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On the monodromies of a polynomial map from C^2 to C

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Abstract

Let $f: C^2 \to C$ be a polynomial function. It is well known that there exists a finite set $A \subset C$ such that the restriction of f to $C^2 - f^{-1}(A)$ is a differentiable fibration onto $C - A$. Following Broughton in (Proc. Symp. Pure Math. 40 (1983) 167.) we call the smallest of such A's the set of atypical values of f and write it A_f. Let F be a generic fiber of f. The main goal of this article is to describe the monodromy on $H_1(F, Z)$ around an atypical value $a \in A_f$. For that purpose we define and study a monodromic filtration on the homology with coefficients in $Z: 0 \subset M_{-1} \subset M_0 \subset M_1 \subset M_2 = H_1(F, Z)$. The term M_{-1} is added to allow for the boundary of F. We introduce a compact model $\hat{L}_a$ for the smooth part of the reduced curve associated to the affine fiber $f^{-1}(a)$. One important result of this article is theorem (8.12) which shows how $H_1(\hat{L}_a, Z)$ gives (via the transfer homomorphism) a precise description of the invariant cycles in $H_1(F, Z)$. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Let $f: C^2 \to C$ be a polynomial function. It is well known that there exists a finite set $A \subset C$ such that the restriction of f to $C^2 - f^{-1}(A)$ is a differentiable fibration onto $C - A$. Following Broughton in [4] we call the smallest of such A's the set of *atypical values* of f and write it A_f. The set A_f contains the set C_f of critical values of f. As f is not a proper map, in general A_f is not equal to C_f. Indeed one has $A_f = C_f \cup I_f$ where I_f is the set of irregular values of f at infinity (see Section 7 for more details).

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Our aim is to describe the monodromy around any atypical value of f. To achieve this goal we prepare two tools.

(1) A twist formula. Its proof relies on Durfee's description of the semi-stable reduction given in [11]. This formula generalizes the local one presented in [12,9].

(2) A homomorphism from $H_1(\hat{L}_a, Z)$ to $H_1(F, Z)$ where F is a generic fiber of f, $a \in A_f$ and $\hat{L}_a$ is a compact model for the smooth part of the reduced curve associated to the affine fiber $f^{-1}(a)$. This homomorphism follows from a transfer homomorphism.

We now state the main results of this article. In Section 8, we produce a morphism $\bar{f}$ from a surface (which is a blow up of $P^2(C)$) to $P^1(C)$. This morphism is a certain compactification of f, and it has the property that $\bar{f}^{-1}(a)$ is a divisor with normal crossings. From the divisor $\bar{f}^{-1}(a)$, one can easily obtain a positive integer m such that the homomorphism $(t^m - 1)^2 : H_1(F, Z) \to H_1(F, Z)$ is equal to zero, where t stands for the homomorphism induced by the monodromy.

We define a monodromic filtration $0 \subset M_{-1} \subset M_0 \subset M_1 \subset M_2 = H_1(F, Z)$ by specifying that $M_{-1} = \{x \mid \text{ such that } I(x, y) = 0 \text{ for all } y \in M_2\}$ where $I(,)$ stands for the intersection pairing on $H_1(F, Z)$, $M_0 = M_{-1} \oplus \mathrm{Im}(t^m - 1)$ and $M_1 = \mathrm{Ker}(t^m - 1)$.

The term M_{-1} which is added to allow for the boundary of F, is computed in Theorem 8.10. The terms M_0 and M_1 are determined in Theorem 8.15, using essentially the topological tools described above.

The monodromies can be computed from a combinatorial data which is summarized in (3.16) and (8.26).

We next apply these results to the determination of the invariant cycles, thus answering a question which was asked to us by F. Pham. Classically, one looks in the proper case at invariant cocyles (see [15]). When f is proper there is a neighbourhood V_a of the special fiber F_a which deformation retracts onto F_a. The invariant cocyle theorem then states that the image of the homomorphism $H^1(F_a, C) \overset{\approx}{\to} H^1(V_a, C) \overset{(\mathrm{incl})^*}{\longrightarrow} H^1(F, C)$ is equal to $\mathrm{Ker}(t-1)$. In the case we consider, f is not proper. When $a \in I_f$, there is nothing comparable to V_a and to the deformation retraction. So the classical point of view does not work to characterize the invariant cocyles in terms of the cohomology of the special fiber $f^{-1}(a)$. That being so, we replace cohomology by homology and use the transfer homomorphism. It is in Section 8, that we introduce a compact model $\hat{L}_a$ for the smooth part of the reduced curve associated to the affine fiber $f^{-1}(a)$. One important result of this article is Theorem 8.12 which shows how $H_1(\hat{L}_a, Z)$ gives (via the transfer homomorphism) a precise description of the invariant cycles in $H_1(F, Z)$. In fact, our point of view gives also a homological version of "the" invariant cycle theorem in the proper case (see Theorem 5.4).

From our description of the geometric monodromy around $a \in A_f$ and from the determination of the invariant cycles obtained from $\hat{L}_a$ we can answer a question of A. Dimca which deals with the characterization of the atypical fibers in terms of the homology monodromy. The answer is the following (for a different approach see [3]).

Theorem 8.18. *Let $c \in C$ and suppose that the fiber $f^{-1}(c)$ is irreducible and that the homology monodromy $H_1(F, Z) \to H_1(F, Z)$ around c is the identity. Then c is a generic value of f i.e. $c \notin A_f$.*

Remark. It is known that after a suitable base change a reducible fiber is always atypical. Hence the preceding theorem gives a new characterization of the atypical values.

The article is organized as follows.

In Section 2 we introduce notations and concepts which will be used in the sequel. The weight filtration is studied in Section 3. The twist formula is proved in Section 4. In Section 5 we briefly recall facts about the transfer homomorphism and use it. In Section 6 we study the monodromy of a germ of holomorphic function defined in a neighbourhood of a normal surface singularity. Now the fiber has a boundary and this requires a modification of the tools used in Sections 2–5 to take account of the boundary components. This kind of modification is useful in the affine case, to deal properly with the contribution to the invariant cycles of the homology coming from infinity.

In Section 7 we make explicit a smooth compactification $\Phi: X \to P^1(C)$ for the given $f: C^2 \to C$. We recall its main properties which have been presented with more details in [18]. An explicit description of the set of atypical values is then quickly obtained.

In Section 8 we specifically deal with the monodromies around an atypical value $a \in A_f$ and also with the monodromy at infinity. By definition the latter is the monodromy associated to a loop which goes once around a small circle centered at $\infty \in P^1(C)$.

In Section 9 we work out an example of monodromy which sheds light on Theorem 8.12. The example is based on specific polynomials of degree 6 due to Artal [2].

2. The monodromy of a proper morphism

Let X be a complex analytic surface, normal and irreducible. Let U be an open disc in C, centered at the origin. Let $f: X \to U$ be a proper and surjective morphism.

Let $U^* = U - \{0\}$ and $X^* = X - f^{-1}(0)$. If U is small enough, we may suppose that the singularities of X are in $f^{-1}(0)$ and that $0 \in C$ is the only critical value of f. Under these conditions, the restriction of f to X^* is a differentiable fibration onto U^*.

Choose $z \in U^*$ and let $F = f^{-1}(z)$. We write S^1_η for the circle of radius η centered at the origin of C with $\eta = |z|$.

Definition 2.1. A monodromy for f is a diffeomorphism $h: F \to F$ such that the restriction of f to $f^{-1}(S^1_\eta)$ is a fibration isomorphic to the obvious fibration of the mapping torus $Fx[0,1]/(x,1) \approx (h(x),0)$ onto S^1.

The monodromy h is defined up to isotopy. We shall construct an explicit monodromy for f and describe its induced action on $H_1(F, Z)$ by following the method introduced by Clemens in [5]. The same path has already been followed by Campo in [1] and by Du Bois-Michel in [8,9] in the local case.

Let $p': X' \to X$ be a resolution of the singularities of X and let $\bar{p}: \bar{X} \to X'$ be a sequence of blow-ups (of points). Let us write $\bar{f}$ for $f \circ p' \circ \bar{p}$.

Definition 2.2. One says that the morphism $p' \circ \bar{p}$ is a very good resolution if $\bar{f}^{-1}(0)$ fulfills the following properties:

(1) The divisor $\bar{f}^{-1}(0)$ has normal crossings.

Let us write D for the reduced divisor associated to $\bar{f}^{-1}(0)$ and let $D_1, \ldots, D_i, \ldots, D_k$ be the irreducible components of D.

(2) Each D_i is a smooth curve.

(3) If $i \neq j$ then $D_i \cap D_j$ either is empty or contains a unique point which will be written P_{ij}.

We assume $i < j$ to avoid redundancy.

It is well know that very good resolutions exist. From now on, we assume that $p' \circ \bar{p}$ is a very good resolution.

Notation 2.3. (i) D_i^0 stands for the subset of smooth points of D in D_i. One has the equality $D_i^0 = D_i - \bigcup_{j \neq i} D_j \cap D_i$.

(ii) For every $P_{ij} = D_i \cap D_j$ we choose a small open disc $U_{i(j)}$ in D_i containing P_{ij} and a disc $U_{j(i)}$ in D_j also containing P_{ij}. Let $\hat{D}_i = D_i - \bigcup_{j \neq i} U_{i(j)}$. The difference set $D_i^0 - \hat{D}_i$ is a disjoint union of small open annuli.

Definition 2.4. A *curvette* Γ_i of D_i is a germ of smooth complex curve, transversal to D_i at a point P_i of D_i^0. The multiplicity m_i of D_i in $\bar{f}^{-1}(0)$ is the order at P_i of the restriction $\bar{f} | \Gamma_i \to C$. One has the equality of divisors $\bar{f}^{-1}(0) = \sum_{i=1}^{k} m_i D_i$.

Let D_η^2 be the closed disc in C, centered at the origin, of radius η. We write $V = \bar{f}^{-1}(D_\eta^2)$ and $\Sigma = \bar{f}^{-1}(S_\eta^1)$. We write f_Σ for $\bar{f} | \Sigma \to S_\eta^1$.

Remark. The differentiable fibration $f_\Sigma : \Sigma \to S_\eta^1$ is isomorphic by $p' \circ \bar{p}$ to the fibration $f | f^{-1}(S_\eta^1) \to S_\eta^1$. Hence we shall also write F for the fiber $f_\Sigma^{-1}(z)$.

From the isomorphism theorem of Durfee [11] and the formulae of du Bois-Michael in [8] we get the next two claims.

