp. $(\alpha AX{}^tB, \beta AYB^{-1})$) and hence, $f(x)=\det({}^tXX)\cdot\det({}^tYY)\cdot\det({}^tXY)^{-2}$ is a nonconstant absolute invariant.
Q.E.D.

SUBLEMMA 2.4.2. *For* $n\gneqq m\geqslant 1$, *the triplet* $(GL(1)^2\times SO(n)\times SL(m), \Lambda_1\otimes 1+\Lambda_1\otimes\Lambda_1, V(n)\oplus M(n,m))$ *is a non-P.V.*

Proof. By pp. 109–110 in [1], it is a P.V. if and only if $(GL(1)\times SO(n-m)\times SO(m), \Lambda_1\otimes\Lambda_1\otimes 1+\Lambda_1\otimes 1\otimes\Lambda_1)$ is a P.V. In this case, a triplet $(SO(m), \Lambda_1, V(m))$ without scalar multiplication must be a P.V., which is a contradiction. Q.E.D.

SUBLEMMA 2.4.3. *For* $m_1, m_2\geqslant n\geqslant 1$, *the triplet* $(SO(m_1)\times SO(m_2)\times GL(n), \Lambda_1\otimes 1\otimes\Lambda_1+1\otimes\Lambda_1\otimes\Lambda_1^{(*)}, M(m_1,n)\oplus M(m_2,n))$ *is a non-P.V.*

Proof. For $x=(X,Y)\in M(m_1,n)\oplus M(m_2,n)$, $g=(A,B,C)\in SO(m_1)\times SO(m_2)\times GL(n)$ and $\rho^{(*)}=\Lambda_1\otimes 1\otimes\Lambda_1+1\otimes\Lambda_1\otimes\Lambda_1^{(*)}$, we have $\rho(g)x=(AX{}^tC, BY{}^tC)$ (resp. $\rho^*(g)x=(AX{}^tC, BYC^{-1})$), and hence $f(x)=\det({}^tXX)\cdot\det({}^tYY)^{-1}$ (resp. $f(x)=\det({}^tXX)\cdot\det({}^tYY)$) is a nonconstant absolute invariant. Q.E.D.

SUBLEMMA 2.4.4. *For* $n \geq m \geq 1$, *the triplet* $(GL(n) \times GL(m), (1 + \Lambda_1 + \Lambda_1^*) \otimes \Lambda_1, V(m) \oplus M(n, m) \oplus M(n, m))$ *is a non-P.V.*

Proof. For $x = (y, X_1, X_2) \in V(m) \oplus M(n, m) \oplus M(n, m)$, $g = (A, B) \in GL(n) \times GL(m)$ and $\rho = (1 + \Lambda_1 + \Lambda_1^*) \otimes \Lambda_1$, we have $\rho(g)x = (By, AX_1{}^tB, {}^tA^{-1}X_2{}^tB)$. Hence, if $m \geq 2$, then $f(x) = \det({}^tX_2X_1 + {}^tX_1X_2) \cdot \det({}^tX_2X_1)^{-1}$ is a nonconstant absolute invariant. If $m = 1$, then $f(x) = ({}^tX_1X_2) \cdot y^{-2}$ is a nonconstant absolute invariant. Q.E.D.

SUBLEMMA 2.4.5. *For* $n \geq m \geq 1$, *the triplet* $(SO(n) \times GL(m), 1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1^{(*)}, V(m) \oplus M(n, m))$ *is a non-P.V.*

Proof. By pp. 109–110 in [1], it is a P.V. if and only if $(SO(m), \Lambda_1, V(m))$ is a P.V. without scalar multiplication, which is a contradiction. Q.E.D.

Now we start to prove the case (32). Note that if $\rho_2' = 1$, we may assume $n \geq 2m$ by a castling transformation. If $\rho_2 \neq 1$, then $\rho_2 \otimes \rho_2'$ must be one of $\Lambda_1 \otimes \Lambda_1^{(*)}$, $\Lambda_1 \otimes 1$, or $\Lambda \otimes 1$ $(n = 9, 11, 14)$ with $\Lambda =$ (half-) spin representation by [1]. If $\rho_2 \otimes \rho_2' = \Lambda_1 \otimes \Lambda_1^{(*)}$ (resp. $\Lambda_1 \otimes 1$), then it is a non-P.V. by Sublemma 2.4.1 (resp. Sublemma 2.4.2). For $n = 9$ and $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$, it is a non-P.V. by p. 127 in [1], $\Lambda(\mathrm{Spin}(7)) \subset SO(8)$, and Sublemma 2.4.5. For $n = 11$ and $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$, it is a non-P.V. by p. 130 in [1] and Sublemma 2.4.4. For $n = 14$ and $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$, it is a non-P.V. by p. 133 in [1] and Sublemma 2.4.3. For (33), we have $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$ ($\Lambda =$ the spin rep.) or $\Lambda_1 \otimes 1$ ($\Lambda_1 =$ the vector rep.) by dimension reason. If $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$, then it is a non-P.V. by $\Lambda(\mathrm{Spin}(7)) \subset SO(8)$ and Sublemma 2.4.2. If $\rho_2 \otimes \rho_2' = \Lambda_1 \otimes 1$, then it is a non-P.V. by (5.35), p. 117 in [1], and Sublemma 2.4.4. For (34), we have $\rho_2 \otimes \rho_2' =$ vector rep. $\otimes\, 1$, and it is a P.V. if and only if its castling transform $(\mathrm{Spin}(7) \times GL(5)$, spin rep. $\otimes\, \Lambda_1 +$ vector rep. $\otimes\, 1)$ is a P.V. which is a contradiction by Theorem 2.1 for (9) in Theorem 1.5. For (35), if $\rho_2 \neq 1$, then we have $\rho_2' = 1$, since otherwise $\dim H + 1 = 10 \geq \deg \rho_2 \otimes \rho_2' \geq 7 \cdot 6 = 42$, which is a contradiction. Hence we can reduce (35) to (33) by the castling transformation. For (36), we have $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$ ($\Lambda =$ half-spin rep.) or $\Lambda_1 \otimes 1$ ($\Lambda_1 =$ the vector rep.) by dimension reason. If $\rho_2 \otimes \rho_2' = \Lambda \otimes 1$, it is a P.V. if and only if $(GL(1) \times (G_2) \times SL(2), \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes \Lambda_1, V(14) \oplus V(2))$ is a P.V. by calculation of weights (cf. p. 123 in [1]). By (5.53), p. 136 in [1], it is a P.V. if and only if $(O(2), \Lambda_1, V(2))$ is a P.V. Since $\dim O(2) = 1 < \dim V(2) = 2$, it is a non-P.V. If $\rho_2 \otimes \rho_2' = \Lambda_1 \otimes 1$, it is a non-P.V. by (5.42), p. 123 in [1], $\Lambda_2(G_2) \subset SO(7)$, and Sublemma 2.4.3. For (38), if $\rho_2 \neq 1$, then $\rho_2' = 1$ by dimension reason, and hence (38) reduces to the case (36) by a castling transformation. For (40), if $\rho_2 \neq 1$, we have $\rho_2 \otimes \rho_2' = \Lambda_2 \otimes 1$ by dimension reason. It is a non-P.V. by $\Lambda_2(G_2) \subset SO(7)$ and Sublemma 2.4.2. Q.E.D.

THEOREM 2.5. *All non-irreducible 2-simple P.V.s which have one of* (III) *in Theorem* 1.5 *as an irreducible component are given as follows*:

$(GL(1)^2 \times SL(5) \times SL(3),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.12)

$(GL(1)^2 \times SL(2m+1) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \rho)$ $(m \geqslant 5)$, *where* $\rho = \Lambda_1,\ 2\Lambda_1$, *or* $3\Lambda_1$. (2.13)

$(GL(1)^3 \times SL(2m+1) \times SL(2),$ $\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + 1 \otimes \rho)$ $(m \geqslant 5)$, *where* $\rho = \Lambda_1$ *or* $2\Lambda_1$. (2.14)

$(GL(1)^4 \times SL(2m+1) \times SL(2),$ $\Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$ $(m \geqslant 5)$ (2.15)

$(GL(1)^2 \times \mathrm{Sp}(n) \times SL(2),\ \Lambda_1 \otimes 2\Lambda_1 + 1 \otimes \Lambda_1)$ $(n \geqslant 2)$ (2.16)

$(GL(1)^2 \times SO(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.17)

$(GL(1)^2 \times \mathrm{Spin}(7) \times SL(2)$, spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1)$ (2.18)

$(GL(1)^2 \times \mathrm{Spin}(7) \times SL(3)$, spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.19)

$(GL(1)^2 \times \mathrm{Spin}(7) \times SL(6)$, spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.20)

$(GL(1)^2 \times \mathrm{Spin}(10) \times SL(2)$, half-spin rep. $\otimes \Lambda_1 + 1 \otimes \rho)$, *where* $\rho = \Lambda_1,\ 2\Lambda_1$, *or* $3\Lambda_1$. (2.21)

$(GL(1)^3 \times \mathrm{Spin}(10) \times SL(2)$, half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1 + 1 \otimes \rho)$, *where* $\rho = \Lambda_1$ *or* $2\Lambda_1$. (2.22)

$(GL(1)^4 \times \mathrm{Spin}(10) \times SL(2)$, half-spin rep. $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$ (2.23)

$(GL(1)^2 \times \mathrm{Spin}(10) \times SL(3)$, half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.24)

$(GL(1)^2 \times \mathrm{Spin}(10) \times SL(14)$, half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.25)

$(GL(1)^2 \times (G_2) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$ (2.26)

$(GL(1)^2 \times (G_2) \times SL(6),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$. (2.27)

Proof. First note that if $(GL(1)^k \times SO(n),\ \rho_1 \oplus \cdots \oplus \rho_k)$ is a P.V., then we have $k = 1$ and $\rho_1 = \Lambda_1$. The $SL(m)$-part of the generic isotropy subgroup of (29) (resp. (31), (32), (33), (34), (37), (39)) is $SO(m)$ by p. 96 (resp. pp. 104, 109, 117, 118, 125, 136) in [1], and hence we have (2.12), (2.16)–(2.19), (2.24), and (2.26). Now, if $(GL(1)^k \times SL(2),\ \rho_1 \oplus \cdots \oplus \rho_k)$ is a P.V., then we have $k \leqslant 3$ and $\rho_1 \oplus \cdots \oplus \rho_k = \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1$ $(k = 3)$; $2\Lambda_1 \oplus \Lambda_1$, $\Lambda_1 \oplus \Lambda_1$ $(k = 2)$; $3\Lambda_1$, $2\Lambda_1$, Λ_1 $(k = 1)$. The $SL(2)$-part of the generic isotropy subgroup of (30) (resp. (36)) is $SL(2)$ by (1.1) in Lemma 1.4 (resp. p. 112 in [1]) and hence, we have (2.13)–(2.15) and (2.21)–(2.23). For (35), (38), and (40), we have $\rho_2 \otimes \rho'_2 + \cdots + \rho_l \otimes \rho'_l = 1 \otimes \Lambda_1^{(*)}$, i.e., $l = 2$ and $\rho'_2 = \Lambda_1^{(*)}$ by dimension reason. Since the generic isotropy subgroups of (35), (38), (40) in Theorem 1.5 are reductive, we may assume that $\rho_2 \otimes \rho_2' = 1 \otimes \Lambda_1$ to see the prehomogeneity. Then, by a castling transformation, (35) (resp. (38), (40)) is reduced to (2.19) (resp. (2.24), (2.26)) and we have (2.20), (2.25), and (2.27). Q.E.D.

LEMMA 2.6. *Let* $(GL(1) \times G \times G', \rho_1 \otimes \rho_1')$ *be one of* (IV) *in Theorem* 1.5. *Then,* (i) $(GL(1)^2 \times G \times G', \rho_1 \otimes \rho_1' + \rho_2 \otimes \rho_2')$ *is a non-P.V. for any* $\rho_2 \neq 1$ *and* $\rho_2' \neq 1$; (ii) $(GL(1)^3 \times G \times G', \rho_1 \otimes \rho_1' + \rho_2 \otimes 1 + 1 \otimes \rho_3')$ *is a non-P.V. for any* $\rho_2 \neq 1$ *and* $\rho_3' \neq 1$.

