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Maximal feebly compact spaces

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Abstract

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Maximal feebly compact spaces (i.e., feebly compact spaces possessing no strictly stronger feebly compact topology) are characterized, as are special classes (countably compact, semiregular, regular) of maximal feebly compact spaces.

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1. Introduction

A topological space X is called *pseudocompact* if every continuous real-valued function with domain X is bounded; it is called *feebly compact* (or, *lightly compact*) if every locally finite collection of open sets is finite. Both classes of spaces have been extensively studied; see [10] or [15], for example. The following well-known relationships link these classes of spaces:

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Theorem 1.1. *A Tychonoff space is feebly compact if and only if it is pseudocompact.*

Theorem 1.2. *Every feebly compact space is pseudocompact, but there are pseudocompact spaces that are not feebly compact.*

A proof of Theorem 1.1 and the first part of Theorem 1.2 can be found in [15, 1.11(d)], examples witnessing the second part of Theorem 1.2 can be found in [17], and one of these appears as [15, problem 1U].

Let τ and σ be two topologies on a set X. If $\tau \subseteq \sigma$ we say that σ is an *expansion* of τ and that τ is a *compression* of σ. If $\sigma \setminus \tau \neq \emptyset$ then σ is a *proper expansion* of τ and τ is a *proper compression* of σ. If $A \subseteq X$, the closure of A in (X, τ) will be denoted by $\mathrm{cl}_\tau A$. If only one topology τ on X is under discussion, we write $\mathrm{cl}\, A$ or $\mathrm{cl}_X A$ instead of $\mathrm{cl}_\tau A$. Similar conventions apply to closures in subspaces.

Let $\mathscr{P}$ be a topological property. A space (X, τ) is said to be a *maximal* $\mathscr{P}$*-space* (respectively, *minimal* $\mathscr{P}$*-space*) if (X, τ) has $\mathscr{P}$ and if σ is a proper expansion (respectively, proper compression) of τ, then (X, σ) does not have $\mathscr{P}$. Various classes of minimal $\mathscr{P}$-spaces have been studied in the past; see [3,5,14] for example. Maximal $\mathscr{P}$-spaces have also been studied; see [16] and the five papers by Cameron listed in the references.

In this paper we study the class of maximal feebly compact spaces and considerably extend the previously known results about these classes. In Section 2 we characterize maximal feebly compact spaces, and then use this characterization to give characterizations of countably compact maximal feebly compact spaces, semiregular maximal feebly compact spaces, and regular maximal feebly compact spaces. We present some examples to indicate limits beyond which our results cannot be extended.

In a companion paper to this [13], we will study maximal pseudocompact spaces, and contrast their properties with those of maximal feebly compact spaces obtained herein.

We conclude this section by listing some known results that we will use in subsequent sections. We assume no separation axioms unless they are explicitly stated. The set of positive integers will be denoted by $\mathbb{N}$.

Theorem 1.3. *The following are equivalent for a space* X:

(1) *X is feebly compact.*

(2) *Every locally finite family of pairwise disjoint open subsets of X is finite.*

(3) *If $\{V_n\colon n \in \mathbb{N}\}$ is a decreasing family of nonempty open subsets of X, then $\bigcap\{cl_X V_n\colon n \in \mathbb{N}\} \neq \emptyset$.*

(4) *Every countable open cover of X has a finite subfamily whose union is dense in X.*

[The equivalence of (1), (2), and (4) in Theorem 1.3 above was proved in [1, Theorem 1], while the equivalence of (1) and (3) was pointed out in [17, Theorem 2.6: (ii)]. A proof of Theorem 1.3 can be found in [15, in 1.11(b)]].

We next list some well-known properties of feebly compact spaces. It is noted in [6, p. 103] that feebly compact spaces possess these properties; also see [15, 1Q].

Theorem 1.4. (a) *Regular closed subsets (i.e., subsets that are closures of open sets) of feebly compact spaces are feebly compact.*

(b) *If A is a feebly compact subspace of X and if $A \subseteq T \subseteq \mathrm{cl}_X A$, then T is feebly compact.*

(c) *The union of finitely many feebly compact subspaces of a space is feebly compact.*

(d) *Continuous images of feebly compact spaces are feebly compact.*

Here are some properties of maximal feebly compact spaces (previously obtained by Cameron and by Raha) that we will need.

Theorem 1.5 [6, Theorem 4]. *A maximal feebly compact space is T_1.*

Definition 1.6. A topological space is called *submaximal* if every dense subset of it is open.

See [4, Exercise 22 of I(8)] for information on this class of spaces.

Theorem 1.7 [6, Corollary 2]. *If X is feebly compact, submaximal, and each feebly compact subspace of X is closed, then X is maximal feebly compact.*

Theorem 1.8 [16, Theorem 14]. *If X is a maximal feebly compact space, then X is submaximal.*

The following is a special case of one direction of [6, Theorem 3].

Theorem 1.9. *If X is a maximal feebly compact space, and if A and $X \setminus \mathrm{int}_X A$ are both feebly compact subspaces of X, then A is closed in X.*

2. Characterizations of maximal feebly compact spaces

We begin by giving a very useful internal characterization of maximal feebly compact spaces. First we need a lemma.

Lemma 2.1 [4, I(8), 22(e)]. *If S is a dense subspace of a submaximal space X, then $X \setminus S$ is a closed discrete subspace of X.*

Proof. Let $A \subseteq X \setminus S$. Then $S \cup A$ is dense in X and hence open in X. Since $A = (X \setminus S) \cap (S \cup A)$, A is open in $X \setminus S$. □

Theorem 2.2. *The following are equivalent for a feebly compact space X:*
(a) *X is a maximal feebly compact space.*
(b) (i) *X is submaximal (see Definition* 1.6), *and*
(ii) *feebly compact subspaces of X are closed in X.*

Proof. By Theorem 1.7, (b) implies (a). By Theorem 1.8, (a) implies (b)(i). It remains to show that (a) implies (b)(ii).

Let A be a feebly compact subset of X. We claim that $\mathrm{int}_X \mathrm{cl}_X A \subseteq A$. For suppose not; let $p \in \mathrm{int}_X \mathrm{cl}_X A \setminus A$. Hence p is not an isolated point, so by Theorem 1.9, $X \setminus \{p\}$ is not feebly compact. However,

$$X \setminus \{p\} = (X \setminus \mathrm{int}_X \mathrm{cl}_X A) \cup [(A \cup \mathrm{int}_X \mathrm{cl}_X A) \setminus \{p\}].$$

Now $X \setminus \mathrm{int}_X \mathrm{cl}_X A$ is feebly compact by Theorem 1.4(a), and $(A \cup \mathrm{int}_X \mathrm{cl}_X A) \setminus \{p\}$ is feebly compact by Theorem 1.4(b). Hence $X \setminus \{p\}$ is feebly compact by Theorem 1.4(c), contradicting what was proved above. Hence our claim holds, i.e., $\mathrm{int}_X \mathrm{cl}_X A \subseteq A$. From this it follows that $\mathrm{int}_X A = \mathrm{int}_X \mathrm{cl}_X A$, and hence by Theorem 1.4(a), $X \setminus \mathrm{int}_X A$ is feebly compact. Thus by Theorem 1.9, A is closed in X. □

Clearly the hypothesis that X is feebly compact is needed in Theorem 2.2, as an infinite discrete space satisfies Theorem 2.2(b) but fails to be feebly compact.