Claim 2.5. *For every $\eta > 0$ there exists a radius $\rho(\eta)$ with $0 < \rho(\eta) \ll \eta$ such that, for every choice of discs $U_{i(j)}$ of radii smaller than $\rho(\eta)$, there exists a deformation retraction $R : V \to D$ with the following properties*:

1. *The restriction of R to $R^{-1}(\hat{D}_i)$ is the analytic projection of a tubular neighborhood of $\hat{D}_i$ in V.*
2. *For every $P \in D_i^0$ (i.e. not only for every $P \in \hat{D}_i$), the inverse image $R^{-1}(P)$ is a differentiable disc transversal to D_i.*
3. *For every $P_{ij} = D_i \cap D_j$ the intersection $\Sigma \cap R^{-1}(P_{ij})$ is a torus, called the plumbing torus.*

Notation 2.6. $F_i = F \cap R^{-1}(D_i)$, $F_i^0 = F \cap R^{-1}(D_i^0)$, $\hat{F}_i = F \cap R^{-1}(\hat{D}_i)$.

Claim 2.7. *One can construct a monodromy h for f_Σ with the following properties*:

1. *The restriction R_i of R to $\hat{F}_i$ is a cyclic covering of $\hat{D}_i$ of order m_i. Note that $\hat{F}_i$ is not necessarily connected.*
2. *The subspaces F_i, F_i^0, $\hat{F}_i$ are invariant by h.*
3. *The restriction $\hat{h}_i$ of h to $\hat{F}_i$ is a generator of the Galois group of R_i (it will not be necessary here to tell which generator it is). As a consequence, $\hat{h}_i$ is a diffeomorphism of finite order m_i of $\hat{F}_i$.*

4. *If* $P_{ij} = D_i \cap D_j$, we write m_{ij} *for the g.c.d. of* m_i *and* m_j (*as before we suppose that* $i < j$). *Let* A_{ij} *be equal to* $R^{-1}(U_{i(j)} \cup U_{j(i)}) \cap F$. *Then* $F_i \cap F_j$ *is the disjoint union of* m_{ij} *boundary components* C_{ij}^{λ} *of* F_i *and* F_j ($\lambda = 1, 2, \dots, m_{ij}$). *Also* A_{ij} *is the disjoint union of* m_{ij} *open annuli* A_{ij}^{λ}, *each* A_{ij}^{λ} *being a tubular neighbourhood of* C_{ij}^{λ} *in* F.
5. *The* m_{ij} *components of* A_{ij} *are permuted cyclically and transitively by* h. *As a consequence,* $h^{m_{ij}}$ *leaves each* A_{ij}^{λ} *invariant.*

Remark 2.8. The number c_i of connected components of F_i divides m_i and all the multiplicities m_j such that $D_i \cap D_j \neq \emptyset$. Hence c_i divides m_{ij}.

3. The weight filtration on the integral homology

Let $C = \bigcup_{i \neq j} F_i \cap F_j$ and let $\gamma : H_1(C, Z) \to H_1(F, Z)$ be the homomorphism induced by the inclusion of C in F. We consider the Mayer–Vietoris sequence associated to the decomposition of F as $F = \bigcup_{i=1}^{k} F_i$ and to the numbering of the components:

$$H_1(C,Z) \xrightarrow{\gamma} \bigoplus_i H_1(F_i,Z) \xrightarrow{\beta} H_1(F,Z) \xrightarrow{\delta} H_0(C,Z)$$

$$\xrightarrow{\gamma_0} \bigoplus_i H_0(F_i,Z) \xrightarrow{\beta_0} H_0(F,Z) \longrightarrow 0$$

Recall that the homomorphisms β and β_0 are the direct sum of the homomorphisms induced by the inclusions $F_i \subset F$.

Definition 3.1. As in [8] we define a filtration on $H_1(F, Z)$ by specifying that:

$$W_0 = \gamma(H_1(C,Z)), \quad W_1 = \beta(\oplus H_1(F_i,Z)), \quad W_2 = H_1(F,Z).$$

One has $0 \subset W_0 \subset W_1 \subset W_2 = H_1(F, Z)$.

Comments. (1) By the way it is constructed, the monodromy h leaves each F_i, F_i^0, $\hat{F}_i$ invariant, as well as $C = \bigcup C_{ij}^{\lambda}$. We write t for the action induced by the monodromy on homology. Hence, $H_1(F, Z)$, the filtration, the graded group associated to it and the Mayer–Vietoris sequence are all equipped with a $Z[t, t^{-1}]$-module structure.

Let I be the intersection form on $H_1(F, Z)$, F being oriented by its complex structure. Let $I^* : H_1(F, Z) \to \mathrm{Hom}_Z(H_1(F, Z), Z)$ be the homomorphism adjoint to I. Let also $u : H^1(F, C) \to \mathrm{Hom}_C(H_1(F, C), C)$ be the homomorphism induced by the evaluation.

(2) One can show, as in [8] that $u^{-1} \circ I^*(W_i \otimes_Z C)$ is the weight filtration of the mixed hodge structure on $H^1(F, C)$. However, this fact will not be needed in this article.

(3) Using the theory of fibrations of Waldhausen manifolds on the circle or the Nielsen–Thurston theory of diffeomorphisms of surfaces, one can show that the filtration on the integral homology of F by the W_i depends only on the homotopy class of the fibration $f_{\Sigma} : \Sigma \to S^1$.

Definitions 3.2. (1) As usual, we associate to D the configuration graph $G(D)$ in the following way. To each irreducible component D_i of D there corresponds a vertex σ_i; and there is an edge between σ_i and σ_j if and only if $D_i \cap D_j$ is nonempty.

(2) By a process which goes back to Nielsen, we associate to the monodromy h constructed in claim (2.7) a graph $G(h)$ as follows. To each connected component F_i^u of F_i we associate a vertex s_i^u. If F_i^u meets F_j^v we associate an edge between s_i^u and s_j^v to each connected component of $F_i^u \cap F_j^v$. From the construction of the graphs, one easily gets the following proposition.

Proposition 3.3. (1) *There exists continuous and surjective maps* $\pi_D : D \to G(D)$ *and* $\pi_F : F \to G(h)$ *such that* $\pi_D^{-1}(\sigma_i) = \hat{D}_i$ *and* $\pi_F^{-1}(s_i^u) = \hat{F}_i^u$ *where* $\hat{F}_i^u = F_i^u \cap \hat{F}_i$.

(2) *The monodromy* h *induces a graph automorphism* $h_G : G(h) \to G(h)$ *such that* $\pi_F \circ h = h_G \circ \pi_F$.

(3) *Let* $\bar{R}$ *be the restriction* $R \mid F \to D$. *Then there exists a unique morphism of graphs* $\rho : G(h) \to G(D)$ *such that the following diagram commutes*:

$$\begin{array}{ccc} F & \xrightarrow{\pi_F} & G(h) \\ \downarrow{\scriptstyle \bar{R}} & & \downarrow{\scriptstyle \rho} \\ D & \xrightarrow{\pi_D} & G(D) \end{array}$$

(4) *The map* ρ *can be identified with the projection of* $G(h)$ *onto its quotient by the action of* h_G.

Corollary 3.4. *The action of* h_G *equips* $H_1(G(h), Z)$ *with a* $Z[t, t^{-1}]$-*module structure. The homomorphism* $(\pi_F)_1 : H_1(F, Z) \to H_1(G(h), Z)$ *is a* $Z[t, t^{-1}]$-*module homomorphism.*

Proposition 3.5. *The homomorphism* $(\pi_F)_1$ *vanishes on* W_1. *The quotient homomorphism* $\bar{\pi} : W_2/W_1 \to H_1(G(h), Z)$ *is a* $Z[t, t^{-1}]$-*module isomorphism.*

Proof. The way π_F has been constructed shows that, for each 1-cycle a in $G(h)$ one can find (noncanonically) a 1-cycle a' in F such that $\pi_F(a') = a$. Thus $\bar{\pi}$ is onto.

Let $\{a_v\}$ be the edges, arbitrarily oriented, of $G(h)$. They form a basis of the 1-chains of $G(h)$. As $G(h)$ is a graph, each homology class in $H_1(G(h), Z)$ can be written in a unique way as $\sum_v n_v a_v$. The construction of π_F provides a bijection between the edges $\{a_v\}$ and the connected components $\{C_v\}$ of C. We orient C_v in such a way that, if x is an element of $H_1(G(h), Z)$ such that $\pi_F(x) = \sum_v n_v a_v$ then $I(x, C_v) = n_v$. We write (C_v) for the class of C_v in $H_0(C, Z)$. Now, the homomorphism $\delta : H_1(F, Z) \to H_0(C, Z)$ is defined by the formula $\delta(x) = \sum_v I(x, C_v)(C_v)$. Hence $\operatorname{Ker}(\pi_F)_1 = \operatorname{Ker} \delta = W_1$ and the proof is completed. □

Corollary 3.6. *Let* $x \in H_1(F, Z)$. *Then* $x \in W_1$ *if and only if* $I(x, y) = 0$ *for all* $y \in W_0$.

Let $r_i : \mathrm{Hom}_Z(W_2, Z) \to \mathrm{Hom}_Z(W_i, Z)$ *be the restriction homomorphism* (*for* $i = 0, 1$) *and let* $I_i = r_i \circ I^*$.

Corollary 3.7. (1) *The kernel of* I_0 *is equal to* W_1 *and the quotient homomorphism* $\bar{I}_0 : W_2/W_1 \to \mathrm{Hom}_Z(W_0, Z)$ *is a* $Z[t, t^{-1}]$-*module isomorphism.*

(2) *The kernel of* I_1 *is equal to* W_0 *and the quotient homomorphism* $\bar{I}_1 : W_2/W_0 \to \mathrm{Hom}_Z(W_1, Z)$ *is a* $Z[t, t^{-1}]$-*module isomorphism.*

Proof. As F is without boundary, I^* is onto and by (3.6) $\operatorname{Ker} I_0 = W_1$, $\operatorname{Ker} I_1 = W_0$. □

Remark 3.8. Let m be the lcm of the multiplicities m_i defined in (2.4). The definition of W_1 implies that $(t^m - 1)W_1 = 0$ and that $W_0 \subset W_1$. Hence Corollary 3.7 shows that $(t^m - 1)(W_2/W_1) \cong (t^m - 1)W_0 = 0$. In particular one has $(t^m - 1)^2(W_2) = 0$.

Corollary 3.9. *From the Mayer–Vietoris sequence, one gets the following exact sequence of* $Z[t,t^{-1}]$*-modules and homomorphisms*:

$$0 \to H_1(G(h),Z) \xrightarrow{\delta\circ\bar{\pi}^{-1}} H_0(C,Z) \xrightarrow{\gamma_0} \bigoplus_i H_0(F_i,Z) \xrightarrow{\beta_0} H_0(F,Z) \to 0.$$

Remark 3.10. Let c (resp. c_i) be the number of connected components of F (resp. F_i) and let $V(i) = \{j \mid \text{s.t. } j \neq i \text{ and } D_j \cap D_i \neq \emptyset\}$ and $m_{ij} = \gcd(m_i, m_j)$. Corollary 3.9 implies that the characteristic polynomial of the action induced by the monodromy on $H_1(G(h),Z)$ is equal to $(\prod_{i<j}(t^{m_{ij}} - 1))(\prod_i (t^{c_i} - 1))^{-1}(t^c - 1)$. By 3.7 one has the same characteristic polynomial for W_0 and W_2/W_1.