Proof. For (41), we have (i) by $\dim GL(1) \times H = 2 < 4 = 2 \times 2 \leqslant \deg \rho_2 \otimes \rho_2'$ and (ii) by $\dim GL(1)^2 \times H = 3 < 4 = 2 + 2 \leqslant \deg(\rho_2 \otimes 1 + 1 \otimes \rho_3')$. For (42), we have (i) by $\dim GL(1) \times H = 9 < 40 = 5 \times 8 \leqslant \deg \rho_2 \otimes \rho_2'$ and (ii) by $\dim GL(1)^2 \times H = 10 < 13 = 8 + 5 \leqslant \deg(\rho_2 \otimes 1 + 1 \otimes \rho_3')$. For (43), we have (i) by $\dim GL(1) \times H = 13 < 18 = 9 \times 2 \leqslant \deg \rho_2 \otimes \rho_2'$. Now if $(GL(1)^2 \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \rho_2 \otimes 1)$ is a P.V., then $(GL(1)^2 \times SL(9) \times SL(34), \rho_2 \otimes \Lambda_1 + \rho_2^* \otimes 1)$ is also a P.V., and hence, by (2.4), we have $\rho_2 = \Lambda_1^*$. If $(GL(1)^3 \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes \rho_3')$ is a P.V., then $(GL(1)^2 \times SL(2), 3\Lambda_1 + \rho_3')$ must be also a P.V. by Lemma 1.4, and hence $\rho_3' = 1$. For (44), we have (i) by $\dim GL(1) \times H = 2m^2 + 3m + 2 < 4m^3 + 4m^2 - m - 1 = (2m+1) \times (2m^2 + m - 1) \leqslant \deg \rho_2 \otimes \rho_2'$ for $m \geqslant 4$. Now assume that $(GL(1)^2 \times SL(2m+1) \times SL(2m^2 + m - 1), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \rho_3')$ is a P.V. We shall see that $\rho_3' = \Lambda_1$ (and $\rho_3' \neq \Lambda_1^*$). Since $\dim GL(1) \times H = 2m^2 + 3m + 2 \geqslant \deg \rho_3' \geqslant 2m^2 + m - 1$, we have $\rho_3' = \Lambda_1$ or Λ_1^*. By calculating the weights, the $SL(2m^2 + m - 1)$ part of the generic isotropy subgroup H of $(GL(1) \times SL(2m+1) \times SL(2m^2 + m - 1)$,

$$\Lambda_2 \otimes \Lambda_1) \text{ is } \left\{ \left(\begin{array}{c|c} \Lambda_2(A) + \varepsilon_1 I & * \\ \hline 0 & \Lambda_1(A) - \varepsilon_2 I \end{array} \right); \ \varepsilon_1 = (2m^2 + m)\varepsilon, \right.$$

$$\left. 2\varepsilon_2 = (m-1)(2m+1)^2 \varepsilon, \ A \in \mathrm{Sp}(m) \right\}$$

or

$$\left\{ \left(\begin{array}{c|c} \Lambda_2(A) + \varepsilon_1 I & 0 \\ \hline * & \Lambda_1(A) - \varepsilon_2 I \end{array} \right) \right\}.$$

Now if $\rho_3' = \Lambda_1$, its castling transform is $(GL(1)^2 \times SL(2m+1) \times SL(2), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$, which is a P.V. by (2.13). Note that it is a P.V. for

$m \geqslant 4$. This implies that the $SL(2m^2+m-1)$-part of H must be of the form

$$\left\{ \begin{pmatrix} \Lambda_2(A)+\varepsilon_1 I & * \\ \hline 0 & \Lambda_1(A)-\varepsilon_2 I \end{pmatrix} ; A \in \mathrm{Sp}(m) \right\},$$

since $(GL(1)\times \mathrm{Sp}(m), \Lambda_2)$ is a non-P.V. for $m \geqslant 3$. Therefore, if $\rho_3' = \Lambda_1^*$, it is a non-P.V. Assume that $(GL(1)^3 \times SL(2m+1)\times SL(2m^2+m-1)$, $\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + \rho_2 \otimes 1)$ is a P.V. Then its castling transform $(GL(1)^3 \times SL(2m+1)\times SL(2)$, $\Lambda_2\otimes\Lambda_1 + 1\otimes\Lambda_1 + \rho_2^*\otimes 1)$ must be a P.V. If $m \geqslant 5$, then we have $\rho_2^* = 1$ by (30) of Lemma 2.4. If $m=4$, by (43) of our Lemma 2.6, we have $\rho_2^*=1$. For (45), assume that $(GL(1)^2\times SO(10)\times SL(m)$, $\Lambda_1\otimes\Lambda_1+\rho_2\otimes\rho_2')$ is a P.V. for $\rho_2\neq 1$ and $\rho_2'\neq 1$. Then we have $\dim G = m^2+46 \geqslant \dim V \geqslant 20m$, i.e., $(m-10)^2 \geqslant 54$ $(2\leqslant m\leqslant 9)$, and hence $m=2$. Thus $\rho_2\otimes\rho_2'$ must be $\Lambda_1\otimes\Lambda_1^{(*)}$ or half-spin rep.$\otimes\Lambda_1$. By Sublemma 2.4.1, $\rho_2\otimes\rho_2'\neq\Lambda_1\otimes\Lambda_1^{(*)}$. If $\rho_2\otimes\rho_2'=$ half-spin rep.$\otimes\Lambda_1$, then $\dim G = 50 < \dim V = 52$, which is a contradiction. Thus we have (i) for (45). Now assume that $(GL(1)^3\times SO(10)\times SL(m)$, $\Lambda_1\otimes\Lambda_1+\rho_2\otimes 1 + 1\otimes\rho_3')$ is a P.V. with $\rho_2\neq 1$ and $\rho_3'\neq 1$. By Sublemma 2.4.2, ρ_2 must be a half-spin representation of Spin(10). Since $SL(m)$-part of the generic isotropy subgroup of $(SO(10)\times GL(m), \Lambda_1\otimes\Lambda_1)$ is $O(m)$, ρ_3' must be Λ_1 or Λ_1^*. The generic isotropy subgroup of $(GL(1)^2\times \mathrm{Spin}(10)\times SL(m)$, vector rep.$\otimes\Lambda_1 + 1\otimes\Lambda_1^{(*)})$ is locally isomorphic to $O(10-m)\times O(m-1)$ (p. 110 in [1]) and by calculation of weights, we see that the restriction of a half-spin representation of Spin(10) to $O(10-m)\times O(m-1)$ is given by (Spin(8), even half-spin rep.$\oplus$odd half-spin rep.) for $m=2, 9$; $(GL(1)\times\mathrm{Spin}(7)$, $(\Lambda_1+\Lambda_1^*)\otimes$spin rep.) for $m=3, 8$; $(SL(2)\times SL(4)$, $\Lambda_1\otimes(\Lambda_1+\Lambda_1^*))$ for $m=4, 7$; $(SL(2)\times SL(2)\times\mathrm{Sp}(2)$, $\Lambda_1\otimes 1\otimes\Lambda_1 + 1\otimes\Lambda_1\otimes\Lambda_1)$ for $m=5, 6$. Since they are not P.V.s even with a scalar multiplication (see the proof of (2.9) in [4] for $m=5, 6$), we have (ii) for (45). For (46), assume that $(GL(1)^2\times\mathrm{Spin}(12)\times SL(m)$, vector rep.$\otimes\Lambda_1 + \rho_2\otimes\rho_2')$ $(m\geqslant 2)$ is a P.V. with $\rho_2\neq 1$ and $\rho_2'\neq 1$. By Theorem 1.5, $\rho_2\otimes\rho_2'$ must be vector rep.$\otimes\Lambda_1$. By Sublemma 2.4.1, it is a non-P.V. and hence we have (i) for (46). Now assume that $(GL(1)^3\times\mathrm{Spin}(12)\times SL(m)$, vector rep.$\otimes\Lambda_1+\rho_2\otimes 1 + 1\otimes\rho_3')$ is a P.V. Then ρ_2 must be a half-spin representation by Sublemma 2.4.2, and $\rho_3' = \Lambda_1$ or Λ_1^* (see the proof for (45)). Since the generic isotropy subgroup of $(GL(1)\times\mathrm{Spin}(12), \rho_2)$ is $SL(6)$ (p. 129 in [1]), $(GL(1)^2\times\mathrm{Spin}(12)\times SL(m)$, vector rep.$\otimes\Lambda_1+\rho_2\otimes 1)$ is a P.V. if and only if $(SL(6)\times GL(m), (\Lambda_1+\Lambda_1^*)\otimes\Lambda_1)$ is a P.V. By the proof of Sublemma 2.4.4 (and by a castling transformation if necessary), it is not a P.V. for $2\leqslant m\leqslant 10$. Since the generic isotropy subgroup of $(GL(1)^2\times\mathrm{Spin}(12)\times SL(11)$, vector rep.$\otimes\Lambda_1+\rho_2\otimes 1)$ is reductive, we may assume that $\rho_3'=\Lambda_1$ as far as we consider the prehomogeneity. Then its castling transform is $(GL(1)^3 \times \mathrm{Spin}(12) \times SL(2)$, vector rep. $\otimes \Lambda_1 + \rho_2 \otimes 1 + 1 \otimes \Lambda_1)$, which is not a P.V. as we have seen above. Thus we have (ii) for (46). For (47), if $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7)$, spin rep. $\otimes \Lambda_1 + \rho_2 \otimes \rho_2')$ is a P.V. for $\rho_2 \neq 1$ and $\rho_2' \neq 1$, then $\dim GL(1) \times H = 15 \geqslant \deg \rho_2 \otimes \rho_2' \geqslant 49$, which is a contradiction, and hence we have (i) for (7). If $(GL(1)^3 \times \mathrm{Spin}(7) \times SL(7)$, spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1 + 1 \otimes \rho_3')$ is a P.V., then ρ_2 must be the vector representation by Theorem 1.3, since a castling transform $(GL(1)^2 \times \mathrm{Spin}(7)$, spin rep. $+ \rho_2)$ of $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7)$, spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1)$ must be a P.V. By dimension reason, we have $\rho_3 = \Lambda_1$ or Λ_1^*. Since the generic isotropy subgroup of $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7)$, spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1)$ is reductive, we may assume $\rho_3 = \Lambda_1$. Then, by a castling transformation, we have $(GL(1)^3 \times \mathrm{Spin}(7) \times SL(2)$, spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1 + 1 \otimes \Lambda_1)$, which is not a P.V. by (33) of Lemma 2.4. Thus we have (ii) for (47). For (48), if $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15)$, half-spin rep. $\otimes \Lambda_1 + \rho_2 \otimes \rho_2')$ is a P.V. for $\rho_2 \neq 1$ and $\rho_2' \neq 1$, then $\dim GL(1) \times H = 31 \geqslant \deg \rho_2 \otimes \rho_2' \geqslant 150$, which is a contradiction, and hence we have (i) for (48). If $(GL(1)^3 \times \mathrm{Spin}(10) \times SL(15)$, half-spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1 + 1 \otimes \rho_3')$ is a P.V., then ρ_2 must be the half-spin representation or the vector representation by [2], and $\rho_3' = \Lambda_1$ or Λ_1^* by dimension reason. Since the generic isotropy subgroup of $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15)$, half-spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1)$ is reductive (see pp. 96, 97 in [2]), we may assume that $\rho_3' = \Lambda_1$. Then, by a castling transformation, we have $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(2)$, half-spin rep. $\otimes \Lambda_1 + \rho_2 \otimes 1 + 1 \otimes \Lambda_1)$, which is not a P.V. by (36) of Lemma 2.4. Thus we have (ii) for (48). Q.E.D.