Corollary 2.3. *Feebly compact subspaces of maximal feebly compact spaces are maximal feebly compact. In particular, maximal feeble compactness is inherited by regular closed subspaces.*

Proof. Clearly Theorem 2.2(b)(ii) is a hereditary property. If S is a feebly compact subspace of the maximal feebly compact space X, and if T is a dense subset of S, then $T \cup (X \setminus \mathrm{cl}_X S)$ is dense in X, and hence by Theorem 2.2 open in X. Its intersection with S is T; hence T is open in S. By Theorem 2.2, S is maximal feebly compact. The second assertion follows from the first assertion and from Theorem 1.4. □

We can use Theorem 2.2 to provide an alternate characterization of maximal feebly compact spaces.

Corollary 2.4. *The following are equivalent for a feebly compact submaximal space:*
(a) *X is maximal feebly compact.*
(b) *Feebly compact subspaces of X are closed.*
(c) *X is T_1 and each feebly compact subset of X is the union of a regular closed subset of X with a finite subset of X.*

Proof. By Theorem 2.2, (a) and (b) are equivalent. Since by hypothesis finite subsets of X are closed, (c) clearly implies (b). To show that (b) implies (c), let A be a feebly compact subset of X. By (b), $\mathrm{cl}_X \mathrm{int}_X A \subseteq A$. Let $S = \mathrm{cl}_A(A \setminus \mathrm{cl}_X \mathrm{int}_X A)$. As S is a regular closed subset of the feebly compact space A, by Theorem 1.4(a), S is feebly compact. As $S \subseteq A \setminus \mathrm{int}_X A$, it follows from Lemma 2.1 that S is discrete. But feebly compact discrete spaces are finite, and since $A = (A \setminus \mathrm{cl}_X \mathrm{int}_X A) \cup (\mathrm{cl}_X \mathrm{int}_X A)$, the desired implication follows. $\square$

Notation 2.5. The set of isolated points of a space (X, τ) will be denoted by $I(X, \tau)$ (or by $I(X)$ if only one topology on X is under discussion).

We now develop another characterization of those spaces whose feebly compact subspaces are closed.

Definition 2.6. (a) A point p of a space X is called an *E_1-point* (respectively, an *almost E_1-point*) of X if p has a countable family $(V_n)_{n \in \mathbb{N}}$ of open neighborhoods such that $\{p\} = \bigcap\{\mathrm{cl}_X V_n \colon n \in \mathbb{N}\}$ (respectively, there is a countable family $(F_n)_{n \in \mathbb{N}}$ of closed sets of X such that $\{p\} = \bigcap\{F_n \colon n \in \mathbb{N}\}$ and for all $n \in \mathbb{N}$, $\emptyset \neq \mathrm{int}_X F_{n+1} \subseteq F_{n+1} \subseteq F_n$).

(b) A space is called an *E_1-space* (respectively, an *almost E_1-space*) if all of its points are E_1-points (respectively, almost E_1-points).

Clearly almost E_1-spaces are T_1 and E_1-spaces are T_2. In [12, 2.1] it is shown that feebly compact subspaces of E_1-spaces are closed. More generally, we have the following

Proposition 2.7. *Let X be a T_1-space.*

(a) *Suppose $p \in X \setminus I(X)$. Then p is an almost E_1-point of X if and only if $X \setminus \{p\}$ is not feebly compact.*

(b) *X is an almost E_1-space if and only if none of its proper dense subspaces is feebly compact.*

(c) *Every feebly compact subspace of X is closed if and only if every closed subspace of X is an almost E_1-space.*

(d) *Every feebly compact subspace of X with a dense interior is closed if and only if every regular closed subspace of X is an almost E_1-space.*

(e) *Let X be submaximal. Then every feebly compact subspace of X is closed if and only if every regular closed subspace of X is an almost E_1-space if and only if every feebly compact subspace of X with a dense interior is closed.*

Proof. (a) Suppose $X \setminus \{p\}$ is not feebly compact. Then by Theorem 1.3 there is a countable open cover $(V_n)_{n \in \mathbb{N}}$ of $X \setminus \{p\}$ such that for all $k \in \mathbb{N}$, $(X \setminus \{p\}) \setminus \mathrm{cl}_{X \setminus \{p\}}[\bigcup\{V_n \colon 1 \leqslant n \leqslant k\}] \neq \emptyset$. Without loss of generality we may assume that $V_n \subseteq V_{n+1}$ for each $n \in \mathbb{N}$. Each V_n is open in X (as X is T_1), and evidently

$\mathrm{cl}_X V_n \neq X\setminus\{p\}$ for each $n \in \mathbb{N}$. Let $F_n = X\setminus V_n$; then $\{p\} = \bigcap\{F_n\colon n \in \mathbb{N}\}$ and $\mathrm{int}_X F_n \neq \emptyset$ for each $n \in \mathbb{N}$. Hence p is an almost E_1-point of X.

Conversely, suppose that $\{p\}$ is an almost E_1-point of X. Let $\{p\} = \bigcap\{F_n\colon n \in \mathbb{N}\}$, where F_n is closed, $F_{n+1} \subseteq F_n$, and $\mathrm{int}_X F_n \neq \emptyset$. Then $X\setminus\{p\} = \bigcup\{X\setminus F_n\colon n \in \mathbb{N}\}$ and each $X\setminus F_n$ is open in $X\setminus\{p\}$. As $p \notin I(X)$, it follows that $(\mathrm{int}_X F_n)\setminus\{p\} \neq \emptyset$ for each $n \in \mathbb{N}$ and so $\mathrm{cl}_{X\setminus\{p\}}[(X\setminus\{p\})\setminus F_n] \neq X\setminus\{p\}$ for all n. As $\{X\setminus F_n\colon n \in \mathbb{N}\}$ is an increasing open cover of $X\setminus\{p\}$, it follows that $X\setminus\{p\}$ is not feebly compact.

(b) Suppose X has a proper dense feebly compact subspace S. Let $p \in X\setminus S$; then by Theorem 1.4(b), $X\setminus\{p\}$ is feebly compact. Hence by (a), p is not an almost E_1-point of X, and X is not an almost E_1-space. Conversely, if p is not an almost E_1-point of X, then $p \notin I(X)$. Hence by (a), $X\setminus\{p\}$ is a proper dense feebly compact subspace of X.

(c) Obviously X has a feebly compact subspace S that is not closed if and only if X has a feebly compact closed subspace $\mathrm{cl}_X S$ that has a proper dense feebly compact subspace S (see Theorem 1.4(b)). Now use (b).

(d) Suppose $\mathrm{int}_X S$ is dense in S and that every regular closed subspace of X is an almost E_1-space. Then $\mathrm{cl}_X S$ is regular closed and $\mathrm{int}_X S$ is dense in $\mathrm{cl}_X S$. If S is feebly compact, then by (b), $S = \mathrm{cl}_X S$. Conversely, suppose that every feebly compact subset of X with dense interior is closed. If G is a proper dense subset of the regular closed set A, let $p \in A\setminus G$ and let $S = A\setminus\{p\}$. As $\mathrm{int}_X S$ is dense in S but S is not closed, by hypothesis S cannot be feebly compact. Hence its dense subset G cannot be either. Thus, by (b), A is an almost E_1-space.