To study the module W_1/W_0 a short way is to glue a disc on each boundary component of $\hat{F}_i$ to obtain a surface $\bar{F}_i$ without boundary. The monodromy $\hat{h}_i$ extends to a diffeomorphism $\bar{h}_i : \bar{F}_i \to \bar{F}_i$ of finite order. The projection onto the quotient is a ramified covering $\bar{R}_i : \bar{F}_i \to D_i$ which extends the cyclic covering R_i. The action of $\bar{h}_i$ equips $H_1(\bar{F}_i, Z)$ with a $Z[t,t^{-1}]$-module structure. From the definition of W_0 and W_1 one gets the next proposition.

Proposition 3.11. *The module* W_1/W_0 *is isomorphic to* $\oplus_i H_1(\bar{F}_i, Z)$.

It is not difficult to describe the $Z[t,t^{-1}]$-module $H_1(\bar{F}_i, Z)$ in terms of numerical invariants which come from D_i and neighboring components (the "halo" of D_i). Here is how. Let g_i be the genus of D_i and let $v(i) = \mathrm{Card}(V(i))$.

Definition 3.12. The halo θ_i of D_i is the disjoint union of the ordered triple $(m_i, v_i + 2g_i - 2, c_i)$ with the set $\{m_{ij}\}_{j\in V(i)}$.

The following proposition is a direct generalization of Proposition 1.7 of [9].

Proposition 3.13. *The characteristic polynomial of the action induced by* $\bar{h}_i$ *on* $H_1(\bar{F}_i, Z)$ *is equal to* $(t^{m_i} - 1)^{v_i + 2g_i - 2}(\prod_{j\in V(i)}(t^{m_{ij}} - 1)^{-1})(t^{c_i} - 1)^2$.

We now interpret geometrically the graded module W_2/W_0 which is isomorphic to $\mathrm{Hom}_Z(W_1, Z)$. To do that we use the chorizo space $\mathrm{Ch}(h)$ introduced by [19]. By definition, $\mathrm{Ch}(h)$ is the quotient of F obtained by identifying each curve C_{ij}^{λ} to a point. Note that $\mathrm{Ch}(h)$ can be identified with D''' which will be defined in Section 4. It is clear that h induces a homeomorphism $h_{\mathrm{Ch}} : \mathrm{Ch}(h) \to \mathrm{Ch}(h)$ which is of finite order m. It equips $H_1(\mathrm{Ch}(h), Z)$ with a $Z[t,t^{-1}]$-module structure.

One has projections $\pi' : F \to \mathrm{Ch}(h)$ and $\pi'' : \mathrm{Ch}(h) \to G(h)$ such that $\pi_F = \pi'' \circ \pi'$.

Proposition 3.14. *One has*

(1) $\operatorname{Ker}(\pi_1' : H_1(F,Z) \to H_1(\operatorname{Ch}(h),Z)) = W_0$ *and hence* W_2/W_0 *is isomorphic to* $H_1(\operatorname{Ch}(h),Z)$.
(2) $\operatorname{Ker}(\pi_1'' : H_1(\operatorname{Ch}(h),Z) \to H_1(G(h),Z)) = \pi_1'(W_1/W_0)$.

The proof of (1) is immediate and (2) is a direct consequence of Proposition 3.5. From Corollary 3.7 we obtain the following corollary.

Corollary 3.15. $H_1(\operatorname{Ch}(h),Z)$ *is isomorphic to* $\operatorname{Hom}_Z(W_1,Z)$.

Conclusion 3.16. The configuration graph $G(D)$ (see (3.2)) together with the halo (see Definition 3.12) associated to each vertex determine the geometric and the homological monodromies. The graph $G(h)$ and the action induced by h on $G(h)$ are easily obtained from this data. The graph $G(D)$ and the halos can be computed from the special divisor D except for the integers c_i. However, in some special cases, the c_i can be determined. If the component D_i is rational, then c_i is equal to the gcd of m_i and of the m_j for $j \in V(i)$. If the component D_i is reduced (i.e. $m_i = 1$) one has $c_i = 1$.

4. The twist formula

Recall that $\sum_{i=1}^k m_i D_i$ is the divisor associated to $\bar{f}^{-1}(0)$ and that m is the lcm of the m_i's. From Claim 2.7 we deduce that h^m is the identity outside the annuli A_{ij}^λ which are tabular neighbourhoods of the curves $C_{ij}^\lambda \subset C$. Therefore there exist rational numbers r_{ij}^λ such that, for all $y \in H_1(F,Z)$ the following equality holds (where $[C_{ij}^\lambda]$ stands for the homology class):

$$t^m(y) = y + m \sum_{i<j} \sum_{\lambda} r_{ij}^\lambda I(y, C_{ij}^\lambda)[C_{ij}^\lambda].$$

Remark 4.1. The sign of the rational numbers r_{ij}^λ depends on the orientation of F (via the intersection form I) but not on the choice of an orientation of the $C_{ij}^\lambda \subset C$. The fiber F is equipped with the orientation induced by its complex structure.

Theorem 4.2 (Twist formula). *One has the equality*: $r_{ij}^\lambda = -1/\operatorname{lcm}(m_i, m_j)$.

Corollary 4.3. *All the twist numbers* mr_{ij}^λ *associated to* h^m *are negative integers.*

Comments. (1) The sign of the twists depends on the direction of the monodromy. It is defined as follows. The loop $\gamma(s) = z\exp(2i\pi s)$ for $0 \leqslant s \leqslant 1$ goes once along the circle S_η^1 in the direction which agrees with the orientation induced by the complex structure of D_η^2 on its boundary S_η^1. One chooses a vector field Θ tangent to $\Sigma = \bar{f}^{-1}(S_\eta^1)$ which projects down by $\mathrm{d}f_\Sigma$ onto $\mathrm{d}\gamma/\mathrm{d}s$. Let $\Phi : \Sigma \times \boldsymbol{R} \to \boldsymbol{R}$ be the flow associated to Θ. A monodromy h is the first return map on F associated to the flow; in other words $h(x) = \Phi(x,1)$ for all $x \in F$. To avoid controversy, it is important to note that such a monodromy is the inverse of monodromies used by topologists in knot theory.

(2) Another way to get the sign of the twists proceeds as follows. One can prove a priori that the ε of Waldhausen (see [21, p. 322] or [23, p. 109]) is equal to minus the sign of the twist. For a proof see [22]. Now when everything is oriented by the complex structure, all Waldhausen ε are equal to $+1$.

Proof of Theorem 4.2. We shall use the semi-stable reduction of $\bar{f}^{-1}(0)$, as it is presented by Durfee in [10]. Let us start from $\bar{f}: \bar{X} \to U$. Let η be the radius of U and let $\eta' = \eta^{1/m}$. Let U' be the disc centered of the origin of C, of radius η' and let $\mu_m : U' \to U$ be defined by $\mu_m(z) = z^m$. Let us consider the pull-back diagram

$$\begin{array}{ccc} \bar{X}' & \xrightarrow{\mu'_m} & \bar{X} \\ \downarrow{\scriptstyle \bar{f}'} & & \downarrow{\scriptstyle \bar{f}} \\ U' & \xrightarrow{\mu_m} & U \end{array}$$

Let D' be the reduced divisor associated to $(\bar{f}')^{-1}(0)$. Let $n : \bar{X}'' \to \bar{X}'$ be the normalization. One gets the following commutative diagram:

$$\begin{array}{ccccc} \bar{X}'' & \xrightarrow{n} & \bar{X}' & \xrightarrow{\mu'_m} & \bar{X} \\ \downarrow{\scriptstyle \bar{f}''} & & \downarrow{\scriptstyle \bar{f}'} & & \downarrow{\scriptstyle \bar{f}} \\ U' & \xrightarrow{=} & U' & \xrightarrow{\mu_m} & U \end{array}$$

As $\bar{f}^{-1}(0)$ is supposed to be with normal crossings, one can determine the nature of the singular points of $\bar{X}''$ (a priori one knows that they are isolated and on $(\bar{f}'')^{-1}(0)$). More precisely, one has the following proposition.

Proposition 4.4. (1) *The restriction $\mu'_m \mid D' \to D$ is a bijection. Let D'_i be the component of D' which sits over D_i. Let D''_i be the inverse image $n^{-1}(D'_i)$. It is not necessarily irreducible.*

(2) *Suppose that $D_i \cap D_j = P_{ij}$ and let m_{ij} be the gcd of m_i and m_j. Then in $\bar{X}''$ one has m_{ij} intersection points between D''_i* and *D''_j, written Q^{λ}_{ij} for $\lambda = 1,2,\ldots,m_{ij}$.*

(3) *The points Q^{λ}_{ij} are the only points in $\bar{X}''$ which are possibly singular. Let n_{ij} be the* lcm *of m_i and m_j.*

(4) *The singularity of $\bar{X}''$ in Q^{λ}_{ij} is of type A_p with $p = (m/n_{ij}) - 1$.*

For a proof see [10].

Proposition 4.5. *Let $\sigma : \bar{X}''' \to \bar{X}''$ be the minimal resolution of the singularities of $\bar{X}''$. We get by composition a morphism $\Pi : \bar{X}''' \to \bar{X}$. This morphism is finite, except possibly over the points P_{ij}. The fiber $\Pi^{-1}(P_{ij})$* has *m_{ij} connected components Γ^{λ}_{ij}. The configuration graph of each Γ^{λ}_{ij} is a bamboo* (*i.e. a tree with vertices of valency inferior or equal to* (2) *with* $(m/n_{ij}) - 1$ *vertices. Each vertex represents a smooth rational curve of self-intersection equal to* -2.

For a proof see [10].

One has the following commutative diagram:

$$\begin{array}{ccccccc} \bar{X}''' & \xrightarrow{\sigma} & \bar{X}'' & \xrightarrow{n} & \bar{X}' & \xrightarrow{\mu'_m} & \bar{X} \\ \downarrow{\scriptstyle \bar{f}'''} & & \downarrow{\scriptstyle \bar{f}''} & & \downarrow{\scriptstyle \bar{f}'} & & \downarrow{\scriptstyle \bar{f}} \\ U' & \xrightarrow{=} & U' & \xrightarrow{=} & U' & \xrightarrow{\mu_m} & U \end{array}$$

Let D''' be the reduced divisor of $(\bar{f}''')^{-1}(0)$.

Lemma 4.6. *The divisor* $(\bar{f}''')^{-1}(0)$ *is reduced* (*i.e.* $D''' = (\bar{f}''')^{-1}(0)$).

For a proof see [10].

Let us choose a point $z' \in U'$ such that $\mu_m(z') = z$. Via Π one can identify the fiber $(\bar{f}''')^{-1}(z')$ with $\bar{f}^{-1}(z) = F$. From the pull-back construction, one can see that h^m is a monodromy for $\bar{f}'''$.