THEOREM 2.7. *All non-irreducible 2-simple P.V.s which have one of* (IV) *in Theorem* 1.5 *as an irreducible component are given as follows*:

$$(GL(1)^2 \times SL(2) \times SL(2), 2\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1) \tag{2.28}$$

$$(GL(1)^2 \times SL(2) \times SL(2), 2\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1) \tag{2.29}$$

$$(GL(1)^{1+s} \times SL(5) \times SL(8), \Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1) \quad (s = 1, 2), \text{ where } \Sigma_1 = \Lambda_1^{(*)} \text{ and } \Sigma_2 = \Lambda_1 \oplus \Lambda_1^{(*)}. \tag{2.30}$$

$$(GL(1)^2 \times SL(5) \times SL(8), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)}) \tag{2.31}$$

$$(GL(1)^2 \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1) \tag{2.32}$$

$$(GL(1)^{1+t} \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 + 1 \otimes T_t) \quad (t = 1, 2, 3), \text{ where } T_1 = \Lambda_1, 2\Lambda_1, 3\Lambda_1;\ T_2 = \Lambda_1 \oplus \Lambda_1, \Lambda_1 \oplus 2\Lambda_1;\ T_3 = \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1. \tag{2.33}$$

$$(GL(1)^{1+s} \times SL(2m+1) \times SL(2m^2+m-1), \Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1) \quad (s = 1, 2, 3), \text{ where } \Sigma_1 = \Lambda_1^{(*)}, \Lambda_2^*;\ \Sigma_2 = \Lambda_1^{(*)} \oplus \Lambda_1^{(*)};\ \Sigma_3 = \Lambda_1^{(*)} \oplus \Lambda_1^{(*)} \oplus \Lambda_1^{(*)} \text{ except for } \Sigma_3 \simeq \Lambda_1 \oplus \Lambda_1^* \oplus \Lambda_1^* \quad (m \geqslant 4). \tag{2.34}$$

$(GL(1)^2 \times SL(2m+1) \times SL(2m^2+m-1),$
$\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1) \quad (m \geqslant 4)$ (2.35)

$(GL(1)^2 \times \mathrm{Spin}(10) \times SL(m)$, vector rep. $\otimes \Lambda_1$ + half-spin rep. $\otimes 1) \quad (2 \leqslant m \leqslant 9,\ m \neq 5)$ (2.36)

$(GL(1)^2 \times SO(10) \times SL(m),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)}) \quad (2 \leqslant m \leqslant 9)$ (2.37)

$(GL(1)^2 \times \mathrm{Spin}(12) \times SL(11)$, vector rep. $\otimes \Lambda_1$ + half-spin rep. $\otimes 1)$ (2.38)

$(GL(1)^2 \times SO(12) \times SL(m),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)}) \quad (2 \leqslant m \leqslant 11)$ (2.39)

$(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7)$, spin rep. $\otimes \Lambda_1$ + vector rep. $\otimes 1)$ (2.40)

$(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7)$, spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$ (2.41)

$(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15),\ \Lambda_0 \otimes \Lambda_1 + \rho \otimes 1)$ *with* $\rho = \Lambda_e$ *or* χ, *where* Λ_0 (*resp.* Λ_e, χ) *is the odd half-spin* (*resp. even half-spin, vector*) *representation of* Spin(10). (2.42)

$(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15),\ \Lambda_0 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$. (2.43)

Note that $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1$ *for* (2.32) *and* $\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$ *for* (2.35) *are non-P.V.s.*

Proof. For (2.28) (resp. (2.29)), we have $\rho_2 = \Lambda_1$ (resp. $\rho_3' = \Lambda_1$), since $\deg \rho_2$ (resp. $\deg \rho_3'$) $\leqslant \dim G - \deg 2\Lambda_1 \otimes \Lambda_1 = 2$. Since the $SL(2)$ part of the generic isotropy subgroup is $O(2)$, (2.28) and (2.29) are actually P.V.s. For (2.30), $(GL(1)^{1+s} \times SL(5) \times SL(8),\ \Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1)$ with $\Sigma_s = \sigma_1 + \cdots + \sigma_s$, is a P.V. if and only if $(GL(1)^{1+s} \times SL(5) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1)$ is a P.V. Since $\dim G \geqslant \dim V$, we have $5s \leqslant \deg \sigma_1^* + \cdots + \deg \sigma_s^* \leqslant s + 8$, and hence $s = 1$ or 2. Thus we have $\sigma_1^* = \Lambda_1^{(*)}$ for $s = 1$ and $\sigma_1^* \oplus \sigma_2^* = \Lambda_1^{(*)} \oplus \Lambda_1^{(*)}$. However, $\sigma_1^* \oplus \sigma_2^* \neq \Lambda_1 \oplus \Lambda_1$ since otherwise $(GL(1)^2 \times SL(2),\ 2\Lambda_1 \oplus 2\Lambda_1)$ becomes a P.V. by (1.1), which is a contradiction by dimension reason. By calculating the isotropy subalgebra at $(X_0, e_5,\ e_1 + e_3 + e_4 + e_5)$ (resp. $(X_0,\ e_5,\ e_1 + e_3 + e_5)$) of $(GL(1)^3 \times SL(5) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1$ (resp. $\Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1) \otimes 1$)) (see Lemma 1.4), we see that they are actually P.V.s. For (2.31), if $\Lambda_2 \otimes \Lambda_1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ is a P.V., then we have $8t \leqslant \deg \tau_1 + \cdots + \deg \tau_t \leqslant 8 + t$ and hence $t = 1$, $\tau_1 = \Lambda_1^{(*)}$. If $\tau_1 = \Lambda_1$, then it is castling-equivalent to (2.12), and hence it is a P.V. If $\tau_1 = \Lambda_1^*$, we identify the representation space of $\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$ with

$$V = \overbrace{V_2 \oplus \cdots \oplus V_2}^{8} \oplus K^8,$$

where $V_2 = \Sigma K e_i \wedge e_j\ (1 \leqslant i < j \leqslant 5)$. Then the action is given by $x \mapsto \alpha \cdot \Lambda_2(A)\ (x_1, \ldots, x_8)^t B + \beta^t B^{-1} y$ for $x = (x_1, \ldots, x_8; y) \in V$ and $g = (\alpha, \beta; A, B) \in GL(1)^2 \times SL(5) \times SL(8)$. By calculating the isotropy subalgebra at $x=(\omega_1, 2\omega_3, 2\omega_2, \omega_{10}, \omega_5-\omega_8, \omega_4-\omega_9, \omega_6, \omega_7; e_2+e_8)$ (see p. 95 in [1]), we see that it is a P.V. For $(GL(1)^{1+s}\times SL(9)\times SL(2), \Lambda_2\otimes\Lambda_1+\Sigma_s\otimes 1)$, we have (2.32) from (2.4) by a castling transformation. For (2.33), since the $SL(2)$ part of the generic isotropy subgroup of $(GL(1)\times SL(9)\times SL(2), \Lambda_2\otimes\Lambda_1)$ is $SL(2)$ by Lemma 1.4, we have our result by [1]. For (2.34), it is castling-equivalent to a simple P.V. $(GL(1)^{1+s}\times SL(2m+1), \Lambda_2\oplus\Sigma_s^*)$, and hence we obtain our result by [2]. For (2.35), if $\Lambda_2\otimes\Lambda_1+1\otimes(\tau_1+\cdots+\tau_t)$ is a P.V., then we have $(2m^2+m-1)t\leqslant\deg\tau_1+\cdots+\deg\tau_t\leqslant t+(2m^2+3m+1)$ and hence $t=1$ and $\tau_1=\Lambda_1^{(*)}$. By the proof of (44) of Lemma 2.6, we have our result. For (2.36), if vector rep.$\otimes\Lambda_1+(\sigma_1+\cdots+\sigma_s)\otimes 1$ $(2\leqslant m\leqslant 9)$ is a P.V., then $\sigma_1, ..., \sigma_s\neq$ the vector representation by Sublemma 2.4.2 and $\sigma_1=\Lambda_e$ or $\sigma_1+\sigma_2=\Lambda_e+\Lambda_e$, $s\leqslant 2$ by [2]. If $\sigma_1+\sigma_2=\Lambda_e+\Lambda_e$, then $\dim G\geqslant\dim V$ implies $(m-5)^2\geqslant 10$ $(2\leqslant m\leqslant 9)$ and hence $m=9$. Then, it is castling-equivalent to $(GL(1)^3\times \mathrm{Spin}(10)$, vector rep.$\oplus\Lambda_e\oplus\Lambda_e)$, which is a non-P.V. by [2], and hence we have $\sigma_1=\Lambda_e$. In this case, it is a P.V. for $m=1, 2, 3$ (and hence $m=9, 8, 7$) by Theorems 3.3 and 5.7 in Kimura *et al.* [4]. For $m=4$ (resp. $m=5$), the restriction of $(GL(1)\times\mathrm{Spin}(10)\times SL(m), \Lambda_e\otimes 1)$ to the generic isotropy subgroup $SO(10-m)\times SL(m)$ is equivalent to $(GL(1)\times SL(2)\times SL(2)\times SL(4), \Lambda_1\otimes\Lambda_1\otimes 1\otimes\Lambda_1+\Lambda_1\otimes 1\otimes\Lambda_1\otimes\Lambda_1^*)$ for $m=4$ (resp. $(GL(1)\times \mathrm{Sp}(2)\times\mathrm{Sp}(2), \Lambda_1\otimes\Lambda_1\otimes\Lambda_1)$ for $m=5$). Thus the case for $m=4$ (and hence $m=6$) is a P.V. (see the corollary of Theorem 1.16 in [3]), and the case $m=5$ is a non-P.V. For (2.37), since the $SL(m)$ part of the generic isotropy subroup of $(GL(1)\times SO(10)\times SL(m), \Lambda_1\otimes\Lambda_1)$ is $SO(m)$, we have our result by [2]. If $(GL(1)^{1+s}\times\mathrm{Spin}(12)\times SL(m)$, vector rep.$\otimes\Lambda_1+\Sigma_s\otimes 1)$ is a P.V. with $2\leqslant m\leqslant 11$, then we have $m=11$ by the proof of Lemma 2.6. Hence it is castling-equivalent to a simple P.V. $(GL(1)^{1+s}\times\mathrm{Spin}(12)$, vector rep.$+\Sigma_s)$. Thus we obtain (2.38) by Theorem 1.3. For (2.39), we have our result similarly as (2.37). For (2.40), it is castling-equivalent to a simple P.V. For (2.41), since the $SL(7)$ part of the generic isotropy subgroup of $(GL(1)\times\mathrm{Spin}(7)\times SL(7)$, spin rep.$\otimes\Lambda_1)$ is $((G_2), \Lambda_2)$, we have our result by [2]. For (2.42), it is castling-equivalent to a simple P.V. and we have our result by Theorem 1.3. Now assume that $(GL(1)^{1+t}\times\mathrm{Spin}(10)\times SL(15), \Lambda_0\otimes\Lambda_1+1\otimes(\tau_1+\cdots+\tau_t))$ is a P.V. Then we have $15t\leqslant\deg\tau_1+\cdots+\deg\tau_t\leqslant 30+t$ and hence $t=1$ or 2. By dimension reason, we have $\tau_1=\Lambda_1^{(*)}$ for $t=1$ and $\tau_1+\tau_2=\Lambda_1^{(*)}+\Lambda_1^{(*)}$ for $t=2$. If $t=1$ and $\tau_1=\Lambda_1$, it is castling-equivalent to $(GL(1)^2\times\mathrm{Spin}(10)\times SL(2), \Lambda_e\otimes\Lambda_1+1\otimes\Lambda_1)$ which is a P.V. by (2.21). If $t=2$ and $\tau_1+\tau_2=\Lambda_1+\Lambda_1$, it is castling-equivalent to $(GL(1)^3\times\mathrm{Spin}(10)\times SL(3), \Lambda_e\otimes\Lambda_1+1\otimes\Lambda_1+1\otimes\Lambda_1)$ which is a non-P.V. by Theorem 2.5 for (37). Let V_e be the vector space spanned by 1, e_ie_j $(1\leqslant i<j\leqslant 5)$, $e_ke_le_se_t$ $(1\leqslant k<l<s<t\leqslant 5)$ over K. Let ρ_1 by the even half-spin representation Λ_e on V_e. Then, the odd half-spin representation