(e) By (c), it suffices to show that if all regular closed subspaces are almost E_1-spaces, then all closed subspaces are almost E_1-spaces. Let S be a closed subspace of X. By Lemma 2.1, $S\setminus \mathrm{cl}_X \mathrm{int}_X S \subseteq I(S)$, each point of which is evidently an almost E_1-point of S. If $p \in \mathrm{cl}_X \mathrm{int}_X S$, then p is an almost E_1-point of $\mathrm{cl}_X \mathrm{int}_X S$ by hypothesis, so there is a decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of $\mathrm{cl}_X \mathrm{int}_X S$ such that $\{p\} = \bigcap\{F_n\colon n \in \mathbb{N}\}$ and $\mathrm{int}_{\mathrm{cl}_X \mathrm{int}_X S} F_n \neq \emptyset$. This latter condition implies that $\mathrm{int}_X F_n \neq \emptyset$ and hence $\mathrm{int}_S F_n \neq \emptyset$. As each F_n is closed in S, clearly p is an almost E_1-point of S. The final equivalence follows from (d). □

Corollary 2.8. *A feebly compact space is maximal feebly compact if and only if it is submaximal and all its regular closed subsets are almost E_1-spaces; equivalently, it is submaximal and all its feebly compact subspaces with dense interior are closed.*

Proof. This follows from Theorem 2.2 and Proposition 2.7. □

An almost E_1, feebly compact, submaximal T_2-space need not be maximal feebly compact; see Example 2.22.

We now investigate the structure of countably compact maximal feebly compact spaces. This leads to the characterization given in Theorem 2.11 below.

Lemma 2.9. *Let $\mathscr{P}$ be a topological property that is inherited by regular closed subspaces. Suppose that each maximal feebly compact space with $\mathscr{P}$ has an isolated point. Then if X is a maximal feebly compact space with $\mathscr{P}$, $I(X)$ is dense in X and $X \setminus I(X)$ is a closed discrete subspace of X.*

Proof. If $I(X)$ were not dense in X, then $\mathrm{cl}_X(X \setminus \mathrm{cl}_X I(X))$ would be a maximal feebly compact space by Corollary 2.3, and would have $\mathscr{P}$ as it is regular closed in X. But $I(\mathrm{cl}_X(X \setminus \mathrm{cl}_X I(X))) = \emptyset$, in contradiction to the hypothesis. Hence $I(X)$ is dense in X. The remainder of the lemma follows from Lemma 2.1. □

Lemma 2.10. *A nonempty countably compact maximal feebly compact space has an isolated point.*

Proof. Let (X, τ) be a nonempty countably compact and maximal feebly compact space. If X is finite, by Theorem 1.5, τ is discrete; hence assume X is infinite.

Observe that every dense open subset of (X, τ) is cofinite; for if V were a dense open subset of (X, τ), by Theorem 2.2 and Lemma 2.1, $X \setminus V$ would be discrete, and hence finite as (X, τ) is countably compact. However, not every nonempty member of τ is cofinite; for if it were, define a new topology σ on X as follows. Let $p \in X$, let every subset of $X \setminus \{p\}$ belong to σ, and let sets containing p belong to σ if and only if they are cofinite. Then (X, σ) is the one-point compactification of the discrete space $X \setminus \{p\}$, and clearly $\tau \subsetneq \sigma$. This contradicts the maximality of τ and hence there exists a nonempty set $V \in \tau$ such that $X \setminus \mathrm{cl}_X V \neq \emptyset$. By Corollary 2.3, $\mathrm{cl}_X V$ and $\mathrm{cl}_X(X \setminus \mathrm{cl}_X V)$ are countably compact maximal feebly compact subspaces of X. If either were finite, it would follow immediately that X has an isolated point. So, assume that both are infinite.

Now repeat the above reasoning with the space $\mathrm{cl}_X V$, and continue inductively. If X has no isolated points, this procedure yields a countably infinite family of pairwise disjoint nonempty open subsets of X. Choose one point from each set, and obtain a countably infinite discrete subspace D of X. If X has no isolated points, then $\mathrm{int}_X D = \emptyset$, so as X is submaximal by Theorem 2.2, it follows that D is closed. This contradicts the countable compactness of X, and the lemma follows. □

Theorem 2.11. (a) *The following are equivalent for a space X.*

(1) *X is countably compact and maximal feebly compact.*

(2) *X is compact T_1, $X \setminus I(X)$ is finite, and if V is open and $p \in \mathrm{cl}\, V \setminus V$, then there is a sequence $(x_n)_{n \in \mathbb{N}} \subseteq I(V)$ such that $\mathrm{cl}_X\{x_n : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\} \cup \{p\}$.*

(b) *An infinite countably compact Hausdorff space is maximal feebly compact if and only if it is the topological sum of a finite number of one-point compactifications of infinite discrete spaces.*

Proof. (a) Let X be a countably compact maximal feebly compact space. By Lemmas 2.9 and 2.10, $I(X)$ is dense in X, and $X\backslash I(X)$ is closed and discrete. Hence $X\backslash I(X)$ is countably compact and therefore finite. Suppose $\mathscr{C}$ is an open cover of X. For each $x \in X\backslash I(X)$, find $C_x \in \mathscr{C}$ such that $x \in C_x$. Denote $X\backslash \bigcup\{C_x: x \in X\backslash I(X)\}$ by T; then T is a closed discrete subspace of the countably compact space X and hence is finite. For each $t \in T$, find $C_t \in \mathscr{C}$ such that $t \in C_t$. Then $\{C_x: x \in X\backslash I(X)\} \cup \{C_t: t \in T\}$ is a finite subcover of $\mathscr{C}$. Hence X is compact. By Theorem 1.5, X is T_1. Finally, let V be open in X and $p \in \mathrm{cl}\, V\backslash V$. By Proposition 2.7(a) and Corollary 2.8, $\mathrm{cl}\, V\backslash\{p\}$ is not feebly compact, so there is a pairwise disjoint, locally finite infinite sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of $\mathrm{cl}\, V\backslash\{p\}$. As $I(X)$ is dense in X, we can choose $x_n \in U_n \cap V \cap I(X)$ for each $n \in \mathbb{N}$. Then $(\{x_n\})_{n\in\mathbb{N}}$ is locally finite in $\mathrm{cl}\, V\backslash\{p\}$, and disjoint from the open set $X\backslash \mathrm{cl}\, V$. Hence $\mathrm{cl}\{x_n: n \in \mathbb{N}\} \subseteq \{x_n: n \in \mathbb{N}\} \cup \{p\}$. But as X is feebly compact $(\{x_n\})_{n\in\mathbb{N}}$ is not locally finite in X; hence $\mathrm{cl}\{x_n: n \in \mathbb{N}\} \supseteq \{x_n: n \in \mathbb{N}\} \cup \{p\}$. This completes the proof that (1) implies (2).