If we take account of the intersections of Γ_{ij}^{λ} with the neighboring components, there are m/n_{ij} intersection points along Γ_{ij}^{λ} in D'''. As D''' is reduced, the monodromy for each such point is a Picard–Lefschetz monodromy. It is well known that the P.L. monodromy is a Dehn twist of sign -1 around the vanishing cycle. For a detailed proof taking great care of the sign, see [16] and [6]. Observe that our sign and direction conventions are the same as the ones in Lamotke and Deligne.

Now, each annulus A_{ij}^{λ} of $\bar{f}^{-1}(z) = F$ can be identified via Π with the intersection B_{ij}^{λ} of $(\bar{f}''')^{-1}(z')$ with the boundary of a well-chosen neighborhood of Γ_{ij}^{λ}. The m/n_{ij} vanishing cycles are all homologous in B_{ij}^{λ}. Therefore the Dehn twists add up and we get that the twist number for h^m along B_{ij}^{λ} is equal to $-m/n_{ij}$. Hence $mr_{ij}^{\lambda} = -m/n_{ij}$ and we obtain eventually the equality $r_{ij}^{\lambda} = -1/n_{ij}$.

We now draw some consequences of the twist formula on the $Z[t,t^{-1}]$-module structure on $H_1(F,Z)$.

From the definition of the filtration $0 \subset W_0 \subset W_1 \subset W_2 = H_1(F,Z)$ given in Section 3 we immediately get the following proposition.

Proposition 4.7. *One has* (1) $(t^m - 1)(W_1) = 0$ *and* (2) $(t^m - 1)(W_2) \subset W_0$.

Theorem 4.8. *One has the equality* $W_1 = \mathrm{Ker}(t^m - 1)$.

Proof of Theorem 4.8. There remains to prove that $W_1 \supset \mathrm{Ker}(t^m - 1)$. To do that we define a map $q: H_1(F,Z) \to Z$ by $q(x) = I(x,(t^m - 1)x)$. One can prove that q is a quadratic form, but we shall not need this fact.

From the definition of r_{ij}^{λ} we obtain the equality: $q(x) = \sum_{i<j}\sum_{\lambda} mr_{ij}^{\lambda}(I(x, C_{ij}^{\lambda}))^2$.

Let us now suppose that $y \in \mathrm{Ker}(t^m - 1)$. From the definition of the map q we have that $q(y) = 0$. The above formula for $q(x)$ applied to $q(y)$ shows that $I(y, C_{ij}^{\lambda}) = 0$ for all C_{ij}^{λ}, because $r_{ij}^{\lambda} < 0$ (Theorem 4.2). By (3.6) we have that $y \in W_1$, which proves the theorem.

Theorem 4.9. *One has* $(t^m - 1)(W_2) \otimes_Z Q = W_0 \otimes_Z Q$.

Proof of Theorem 4.9. Let us consider the homomorphism $(t^m - 1): W_2 \to W_2$. From Proposition 4.7 we know that $(t^m - 1)(W_2) \subset W_0$. From Theorem 4.8 we get an injective homomorphism $\psi: W_2/W_1 \to W_0$ induced by $(t^m - 1)$. From Corollary 3.7 we deduce that W_2/W_1 and W_0 have the same rank over Q, which proves the theorem. □

Remark 4.10. The last theorem shows that $\psi(W_2/W_1)$ is a subgroup of finite index in W_0. The finite abelian group $W_0/\mathrm{Im}(\psi)$ is an interesting invariant. It has been used by Du Bois-Michel in the local case to distinguish topologically singularities which have the same monodromy over Q.

We have obtained so far a detailed description of the integral monodromy. After tensorization with C, Remark 3.8, Corollary 3.15 and Theorem 4.9 give the Jordan structure on $H_1(F, C)$.

Corollary 4.11. *Let λ be a mth root of unity. Then*:

(1) *The dimension of the λ-eigenspace of $H_1(F, C)$ (i.e. the number of Jordan n-blocks for $n = 1$ or 2 which correspond to λ) is equal to the dimension of the λ-eigenspace of $H_1(\mathrm{Ch}(h), C)$.*

(2) *The number of Jordan 2-blocks of $H_1(F, C)$ which correspond to λ is equal to the dimension of the λ-eigenspace of $H_1(G(h), C)$.*

5. Transfer and invariant cycles

We briefly recall some facts about the transfer homomorphism. For details and proofs see [13].

Let K be a (finite) simplicial complex and let G be a finite group acting simplicially on K (on the left). Without loss of generality, we may suppose that:

(i) If τ is a simplex of K such that $g\tau = \tau$ for some $g \in G$ then $g \mid \tau = \mathrm{id} \mid \tau$.

(ii) The quotient space $\bar{K} = K/G$ is a simplicial complex and the projection $\psi : K \to \bar{K}$ is simplicial.

Let A be an abelian group and let $C_n(K, A)$ be the group of simplicial n-chains on K with coefficients in A. One defines a homomorphism. $T_n : C_n(\bar{K}, A) \to C_n(K, A)$ as follows. Let $\bar{\tau}$ be an n-simplex in $\bar{K}$ and let τ be a simplex in K such that $\psi(\tau) = \bar{\tau}$. By definition $T_n(\bar{\tau}) = \sum_{g \in G} g\tau$.

It is easily checked that the homomorphisms $\{T_n\}_{n \geqslant 0}$ make up a chain homomorphism (the transfer) $T : C_*(\bar{K}, A) \to C_*(K, A)$.

Proposition 5.1. (1) *The composition $\psi_* \circ T_* : H_*(\bar{K}, A) \to H_*(\bar{K}, A)$ is equal to multiplication by $r = \mathrm{Card}(G)$.*

(2) *The composition $T_* \circ \psi_* : H_*(K, A) \to H_*(K, A)$ is equal to $\sigma = \sum_{g \in G} g_*$ where g_* is the homomorphism induced by $g : K \to K$.*

From the proposition it is not hard to deduce the transfer theorem which reads as follows.

Theorem 5.2. *Suppose that multiplication by r is an automorphism of A. Then*:

(1) *$T_* : H_*(\bar{K}, A) \to H_*(K, A)$ is injective.*

(2) *$\mathrm{Im}(T_*)$ is equal to the invariant subgroup of the action of G on $H_*(K, A)$.*

Let $T_* : H_1(\hat{D}_i, A) \to H_1(\hat{F}_i, A)$ be the transfer homomorphism associated to the cyclic covering R_i defined in (2.7).

Theorem 5.3. *Suppose that multiplication by m is an automorphism of A. One has*

$$\mathrm{Ker}((t-1) : H_1(F, A) \to H_1(F, A)) = \beta\left(\bigoplus_i T_*(H_1(\hat{D}_i, A)) \right).$$

Proof of Theorem 5.3. One has $\mathrm{Ker}(t^m - 1) = W_1$ by Theorem 4.7. Hence $\mathrm{Ker}(t-1) \subset W_1$. The conclusion follows from the definition of W_1 and the transfer theorem.

In principle, Theorem 5.3 is not true if integer coefficients are used. To deal with this case, one can proceed as follows. First, one remarks that $H_1(F, Z)$ is torsion-free and hence $\mathrm{Ker}((t-1): H_1(F,Z) \to H_1(F,Z))$ is a pure subgroup of $H_1(F,Z)$. Then, the proof of the transfer theorem, without the hypothesis on A, shows that $\mathrm{Im}(T_*)$ is contained in $\mathrm{Ker}(t-1)$. After tensorization with Q, one finally obtains the following theorem.

Theorem 5.4. *The subgroup* $\mathrm{Ker}((t-1): H_1(F,Z) \to H_1(F,Z))$ *of* $H_1(F,Z)$ *is the smallest pure subgroup of* $H_1(F,Z)$ *which contains* $\beta(\oplus_i T_* H_1(\hat{D}_i, Z))$.

Now, the homeomorphism $h_{\mathrm{Ch}}: \mathrm{Ch}(h) \to \mathrm{Ch}(h)$ is of finite order and its orbit space is D. The homeomorphism $h_G: G(h) \to G(h)$ is also of finite order and its orbit space is $G(D)$. Hence there are transfers associated to them. The next theorem is a consequence of the transfer theorem and of Corollary 4.11 applied to the case $\lambda = 1$.

Theorem 5.5. (1) *The invariant space of* $H_1(\mathrm{Ch}(h), C)$ *is equal to* $T_*(H_1(D,C))$. *Hence the dimension of the invariant space of* $H_1(F,C)$ *is equal to* $\dim(H_1(D,C))$.

(2) *The invariant space of* $H_1(G(h),C)$ *is equal to* $T_*(H_1(G(D),C))$. *Hence the number of Jordan 2-blocks of* $H_1(F,C)$ *which correspond to* $\lambda = 1$ *is equal to* $\dim(H_1(G(D),C))$.

Comment. The value for the dimension of the invariant space of $H_1(F,C)$ is well known. See [15, Section 15].

Corollary 5.6. (1) *There are no invariant cycles in* $H_1(F,C)$ *if and only if the configuration of D is a disjoint union of trees of rational curves.*

(2) *There are no Jordan 2-blocks in* $H_1(F,C)$ *which correspond to* $\lambda = 1$ *if and only if the configuration graph of D is a disjoint union of trees.*

6. The local case

Let (Z,P) be a representative of a germ of complex analytic normal surface at $P \in Z$ and let $f:(Z,P) \to (C,0)$ be a holomorphic function. One chooses Z small enough to have $Z - \{P\}$ smooth. Using Durfee's theory of neighborhoods, one can find a compact sub-analytic neighborhood N of P in Z, adapted to $f^{-1}(0)$, and a small enough disc $U \subset C$ centered at $0 \in C$ such that the restriction $f|(f^{-1}(U) \cap N - f^{-1}(0) \cap N) \to U - \{0\}$ is a differentiable fibration.

It is proved in [11] and in [17] that the topology of the construction does not depend on the choices.

We write X for $N \cap f^{-1}(U)$ and keep f for $f|X \to U$. We proceed as in Section 2 to define $\bar{X}, \bar{f}$ and D. The new feature is that some components of D have a boundary. From a differentiable point of view, each such component is a compact two-dimensional disc. Let $D_1, \ldots, D_k$ be the components of D which have no boundary and let $D_{-1}, \ldots, D_{-r}$ be those which have one. One has the equality for divisors $\bar{f}^{-1}(0) = \sum m_i D_i$ with $i = -r, \ldots, -1, 1, \ldots, k$. The m_i are positive integers. The 3-manifold $\Sigma = \bar{f}^{-1}(S^1_\eta)$ is constructed as in Section 2.

Claims 2.5 and 2.7 are true with the added property that can one find a deformation retraction R such that $\bar{R}^{-1}(bD) = bF$ where $b()$ stands for "boundary".