Λ_0 is the dual ρ_1^* of ρ_1. Now the representation space V of $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15), \Lambda_0 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$ is identified with

$$V = \overbrace{V_e \oplus \cdots \oplus V_e}^{15} \oplus K^{15}.$$

The action is given by $x \to (\alpha_1^*(A)(X_1, ..., X_{15})^t B;\ \beta^t B^{-1} y)$ for $x = (X_1, ..., X_{15}; y) \in V$, $g = (\alpha, \beta; A, B) \in GL(1)^2 \times \mathrm{Spin}(10) \times SL(15)$. Put $x_0 = (e_1e_5, e_2e_5, e_3e_5, e_4e_5, e_2e_3e_4e_5, -e_1e_3e_4e_5, e_1e_2e_4e_5, -e_1e_2e_3e_5, -1 + e_1e_2e_3e_4, e_1e_2, e_1e_3, e_1e_4, -e_3e_4, e_2e_4, -e_2e_3; e_9) \in V$. The isotropy subalgebra of $\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{o}(10) \oplus \mathfrak{sl}(15)$ at x_0 is given by $\{(16\varepsilon), (\varepsilon), \{A \oplus (-30\varepsilon) \oplus (-{}^tA) \oplus (30\varepsilon)\}, \{(A - 14\varepsilon I_4) \oplus (-{}^tA - 14\varepsilon I_4) \oplus (16\varepsilon) \oplus (\Lambda_2(A) + 16\varepsilon)\} \mid A \in \mathfrak{sl}(4),\ \varepsilon \in \mathfrak{gl}(1)\} \simeq \mathfrak{gl}(1) \oplus \mathfrak{sl}(4)$. Hence it is a P.V. Since $(GL(1) \times SL(4), \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_2^{(*)})$ is a non-P.V., $(GL(1)^3 \times \mathrm{Spin}(10) \times SL(15), \Lambda_0 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* + 1 \otimes \Lambda_1^{(*)})$ is a non-P.V. Q.E.D.

THEOREM 2.8. *All non-irreducible 2-simple P.V.s which have* $(SL(5) \times GL(2), \Lambda_2 \otimes \Lambda_1)$ ((49) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$$(GL(1)^{1+s} \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1) \quad (s = 1, 2),$$
$$\text{where } \Sigma_1 = \Lambda_1^{(*)} \text{ and } \Sigma_2 = \Lambda_1^* + \Lambda_1^{(*)} \quad (\Sigma_2 \neq \Lambda_1 + \Lambda_1). \tag{2.44}$$

$$(GL(1)^{1+t} \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + 1 \otimes T_t) \quad (t = 1, 2, 3),$$
$$\text{where } T_1 = \Lambda_1, 2\Lambda_1, 3\Lambda_1;\ T_2 = \Lambda_1 + \Lambda_1, 2\Lambda_1 + \Lambda_1;$$
$$T_3 = \Lambda_1 + \Lambda_1 + \Lambda_1. \tag{2.45}$$

$$(GL(1)^3 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1) \tag{2.46}$$

$$(GL(1)^{2+t} \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes T_t)$$
$$(t = 1, 2), \text{ where } T_1 = 2\Lambda_1;\ T_2 = \Lambda_1 + \Lambda_1. \tag{2.47}$$

Proof. By dimension reason, $\Lambda_2 \otimes \Lambda_1 + \rho_2 \otimes \rho_2' (\rho_2 \neq 1, \rho_2' \neq 1)$ is a non-P.V. If $\Lambda_2 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V., then its castling transform $(GL(1)^{s+1} \times SL(5) \times SL(8), \Lambda_2 \otimes \Lambda_1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1)$ is also a P.V., and hence, by (2.30), we have $\sigma_1^* = \Lambda_1^{(*)}$ and $\sigma_1^* + \sigma_2^* = \Lambda_1 + \Lambda_1^{(*)}$, i.e., (2.44). We have (2.45) similarly as (2.13)–(2.15). By dimension reason, $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + \Lambda_1^{(*)} \otimes 1 + \rho_4 \otimes \rho_4'$ is a non-P.V. for any $\rho_4 \otimes \rho_4' \neq 1$. Assume that $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ is a P.V. Then, by (1, 1) of Lemma 1.4, $(GL(1)^{t+1} \times SL(2), 2\Lambda_1 + \tau_1 + \cdots + \tau_t)$ must be a P.V., and hence we have $t = 1$ and $\tau_1 = \Lambda_1$. Next assume that $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ is a P.V. Then, by (1.1) of Lemma 1.4, $(GL(1)^{t+1} \times SL(2), \Lambda_1 + \tau_1 + \cdots + \tau_t)$ must be a P.V., and hence $t = 1$, $\tau_1 = \Lambda_1, 2\Lambda_1$; $t = 2$, $\tau_1 + \tau_2 = \Lambda_1 + \Lambda_1$. Thus it is enough to prove that (2.46) and (2.47) are actually P.V.s. (2.46) is a F.P. (see (5.19) in [4]) and hence a P.V. For (2.47), the generic isotropy subgroup of $(GL(1)^2 \times SL(2), \Lambda_1 + \Lambda_1)$ or

$(GL(1)\times SL(2), 2\Lambda_1)$ is $O(2)$,and hence (2.47) is a P.V. if and only if $(GL(1)^3\times SL(5), \Lambda_2\oplus\Lambda_2\oplus\Lambda_1^*)$ is a P.V. By (5) of Theorem 1.3, we have our results. Q.E.D.

THEOREM 2.9. *All non-irreducible 2-simple P.V.s which have* $(SL(5)\times GL(9), \Lambda_2\otimes\Lambda_1)$ ((50) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$$(GL(1)^{1+s}\times SL(5)\times SL(9), \Lambda_2\otimes\Lambda_1+\Sigma_s\otimes 1) \qquad (s=1,2,3),$$
$$\text{where } \Sigma_1=\Lambda_1^{(*)}, \Lambda_2^*;\ \Sigma_2=\Lambda_1^{(*)}+\Lambda_1^{(*)}, \Lambda_2^*+\Lambda_1;$$
$$\Sigma_3=\Lambda_1^{(*)}+\Lambda_1^{(*)}+\Lambda_1^{(*)} \text{ except for } \Sigma_3\simeq\Lambda_1^*+\Lambda_1^*+\Lambda_1. \tag{2.48}$$
$$(GL(1)^2\times SL(5)\times SL(9), \Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^{(*)}) \tag{2.49}$$
$$(GL(1)^3\times SL(5)\times SL(9), \Lambda_2\otimes\Lambda_1+\Lambda_1^{(*)}\otimes 1+1\otimes\Lambda_1^{(*)}). \tag{2.50}$$

Proof. By dimension reason, $\Lambda_2\otimes\Lambda_1+\rho_2\otimes\rho_2'(\rho_2\neq 1, \rho_2'\neq 1)$ is a non-P.V. Since $\Lambda_2\otimes\Lambda_1+\Sigma_s\otimes 1$ $(\Sigma_s=\sigma_1+\cdots+\sigma_s)$ is castling-equivalent to $(GL(1)^{1+s}\times SL(5), \Lambda_2+\Sigma_s^*)$, we have our result by Theorem 1.3. If $\Lambda_2\otimes\Lambda_1+1\otimes T_t(T_t=\tau_1+\cdots+\tau_t)$ is a P.V., then $t=1$ and $\tau_1=\Lambda_1^{(*)}$ by dimension reason. The prehomogeneity of (2.49) comes from that of (2.50). If $\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^{(*)}+\Sigma_s\otimes 1$ $(\Sigma_s\neq 1)$ is a P.V., then we have $s=1$ and $\sigma_1=\Lambda_1^{(*)}$ by dimension reason. Now $\Lambda_2\otimes\Lambda_1+\Lambda_1^{(*)}\otimes 1+1\otimes\Lambda_1$ is castling-equivalent to (2.46) and hence it is a P.V. Since $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1$ is castling-equivalent to a regular P.V. $(GL(1)^2\times SL(5), \Lambda_2\oplus\Lambda_1)$, its generic isotropy subgroup is reductive. Since $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1+1\otimes\Lambda_1$ is a P.V., $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1+1\otimes\Lambda_1^*$ is also a P.V. By Lemma 2.10, it is castling-equivalent to $\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^*$, and hence (2.50) is actually a P.V. Q.E.D.

LEMMA 2.10. *Assume that G is reductive and* $\deg\rho_1=m\geqslant 3$. *Then* $(GL(1)^3\times G\times SL(m-1), \rho_1\otimes\Lambda_1+\rho_2\otimes 1+1\otimes\Lambda_1^*)$ *is castling-equivalent to* $(GL(1)^3\times G\times SL(m-1), \rho_1\otimes\Lambda_1+\rho_2^*\otimes 1+1\otimes\Lambda_1^*)$.

Proof. It is equivalent to $((GL(1)\times G)\times GL(m-1)\times GL(1), \rho_1\otimes\Lambda_1\otimes 1+\rho_2\otimes 1\otimes 1+1\otimes\Lambda_1^*\otimes\Lambda_1)$ $\sim^c$ $((GL(1)\times G)\times GL(m-1)\times GL(m-2), \rho_1\otimes\Lambda_1\otimes 1+\rho_2\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1)$ $\sim^c$ $\rho_1^*\otimes\Lambda_1\otimes 1+\rho_2\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1^*$ $\sim^R$ $\rho_1\otimes\Lambda_1\otimes 1+\rho_2^*\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1^*$ $\sim^c$ $(GL(1)^3\times G\times SL(m-1), \rho_1\otimes\Lambda_1+\rho_2^*\otimes 1+1\otimes\Lambda_1^*)$, where $\sim^c$(resp. $\sim^R$) denotes the castling- (resp. reductive-)equivalence. Q.E.D.

THEOREM 2.11. *All non-irreducible 2-simple P.V.s which have* $(SL(7)\times GL(2), \Lambda_2\otimes\Lambda_1)$ ((51) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$(GL(1)^2 \times SL(7) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1)$ (2.51)

$(GL(1)^{1+t} \times SL(7) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes T_t)$ $(t = 1, 2, 3)$,
where $T_1 = \Lambda_1,\ 2\Lambda_1,\ 3\Lambda_1;\ T_2 = \Lambda_1 + \Lambda_1,\ \Lambda_1 + 2\Lambda_1;$
$T_3 = \Lambda_1 + \Lambda_1 + \Lambda_1$. (2.52)

$(GL(1)^3 \times SL(7) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes \Lambda_1)$. (2.53)

Note that $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1$ is a non-P.V. for (2.53).

Proof. $\Lambda_2 \otimes \Lambda_1 + \rho_2 \otimes \rho_2'(\rho_2 \neq 1,\ \rho_2' \neq 1)$ is a non-P.V. by dimension reason. $\Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1$ is castling-equivalent to $(GL(1)^{1+s} \times SL(7) \times SL(19),\ \Lambda_2^* \otimes \Lambda_1 + \Sigma_s \otimes 1)$ and hence we obtain (2.51) from (2.3). If $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes T_t$ is a P.V., then $(GL(1)^{1+t} \times SL(2),\ 3\Lambda_1 + T_t)$ must be a P.V. by Lemma 1.4. Hence we have $t = 0$, $T_t = 1$. If $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes T_t(T_t \neq 1)$ is a P.V., then $(GL(1)^{1+t} \times SL(2),\ 2\Lambda_1 + T_t)$ must be a P.V. and hence $t = 1$, $T_t = \Lambda_1$, i.e., (2.53). It is actually a P.V. For example, $(X_0;\ {}^t(0000010), {}^t(1, 1))$ (see Lemma 1.4 for X_0) is a generic point. Q.E.D.