To show that (2) implies (1), observe that as X is compact T_1 and $X\backslash I(X)$ is finite, X is countably compact (and hence feebly compact) and submaximal. If U is open in X, let $p \in \mathrm{cl}\, U\backslash I(\mathrm{cl}\, U)$ and let $V = U\backslash\{p\}$. Then V is open as X is T_1 and $p \in \mathrm{cl}\, V\backslash V$. Hence by hypothesis there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq I(V)$ such that $\mathrm{cl}\{x_n: n \in \mathbb{N}\} = \{x_n: n \in \mathbb{N}\} \cup \{p\}$. Thus $(\{x_n\})_{n\in\mathbb{N}}$ is an infinite locally finite collection of open sets of $\mathrm{cl}\, U\backslash\{p\}$, and so $\mathrm{cl}\, U\backslash\{p\}$ is not feebly compact. Hence by Proposition 2.7(a) and Corollary 2.8, X is maximal feebly compact.

(b) If X is an infinite countably compact maximal feebly compact Hausdorff space, then $X\backslash I(X)$ is a finite set F and X is compact. As X is T_2 for each $x \in F$ there exists an open set $V(x)$ of X such that $x \in V(x)$ and $V(x) \cap V(y) = \emptyset$ if x, $y \in F$ and $x \neq y$. Then $V(x) = X\backslash[\bigcup\{V(y): y \in F\backslash\{x\}\} \cup (I(X)\backslash V(x))]$, so $V(x)$ is closed in X and hence compact. Thus $V(x)$ is the one-point compactification of the discrete space $V(x)\backslash\{x\}$. Arguing as in (a), we see that $X\backslash \bigcup\{V(x): x \in F\}$ is finite; attach it to a $V(x)$, and then observe that $X = \oplus\{V(x): x \in F\}$. Conversely, a topological sum of finitely many one-point compactifications of discrete spaces is compact Hausdorff, and satisfies the conditions of Theorem 2.2(b); hence it is maximal feebly compact. □

Example 2.12(a) below shows that countably compact maximal feebly compact spaces need not be Hausdorff; thus "T_1" cannot be replaced by "T_2" in statement (2) of Theorem 2.11(a). This example was constructed by Balachandran [2] as an example of a non-Hausdorff maximal compact space which is countable. Example 2.12(b) shows that an almost E_1 countable, compact semiregular T_1-space with two nonisolated points need not be maximal feebly compact. This space also has the property that if A is a regular closed subset of it and if $p \in A\backslash I(A)$, then there is a sequence of points of $I(A)$ that converges to p. Recall that a space (X, τ) is *semiregular* if its set $\mathrm{RO}(X, \tau)$ of regular open sets (i.e., complements of regular closed sets) forms a base for τ. Thus if the condition on p in the second part of

Theorem 2.11(a)(2) is weakened only slightly, the resulting statement is not equivalent to Theorem 2.11(a)(1).

Example 2.12. (a) Let $X = (\mathbb{N} \times \mathbb{N}) \cup \{a, b\}$ where $a, b \notin \mathbb{N} \times \mathbb{N}$. Give X a topology τ defined as follows: each subset of $\mathbb{N} \times \mathbb{N}$ belongs to τ, as do $\emptyset$ and X. If $a \in V \subseteq X$, then V is a τ-neighborhood of a if $(\{n\} \times \mathbb{N}) \setminus V$ is finite for each $n \in \mathbb{N}$; if $b \in U \subseteq X$, then U is a τ-neighborhood of b if $\{n\} \times \mathbb{N} \subseteq U$ for all but finitely many $n \in \mathbb{N}$. It is routine to show that τ is a compact submaximal T_1-topology on X, and that feebly compact subspaces of (X, τ) are closed. It now follows from Theorem 2.2 that (X, τ) is a maximal feebly compact space. It is not Hausdorff, and in fact cannot be written as a union of finitely many compact Hausdorff subspaces. If we choose V so that $(\{n\} \times \mathbb{N}) \setminus V \neq \emptyset$ for infinitely many n and $a \in V$, then $\operatorname{int}_\tau \operatorname{cl}_\tau V = V$. This and a proper choice of U for b show that (X, τ) is semiregular.

(b) Partition $\mathbb{N}$ into three infinite sets A, B, and C. Let p and q be two points not in $\mathbb{N}$, and let $X = \{p, q\} \cup A \cup B \cup C$. Define a topology τ as follows: $\tau = \{V \subseteq X$: if $p \in V$ then $(A \cup C) \setminus V$ is finite, and if $q \in V$ then $(B \cup C) \setminus V$ is finite$\}$. The reader can easily verify that (X, τ) has the properties advertised above.

Recall that we can associate with each space (X, τ) its *semiregularization* $(X, \tau(s))$; $\tau(s)$ is the compression of τ for which $\mathrm{RO}(X, \tau)$ (defined preceding Example 2.12) forms an open base. The elementary properties of $(X, \tau(s))$ with which we will be concerned are summarized in Proposition 2.13 below. All of Proposition 2.13 except for (d) and (g) appear in [15, 2.2(f), (g) and (i)].

Proposition 2.13. *Let (X, τ) be a space.*

(a) *If (X, τ) is Hausdorff, then so is $(X, \tau(s))$.*

(b) $\mathrm{RO}(X, \tau) = \mathrm{RO}(X, \tau(s))$; *in particular $(X, \tau(s))$ is semiregular and if $(X, \tau(s))$ is T_1 then $I(X, \tau) = I(X, \tau(s))$.*

(c) *(X, τ) is semiregular iff $\tau = \tau(s)$.*

(d) *(X, τ) is feebly compact iff $(X, \tau(s))$ is feebly compact.*

(e) *Let Z be a dense subset of (X, τ). Then $(\tau \mid Z)(s) = \tau(s) \mid Z$; in particular, dense subspaces of semiregular spaces are semiregular.*

(f) *Regular spaces are semiregular.*

(g) *(X, τ) is an almost E_1-space iff $(X, \tau(s))$ is an almost E_1-space.*

As is well known (and illustrated by Examples 2.12(b) and 2.19) the converse of (f) fails.

Proof. (d) If (X, τ) is feebly compact then so is $(X, \tau(s))$ by Theorem 1.4(d) since the identity map from (X, τ) onto $(X, \tau(s))$ is clearly continuous. The converse follows easily from the fact that if $V, W \in \tau$, then $V \cap W = \emptyset$ iff $\operatorname{int}_\tau \operatorname{cl}_\tau V \cap \operatorname{int}_\tau \operatorname{cl}_\tau W = \emptyset$.