One has points P_{ij} and curves C^λ_{ij} as in Section 2, with $i < j$. Here i can be < 0 but not j. If i is < 0, let $j(i)$ be the unique j such that $D_i \cap D_j \neq \emptyset$.

We write C_- for the union of the C^λ_{ij} with $i < 0$ and C_+ for the union of those with $i > 0$. Let $C = C_- \cup C_+$, The inclusions $C \subset F$ and $F_i \subset F$ induce the homomorphisms γ and β as in Section 3.

We define a weight filtration $0 \subset W_{-1} \subset W_0 \subset W_1 \subset W_2 = H_1(F, Z)$ by specifying that: $W_{-1} = \gamma H_1(C_-, Z)$, $W_0 = \gamma H_1(C, Z)$, $W_1 = \beta(\oplus_i H_1(F_i, Z))$.

Remark 6.1. For $i < 0$ the surface F_i is a disjoint union of annuli $A^\lambda_{ij(i)}$ for $\lambda = 1, \ldots, m_{ij(i)}$. One boundary component of such an annulus is the curve $C^\lambda_{ij(i)}$ and the other boundary component $B^\lambda_{ij(i)}$ belongs to bF. As a consequence W_1 is also equal to $\beta(\oplus_{i>0} H_1(F_i, Z))$.

The twist formula (4.2) is valid. The proof is essentially the same. As in Section 4 a consequence is the following theorem.

Theorem 6.2. *One has the equality* $\mathrm{Ker}((t^m - 1): H_1(F, Z) \to H_1(F, Z)) = W_1$.

To describe the $Z[t, t^{-1}]$-module structure on $H_1(F, Z)$ one studies first W_{-1}. Notice that it can be defined intrisically as a submodule of $H_1(F, Z)$. Indeed, it consists of the elements $x \in H_1(F, Z)$ such that $I(x, y) = 0$ for all $y \in H_1(F, Z)$.

Let D_i be a component of D with $i < 0$. The curves $C^\lambda_{ij(i)}$ for $\lambda = 1, \ldots, m_{ij(i)}$ (or equivalently the curves $B^\lambda_{ij(i)}$) are permuted cyclically and transitively by the monodromy h. One orients them as bF with F oriented from its complex structure. It is now easy to see that $W_{-1} \otimes C$ is semi-simple and that its characteristic polynomial is equal to $(t^c - 1)^{-1} \prod_{i=-1}^{-r} (t^{m_{ij(i)}} - 1)$ where c is the number of connected components of F.

The structure of the submodule of invariant cycles $\mathrm{Ker}(t-1) \cap W_{-1}$ of W_{-1} is now easily understood. The multiplicity of the root $\lambda = 1$ in the above polynomial is equal to $(r-1)$. There is no problem going from C to Z and one gets the following proposition.

Proposition 6.3. *The subgroup* $\mathrm{Ker}(t-1) \cap W_{-1}$ of W_{-1} *is torsion-free and its rank is equal to* $(r-1)$ *where* r *is the number of connected components of* $f^{-1}(0) - \{P\}$.

To further study the module structure on $H_1(F, Z)$ the shortest way is to glue a disc on each boundary component of bF. One obtains a surface $\bar{F}$ without boundary. The inclusion $F \subset \bar{F}$ induces a surjective homomorphism $\omega: H_1(F, Z) \to H_1(\bar{F}, Z)$. The image by ω of the filtration of $H_1(F, Z)$ is equal to $0 \subset \bar{W}_0 \subset \bar{W}_1 \subset \bar{W}_2 = H_1(\bar{F}, Z)$ where $\bar{W}_j = W_j / W_{-1}$.

One then proceeds as in Sections 3, 4, 5. The monodromy h gives rise to a monodromy $\bar{h}: \bar{F} \to \bar{F}$. The chorizo space $\mathrm{Ch}(\bar{h})$ has only components without boundary. One defines $\bar{D}$ as the part of D which consists of the components which have no boundary. The graphs $G(\bar{D})$ and $G(\bar{h})$ are defined as before. As P is a normal point in Z, D and $G(\bar{D})$ are connected. The analogue of Corollary 4.11 is true for the filtration on $H_1(\bar{F}, Z)$.

We conclude this section by stating what happens to the eigenvalue $\lambda = 1$ in $H_1(F, C)$.

Theorem 6.4. (1) *The dimension of the subspace of invariant cycles of* $H_1(F,C)$ *is equal to* $(r-1) + \dim H_1(\bar{D}, C)$.

(2) *The number of Jordan* 2-*blocks of* $H_1(F,C)$ *which correspond to the eigenvalue* $\lambda = 1$ *is equal to* $\dim H_1(G(\bar{D}), C)$.

7. Compactifications for a polynomial function

Let us start from a polynomial function $f: C^2 \to C$. If (X, Y) are coordinates on C^2, then f is expressed by a polynomial $f(X, Y) = \sum_{\alpha+\beta \leqslant d} c_{\alpha\beta} X^\alpha Y^\beta$ where d is assumed to be the degree of the polynomial. If $(X:Y:Z)$ are homogeneous coordinates on $P^2(C)$, then f gives rise to a meromorphic function $\varphi: P^2(C) - \to P^1(C)$ defined as usual by $\varphi(X:Y:Z) = (\tilde{\varphi}(X;Y;Z) : Z^d)$ where $\tilde{\varphi}(X:Y:Z) = \sum c_{\alpha\beta} X^\alpha Y^\beta Z^{d-(\alpha+\beta)}$.

We write L_∞ for the line at infinity of $P^2(C)$, defined by $Z = 0$. Hence $P^2(C) - L_\infty$ can be identified with C^2 equipped with the coordinates (X, Y).

We shall call modification of $P^2(C)$ a morphism $\psi: X \to P^2(C)$ which is a composition of blowing down (of exceptional curves of first kind) whose center project to L_∞ (it should properly be called a modification over L_∞).

Definition 7.1. A resolution of the meromorphic function $\varphi: P^2(C) -\!\!\to P^1(C)$ is a modification $\psi: X \to P^2(C)$ such that $\Phi = \varphi \circ \psi$ is a morphism from X to $P^1(C)$.

It is well known that resolutions do exist. Let us call minimal a resolution such that there is no contractible $P^1(C)$ in X which is vertical both for Φ and ψ. Classical theorems about birational morphisms between smooth surfaces imply that a minimal resolution is unique.

For the moment let ψ be any resolution of φ. Let $\tilde{E}$ (resp. D_∞) be the reduced divisor associated to $\psi^{-1}(L_\infty)$ (resp. $\Phi^{-1}(\infty)$). Thus $X - \tilde{E}$ is canonically isomorphic to $P^2(C) - L_\infty$ and can be identified with C^2. Note that $\psi^{-1}(L_\infty)$ is a curve with normal crossings whose configuration graph is a tree of $P^1(C)$.

Theorem 7.2. D_∞ *is connected.*

Comment. This theorem is true with no extra assumption on f. One does not need to assume that the generic fiber of Φ is connected. For a proof see [18]. We now draw important consequences of the theorem.

Let $A_1, \ldots, A_s$ be the connected components of the closure of $\tilde{E} - D_\infty$. A priori the configuration graph of each A_j is a tree of $P^1(C)$. Let E_j be the unique component of A_j which meets D_∞.

Definition 7.3. An irreducible component E of $\tilde{E}$ is called *dicritical* if the restriction $\Phi \mid E \to P^1(C)$ is non-constant. As this restriction is holomorphic and proper, it is necessarily surjective.

Corollary 7.4. *Each* E_j *is a dicritical component. It is the unique dicritical component of* A_j.

Proof. The restriction $\Phi \mid E_j \to P^1(C)$ cannot be constant, otherwise we would have $E_j \subset D_\infty$ which is impossible. If another component of A_j were dicritical, the subset D_∞ of $\tilde{E}$ would not be connected.

Corollary 7.5. *The restriction $\Phi \mid (A_j - E_j) \to P^1(C)$ takes values in C and is constant on each connected component of $(A_j - E_j)$.*

From now on, we suppose that ψ is the minimal resolution of φ.

Theorem 7.6. *The configuration graph of each A_j in the minimal resolution is a bamboo. One extremity of the bamboo represents the dicritical component E_j.*

Corollary 7.7. *Let B_j be the closure of $(A_j - E_j)$ in the minimal resolution. Then the configuration graph of B_j is a bamboo, possibly empty.*

For a proof see [18].

We now describe explicitly the atypical set of f, i.e. the minimal set A_f such that $f|(C^2 - f^{-1}(A_f)) \to (C - A_f)$ is a differentiable fibration. Let C_f be the finite set of critical values of f. Let E_j be a dicritical component of the minimal resolution. We define I'_j to be the set of critical values (different from ∞) of the restriction $\Phi \mid E_j \to P^1(C)$. And we define I_j to be equal to $I'_j \cup \{\Phi(B_j)\}$. Finally let $I_f = \bigcup_{j=1}^{s} I_j$.

Theorem 7.8. *One has the equality $A_f = C_f \cup I_f$.*

Comment. The union is not necessarily disjoint. It is not hard to see that $A_f \subset C_f \cup I_f$. The converse requires some work. For a proof see [18].

Definition 7.9. If $z \notin A_f$ the fiber $f^{-1}(z)$ is said to be *generic*. If $z \notin I_f$ the fiber $f^{-1}(z)$ is said to be *generic at infinity*.

8. The affine monodromies

Let $f: C^2 \to C$ be a polynomial function. Let ψ be the minimal resolution of φ defined in Section 7. We consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad \Phi \quad} & P^1(C) \\ \downarrow \psi & & \downarrow = \\ P^2(C) & \xrightarrow{\quad \varphi \quad} & P^1(C) \\ \cup & & \cup \\ C^2 & \xrightarrow{\quad f \quad} & C \end{array}$$

According to Definition 2.2, there exists a sequence of blowing ups of points $p:\bar{X}\to X$ such that p is a very good resolution for all the fibers of $\bar{f}=\Phi\circ p:\bar{X}\to P^1(C)$. We ask moreover that each fiber cuts transversally the dicritical components. The dicritical components of $\bar{f}$ will be written E_j (for $j=1,\dots,s$).

Definition 8.1. The order d_j of the dicritical component E_j is the degree of the restriction $\bar{f}|E_j\to P^1(C)$.

Let $a\in A_f\cup\{\infty\}$ and D_a be the reduced divisor associated to $\bar{f}^{-1}(a)$. Let D^2_η be the closed disc centered at a of radius η. One chooses the radius small enough in order that $D^2_\eta\cap(A_f\cup\{\infty\})=\{a\}$. As in Section 2, we write $V=\bar{f}^{-1}(D^2_\eta)$ and $\Sigma=\bar{f}^{-1}(S^1_\eta)$. We write f_Σ for the restriction $\bar{f}|\Sigma\to S^1_\eta$. Let $\bar{F}$ be a fiber of f_Σ. As $\bar{f}$ is a proper map defined on a smooth complex manifold, f_Σ is a differentiable fibration. Its monodromy can be described as in Sections 2–5. In particular, for a sufficiently small η we can find a deformation retraction $R:V\to D_a$ fulfilling the conditions stated in (2.5) and a monodromy h associated to R as in (2.7).