THEOREM 2.12. *All non-irreducible 2-simple P.V.s which have* $(SL(7) \times GL(20),\ \Lambda_2 \otimes \Lambda_1)$ ((52) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$(GL(1)^{1+s} \times SL(7) \times SL(20),\ \Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1)$ $(s = 1, 2, 3)$,
where $\Sigma_1 = \Lambda_1^{(*)},\ \Lambda_2^*;\ \Sigma_2 = \Lambda_1^{(*)} + \Lambda_1^{(*)};$
$\Sigma_3 = \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)}$ *except for* $\Sigma_3 \simeq \Lambda_1 + \Lambda_1^* + \Lambda_1^*$. (2.54)

$(GL(1)^2 \times SL(7) \times SL(20),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$ (2.55)

$(GL(1)^3 \times SL(7) \times SL(20),\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)$. (2.56)

Note that $\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*$ for (2.55), and $\Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes \Lambda_1$ for (2.56), are not P.V.s.

Proof. If $\Lambda_2 \otimes \Lambda_1 + \rho_2 \otimes \rho_2'$ $(\rho_2 \neq 1,\ \rho_2' \neq 1)$ is a P.V., then we have $\dim G = 449 \geqslant \dim V \geqslant 420 + 7 \times 20 = 560$, which is a contradiction. Similarly as (2.34) and (2.35), we have (2.54) and (2.55). Since $\Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + \Sigma_s \otimes 1$ is castling-equivalent to $(GL(1)^{2+s} \times SL(7) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + \Sigma_s^* \otimes 1)$, we have $s = 1$ and $\Sigma_1 = \Lambda_1$ by (2.53). Q.E.D.

THEOREM 2.13. *All non-irreducible 2-simple P.V.s which have* $(SO(5) \times GL(m),\ \Lambda_1 \otimes \Lambda_1) \simeq (\mathrm{Sp}(2) \times GL(m),\ \Lambda_2 \otimes \Lambda_1)$ $(m = 2, 3, 4)$ ((53) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$(GL(1)^2 \times \mathrm{Sp}(2) \times SL(m),\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$ (2.57)

$$(GL(1)^2\times \mathrm{Sp}(2)\times SL(m),\ \Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^{(*)}) \tag{2.58}$$

$$(GL(1)^3\times \mathrm{Sp}(2)\times SL(m),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^{(*)}),\ \text{for } m=2,4\ (m\neq 3). \tag{2.59}$$

Proof. If $\Lambda_2\otimes\Lambda_1+\rho_2\otimes\rho_2'$ $(\rho_2\neq 1,\ \rho_2'\neq 1)$ is a P.V., then we have $\dim G=m^2+11\geqslant \dim V\geqslant 5m+4m$ $(m=2,3,4)$, which is a contradiction. First note that $\Lambda_2\otimes\Lambda_1+\Lambda_2\otimes 1$ is a non-P.V. by Sublemma 2.4.2. Hence if $\Lambda_2\otimes\Lambda_1+\Sigma_s\otimes 1$ is a P.V., then $s=1$, $\Sigma_1=\Lambda_1$ or $s\leqslant 2$, $\Sigma_2=\Lambda_1+\Lambda_1$ for $m=4$, by dimension reason. However, a castling transform $(GL(1)^3\times \mathrm{Sp}(2),\ \Lambda_2+\Lambda_1+\Lambda_1)$ of $(GL(1)^3\times\mathrm{Sp}(2)\times SL(4),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+\Lambda_1\otimes 1)$ is a non-P.V., we have (2.57). Actually (2.57) is a P.V. by (5.10) in [4]. Since the $SL(m)$ part of the generic isotropy subgroup of $(GL(1)\times \mathrm{Sp}(2)\times SL(m),\ \Lambda_2\otimes\Lambda_1)$ $(m=2,3,4)$ is $O(m)$, we have (2.58). For (2.59), we have $m\neq 3$ by dimension reason. Since the generic isotropy subalgebra of

$$(\mathrm{Sp}(2)\times GL(2),\ \Lambda_2\otimes\Lambda_1) \text{ is } \left\{\begin{pmatrix} A & 0\\ 0 & -{}^tA\end{pmatrix}\oplus\begin{pmatrix}-\mathrm{Tr}\,A & 0\\ 0 & \mathrm{Tr}\,A\end{pmatrix};\ A\in\mathfrak{gl}(2)\right\}$$

(see p. 455 in Kimura and Kasai [5]), $(GL(1)^3\times\mathrm{Sp}(2)\times SL(2),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1)$ is a P.V., and hence $(GL(1)^3\times\mathrm{Sp}(2)\times SL(4),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1)$ is a P.V. by a castling transformation. Since the $SL(4)$ part of the generic isotropy subgroup of $(GL(1)\times\mathrm{Sp}(2)\times SL(4),\ \Lambda_2\otimes\Lambda_1)$ is $O(4)$, $(GL(1)^2\times\mathrm{Sp}(2)\times SL(4),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^*)$ is also a P.V. Q.E.D.

THEOREM 2.14. *All non-irreducible 2-simple P.V.s which have* $(SO(6)\times GL(m),\ \Lambda_1\otimes\Lambda_1)\simeq(SL(4)\times GL(m),\ \Lambda_2\otimes\Lambda_1)$ $(2\leqslant m\leqslant 5)$ ((54) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$$(GL(1)^2\times SL(4)\times SL(2),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes\Lambda_1) \tag{2.60}$$

$$(GL(1)^{1+s}\times SL(4)\times SL(m),\ \Lambda_2\otimes\Lambda_1+\Sigma_s\otimes 1)\quad (s=1,2,3),$$
where $\Sigma_1=\Lambda_1$; $\Sigma_2=\Lambda_1+\Lambda_1$ $(m\neq 3)$, $\Sigma_2=\Lambda_1+\Lambda_1^*$ $(m=5)$; $\Sigma_3=\Lambda_1+\Lambda_1^{(*)}+\Lambda_1^{(*)}$ $(m=5)$ (2.61)

$$(GL(1)^2\times SL(4)\times SL(m),\ \Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^{(*)}) \tag{2.62}$$

$$(GL(1)^3\times SL(4)\times SL(m),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^{(*)}). \tag{2.63}$$

Note that $\Lambda_2=\Lambda_2^*$ for $SL(4)$.

Proof. If $\Lambda_2\otimes\Lambda_1+\rho_2\otimes\rho_2'$ $(\rho_2\neq 1,\ \rho_2'\neq 1)$ is a P.V., we have $\dim G=16+m^2\geqslant\dim V\geqslant 6m+4m$ $(2\leqslant m\leqslant 5)$ and hence $m=2$, $\rho_2\otimes\rho_2'=$

$\Lambda_1 \otimes \Lambda_1$. Then it is acturally a P.V., since $((e_1 \wedge e_2, e_3 \wedge e_4), (e_1 + e_3, e_2 + e_4))$ is a generic point. Note that $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1 + \rho_3 \otimes \rho'_3$ is not a P.V. for any $\rho_3 \otimes \rho'_3 \neq 1$ by dimension reason. Thus we have (2.60). Let us consider $\Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1$ with $\Sigma_s = \sigma_1 + \cdots + \sigma_s$. If $m = 5$, it is castling-equivalent to a simple P.V. $(GL(1)^{1+s} \times SL(4), \Lambda_2 + \sigma_1 + \cdots + \sigma_s)$ and hence we have $s = 1, 2, 3$; $\sigma_1 = \Lambda_1$; $\sigma_1 + \sigma_2 = \Lambda_1 + \Lambda_1^{(*)}$; $\sigma_1 + \sigma_2 + \sigma_3 = \Lambda_1 + \Lambda_1^{(*)} + \Lambda_1^{(*)}$. For $m = 2, 3, 4$, $\Lambda_2 \otimes \Lambda_1 + \sigma_1 \otimes 1$ with $\sigma_1 = \Lambda_2$ (resp. $\sigma_1 = 2\Lambda_1$) is not a P.V. by Sublemma 2.4.2 (resp. by dimension reason), and hence

$$\Sigma_s = \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}.$$

Since $\dim G = 15 + s + m \geqslant \dim V = 6m + \deg \Sigma_s \geqslant 6m + 4s$, i.e., $(m-3)^2 + 6 \geqslant 3s$ $(2 \leqslant m \leqslant 4)$, we have $s = 1$ or 2. Since the $SL(4)$ part of the generic isotropy subgroup of $(SL(4) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(6) \oplus V(6) \oplus V(6))$ at $(e_1 \wedge e_2,\ e_3 \wedge e_4,\ e_1 \wedge e_3 + e_2 \wedge e_4)$ is $SO(4)$, $(GL(1)^2 \times SL(4) \times SL(3), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$ is a P.V. and $(GL(1)^3 \times SL(4) \times SL(3), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1^{(*)} \otimes 1)$ is a non-P.V. Since the generic isotropy subalgebra of $(SL(4) \times GL(2), \Lambda_2 \otimes \Lambda_1)$ at $(e_1 \wedge e_2, e_3 \wedge e_4)$ is given by

$$\left\{ \left(\begin{array}{c|c} A + \alpha I_2 & 0 \\ \hline 0 & B - \alpha I_2 \end{array} \right), \begin{pmatrix} -2\alpha & 0 \\ 0 & 2\alpha \end{pmatrix}; A, B \in \mathfrak{sl}(2), \alpha \in \mathfrak{gl}(1) \right\},$$

one can check easily that $(GL(1)^3 \times SL(4) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1$ (resp. $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1^* \otimes 1$)) is a P.V. (resp. is a non-P.V.), and so is the case for $m = 4$ by a castling transformation. Thus we have (2.61). Since the $SL(m)$-part of the generic isotropy subgroup of $(GL(1) \times SL(4) \times SL(m), \Lambda_2 \otimes \Lambda_1)$ is $SO(m)$, we have (2.62). Assume that $\Lambda_2 \otimes \Lambda_1 + \Sigma_s \otimes 1 + 1 \otimes \Lambda_1^{(*)}$ is a P.V. Then we have $\dim G = s + 16 + m^2 \geqslant 7m + \deg \Sigma_s \geqslant 7m + 4s$, and hence $s = 1$; $s = 2$ $(m = 2, 5)$. We shall see that $s \neq 2$. Since the $SL(2)$ part of the generic isotropy subalgebra of $(GL(1)^3 \times SL(4) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1)$ is zero, we have $s \neq 2$ for $m = 2$. By p. 94 in [2], the generic isotropy subgroup of $(GL(1)^3 \times SL(4) \times SL(5), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1^{(*)} \otimes 1)$ is reductive, $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1^*$ and $\Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1$ are P.V.-equivalent. However, its castling transform $(GL(1)^4 \times SL(4) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1)$ is a non-P.V. as above, we have $s \neq 2$ for $m = 5$. If $s = 1$, it is a F.P. by Theorem 5.17 in [4], and hence it is a P.V. Thus we have (2.63). Q.E.D.