(g) (X, τ) is almost E_1 iff $(X\setminus\{p\}, \tau | X\setminus\{p\})$ fails to be feebly compact for each $p \notin I(X, \tau)$ by Proposition 2.7(a). By (b) and (a) above, this happens iff $(X\setminus\{p\}, (\tau | X\setminus\{p\})(s))$ fails to be feebly compact for each $p \notin I(X, \tau(s))$. But $(\tau | X\setminus\{p\})(s) = \tau(s) | X\setminus\{p\}$ by (e), so this is equivalent to saying that $(X\setminus\{p\}, \tau(s) | X\setminus\{p\})$ fails to be feebly compact for each $p \in X\setminus I(X, \tau(s))$. By Proposition 2.7(a) again, this is equivalent to $(X, \tau(s))$ being almost E_1. □

Lemma 2.14. *Let (X, τ) be feebly compact and almost E_1. Suppose that $p \in X\setminus I(X, \tau)$. Then there is a countably infinite pairwise disjoint family $(U_n)_{n \in N}$ of nonempty open subsets of $X\setminus\{p\}$ that is locally finite in $X\setminus\{p\}$ such that if $p \in V \in \tau$, then $A_V = \{n \in \mathbb{N}: U_n \cap V$ is dense in $U_n\}$ is cofinite.*

Proof. By Proposition 2.7(a), $X\setminus\{p\}$ is not feebly compact, so there is a countably infinite pairwise disjoint family $(U_n)_{n \in \mathbb{N}}$ of nonempty open subsets of $X\setminus\{p\}$ that is locally finite in $X\setminus\{p\}$. As X is almost E_1, X is a T_1-space and so each $U_n \in \tau$. If $\mathbb{N}\setminus A_V$ were infinite for some $V \in \tau$ for which $p \in V$, then $\{U_n \setminus \mathrm{cl}_\tau V: n \in \mathbb{N}\setminus A_V\}$ would be an infinite pairwise disjoint locally finite family of nonempty members of τ, in contradiction to the feeble compactness of (X, τ). □

A *π-base* at a point p of a space (X, τ) is a family $\mathscr{F} \subseteq \tau\setminus\{\emptyset\}$ such that if $p \in V \in \tau$, then there exists $F \in \mathscr{F}$ such that $F \subseteq V$. A space has *countable π-character* if each of its points has a countable π-base. See [11] for more details.

Lemma 2.15. *Let (X, τ) be a feebly compact almost E_1-space and suppose $p \in X\setminus I(X, \tau)$. Then:*

(1) *There exists a countably infinite pairwise disjoint family $(U_n)_{n \in \mathbb{N}} \subseteq \tau(s)\setminus\{\emptyset\}$ such that $(U_n)_{n \in \mathbb{N}}$ is a locally finite family of subsets of the subspace $X\setminus\{p\}$ of $(X, \tau(s))$, and if $p \in V \in \tau(s)$ then $\{n \in \mathbb{N}: U_n \subseteq V\}$ is a cofinite subset of $\mathbb{N}$.*

(2) *If $x_n \in U_n$ for each $n \in \mathbb{N}$, then $\mathrm{cl}_{\tau(s)}\{x_n: n \in \mathbb{N}\} = \{x_n: n \in \mathbb{N}\} \cup \{p\}$.*

(3) *$(X, \tau(s))$ has countable π-character.*

Proof. (1) By Proposition 2.13(b), (d), (g), $(X, \tau(s))$ is feebly compact and almost E_1, and $p \in X\setminus I(X, \tau(s))$. Thus by Lemma 2.14 there is a countably infinite pairwise disjoint family $(U_n)_{n \in \mathbb{N}}$ of nonempty open subsets of the subspace $X\setminus\{p\}$ of $(X, \tau(s))$, $(U_n)_{n \in \mathbb{N}}$ is a locally finite family in this subspace, and if $p \in V \in \tau(s)$, then A_V (as defined in Lemma 2.14) is cofinite. But if $p \in V \in \tau(s)$, there exists $W \in \mathrm{RO}(X, \tau(s))$ such that $p \in W \subseteq V$. Then A_W is also cofinite, and if $n \in A_W$ then $U_n \cap W$ is dense in U_n (as a subspace of $(X, \tau(s))$) and so $U_n \subseteq \mathrm{int}_{\tau(s)}\mathrm{cl}_{\tau(s)}W = W \subseteq V$.

(2) As $(X, \tau(s))$ is T_1 and $(U_n)_{n \in \mathbb{N}}$ is a locally finite family in the subspace $X\setminus\{p\}$ of $(X, \tau(s))$, no point of $X\setminus\{p\}$ is a $\tau(s)$-limit point of $\{x_n: n \in \mathbb{N}\}$. If $p \in V \in \tau(s)$ then V contains all but finitely many x_n; hence the result follows.

(3) As $(X, \tau(s))$ is T_1, each U_n is in $\tau(s)$. Thus $(U_n)_{n\in\mathbb{N}}$ is a π-base at p in $(X, \tau(s))$. As $\{\{x\}\}$ is a π-base at an isolated point x, $(X, \tau(s))$ has countable π-character. □

Proposition 2.16. *The following are equivalent for a semiregular feebly compact space* X:

(a) X *is submaximal and almost* E_1.

(b) X *is* T_1, $I(X)$ *is dense in* X, $X \setminus I(X)$ *is discrete, and if* $p \in X \setminus I(X)$, *then there is a sequence* $(x_n)_{n\in\mathbb{N}} \subseteq I(X)$ *for which* $\mathrm{cl}\{x_n\colon n \in \mathbb{N}\} = \{x_n\colon n \in N\} \cup \{p\}$.

Proof. (a) implies (b): As X is almost E_1, it is T_1. Let $p \in X \setminus I(X)$; by Proposition 2.13(c) and Lemma 2.15 there is a pairwise disjoint family $(U_n)_{n\in\mathbb{N}}$ of nonempty open sets of $X \setminus \{p\}$ such that if $p \in V$ and V is open, then $J_V = \{n \in \mathbb{N}\colon U_n \subseteq V\}$ is cofinite. Choose $x_n \in U_n$ for each $n \in N$, and let $\{n \in \mathbb{N}\colon x_n$ is not isolated in $X\} = B$. As $\{x_n\colon n \in B\}$ is discrete, it has empty interior in X and hence is closed since X is submaximal. If B were infinite, then as J_V is cofinite, p would belong to $\mathrm{cl}_X\{x_n\colon n \in B\} \setminus \{x_n\colon n \in B\}$, which would be a contradiction. Hence B is finite, and so x_n is isolated in X for all but finitely many n. Then $\mathrm{cl}\{x_n\colon n \in \mathbb{N} \setminus B\} = \{x_n\colon n \in \mathbb{N} \setminus B\} \cup \{p\}$ by Lemma 2.15(2). It follows immediately that $I(X)$ is dense in X. Since $X \setminus I(X)$ is nowhere dense, it is discrete as it is submaximal.

(b) implies (a): As $I(X)$ is dense and $X \setminus I(X)$ is discrete, it follows immediately that X is submaximal. If $p \in X \setminus I(X)$, let $(x_n)_{n\in\mathbb{N}} \subseteq I(X)$ such that $\mathrm{cl}\{x_n\colon n \in \mathbb{N}\} = \{x_n\colon n \in \mathbb{N}\} \cup \{p\}$. Define F_n to be $\{x_j\colon j \geqslant n\} \cup \{p\}$; then $(F_n)_{n\in\mathbb{N}}$ witnesses that $\{p\}$ is an almost E_1-point (see Definition 2.6(a)). As isolated points are obviously almost E_1-points, X is an almost E_1-space. □

Now we characterize semiregular maximal feebly compact spaces.

Theorem 2.17. *The following are equivalent for a feebly compact semiregular space* (X, τ).