We study first the affine monodromy at infinity. Accordingly, we let $a=\infty$ and we consider the monodromy $h:\bar{F}\to\bar{F}$ constructed with the help of $R:V\to D_\infty$. Let $P_j=D_\infty\cap E_j$. The intersection $R^{-1}(P_j)\cap\bar{F}$ is made of d_j points which are cyclically permuted by h. The space $\mathring{F}=\bar{F}-\bigcup_{j=1}^s R^{-1}(P_j)\cap\bar{F}$ is isomorphic via $\psi\circ p$ to the generic affine fiber. To deal with compact fibers with boundary, we choose (for $j=1,\dots,s$) a small open disc U_j around P_j in D_∞ and we consider $F=\bar{F}-\bigcup_{j=1}^s(R^{-1}(U_j)\cap\bar{F})$.

Definition 8.2. The diffemorphism $h_\infty:F\to F$ obtained by restriction of h is the affine monodromy of f at infinity.

The inclusion of F in $\bar{F}$ induces a surjective homomorphism $\omega:H_1(F,Z)\to H_1(\bar{F},Z)$. Let $W_{-1}=\operatorname{Ker}\omega$. Let $0\subset\bar{W}_0\subset\bar{W}_1\subset\bar{W}_2=H_1(\bar{F},Z)$ be the filtration on $H_1(\bar{F},Z)$ defined as in (3.1). For $i=0,1,2$ we write $W_i=\omega^{-1}(\bar{W}_i)$. Hence the quotients W_{i+1}/W_i are isomorphic as $Z[t,t^{-1}]$-modules to $\bar{W}_{i+1}/\bar{W}_i$ and the following proposition is a direct consequence of Sections 4 and 5.

Proposition 8.3. *Let m be* 1 cm *of the multiplicities of the irreducible components of D_∞. Then*:

(1) $(t^m-1)^2H_1(F,Z)=0$.
(2) $\operatorname{Ker}(t^m-1)=W_1$.
(3) W_2/W_1 *is a semi-simple $Z[t,t^{-1}]$-module isomorphic to $H_1(G(h),Z)$.*
(4) $(t^m-1)H_1(F,Z)\subset W_0$ *and* $M_0\overset{\text{def}}{=}W_{-1}\oplus(t^m-1)H_1(F,Z)$ *is of finite index in W_0.*

Remark 8.4. The index of M_0 in W_0 can be computed with the twist formula (4.2).

Theorem 8.5. *One has* $\operatorname{Ker}(t-1)\subset W_{-1}$ *and the characteristic polynomial of the semi-simple module W_{-1} is equal to $(t^c-1)^{-1}\prod_{j=1}^s(t^{d_j}-1)$ where c is the number of connected components of F.*

Corollary 8.6. *The rank of* $\operatorname{Ker}(t-1)$ *is* $(s-1)$.

Proof of Theorem 8.5. From Section 7 we know that the divisor D_∞ has only rational components and its configuration graph is a tree. Hence $\bar{W}_2 \cong W_2/W_{-1}$ has no invariant cycle by (5.6).

Remark 8.7. As $\bar{f}^{-1}(\infty) = \sum m_i D_i$ is a divisor with rational components, the number c of connected components of the fibers F and $\bar{F}$ is equal to the gcd of the multiplicities m_i.

Partial results about h_∞ have been obtained by Dimca in [7].

We now study the affine monodromy around $a \in A_f$. Hence $h: \bar{F} \to \bar{F}$ is the monodromy constructed with the help of $R: V \to D_a$. For every dicritic component E_j let $\{Q_{ij} \mid i = 1, \dots, n(j)\} \overset{\text{def}}{=} D_a \cap E_j$ and let d_{ji} be the multiplicity of the restriction $\bar{f} | E_j$ at Q_{ji}.

Remark 8.8. One has the equality $d_j = \sum_{i=1}^{n(j)} d_{ji}$.

Let $\mathring{F} = \bar{F} - \bigcup_{j=1}^{s} \bigcup_{i=1}^{n(j)} (R^{-1}(Q_{ji}) \cap \bar{F})$. It is isomorphic via $\Phi \circ p$ to the generic fiber of f. To deal again with compact fibers (with boundary) we choose small open discs U_{ji} around Q_{ji} in D_a and we define $F = \bar{F} - \bigcup_{j=1}^{s} \bigcup_{i=1}^{n(j)} (R^{-1}(U_{ji}) \cap \bar{F})$.

Definition 8.9. The restriction $h_a: F \to F$ of h on F is the affine monodromy of f around a.

Let $\omega: H_1(F, Z) \to H_1(\bar{F}, Z)$ be the surjective homomorphism induced by the inclusion of F in $\bar{F}$.

Let $0 \subset \bar{W}_0 \subset \bar{W}_1 \subset \bar{W}_2 = H_1(\bar{F}, Z)$ be the filtration on $H_1(\bar{F}, Z)$ defined as in (3.1). We equip $H_1(F, Z)$ with a filtration defined as follows: $W_{-1} = \operatorname{Ker} \omega$ and $W_i = \omega^{-1}(\bar{W}_i)$ for $i = 0,1,2$.

Theorem 8.10. *The $Z[t, t^{-1}]$-module W_{-1} is semi-simple and its characteristic polynomial is equal to $(t^c - 1)^{-1} \prod_{j=1}^{s} \prod_{i=1}^{n(j)} (t^{d_{ji}} - 1)$ where c is the number of connected components of F.*

Proof. The monodromy h_a permutes cyclically the d_{ji} connected components of the boundary of $R^{-1}(U_{ji}) \cap \bar{F}$.

Now, let L_a be the union of the irreducible components of D_a such that the subset of smooth points $\mathring{L}_a$ of L_a is isomorphic via $\Phi \circ p$ to the subset of smooth points of the reduced curve associated to the affine fiber $f^{-1}(a)$. Without loss of generality, we may assume that L_a is smooth (just perform some more blow-ups). Let us write $D_a = L_a \bigcup_i D_i$ where the D_i's are the irreducible components which are not in L_a. Let us define $\hat{D}_i$ and $\hat{L}_a$ as in Section 2.

Remark 8.11. We interpret $\hat{L}_a$ as a compact model for the smooth part of the affine special fiber. In our description of h_a, an essential part will be played by $\hat{L}_a$ to determine the invariant cycles.

The difference $\mathring{L}_a - \hat{L}_a$ is a disjoint union of small open annuli. One has the following inclusions: $\hat{F}_a = R^{-1}(\hat{L}_a) \cap F \subset \mathring{F}_a = R^{-1}(\mathring{L}_a) \cap \bar{F} \subset F_a = R^{-1}(L_a) \cap \bar{F}$.

Let $\rho: H_1(\hat{F}_a, Z) \to H_1(F, Z)$ be the homomorphism induced by the inclusion $\hat{F}_a \subset F$. Let also $T: H_1(\hat{L}_a, Z) \to H_1(F, Z)$ be the transfer homomorphism defined in Section 5. One of the main results of this section is the following.

Theorem 8.12. $\operatorname{Ker}((t-1): H_1(F, Z) \to H_1(F, Z))$ *is the smallest pure subgroup of* $H_1(F, Z)$ *which contains* $\rho(T(H_1(\hat{L}_a, Z)))$.

Proof of Theorem 8.12. The theorem is not an immediate consequence of (5.4), because the transfer on the irreducible components of $D_a - L_a$ does not appear in the statement. We shall prove that

$\mathrm{Ker}(t-1)\cap H_1(F,Z)=\mathrm{Ker}(t-1)\cap\rho H_1(\hat{F}_a,Z)$. To do that we need to study D_a more carefully. Let $P(a)$ be the set of singular points of the reduced curve associated to the affine curve $f^{-1}(a)$ and let $Q(a)=D_a\cap(\bigcup_{j=1}^s E_j)=\bigcup_{j=1}^s\bigcup_{i=1}^{n(j)}\{Q_{ji}\}$. Recall that $\bar{p}=\psi\circ p:\bar{X}\to P^2(C)$ is a sequence of blow ups of points.

For any $P\in P(a)$ let $D(P)$ be the divisor $\bar{p}^{-1}(P)$. The theory of the resolution of plane curve singularities implies that the configuration graph of $D(P)$ is a tree of rational curves.

If $Q_{ji}\in L_a$ then $D(Q_{ji})$ is empty. Otherwise $D(Q_{ji})$ is the unique connected component of $D_a-\mathring{L}_a$ which meets E_i at Q_{ji}. The construction of the resolution at infinity implies that if $D(Q_{ji})$ is not empty, its configuration graph is a tree of rational curves.

Theorem 5.2 implies that $\mathrm{Ker}(t-1)\cap\rho H_1(\hat{F}_a,Z)$ is the image by ρ of the smallest pure subgroup of $H_1(\hat{F}_a,Z)$ which contains $T(H_1(\hat{L}_a,Z))$. Let $F(P)=R^{-1}(D(P))\cap F$ and $F(Q_{ij})=R^{-1}(D(Q_{ij}))\cap F$. We write $\rho_P:H_1(F(P),Z)\to H_1(F,Z)$ and $\rho_{ji}:H_1(F(Q_{ji}),Z)\to H_1(F,Z)$ for the homomorphisms induced by the inclusions. As $F=F_a\bigcup_P F(P)\bigcup_{j,i}F(Q_{ji})$ the following two lemmas imply Theorem 8.12.

Lemma 8.13. *One has* $\mathrm{Ker}(t-1)\cap\mathrm{Im}\,\rho_P\subset\mathrm{Im}\,\rho$.

Lemma 8.14. *One has* $\mathrm{Ker}(t-1)\cap\mathrm{Im}\,\rho_{ji}\subset\mathrm{Im}\,\rho$.

Proof of Lemma 8.13. Let $C(P)$ be the boundary of $F(P)$. One has $C(P)=F(P)\cap F_a$. As the configuration graph of $D(P)$ is a tree of rational curves one deduces from (5.6) that one has the inclusion $\mathrm{Ker}(t-1)\cap H_1(F(P),Z)\subset\gamma_P(H_1(C(P),Z))$ where γ_P is induced by the inclusion of $C(P)$ in $F(P)$. The equality $C(P)=F(P)\cap F_a$ finishes the proof of Lemma 8.13.