THEOREM 2.15. *Let Λ (resp. Λ_1) be the spin (resp. the vector) representation of* Spin(7). *All non-irreducible 2-simple P.V.s which have*

$(SO(7)\times GL(m),\ \Lambda_1\otimes\Lambda_1)\simeq(\mathrm{Spin}(7)\times GL(m),\ \Lambda_1\otimes\Lambda_1)$ $(2\leqslant m\leqslant 6)$ ((55) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$$(GL(1)^2\times\mathrm{Spin}(7)\times SL(m),\ \Lambda_1\otimes\Lambda_1+\Lambda\otimes 1)\ (m=2,5,6) \tag{2.64}$$

$$(GL(1)^2\times\mathrm{Spin}(7)\times SL(m),\ \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_1^{(*)})\ (2\leqslant m\leqslant 6) \tag{2.65}$$

$$(GL(1)^3\times\mathrm{Spin}(7)\times SL(m), \Lambda_1\otimes\Lambda_1+\Lambda\otimes 1+1\otimes\Lambda_1^{(*)})\ (m=2,6). \tag{2.66}$$

Proof. If $\Lambda_1\otimes\Lambda_1+\rho_2\otimes\rho_2'$ $(\rho_2\neq 1,\ \rho_2'\neq 1)$ is a P.V., then we have $\dim G=22+m^2\geqslant\dim V\geqslant 7m+7m$, i.e., $(m-7)^2\geqslant 27$ $(2\leqslant m\leqslant 6)$, which is a contradiction. If $\Lambda_1\otimes\Lambda_1+(\sigma_1+\cdots+\sigma_s)\otimes 1$ is a P.V., then we have $s=1$ and $\sigma_1=\Lambda$ by Sublemma 2.4.2 and [2]. Since the restriction of $(GL(1)\times\mathrm{Spin}(7),\ \Lambda_1)$ to a generic isotropy subgroup of $(GL(1)\times\mathrm{Spin}(7), \Lambda)$ is equivalent to $((G_2),\ \Lambda_2,\ V(7))$ (see p. 116 in [1]), we have (2.64). Since the $SL(m)$ part of a generic isotropy subgroup of $(GL(1)\times\mathrm{Spin}(7)\times SL(m),\ \Lambda_1\otimes\Lambda_1)$ is $SO(m)$, we have (2.65). Now $\Lambda_1\otimes\Lambda_1+\Lambda\otimes 1+1\otimes\Lambda_1^{(*)}$ is a P.V. if and only if $(GL(1)^2\times(G_2)\times SL(m),\ \Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^{(*)})$ $(m=2,5,6)$ is a P.V. Thus we have $m\neq 5$ by (14) of Theorem 2.1. It is a P.V. for $m=2$, 6 by (2.26) and (2.27). Thus we have our result. Q.E.D.

Let Λ_e(resp. Λ_0, Λ_1) be the even half-spin (resp. the odd half-spin, the vector) representation of Spin(8). Then it is well known that $(\mathrm{Spin}(8), \Lambda_e)\simeq(\mathrm{Spin}(8), \Lambda_0)\simeq(\mathrm{Spin}(8), \Lambda_1)\simeq(SO(8),\ \Lambda_1)$ as triplets (see p. 36 in [1]).

THEOREM 2.16. *All non-irreducible* 2-*simple P.V.s which have* $(SO(8)\times GL(m),\ \Lambda_1\otimes\Lambda_1)(2\leqslant m\leqslant 7)$ ((56) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$$(GL(1)^2\times\mathrm{Spin}(8)\times SL(m), \Lambda_e\otimes\Lambda_1+\Lambda_1\otimes 1)\ (2\leqslant m\leqslant 7, m\neq 4) \tag{2.67}$$

$$(GL(1)^2\times\mathrm{Spin}(8)\times SL(m),\ \Lambda_e\otimes\Lambda_1+1\otimes\Lambda_1^{(*)})\qquad(2\leqslant m\leqslant 7) \tag{2.68}$$

$$(GL(1)^3\times\mathrm{Spin}(8)\times SL(m),\ \Lambda_e\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^{(*)})\ \textit{for}\ m=2,3,6,7. \tag{2.69}$$

Proof. If $\Lambda_e\otimes\Lambda_1+\rho_2\otimes\rho_2'$ $(\rho_2\neq 1,\ \rho_2'\neq 1)$ is a P.V., then we have $\dim G=29+m^2\geqslant\dim V\geqslant 8m+8m$, i.e., $(m-8)^2\geqslant 35$ $(2\leqslant m\leqslant 7)$, and hence $m=2$. Note that $(\mathrm{Spin}(8)\times SL(2),\ \rho_2\otimes\rho_2')\simeq(SO(8)\times SL(2),\ \Lambda_1\otimes\Lambda_1,\ V(8)\otimes V(m))$ as triplets if $\rho_2\neq 1$ and $\rho_2'\neq 1$. Hence the $SL(2)$ part of a generic isotropy subgroup of $(GL(1)\times\mathrm{Spin}(8)\times SL(2),\ \rho_2\otimes\rho_2')$ is $O(2)=\{(^{\alpha}_{\ \beta});\ \alpha\beta=\pm 1\}$. Thus $(GL(1)^2\times\mathrm{Spin}(8),\ \Lambda_e+\Lambda_e)$ must be a P.V., which is a contradiction by Theorem 1.3. Assume that $\Lambda_e\otimes\Lambda_1+$

$(\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V. Then, by Sublemma 2.4.2 and Theorem 1.3, we have $s = 1, 2$; $\sigma_1 = \Lambda_1$; $\sigma_2 = \Lambda_1 + \Lambda_0$. Since the restriction of Λ_e and Λ_0 of Spin(8) to a generic isotropy subgroup of $(GL(1) \times \mathrm{Spin}(8), \Lambda_1)$ gives both the spin representation Λ of Spin(7) and $\Lambda(\mathrm{Spin}(7)) \subset SO(8)$, we have $s \neq 2$, i.e., $s = 1$ by Sublemma 2.4.2. Since $\Lambda_e \otimes \Lambda_1 + \Lambda_1 \otimes 1$ is P.V.-equivalent to $(GL(1) \times \mathrm{Spin}(7) \times SL(m), \Lambda \otimes \Lambda_1)$, we have (2.67) by Theorem 1.5. Since the $SL(m)$ part of a generic isotropy subgroup of $(GL(1) \times \mathrm{Spin}(8) \times SL(m), \Lambda_e \otimes \Lambda_1)$ is $O(m)$, we have (2.68). For (2.69), $\Lambda_e \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)}$ $(2 \leqslant m \leqslant 7,\ m \neq 4)$ is P.V.-equivalent to $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(m), \Lambda \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$. Hence we have $m \neq 5$ by (9) of Theorem 2.1. By (2.18)–(2.20) and (2.41), we have (2.69). Q.E.D.

LEMMA 2.17. *For* $2n > m \geqslant 2$, $(GL(1)^2 \times \mathrm{Sp}(n) \times SL(m), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1^{(*)})$ *is a non-P.V.*

Proof. The representation space of $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1$ (resp. $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1^*$) is identified with $V = M(2n, m) \oplus M(2n, m)$, where the action is given by $(X, Y) \to (\alpha A X {}^t B, \beta A Y {}^t B)$ (resp. $(\alpha A X {}^t B, \beta A Y B^{-1})$) for $g = (\alpha, \beta; A, B) \in GL(1)^2 \times \mathrm{Sp}(n) \times SL(m)$ and $x = (X, Y) \in V$. Then a rational function $f(x) = \det({}^t X J Y - {}^t Y J X) \cdot \det({}^t X J Y)^{-1}$(resp. $\mathrm{Tr}({}^t X J Y)^m \cdot \det({}^t X J Y)^{-1}$) is a nonconstant absolute invariant for $m \geqslant 2$, where

$$J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right). \qquad \text{Q.E.D.}$$

LEMMA 2.18. *All 2-simple P.V.s which contain* $(GL(1)^2 \times \mathrm{Sp}(2) \times SL(m), \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$ $(m = 2, 3)$ *as a component, are given as follows*:

$$(GL(1)^2 \times \mathrm{Sp}(2) \times SL(3), \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1) \tag{2.70}$$

$$(GL(1)^3 \times \mathrm{Sp}(2) \times SL(3), \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 + 1 \otimes \Lambda_1^*). \tag{2.71}$$

Proof. By Lemma 4.6 in [4], $(GL(1)^2 \times \mathrm{Sp}(2) \times SL(2), \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1)$ is a non-P.V. Now (2.70) is actually a P.V., since it is castling-equivalent to (9) in Theorem 1.3. If $\Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 + \sigma_1 \otimes \tau_1 + \cdots + \sigma_k \otimes \tau_k$ is a P.V., then we have $\dim G = k + 20 \geqslant \dim V \geqslant 17 + 3k$, we have $k = 1$. In this case, we have $\deg(\sigma_1 \otimes \tau_1) \leqslant 4$, and hence $\sigma_1 \otimes \tau_1 = \Lambda_1 \otimes 1$or $1 \otimes \Lambda_1^{(*)}$. If $\sigma_1 \otimes \tau_1 = \Lambda_1 \otimes 1$, it is castling-equivalent to $(GL(1)^3 \times \mathrm{Sp}(2), \Lambda_2 + \Lambda_1 + \Lambda_1)$ which is a non-P.V. by Theorem 1.3. If $\sigma_1 \otimes \tau_1 = 1 \otimes \Lambda_1$, then it is castling-equivalent to $(GL(1)^3 \times \mathrm{Sp}(2) \times SL(2), \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 + 1 \otimes \Lambda_1)$ which is a non-P.V. as we have seen above. If $\sigma_1 \otimes \tau_1 = 1 \otimes \Lambda_1^*$, then it is a P.V., since

$$x=\left(\begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix},\left(\begin{array}{c|c}0 & \begin{matrix}1 & \\ & -1\end{matrix}\\ \hline \begin{matrix}1 & \\ & -1\end{matrix} & 0\end{array}\right),\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$$

is a generic point. Q.E.D.

LEMMA 2.19. *All 2-simple P.V.s which contain* $(GL(1)^2\times \mathrm{Sp}(3)\times SL(m),$ $\Lambda_1\otimes\Lambda_1+\Lambda_3\otimes 1)(2\leqslant m\leqslant 5)$ *as a component, are given as follows*:

$$(GL(1)^2\times \mathrm{Sp}(3)\times SL(5), \Lambda_1\otimes\Lambda_1+\Lambda_3\otimes 1). \tag{2.72}$$

Proof. Since the generic isotropy subalgebra of $(GL(1)\times\mathrm{Sp}(3), \Lambda_3)$ at $e_1\wedge e_2\wedge e_3+e_4\wedge e_5\wedge e_6$ is given by $\{(0)\oplus\left(\begin{smallmatrix}A & 0\\ 0 & -{}^tA\end{smallmatrix}\right); A\in\mathfrak{sl}(3)\}$ (see [1]), $\Lambda_1\otimes\Lambda_1+\Lambda_3\otimes 1$ is P.V.-equivalent to $(SL(3)\times GL(m), (\Lambda_1+\Lambda_1^*)\otimes\Lambda_1)$ which is a P.V. (resp. a non-P.V.) for $m=5$ (resp. $m=2, 3, 4$) by the proof of Sublemma 2.4.4 (and a castling transformation for $m=4$). If $(GL(1)^3\times \mathrm{Sp}(3)\times SL(5), \Lambda_1\otimes\Lambda_1+\Lambda_3\otimes 1+\rho_3\otimes\rho_3')(\rho_3\otimes\rho_3'\neq 1)$ is a P.V., then we have $\dim G=48\geqslant\dim V=44+\deg\rho_3\otimes\rho_3'\geqslant 49$, which is a contradiction. Q.E.D.

LEMMA 2.20. *For* $n>m\geqslant 1$, *a triplet* $(GL(1)^3\times\mathrm{Sp}(n)\times SL(2m),$ $\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+\Lambda_1\otimes 1, M(2n, 2m)\oplus K^{2n}\oplus K^{2n})$ *is a non-P.V.*

Proof. The action is given by $x\to(\alpha AX{}^tB; \beta Ay, \gamma Az)$ for $g=(\alpha,\beta,\gamma; A, B)\in GL(1)^3\times\mathrm{Sp}(n)\times SL(2m)$ and $x=(X; y, z)\in M(2n, 2m)\oplus K^{2n}\oplus K^{2n}$. Then a rational function $f(x)=({}^tyJz)\cdot\mathrm{Pf}({}^tXJX)$. $\mathrm{Pf}({}^tX'JX')^{-1}$ is a nonconstant absolute invariant, where $X'=(X, y, z)\in M(2n, 2m+2)$ and Pf denotes the Pfaffian. Q.E.D.