(a) (X, τ) *is maximal feebly compact.*

(b) $I(X)$ *is dense in* X, $X \setminus I(X)$ *is discrete*, X *is* T_1, *and if* $V \in \tau$ *and* $p \in \mathrm{cl}_\tau V \setminus V$, *there is a sequence* $(x_n)_{n\in\mathbb{N}} \subseteq V \cap I(X)$ *such that* $\mathrm{cl}_\tau\{x_n\colon n \in \mathbb{N}\} = \{x_n\colon n \in \mathbb{N}\} \cup \{p\}$.

Proof. (a) $\Rightarrow$ (b): By Corollary 2.8, (X, τ) is submaximal and almost E_1, so the first three assertions follow from Proposition 2.16. If $V \in \tau$, then by Corollary 2.3, $\mathrm{cl}_\tau V$ is maximal feebly compact. If $p \in \mathrm{cl}_\tau V \setminus V$ then $p \notin I(\mathrm{cl}_\tau V)$, so by applying the proof of Lemma 2.15 to $(\mathrm{cl}_\tau V, \tau | \mathrm{cl}_\tau V)$ there exists $(x_n)_{n\in\mathbb{N}} \subseteq I(\mathrm{cl}_\tau V)$ such that $\mathrm{cl}_\tau\{x_n\colon n \in \mathbb{N}\} = \{x_n\colon n \in \mathbb{N}\} \cup \{p\}$. But evidently $I(\mathrm{cl}_\tau V) \subseteq V \cap I(X)$.

(b) $\Rightarrow$ (a): By Proposition 2.16, (X, τ) is submaximal. Let $V \in \tau$, and suppose $p \in \mathrm{cl}_\tau V \setminus I(\mathrm{cl}_\tau V)$. Let $U = V \setminus \{p\}$; as X is T_1, $U \in \tau$ and $p \in \mathrm{cl}_\tau U \setminus U$. Hence by (b) there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq U \cap I(X)$ such that $\mathrm{cl}\{x_n\colon n \in \mathbb{N}\} = \{x_n\colon n \in \mathbb{N}\} \cup \{p\}$. Then $(\{x_n\})_{n\in\mathbb{N}}$ is an infinite locally finite family of nonempty open subsets of $\mathrm{cl}_\tau V\backslash\{p\}$, and so $\mathrm{cl}_\tau V\backslash\{p\}$ is not feebly compact. It follows from Proposition 2.7(b) that $\mathrm{cl}_\tau V$ is an almost E_1-subspace of (X, τ). Hence by Corollary 2.8, (X, τ) is maximal feebly compact. □

Remark 2.18. (a) Note that if (X, τ) is Hausdorff then the condition "$\mathrm{cl}\{x_n: n \in \mathbb{N}\} = \{x_n: n \in \mathbb{N}\} \cup \{p\}$" can be replaced by the condition "the sequence $(x_n)_{n\in\mathbb{N}}$ converges to p" in Lemma 2.15, Proposition 2.16, and Theorem 2.17. However, this replacement cannot be made in general, as witnessed by the fact that Example 2.12(b) satisfies Theorem 2.17(b) but not Theorem 2.17(a).

(b) Example 2.12(a) witnesses the fact that semiregular maximal feebly compact spaces need not be Hausdorff.

Recall that a space is *Urysohn* if distinct points can be put inside disjoint closed neighborhoods; obviously Urysohn spaces are Hausdorff.

Example 2.19. A countable semiregular maximal feebly compact Hausdorff space need not be regular, and a countable Urysohn maximal feebly compact space need not be regular. To see this, we consider the space presented in [15, 4.8]; we describe it briefly here. Let $U = \{(1/n, 1/k): n, k \in \mathbb{N}\}$, $L = \{(1/n, -1/k): n, k \in \mathbb{N}\}$, and $A = \{(1/n, 0): n \in \mathbb{N}\}$. Let $Y = U \cup L \cup A$ with the subspace topology inherited from the Euclidean plane. Let $X = Y \cup \{p, q\}$, where $p, q \notin Y$, and define the topology τ on X as follows: $\tau = \{V \subseteq X: V \cap Y$ is open in Y and $p \in V$ (respectively, $q \in V$) implies that there exists $j \in \mathbb{N}$ such that $\{(1/n, 1/k): n \geqslant j, k \in \mathbb{N}\}$ (respectively, $\{(1/n, -1/k): n \geqslant j, k \in \mathbb{N}\}$) is a subset of V$\}$. Then (X, τ) is a Hausdorff space, and as it is H-closed and countable, its feebly compact subspaces are H-closed and hence closed (see [15, 4.8(d)] for details). Clearly $I(X) = U \cup L$, and $X\backslash I(X)$ is discrete. Hence (X, τ) is submaximal and therefore maximal feebly compact by Theorem 2.2. As each point of Y has a neighborhood base of compact open sets, and as the neighborhoods of p and q described above are easily verified to be regular open, it follows that (X, τ) is semiregular. However, it is not regular as p and A cannot be put inside disjoint open sets.

Let $Z = X\backslash(L \cup \{q\})$; then Z is a regular closed subspace of X so by Corollary 2.3, Z is maximal feebly compact. Note however that Z is not semiregular as p does not have a neighborhood base of regular open sets. It is easy to verify that Z is completely Hausdorff (i.e., distinct points can be separated by a continuous real-valued function); however, X is not even Urysohn as p and q cannot be separated by disjoint closed neighborhoods.

Recall that if D is a set, an *almost disjoint family* of subsets of D is a family $\mathscr{M}$ of infinite subsets of D such that distinct members of $\mathscr{M}$ have finite intersection. A *maximal almost disjoint family* (henceforth abbreviated m.a.d. family) of subsets of D is an almost disjoint family that is maximal in the partial ordering by inclusion of sets of subsets of D. We do not require that m.a.d. families be infinite;

for example the family $\{E, O\}$ is a m.a.d. family of subsets of $\mathbb{N}$, where E is the set of even positive integers and $O = \mathbb{N} \setminus E$.

As is well known, associated with each m.a.d. family $\mathcal{M}$ on an infinite set D, one can construct a feebly compact locally compact Hausdorff space $\Psi(\mathcal{M})$ as follows: The underlying set of $\Psi(\mathcal{M})$ is $D \cup \{p(M): M \in \mathcal{M}\}$ where $M_1 \neq M_2$ implies $p(M_1) \neq p(M_2)$ and $D \cap \{p(M): M \in \mathcal{M}\} = \emptyset$. A topology τ on $\Psi(\mathcal{M})$ is defined as follows: $S \in \tau$ if and only if whenever $M \in \mathcal{M}$ and $p(M) \in S$, then $M \setminus S$ is finite. Thus topologized, $\Psi(\mathcal{M})$ is a locally compact Hausdorff feebly compact space, $I(\Psi(\mathcal{M})) = D$, and $\Psi(\mathcal{M}) \setminus I(\Psi(\mathcal{M}))$ is a closed discrete space. (See, for example [10, 5I] or [15, 1N] for proofs of these assertions; although D is taken to be countable in these problems, and $\mathcal{M}$ is taken to be infinite, the proofs used there generalize readily to the situation that we are considering.) Note that $\Psi(\mathcal{M})$ is compact if and only if $\mathcal{M}$ is finite, in which case $\Psi(\mathcal{M})$ is the free union of finitely many one-point compactifications of discrete spaces. It is easy to use Theorem 2.2 to show that $\Psi(\mathcal{M})$ is a maximal feebly compact space; as it is locally compact and Hausdorff, it is regular.