Proof of Lemma 8.14. Let C_{ji} denote $\overline{R^{-1}(U_{ji})}\cap F$ and let $C(Q_{ji})=F_a\cap F(Q_{ji})$. The union of C_{ji}'s and $C(Q_{ji})$ is equal to the boundary $bF(Q_{ji})$. As the configuration graph of $D(Q_{ji})$ is a tree of rational curves one has by Corollary 5.6 the inclusion $\mathrm{Ker}(t-1)\cap H_1(F(Q_{ji}),Z)\subset\gamma_{ji}(H_1(bF(Q_{ji}),Z))$. But $C(Q_{ji})$ is contained in bF_a and h_a permutes cyclically the connected components of C_{ji}. Moreover, the element in $H_1(F,Z)$ represented by the union of the components of C_{ji} (oriented as bF) is equal to the element represented by the union of the connected components of $C(Q_{ji})$ (oriented as bF_a). This ends the proof of Lemma 8.14 and hence of Theorem 8.12.

In $C[X,Y]$ the polynomial $(f(X,Y)-a)$ can be written as the product of irreducible factors $\prod_{j=1}^r f_j^{r_j}(X,Y)$. Accordingly, one has the equalities among divisors $\bar{f}^{-1}(a)=\sum_{j=1}^r r_jL_j+\sum_{i=1}^k m_iD_i$. Now let m be the lcm of the multiplicities $\{m_i\}_i$ and $\{r_j\}_j$. As $H_1(F,Z)/W_{-1}$ is isomorphic to $H_1(\bar{F},Z)$ the structure of $Z[t,t^{-1}]$-module on $H_1(F,Z)$ induced by h_a is determined by the following proposition, which is a direct consequence of Sections 4 and 5.

Proposition 8.15. *One has*
(1) $(t^m-1)^2(H_1(F,Z))=0$.
(2) $\mathrm{Ker}(t^m-1)=W_1$.
(3) W_2/W_1 *is a semi-simple module isomorphic to* $H_1(G(h),Z)$.
(4) $(t^m-1)H_1(F,Z)\subset W_0$.
(5) $M_0=W_{-1}\oplus(t^m-1)H_1(F,Z)$ *is of finite index in* W_0.

We now turn to the study of the homology of the graph $G(D_a)$ associated to D_a when $(f(X,Y)-a)$ is irreducible. Let us recall that (see Theorems 5.5 and 6.4) the rank of $H_1(G(D_a),Z)$ is equal to the number of 2-blocks in the Jordan decomposition of $H_1(F,Z)$ which correspond to the eigenvalue 1, for the action induced by the monodromy h_a.

Notations 8.16. We write $b(P)$ for the number of branches of $f^{-1}(a)$ at $P \in P(a)$. If the reduced divisor B_j defined in (7.7) is empty or if $\Phi(B_j) \neq a$, we say that the dicritical component E_j has no bamboo for a. Otherwise, for convenience's sake, we choose ψ minimal among the resolutions such that the closure of $\psi^{-1}(f^{-1}(a))$ intersects the bamboo B_j only at smooth points of $\tilde{E}$. We write B_j also for the strict transform in $\bar{X}$ (via p) of B_j, and we call it the bamboo of E_j. Moreover, we index the elements $\{Q_{ji}\}$ of $E_j \cap D_a$ such that $Q_{j1} = B_j \cap E_j$. If $Q_{ji} \in L_a$ we set $b_{ji} = 1$. Otherwise, b_{ji} is the number of intersection points of $L_a \cap D(Q_{ji})$.

Proposition 8.17. *If $(f(X,Y)-a)$ is irreducible in $C[X,Y]$, the graph $G(D_a)$ has the homotopy type of a wedge of b circles, where $b = \sum_P (b(P)-1) + \sum_{j,i}(b_{ji}-1)$.*

Proof. The affine curve $f^{-1}(a)$ is irreducible if and only if L_a is connected and has multiplicity 1. Then L_a is represented by only one vertex in $G(D_a)$. As $G(D(P))$ and $G(D(Q_{ji}))$ are trees, the quotient of the graph $G(D_a)$ obtained by identifying each tree $G(D(P))$ and $G(D(Q_{ji}))$ to a point has the homotopy type of $G(D_a)$. The edges of this quotient graph are in bijection with the intersection points $L_a \cap G(D(P))$ and $L_a \cap G(D(Q_{ji}))$ and the result follows now by computing the Euler characteristic of this quotient graph.

From the preceeding proposition we shall now deduce the proof of a conjecture of Dimca.

Theorem 8.18. *Let $a \in C$. Suppose that $f^{-1}(a)$ is irreducible and that the homology monodromy t around a acts as the identify on $H_1(F,Z)$. Then $a \notin A_f$.*

Remark 8.19. We assume that p is a minimal very good resolution of Φ. Hence, to say that $a \notin A_f$ is equivalent to say that $D_a = L_a$. Moreover $D_a = L_a$ is equivalent to $\mathrm{P}(a) = \emptyset$ and $Q_{ji} \in L_a$ for all (ji).

Proof of Theorem 8.18. From the hypothesis $H_1(F,Z) = \mathrm{Ker}(t-1)$ we deduce from Theorem 8.10 that $d_{ji} = 1$ for all (ji). Hence if Q_{ji} is not the intersection of a bamboo B_j with E_j one has that $D(Q_{ji})$ is empty and that $Q_{ji} \in L_a$.

Lemma 8.20. *If $f^{-1}(a)$ is irreducible and if $(t-1)H_1(F,Z) = 0$ then $P(a)$ is empty.*

Proof of Lemma 8.20. We argue by contradiction and suppose that $P \in P(a)$. As $(t-1)H_1(F,Z) = 0$ the module $H_1(F,Z)$ has no Jordan 2-blocks. In particular one has $H_1(G(D_a),Z) = 0$. By Proposition 8.17 this implies $b(P) = 1$. Hence $D(P)$ is the exceptional divisor of the resolution of an irreducible non-smooth plane curve germ. The theory of the resolutions of such germs implies that $D(P)$ has at least an irreducible component D_i, such that its halo (see Definition 3.12) fulfills the inequalities $v_i \geqslant 3$ and $1 \leqslant m_{ik} < m_i$ for all $k \in V(i)$. Then Propositions 3.11 and 3.13 show that $\exp(2i\pi/m_i)$ is an eigenvalue for the action of the monodromy on $(W_1/W_0)\otimes C$. As $(t-1)H_1(F,Z) = 0$ this is impossible. This ends the proof of Lemma 8.20.

Going back to the proof of Theorem 8.18, we are left to prove that it is impossible for a dicritical component E_j to have a non empty bamboo.

Arguing by contradiction, let us suppose that $B_j = \sum_{\sigma=1}^{e(j)} D_\sigma$ is a nonempty bamboo for a. We index the irreducible components D_σ of B_j in such a way that $D_1 \cap E_j = Q_{j1}$ and that $D_\sigma \cap D_{\sigma+1} \neq \emptyset$.

As $(t-1)H_1(F,Z) = 0$ we must have $H_1(G(D_a),Z) = 0$. As $f^{-1}(a)$ is irreducible, Proposition 8.17 implies that $b_{j1} = 1$. Let A_{j1} be the closure of $D(Q_{j1}) - B_j$. As $b_{j1} = 1$, A_{j1} intersects B_j in a unique point which we write Q. From Notation 8.16, Q is a smooth point of B_j. Let $D_{\sigma'}$ be the irreducible component of B_j which contains Q. There exists an open neighborhood U of $p(Q)$ in X and local coordinates (u,v) in U such that $\Phi(u,v) = u^{m_{\sigma'}} g(u,v)$ and $D_{\sigma'} \cap U = \{u=0\} \cap U$ and furthermore $\{g=0\} \cap \{u=0\} = p(Q)$. As $b_{j1} = 1$ the germ g is irreducible at $p(Q)$.

Lemma 8.21. *If* $(t-1)H_1(F,Z) = 0$, *one has that* A_{j1} *is empty.*

Proof. By definition, A_{j1} is the reduced divisor associated to the exceptional divisor of the resolution of the plane curve germ $u^{m_{\sigma'}} g(u,v)$. Hence, if g is smooth and transverse to $D_{\sigma'}$ at $p(Q)$ one has that A_{j1} is empty. If not, there exists an irreducible component D_i of A_{j1} with a halo in $D(Q_{j1})$ satisfying the inequalities $v_i \geqslant 3$ and $1 \leqslant m_{ik} < m_i$. If we argue as in the end of the proof of (8.20), we get the desired conclusion.

Lemma 8.22. *If* $(t-1)H_1(F,Z) = 0$, *one has* $\sigma' = e(j)$.

Proof. We argue by contradiction. If $\sigma' < e(j)$ then $m_{\sigma'} = k m_{e(j)}$ with $k > 1$. As $d_{j1} = m_1 = 1$ we cannot have $\sigma' = 1$. Now if $1 < \sigma' < e(j)$ the halo of $D_{\sigma'}$ in D_a is given by $\theta_{\sigma'} = (m_{\sigma'},1,1) \cup \{1,1,m_{e(j)}\}$. As in Proposition 3.11 we consider $H_1(\bar{F}_{\sigma'},Z)$. By Proposition 3.13 its characteristic polynomial is equal to $\Delta(t) = (t^{m_{\sigma'}}-1)(t-1)^2/(t^{m_{e(j)}}-1)(t-1)^2$. As $H_1(\bar{F}_{\sigma'},Z)$ is a direct summand of W_1/W_0 the polynomial $\Delta(t)$ divides the characteristic polynomial of h_a. As $m_{\sigma'} = k m_{e(j)}$ with $k > 1$ and as $(t-1)H_1(F,Z) = 0$ this is impossible and the proof of Lemma 8.22 is completed. □

Definition 8.23. A fiber D_b of $\bar{f}$ will be called a "bamboo extremity fiber" if $f^{-1}(b)$ is reduced, if $D_b - L_b$ is a union of bamboos and if everytime L_b intersects a bamboo B_j, it intersects it in exactly one point which belongs to the extremity component $D_{e(j)}$ of the bamboo.

Lemma 8.24. *If* D_b *is a bamboo extremity fiber, then* $D_b = L_b$ *and thus* $b \notin A_f$.

Remark 8.25. The lemma means that indeed a bamboo extremity fiber intersects no bamboo at all. We have just shown that the hypothesis of Theorem 8.18 imply that D_a is a bamboo extremity fiber. Thus Lemma 8.24 implies Theorem 8.18.

Proof of Lemma 8.24. We argue by contradiction and suppose that D_b intersects at least one bamboo. We choose one and call it B_1. Let $P = L_\infty \cap p(\psi(B_1))$ (see the beginning of this section for the notations). We choose two distinct generic values c_1 and c_2 of f. We also choose a small compact ball V around P in $P^2(C)$ which is a Milnor ball for $\varphi^{-1}(b)$,$\varphi^{-1}(c_1)$ and $\varphi^{-1}(c_2)$. Let

$K_b = \varphi^{-1}(b) \cap bV$ and let $K_i = \varphi^{-1}(c_i) \cap bV$ for $i = 1,2$. Let $L(\text{-},\text{-})$ stand for the linking coefficient in bV. Because any two germs in a local pencil have the same intersection number, we must have $L(K_1, K_2) = L(K_i, K_b)$. But as D_b is a bamboo extremity fiber intersecting at least one bamboo, an easy computation shows that $L(K_1, K_b) < L(K_1, K_2)$ which is a contradiction.