THEOREM 2.21. *All non-irreducible 2-simple P.V.s which have* $(\mathrm{Sp}(n)\times GL(2m), \Lambda_1\otimes\Lambda_1, V(2n)\otimes V(2m))$ $(n>m\geqslant 1)$ ((57) *in Theorem* 1.5) *as an irreducible component are given as follows*:

$$(GL(1)^2\times\mathrm{Sp}(n)\times SL(2m), \Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1) \tag{2.73}$$

$(GL(1)^{1+t}\times\mathrm{Sp}(n)\times SL(2m), \Lambda_1\otimes\Lambda_1+1\otimes T_t)$ $(t=1, 2, 3)$,
where $T_1=\Lambda_1^{(*)}$, $T_1=2\Lambda_1(m=1)$, $T_1=3\Lambda_1$ $(m=1)$; $T_2=\Lambda_1^{(*)}+\Lambda_1^{(*)}$, $T_2=2\Lambda_1+\Lambda_1$ $(m=1)$; $T_3=\Lambda_1^{(*)}+\Lambda_1^{(*)}+\Lambda_1^{(*)}$. (2.74)

$(GL(1)^{2+t}\times\mathrm{Sp}(n)\times SL(2m), \Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes T_t)$,
where $t=1, 2$; $T_1=\Lambda_1^{(*)}$, $T_1=2\Lambda_1$ $(m=1)$; $T_2=\Lambda_1^{(*)}+\Lambda_1^{(*)}$. (2.75)

Proof. If $\Lambda_1 \otimes \Lambda_1 + \rho_2 \otimes \rho_2'$ ($\rho_2 \neq 1$, $\rho_2' \neq 1$) is a P.V., then we have $\rho_2 \otimes \rho_2' = \Lambda_1 \otimes \Lambda_1^{(*)}$ by (2.16), Theorem 2.13, and Theorem 1.5. By Lemma 2.17, it is a contradiction. Now assume that $\Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V. By Lemmas 2.18–2.20, we have $s = 1$ and $\sigma_1 = \Lambda_1$, i.e., (2.73). Now $\Lambda_1 \otimes \Lambda_1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ is P.V.-equivalent to $(GL(1)^t \times \mathrm{Sp}(m), \tau_1 + \cdots + \tau_t)$, and hence we have (2.74) by Theorem 1.3. By p. 40 in [1], $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ is P.V.-equivalent to $(GL(1)^{2+t} \times SL(2m), \Lambda_2(\Lambda_1 + 1) + \tau_1 + \cdots + \tau_t)$. By a careful check for scalar multiplications, we see that the latter is also P.V.-equivalent to $(GL(1)^{2+t} \times SL(2m), \Lambda_2 + \Lambda_1 + \tau_1 + \cdots + \tau_t)$, and hence we have (2.75). Note that $(GL(1) \times SL(2), \Lambda_1 \otimes \Lambda_2) \simeq (GL(1), \Lambda_1)$, and that the prehomogeneity of (2.73) has been also proved. Q.E.D.

LEMMA 2.22. *For $n > m \geqslant 1$, a triplet $(GL(1)^4 \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes (\Lambda_1 + 1 + 1 + 1), M(2n, 2m+1) \oplus K^{2n} \oplus K^{2n} \oplus K^{2n})$ is a non-P.V.*

Proof. The action is given by $x \to (\alpha A X {}^t B;\ \beta_1 A y_1,\ \beta_2 A y_2,\ \beta_3 A y_3)$ for $g = (\alpha, \beta_1 \beta_2, \beta_3;\ A, B) \in GL(1)^4 \times \mathrm{Sp}(n) \times SL(2m+1)$ and $x = (X; y_1, y_2, y_3) \in M(2n, 2m+1) \oplus K^{2n} \oplus K^{2n} \oplus K^{2n}$. Then the polynomials $f_i(x) = \mathrm{Pf}({}^t X_i J X_i)$ $(i = 1, 2, 3)$ with $X_i = (X, y_i) \in M(2n, 2m+2)$ and $g_{ij}(x) = {}^t y_i J y_j$ $(1 \leqslant i < j \leqslant 3)$ are relative invariants corresponding to the characters $\chi_i(g) = \alpha^{2m+1} \beta_i (i = 1, 2, 3)$ and $\chi_{ij}(g) = \beta_i \beta_j (1 \leqslant i < j \leqslant 3)$, respectively. Now assume that $n \geqslant m + 2$. Then we have $2n \geqslant (2m+1) + 3$ and hence $h(x) = \mathrm{Pf}({}^t X' J X')$ with $X' = (X, y_1, y_2, y_3) \in M(2n, 2m+4)$ is a nonzero relative invariant corresponding to the character $\chi(g) = \alpha^{2m+1} \beta_1 \beta_2 \beta_3$. Hence, $f(x) = f_1 f_2 f_3 g_{12} g_{23} g_{13} h^{-3}(x)$ is a nonconstant absolute invariant. Thus our triplet is a non-P.V. for $n \geqslant m+2$. If $n = m+1$, then we have $2m+1 = 2n-1$, and it is castling-equivalent to $(GL(1)^4 \times \mathrm{Sp}(n), \Lambda_1 + \Lambda_1 + \Lambda_1 + \Lambda_1)$, which is a non-P.V. by Theorem 1.3. Q.E.D.

LEMMA 2.23. *For $n > m \geqslant 1$, a triplet $(GL(1)^4 \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)}, M(2n, 2m+1) \oplus K^{2n} \oplus K^{2n} \oplus {}^{2m+1})$ is a non-P.V.*

Proof. Since $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1$ is castling-equivalent to $(GL(1)^4 \times \mathrm{Sp}(n) \times SL(2n-2m), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)$, it is a non-P.V. by Lemma 2.20. For $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*$, the action is given by $x \to (\alpha A X {}^t B;\ \beta_1 A y_1,\ \beta_2 A y_2,\ \gamma {}^t B^{-1} z)$ for $g = (\alpha, \beta_1, \beta_2, \gamma;\ A, B) \in GL(1)^4 \times \mathrm{Sp}(n) \times SL(2m+1)$ and $x = (X; y_1, y_2, z) \in M(2n, 2m+1) \oplus K^{2n} \oplus K^{2n} \oplus K^{2m+1}$. Then the polynomials $f_i(x) = \mathrm{Pf}({}^t X_i J X_i)$ with $X_i = (X, y_i) \in M(2n, 2m+2)$ $(i = 1, 2)$ and $g_j(x) = {}^t y_j J X z$ $(j = 1, 2)$ are relative invariants corresponding to the characters $\chi_i(g) = \alpha^{2m+1}\beta_i (i = 1, 2)$ and $\chi'_j(g) = \alpha\gamma\beta_j (j = 1, 2)$, respectively, where

$$J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$

Then a rational function $f(x) = (g_1 f_2) \cdot (g_2 f_1)^{-1}(x)$ is a nonconstant absolute invariant, and hence it is a non-P.V. Q.E.D.

THEOREM 2.24. *All non-irreducible 2-simple P.V.s which have* $(\mathrm{Sp}(n) \times GL(2m+1), \Lambda_1 \otimes \Lambda_1)$ $(n > m \geqslant 1)$ *((58) in Theorem 1.5) as an irreducible component are given by (2.70)–(2.72) and the following (2.76)–(2.78):*

$$(GL(1)^{1+s} \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Sigma_s \otimes 1),$$
where $s = 1, 2;\ \Sigma_1 = \Lambda_1,\ \Sigma_2 = \Lambda_1 + \Lambda_1.$ (2.76)

$$(GL(1)^{1+t} \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes T_t),$$
where $t = 1, 2, 3;\ T_1 = \Lambda_1^{(*)}, \Lambda_2,\ T_1 = 2\Lambda_1\ (m = 1);$
$T_2 = \Lambda_1^{(*)} + \Lambda_1^{(*)};\ T_2 = \Lambda_2 + \Lambda_1^*\ (m = 2);$
$T_3 = \Lambda_1^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)};$ *except for* $T_3 \simeq \Lambda_1 + \Lambda_1 + \Lambda_1^*.$ (2.77)

$$(GL(1)^{2+t} \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes T_t),$$
where $t = 1, 2;\ T_1 = \Lambda_1^{(*)};\ T_2 = \Lambda_1 + \Lambda_1,\ \Lambda_1^* + \Lambda_1^*.$ (2.78)

Proof. If $\Lambda_1 \otimes \Lambda_1 + \rho_2 \otimes \rho'_2$ $(\rho_2 \neq 1,\ \rho'_2 \neq 1)$is a P.V., then we have $\rho_2 \otimes \rho'_2 = \Lambda_1 \otimes \Lambda_1^{(*)}$ by Theorem 1.5, which is a contradiction by Lemma 2.17. Now assume that $\Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1$ is a P.V. Then, by Lemmas 2.18–2.20, we have $s = 1, 2;\ \sigma_1 = \Lambda_1,\ \sigma_1 = \Lambda_2\ (n = 2, m = 1),\ \sigma_1 = \Lambda_3\ (n = 3, m = 2)$ and $\sigma_1 + \sigma_2 = \Lambda_1 + \Lambda_1$. We shall show that $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1$ is actually a P.V. If $n = m + 1$, then it is castling-equivalent to a simple P.V. $(GL(1)^3 \times \mathrm{Sp}(n), \Lambda_1 + \Lambda_1 + \Lambda_1)$. If $n \geqslant m + 2$, we can use Proposition 13 in p. 40 in [1], and it is P.V.-equivalent to $(GL(1) \times GL(1) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_2 + \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1)$ which is a P.V. with a generic point

$$\left(\left(\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right), {}^t(0 \cdots 0\ 1), {}^t(1\ 0 \cdots 0\ 1)\right).$$

One can also show the prehomogeneity of $\Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1$ by the calculation of the isotropy subalgebra at

$$\left({}^t\left(\begin{array}{cc|c} I_m & 0 & O_{m,n} \\ \hline O_{m+1,n} & I_{m+1} & 0 \end{array}\right), e_{m+1}, e_1 + e_{m+1} + e_{m+2} + e_{n+m+1}\right).$$

Since $(GL(1)^{1+t}\times \mathrm{Sp}(n)\times SL(2m+1),\ \Lambda_1\otimes\Lambda_1+1\otimes(\tau_1+\cdots+\tau_t))$ is P.V.-equivalent to $(GL(1)^{1+t}\times SL(2m+1),\ \Lambda_2+\tau_1+\cdots+\tau_t))$ by p. 40 in [1], we have (2.77) by Theorem 1.3. Similarly $(GL(1)^{2+t}\times \mathrm{Sp}(n)\times SL(2m+1),\ \Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes(\tau_1+\cdots+\tau_t))$ is P.V.-equivalent to $(GL(1)^{2+t}\times SL(2m+1),\ \Lambda_2(\Lambda_1+1)+\tau_1+\cdots+\tau_t)\simeq(GL(1)^{2+t}\times SL(2m+1),\ \Lambda_2+\Lambda_1+\tau_1+\cdots+\tau_t)$ and hence we have (2.78) by Theorem 1.3. Now assume that $n\geqslant m+2$. Then, by p. 40 in [1], $\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+\Lambda_1\otimes 1+1\otimes(\tau_1+\cdots+\tau_t)$ is a P.V., then $(GL(1)^{3+t}\times SL(2m+1),\ \Lambda_2+\Lambda_1+\Lambda_1+\tau_1+\cdots+\tau_t)$ must be a P.V., and hence $t=1$, $\tau_1=\Lambda_1$. However, in this case, it is a non-P.V. by Lemma 2.23. Finally, assume that $n=m+1$, i.e., $2m+1=2n-1$, and $\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+\Lambda_1\otimes 1+1\otimes\tau$ is a P.V. Then τ must be one of $\Lambda_1^{(*)}$, $\Lambda_2^{(*)}$, $(2\Lambda_1)^{(*)}$, $\Lambda_3^{(*)}$ $(n=4)$. However, we have $\tau\neq\Lambda_1^{(*)}$ by Lemma 2.23 and $\tau\neq\Lambda_2^{(*)}$, $(2\Lambda_1)^{(*)}$, $\Lambda_3^{(*)}(n=4)$ by dimension reason. Q.E.D.

Thus we obtain the following theorem.

THEOREM 2.25. *All non-irreducible 2-simple P.V.s of type I are given by* (2.1)–(2.78).