We now give a characterization of regular maximal feebly compact spaces.

Theorem 2.20. *The following are equivalent for a regular space X:*

(a) *X is maximal feebly compact.*

(b) *X is homeomorphic to $\Psi(\mathcal{M})$ for some m.a.d. family $\mathcal{M}$ on a discrete set D.*

Proof. The remarks above show that (b) implies (a). Conversely, let X be a regular maximal feebly compact space X. Then by Theorem 2.17, $I(X)$ is dense in X, and $X \setminus I(X)$, which we will denote by E, is closed and discrete. As X is T_1 (see Theorem 1.5), for each $p \in E$ there is an open set $V(p)$ of X such that $p \in V(p)$ and $E \cap \mathrm{cl}_X V(p) = \{p\}$. By Theorem 1.4(a), $\mathrm{cl}_X V(p)$ is feebly compact, and evidently p is its only nonisolated point. Let $\mathcal{M} = \{V(p) \cap I(X): p \in E\}$. As each $p \in E$ is nonisolated, each set in $\mathcal{M}$ is infinite.

Let p_1 and p_2 be distinct points of E, and let $S = V(p_1) \cap V(p_2) \cap I(X)$. If S is infinite then as $\{\{d\}: d \in S\}$ is an infinite family of open sets of the feebly compact space X, by Theorem 1.3 it has a limit point q. Evidently $q \in E$ and so $q \in E \cap \mathrm{cl}_X V(p_1) \cap \mathrm{cl}_X V(p_2)$. Hence $p_1 = q = p_2$, a contradiction. Thus $V(p_1) \cap V(p_2) \cap I(X)$ is finite, and so $\mathcal{M}$ is an almost disjoint family. If $A \subseteq I(X)$, $A \notin \mathcal{M}$, and $\mathcal{M} \cup \{A\}$ were an almost disjoint family of subsets of $I(X)$, then A would be infinite and so $\{\{a\}: a \in A\}$ would have a limit point $q \in E$ (as X is feebly compact). But then $V(q) \cap A$ would be infinite, contradicting the almost disjointness of $\mathcal{M} \cup \{A\}$. Hence $\mathcal{M}$ is a m.a.d. family on $I(X)$. It is now routine to show that if we define $\lambda : \Psi(\mathcal{M}) \to X$ by:

$$\lambda(x) = x, \quad \text{if } x \in I(X),$$

$$\lambda(p(M)) = d, \quad \text{where } M = V(d) \cap I(X),$$

then λ is a homeomorphism from $\Psi(\mathcal{M})$ onto X. $\square$

Theorem 2.21. *Every feebly compact almost E_1 submaximal regular space X is maximal feebly compact.*

Proof. By Theorem 2.2 it suffices to prove that feebly compact subspaces are closed. Let X be as hypothesized, let F be a feebly compact subspace of X, and suppose $p \in \text{cl } F \setminus F$. Let $B = F \setminus I(X)$. By Proposition 2.16, $F \setminus I(X)$ is nowhere dense, so as X is submaximal, B is closed in X. By regularity there is an open set V such that $p \in V$ and $B \subseteq X \setminus \text{cl } V$. Now $V \cap I(X) \cap F = V \cap F$, and as $p \in \text{cl } F \setminus B$, evidently $p \in \text{cl}(V \cap F) \setminus F$. Thus $V \cap F$ is an infinite set of isolated points of F, and as F is feebly compact $\{\{x\}: x \in V \cap F\}$ must have a limit point in F. This limit point must be in cl V, and hence cannot be in B. Thus it must be an isolated point of X, which is a contradiction. Our theorem follows. □

The next example shows that "regular" cannot be replaced by "semiregular" in Theorem 2.21.

Example 2.22. A feebly compact almost E_1 submaximal semiregular Hausdorff space need not be maximal feebly compact. To see this, let D be a countably infinite set and partition it into two infinite subsets D_1 and D_2. Let $\mathscr{M}_1$ be an infinite m.a.d. family of infinite subsets of D_1, let $f: D_1 \to D_2$ be a bijection, and let $\mathscr{M} = \{M \cup f[M]: M \in \mathscr{M}_1\}$. It is straightforward to show that these definitions produce a m.a.d. family $\mathscr{M}$ on D. Let A be countably infinite set disjoint from D, let $p \notin \Psi(\mathscr{M}) \cup A$ and let $X = \Psi(\mathscr{M}) \cup \{p\} \cup A$, topologized by stipulating that $\Psi(\mathscr{M})$ is an open subset of X, points of A are isolated, and $\{(D_1 \setminus \bigcup \mathscr{F}) \cup S \cup \{p\}$: $\mathscr{F}$ is a finite subfamily of $\mathscr{M}_1$, S is a cofinite subset of $A\}$ is a neighborhood base at p. Then X is easily seen to be Hausdorff, and as it is the union of the feebly compact space $\Psi(\mathscr{M})$ and the compact space $A \cup \{p\}$, it is feebly compact. Clearly each point of $X \setminus \{p\}$ has a neighborhood base of compact open sets, and hence a neighborhood base of regular open sets. If $(D_1 \setminus \bigcup \mathscr{F}) \cup S \cup \{p\}$ is a basic open neighborhood V of p as described above, note that cl $V = V \cup \{p(M)$: $M \in \mathscr{M}$ and $M \cap (D_1 \setminus \bigcup \mathscr{F})$ is infinite$\}$. But $M \setminus D_1$ is nonempty for each $M \in \mathscr{M}$ by our choice of $\mathscr{M}$, so $p(M) \notin \text{int cl } V$. Hence int cl $V = V$, and so X is semiregular. Clearly $I(X) = D \cup A$ is dense in X and $X \setminus I(X)$ is discrete, so it is easy to use Proposition 2.16 to show that X is submaximal and almost E_1. Since $\Psi(\mathscr{M}) \setminus D$ and p cannot be put inside disjoint open sets, X is not regular. As $\Psi(\mathscr{M})$ is a feebly compact subspace of X that is not closed, by Theorem 2.2, X is not maximal feebly compact.

So far every example of a maximal feebly compact space that we have produced has had a dense set of isolated points. Now we will produce examples of completely Hausdorff maximal feebly compact spaces with no isolated points. We need some preliminary results.

It is known that every topology τ on a set X is contained in a topology σ on X such that (X, σ) is submaximal and the regular closed subsets of (X, σ) are the same as those of (X, τ) (see [4, Exercise 22, p. 139]). Specifically, we have the following lemma which combines [4, Exercises 20 and 22, pp. 138–139], and immediate consequences thereof.