Conclusion 8.26. A statement analogous to (3.16) holds for a polynomial map $f: C^2 \to C$. Again the divisor D_a determines the geometric and the homological monodromies except for the integers c_i.

9. An example

In this short section we show on an example how the methods developed in this article can be used to obtain a description of the affine monodromy and of its filtration.

Before getting to details, a word of comment is in order. Let $f: X \to U$ be a proper morphism as in Section 2. Suppose that f has only one critical point on $f^{-1}(0)$ and that it is an ordinary double point. Suppose that the vanishing cycle does not separate the fiber F. Then the homological monodromy has a Jordan 2-block for the eigenvalue 1. Now, it is easy to reproduce this phenomenon in the case of polynomial mappings $C^2 \to C$. As the referee has kindly pointed out, the polynomial $f(X, Y) = X^2 + X^3 + Y^2$ does the trick. This polynomial is equisingular at infinity (i.e. $I(f)$ is empty). Hence, as far as monodromies are concerned, the polynomial f behaves as if it were a proper mapping. The example we propose is very different. The value $0 \in C$ is not a critical value, but it belongs to $I(f)$. So, the Jordan 2-block for the eigenvalue 1 is due to the behavior of the polynomial at infinity.

The example stems from computations performed by E. Artal in his thesis and which have been used by him in [2]. Artal studies plane projective curves of degree 6 which are irreducible and have one singularity of type A_{17} and no other. In what follows we shall only use the (nontrivial) fact that such curves do exist. Analogous examples could be obtained in degrees 4 and 5 using some of the curves described in Sections 2.2 and 2.3 of [20].

Let Γ be any one of the Artal projective curves of degree 6. One chooses a line L which intersects Γ transversally at the singular point Q_1 and also transversally at four smooth points Q_2, Q_3, Q_4 and Q_5. Let $(X; Y; Z)$ be projective coordinates on $P^2(C)$ such that $L = \{Z = 0\}$. Let $\tilde{\varphi}(X, Y, Z) = 0$ be the equation of Γ. Then $f: C^2 \to C$ defined by $f(X, Y) = \tilde{\varphi}(X, Y, 1)$ is the polynomial map we are looking for.

We are interested in the affine monodromy around $a = 0 \in C$. In the resolution $\Psi: X \to P^2(C)$ the situation at infinity is as follows.

(1) No dicritic component has a bamboo.

(2) For $j = 2,3,4,5$ there is one dicritic component E_j over each Q_j. The degree of the restriction $\Phi | E_j \to P^1(C)$ is equal to 1. Hence each fiber of Φ cuts E_j transversally at one point.

(3) For $j = 1$ there is one dicritic component E_1 over Q_1. The degree of the restriction $\Phi | E_1 \to P^1(C)$ is equal to 2. The fiber $\Phi^{-1}(0)$ cuts E_1 in one point Q. At Q the fiber $\Phi^{-1}(0)$ has a singularity of type A_{11} which is transversal to E_1. All other fibers $\Phi^{-1}(c)$ for $c \notin \{0, \infty\}$ cut E_1 transversally in two distinct points.

In $\bar{X}$ the strict transform of Γ is a smooth curve of genus 1. The generic fiber $\bar{F}$ is a smooth curve of genus 7 which cuts E_1 transversally in two distinct points. The fiber $f^{-1}(0)$ has no singular point

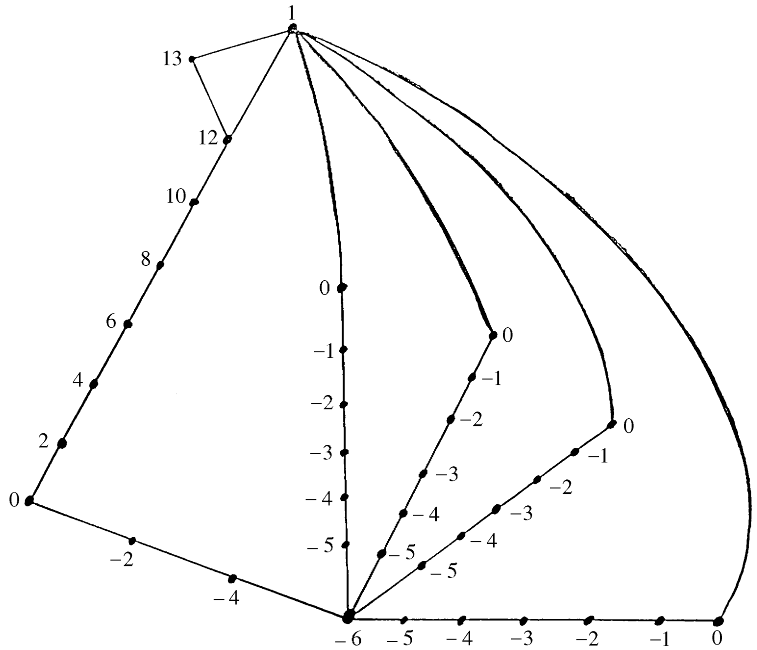

Fig. 1.

in C^2. Fig. 1 produces the configuration graph with multiplicities for $\tilde{E}\cup D_0$ in a minimal very good resolution $p: \bar{X} \to X$ as defined in 2.2 and used in Section 8. In this figure, the vertex weighted by 1 represents the closure L_0 of $\bar{f}^{-1}(0)$ in D_0, the vertex weighted by -6 represents L_∞, and the vertices weighted by 0 represent the dicritical components E_j, $j = 1,\dots,5$. The graph $G(D_0)$ weighted by multiplicities is drawn in Fig. 2.

One sees from Fig. 2 that the rank of $H_1(G(D_0), Z)$ is equal to 1. From Fig. 1 and Theorems 8.10 and 8.15 we obtain the Jordan structure of $H_1(F, Z)$ for the monodromy around 0. For $i = -1,0,1,2$ the characteristic polynomial Δ_i associated to W_i/W_{i-1} is given as follows:

$$\Delta_{-1} = (t^2-1)(t-1)^3, \quad \Delta_0 = \Delta_2 = (t-1), \quad \Delta_1 = (t+1)^{-1}(t-1)(t^{12}-1).$$

One can see that L_0 has genus 1, that $\hat{L}_0$ has six boundary components and that $\rho \circ T$ is injective. The rank of the proper space for the eigenvalue 1 in $H_1(F, Z)$ is the same as the rank of $H_1(\hat{L}_0, Z)$ and is equal to 7 (compare with Theorem 8.12). Hence we get.

Proposition 9.1. *For the action induced by the monodromy h_0 around* 0, *there is one Jordan* 2*-block in $H_1(\bar{F}, Z)$ for the eigenvalue* 1.

See also [3].

Proposition 9.2. *The filtration induced by the filtration W on the eigenspace for the eigenvalue* 1 *has three nontrivial graded quotients. The dimension of each of them is given by the above computation for the characteristic polynomials.*

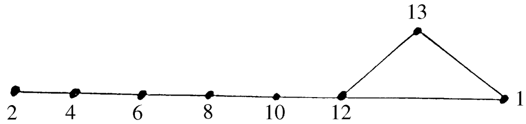

Fig. 2.

Remark 9.3. In the example I_f is equal to $\{0\}$. In [2] Artal shows that f has seven critical values in $C - \{0\}$. Hence the cardinal of A_f is equal to 8. The monodromy h_∞ is not isotopic to h_0^{-1}. Using Fig. 1 and Section 8, the reader can easily compute the characteristic polynomials Δ_i for h_∞.

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References

[1] N. A'Campo, La fonction zeta d'une monodromie, Comment. Math. Helv. 50 (1975) 233–248.
[2] E. Artal-Bartolo, Combinatoire et type topologique des applications polynomiales de C^2 dans C, L'Ens. Math. 39 (1993) 211–224.
[3] E. Artal-Bartolo, P. Cassou-Nogues, A. Dimca, Sur la topologie des polynômes complexes, Singularities, The Briekorn Anniversary Volume, Birkhäuser, Basel, 1998, pp. 317–343.
[4] S.A. Broughton, On the topology of polynomial hypersurfaces, Proc. Symp. Pure Math. 40 (1983) 167–178.
[5] C.H. Clemens Jr., Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities, Trans. Amer. Math. Soc. 136 (1969) 93–108.
[6] P. Deligne, Comparaison avec la théorie transcendante, SGA7 Springer Lecture Notes, vol. 340, 1973, pp. 116–164.
[7] A. Dimca, Monodromy at infinity for polynomials of 2 variables, J. Algebraic Geom. 7 (1998) 771–779.
[8] P. Du Bois, F. Michel, Filtration par le poids et monodromie entière, Bull. Soc. Math. France 120 (1992) 129–167.
[9] P. Du Bois, F. Michel, The integral Seifert form does not determine the topology of plane curve germs, J. Algebraic Geom. 3 (1994) 1–38.
[10] A.H. Durfee, The monodromy of a degenerating family of curves, Invent. Math. 28 (1975) 231–241.
[11] A.H. Durfee, Neighborhoods of algebraic sets, Trans. Amer. Math. Soc. 276 (1983) 517–530.
[12] D. Eisenbud, W. Neumann, Three dimensional link theory and invariants of plane curve singularities, Annals of Mathematical Studies, vol. 110, Princeton Univ. Press, Princeton, NJ, 1985.
[13] E.E. Floyd, Orbit spaces of finite transformation groups I, Duke Math. J. 20 (1953) 563–567.
[14] L. Fourrier, Topologie d'un polynôme de deux variables complexes au voisinage de l'infini, Thèse de l'Universitè de Toulouse, 1993.
[15] P.A. Griffiths, Periods of integrals on algebraic manifolds, Bull. Amer. Math. Soc. 76 (1970) 228–296.
[16] K. Lamotke, Die Homologie isolierter Singularitäten, Math. Z. 143 (1975) 27–44.
[17] D.T. Lê, Some remarks on relative monodromy, Proceedings of Nordic Summer School, Oslo, 1976, pp. 397–403.
[18] D.T. Lê, C. Weber, A geometrical approach to the Jacobian conjecture, Kodai Math. J. 17 (1994) 374–381.
[19] Y. Matsumoto, J.M. Montesinos, Pseudo-periodic homeomorphisms and degeneration of Riemann surfaces, Bull. Amer. Math. Soc. (1994) 70–75.

[20] M. Namba, Geometry of projective algebraic curves, Monographs and Textbooks in Pure and Applied Mathematics, vol. 88, Marcel Dekker Publ, New York, 1984.
[21] W.D. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Amer. Math. Soc. 268 (1981) 299–344.
[22] A. Pichon, Sur les singularités normales d'équation $z^k - f(x, y) = 0$, Thèse, Genève, 1996.
[23] F. Waldhausen, Uber eine Klasse von 3-dimensionalen Mannigfaltigkeiten II, Invent. Math. 4 (1967) 87–117.