3. LIST OF 2-SIMPLE P.V.S OF TYPE I

By Theorem 2.25, any 2-simple P.V.s of type I is castling-equivalent (cf. [1]) to a simple P.V. in Theorem 1.3 or to one of the 2-simple P.V.s in the following list. For example, a 2-simple P.V. $(GL(1)^3\times SL(4)\times SL(4),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1)$ is castling-equivalent to (4) in (I) with $\Lambda_1^{(*)}=\Lambda_1$ in the list. Here H denotes the generic isotropy subgroup and $H\sim H_1$ implies that H is locally isomorphic to H_1. The number of the basic relative invariants is denoted by N and $\Lambda_1^{(*)}$ stands for Λ_1 or its dual Λ_1^*.

Notation. $\Lambda=$ the spin representation of $\mathrm{Spin}(2n+1)$.

$\Lambda'=$ a half-spin representation of $\mathrm{Spin}(2n)$.

$\chi=$ the vector representation of $\mathrm{Spin}(n)$, so that $(\mathrm{Spin}(n),\chi)=(SO(n),\Lambda_1)$.

List

(I) *Regular 2-Simple P.V.s of Type I*

(1) $(GL(1)^2\times SL(4)\times SL(2),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes\Lambda_1)$, $H\sim\{1\}$, $N=2$.

(2) $(GL(1)^3 \times SL(4) \times SL(2), \Lambda_2 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1)$, $H \sim GL(1)$, $N = 2$.

(3) $(GL(1)^2 \times SL(4) \times SL(3), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$, $H \sim SO(3)$, $N = 2$.

(4) $(GL(1)^3 \times SL(4) \times SL(3), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)})$, $H \sim SO(2)$, $N = 3$.

(5) $(GL(1)^3 \times SL(4) \times SL(4), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*)$, $H \sim SO(2)$, $N = 3$.

(6) $(GL(1)^3 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^{(*)}) \otimes 1)$, $H \sim \{1\}$, $N = 3$.

(7) $(GL(1)^2 \times SL(5) \times SL(3), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$, $H \sim SO(2)$, $N = 2$.

(8) $(GL(1)^2 \times SL(5) \times SL(8), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$, $H \sim SO(2)$, $N = 2$.

(9) $(GL(1)^2 \times SL(5) \times SL(9), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$, $H \sim GL(1) \times SL(2) \times SL(2)$, $N = 1$.

(10) $(GL(1)^3 \times \mathrm{Sp}(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}))$, $H \sim GL(1) \times \mathrm{Sp}(n-m) \times \mathrm{Sp}(m-1)$, $N = 2$.

(11) $(GL(1)^2 \times \mathrm{Sp}(n) \times SL(2), \Lambda_1 \otimes \Lambda_1 + 1 \otimes 2\Lambda_1)$, $H \sim \mathrm{Sp}(n-1) \times SO(2)$, $N = 2$.

(12) $(GL(1)^2 \times \mathrm{Sp}(n) \times SL(2), \Lambda_1 \otimes \Lambda_1 + 1 \otimes 3\Lambda_1)$, $H \sim \mathrm{Sp}(n-1)$, $N = 2$.

(13) $(GL(1)^3 \times \mathrm{Sp}(n) \times SL(2), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (2\Lambda_1 + \Lambda_1))$, $H \sim \mathrm{Sp}(n-1)$, $N = 3$.

(14) $(GL(1)^2 \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$, $H \sim GL(1) \times \mathrm{Sp}(m) \times \mathrm{Sp}(n-m-1)$, $N = 1$.

(15) $(GL(1)^4 \times \mathrm{Sp}(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)})$, $H \sim \mathrm{Sp}(m-1) \times \mathrm{Sp}(n-m-1)$, $N = 4$.

(16) $(GL(1)^3 \times \mathrm{Sp}(2) \times SL(3), \Lambda_1 \otimes \Lambda_1 + \Lambda_2 \otimes 1 + 1 \otimes \Lambda_1^*)$, $H \sim GL(1)$, $N = 2$.

(17) $(GL(1)^2 \times \mathrm{Sp}(2) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$, $H \sim SO(2)$, $N = 2$.

(18) $(GL(1)^3 \times \mathrm{Sp}(2) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)$, $H \sim \{1\}$, $N = 3$.

(19) $(GL(1)^3 \times \mathrm{Sp}(2) \times SL(4), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*)$, $H \sim \{1\}$, $N = 3$.

(20) $(GL(1)^2 \times SO(n) \times SL(m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$, $H \sim SO(m-1) \times SO(n-m)$, $N = 2$.

(21) $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(2), \Lambda \otimes \Lambda_1 + 1 \otimes \Lambda_1)$, $H \sim SL(3)$, $N = 2$.

(22) $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(3), \Lambda \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)})$, $H \sim SL(2) \times SO(2)$, $N = 2$.

(23) $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(6),\quad \Lambda \otimes \Lambda_1 + 1 \otimes \Lambda_1^*),\quad H \sim SL(2) \times SO(2)$, $N = 2$.

(24) $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7), \Lambda \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$, $H \sim SL(3)$, $N = 2$.

(25) $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(2), \chi \otimes \Lambda_1 + \Lambda \otimes 1)$, $H \sim GL(2)$, $N = 2$.

(26) $(GL(1)^3 \times \mathrm{Spin}(7) \times SL(2), \chi \otimes \Lambda_1 + \Lambda \otimes 1 + 1 \otimes \Lambda_1)$, $H \sim SL(2)$, $N = 3$.

(27) $(GL(1)^3 \times \mathrm{Spin}(7) \times SL(6), \chi \otimes \Lambda_1 + \Lambda \otimes 1 + 1 \otimes \Lambda_1^*)$, $H \sim SL(2)$, $N = 3$.

(28) $(GL(1)^2 \times \mathrm{Spin}(8) \times SL(2), \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$, $H \sim SL(3) \times SO(2)$, $N = 2$.

(29) $(GL(1)^2 \times \mathrm{Spin}(8) \times SL(3), \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$, $H \sim SL(2) \times SO(3)$, $N = 2$.

(30) $(GL(1)^3 \times \mathrm{Spin}(8) \times SL(2), \chi \otimes \Lambda_1 + \Lambda' \otimes 1 + 1 \otimes \Lambda_1)$, $H \sim SL(3)$, $N = 3$.

(31) $(GL(1)^3 \times \mathrm{Spin}(8) \times SL(3),\quad \chi \otimes \Lambda_1 + \Lambda' \otimes 1 + 1 \otimes \Lambda_1^{(*)}),\quad H \sim SL(2) \times SO(2)$, $N = 3$.

(32) $(GL(1)^3 \times \mathrm{Spin}(8) \times SL(6), \chi \otimes \Lambda_1 + \Lambda' \otimes 1 + 1 \otimes \Lambda_1^*)$, $H \sim SL(2) \times SO(2)$, $N = 3$.

(33) $(GL(1)^3 \times \mathrm{Spin}(8) \times SL(7), \chi \otimes \Lambda_1 + \Lambda' \otimes 1 + 1 \otimes \Lambda_1^*)$, $H \sim SL(3)$, $N = 3$.

(34) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(2),\quad \Lambda' \otimes \Lambda_1 + 1 \otimes 2\Lambda_1),\quad H \sim (G_2) \times SO(2)$, $N = 2$.

(35) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(2),\quad \Lambda' \otimes \Lambda_1 + 1 \otimes 3\Lambda_1),\quad H \sim (G_2)$, $N = 2$.

(36) $(GL(1)^3 \times \mathrm{Spin}(10) \times SL(2), \Lambda' \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$, $H \sim GL(1) \times (G_2)$, $N = 2$.

(37) $(GL(1)^3 \times \mathrm{Spin}(10) \times SL(2), \Lambda' \otimes \Lambda_1 + 1 \otimes (2\Lambda_1 + \Lambda_1))$, $H \sim (G_2)$, $N = 3$.

(38) $(GL(1)^4 \times \mathrm{Spin}(10) \times SL(2), \Lambda' \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$, $H \sim (G_2)$, $N = 4$.

(39) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(3),\quad \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)}),\quad H \sim SL(2) \times SO(2)$, $N = 2$.

(40) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(14),\quad \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*),\quad H \sim SL(2) \times SO(2)$, $N = 2$.

(41) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15),\quad \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*),\quad H \sim GL(1) \times SL(4)$, $N = 1$.

(42) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(2), \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$, $H \sim (G_2)$, $N = 2$.

(43) $(GL(1)^2 \times \text{Spin}(10) \times SL(3), \ \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$, $H \sim SL(3) \times SO(2)$, $N=2$.

(44) $(GL(1)^2 \times \text{Spin}(10) \times SL(4), \ \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$, $H \sim SL(2) \times SL(2)$, $N=2$.

(45) $(GL(1)^2 \times (G_2) \times SL(2), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$, $H \sim SL(2)$, $N=2$.

(46) $(GL(1)^2 \times (G_2) \times SL(6), \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$, $H \sim SL(2)$, $N=2$.

(II) *Nonregular 2-Simple P.V.s of Type I*

(1) $(GL(1)^2 \times SL(2m+1) \times SL(2), \Lambda_2 \otimes \Lambda_1 + 1 \otimes t\Lambda_1)$ $(t=1, 2, 3)$.

(2) $(GL(1)^3 \times SL(2m+1) \times SL(2), \ \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + t\Lambda_1))$ $(t = 1, 2)$.

(3) $(GL(1)^4 \times SL(2m+1) \times SL(2), \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$.

(4) $(GL(1)^2 \times SL(4) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$.

(5) $(GL(1)^3 \times SL(4) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)$.

(6) $(GL(1)^3 \times SL(4) \times SL(5), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*)$.

(7) $(GL(1)^2 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1)$.

(8) $(GL(1)^3 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1)$.

(9) $(GL(1)^3 \times SL(5) \times SL(9), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1^*)$.

(10) $(GL(1)^3 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes 2\Lambda_1)$.

(11) $(GL(1)^4 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1))$.

(12) $(GL(1)^2 \times SL(6) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1)$.

(13) $(GL(1)^2 \times SL(7) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1)$.

(14) $(GL(1)^3 \times SL(7) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes \Lambda_1)$.

(15) $(GL(1)^2 \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1)$.

(16) $(GL(1)^2 \times \text{Sp}(n) \times SL(2m), \ \Lambda_1 \otimes \Lambda_1 + T)$ with $T = \Lambda_1 \otimes 1, \ 1 \otimes \Lambda_1, \ 1 \otimes \Lambda_1^*$.

(17) $(GL(1)^3 \times \text{Sp}(n) \times SL(2m), \ \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)})$.

(18) $(GL(1)^4 \times \text{Sp}(n) \times SL(2m), \ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}) + T)$ with $T = \Lambda_1 \otimes 1, \ 1 \otimes \Lambda_1, \ 1 \otimes \Lambda_1^*$.

(19) $(GL(1)^3 \times \text{Sp}(n) \times SL(2), \ \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes 2\Lambda_1)$.

(20) $(GL(1)^2 \times \text{Sp}(n) \times SL(2m+1), \ \Lambda_1 \otimes \Lambda_1 + 1 \otimes T)$ with $T = \Lambda_1$, Λ_1^*, Λ_2.

(21) $(GL(1)^3 \times \text{Sp}(n) \times SL(2m+1), \ \Lambda_1 \otimes \Lambda_1 + S + T)$ with $S, T = \Lambda_1 \otimes 1, \ 1 \otimes \Lambda_1, \ 1 \otimes \Lambda_1^*$.

(22) $(GL(1)^4 \times \text{Sp}(n) \times SL(2m+1), \ \Lambda_1 \otimes \Lambda_1 + T)$ with $T = 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)}, \ 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1^*)$.

(23) $(GL(1)^2 \times \mathrm{Sp}(n) \times SL(3),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes 2\Lambda_1)$.

(24) $(GL(1)^3 \times \mathrm{Sp}(n) \times SL(5),\ \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1^*))$.

(25) $(GL(1)^2 \times \mathrm{Sp}(n) \times SL(2),\ \Lambda_1 \otimes 2\Lambda_1 + 1 \otimes \Lambda_1)$.

(26) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(2),\ \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1)$.

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