Lemma 2.23. *Let τ be a topology on a set X, let $\mathscr{D}$ be the set of τ-dense subsets of X, partially ordered by inclusion, and let $\mathscr{U}$ be an ultrafilter on the poset $\mathscr{D}$. Let $\tau(\mathscr{U}) = \langle \tau \cup \mathscr{U} \rangle$, i.e., let $\tau(\mathscr{U})$ be the topology on X for which $\{V \cap S\colon V \in \tau,\ S \in \mathscr{U}\}$ is a base. Then*:

(a) *$\tau \subseteq \tau(\mathscr{U})$.*

(b) *$(X, \tau(\mathscr{U}))$ is submaximal.*

(c) *The regular closed subsets of (X, τ) are precisely the regular closed subsets of $(X, \tau(\mathscr{U}))$, and $\tau(\mathscr{U})(s) = \tau(s)$.*

(d) *(X, τ) is feebly compact if and only if $(X, \tau(\mathscr{U}))$ is feebly compact.*

(e) *If (X, τ) is T_1, a point p is isolated in (X, τ) if and only if it is isolated in $(X, \tau(\mathscr{U}))$.*

(f) *If σ is a submaximal expansion of τ, and if $\tau = \sigma(s)$, then there is an ultrafilter $\mathscr{U}$ of dense subsets of (X, τ) such that $\sigma = \tau(\mathscr{U})$.*

Observe that if τ, $\mathscr{D}$, and $\mathscr{U}$ are as above, then S is dense in $(X, \tau(\mathscr{U}))$ if and only if $S \in \mathscr{U}$. Also, if τ is a submaximal topology then the only ultrafilter on $\mathscr{D}$ is $\mathscr{D}$ itself, and $\tau = \tau(\mathscr{D})$.

Lemma 2.24. *Let (X, τ) be feebly compact, let $\mathscr{U}$ be an ultrafilter on the poset $\mathscr{D}$ of dense subsets of (X, τ), and let A be a regular closed subset of (X, τ). Then A is an almost E_1-subspace of (X, τ) if and only if A is an almost E_1-subspace of $(X, \tau(\mathscr{U}))$.*

Proof. Denote A with the topology inherited from τ by A_τ, and A with the topology inherited from $\tau(\mathscr{U})$ by $A_{\mathscr{U}}$.

Let p be an almost E_1-point of A_τ. Then there exists a decreasing sequence $(F_n)_{n \in \mathbb{N}}$ of closed sets of A_τ such that $\{p\} = \bigcap\{F_n\colon n \in \mathbb{N}\}$ and $\operatorname{int}_{A_\tau} F_n \neq \emptyset$. Now F_n is closed in (X, τ) (as A_τ is), hence in $(X, \tau(\mathscr{U}))$, and hence in $A_{\mathscr{U}}$. Furthermore, as the topology of $A_{\mathscr{U}}$ contains that of A_τ, it follows that $\operatorname{int}_{A_{\mathscr{U}}} F_n \neq \emptyset$. Hence p is an almost E_1-point of $A_{\mathscr{U}}$.

Conversely, let p be an almost E_1-point of $A_{\mathscr{U}}$. Hence there exists a decreasing sequence $(F_n)_{n \in \mathbb{N}}$ of closed sets of $A_{\mathscr{U}}$ such that $\{p\} = \bigcap\{F_n\colon n \in \mathbb{N}\}$ and $\operatorname{int}_{A_{\mathscr{U}}} F_n \neq \emptyset$. As $A_{\mathscr{U}}$ is closed in $(X, \tau(\mathscr{U}))$, $X \setminus F_n \in \tau(\mathscr{U})$. Also by Lemma 2.23(c), $A_{\mathscr{U}}$ is a regular closed set of $(X, \tau(\mathscr{U}))$, and hence the interior of each F_n in the space $(X, \tau(\mathscr{U}))$ is nonempty. For each $n \in \mathbb{N}$ we can write $H_n = [\operatorname{cl}_{\tau(\mathscr{U})} \operatorname{int}_{\tau(\mathscr{U})} F_n] \cup \{p\}$. By Lemma 2.23(c), H_n is closed in (X, τ) and $\operatorname{int}_\tau H_n \neq \emptyset$. Clearly $\{p\} = \bigcap\{H_n\colon n \in \mathbb{N}\}$. $\square$

Theorem 2.25. *Let (X, τ) be a feebly compact T_1-space. The following are equivalent:*

(a) *Each regular closed subset of (X, τ) is an almost E_1-space.*

(b) *Each feebly compact subspace of (X, τ) with dense interior is closed.*

(c) *There is an ultrafilter $\mathscr{U}$ on the poset of τ-dense subsets of X such that $\tau(\mathscr{U})$ is a maximal feebly compact topology on X.*

(d) *For all ultrafilters $\mathscr{U}$ on the poset of τ-dense subsets of X, $\tau(\mathscr{U})$ is a maximal feebly compact topology on X.*

Proof. By Proposition 2.7(d), (a) and (b) are equivalent.

As there exist (by Zorn's lemma) ultrafilters on the poset of τ-dense subsets of X, clearly (d) implies (c).

If (c) holds, by Corollary 2.8 all the regular closed subsets of $(X, \tau(\mathscr{U}))$ are almost E_1-spaces. By Lemmas 2.24 and 2.23(c), it follows that all the regular closed subsets of (X, τ) are almost E_1-spaces, i.e., (a) holds.

If (a) holds, let $\mathscr{U}$ be an ultrafilter on the poset of τ-dense subsets of X. By Lemma 2.24 each regular closed subset of $(X, \tau(\mathscr{U}))$ is an almost E_1-space. By Lemma 2.23(b) and Corollary 2.8, $(X, \tau(\mathscr{U}))$ is a maximal feebly compact space. $\square$

Corollary 2.26. *If (X, τ) is a feebly compact E_1-space, and if $\mathscr{U}$ is an ultrafilter on the poset of τ-dense subsets of X, then $\tau(\mathscr{U})$ is a maximal feebly compact topology on X such that $\tau \subseteq \tau(\mathscr{U})$.*

Proof. Use Theorem 2.25, noting that every subspace of an E_1-space is an almost E_1-space. $\square$

Example 2.27. Let (X, τ) be a compact E_1-space without isolated points; note that such a space is of necessity Hausdorff. By Corollary 2.26, if $\mathscr{U}$ is any ultrafilter of τ-dense subsets of X, then $(X, \tau(\mathscr{U}))$ is a maximal feebly compact space, and by Lemma 2.23(e) it has no isolated points. Clearly since (X, τ) is Tychonoff, distinct points of $(X, \tau(\mathscr{U}))$ can be completely separated by a $\tau(\mathscr{U})$-continuous real-valued function; in other words $(X, \tau(\mathscr{U}))$ is completely Hausdorff (and hence Urysohn). Thus if (X, τ) is a compact metric space without isolated points (for example), we can produce from it a maximal feebly compact completely Hausdorff space without isolated points.

We conclude by showing that product spaces cannot be maximal feebly compact except in trivial situations.

Theorem 2.28. *If a product space $X = \prod\{X_i\colon i \in I\}$ is maximal feebly compact, then all of its factors are maximal feebly compact, all but finitely many of its factors are one-point spaces, and all but one of its factors are finite discrete spaces.*

Proof. As each X_i is the continuous image of X, it is feebly compact. Let $X = P \times Q$, and suppose that $p \in P \setminus I(P)$ and $q \in Q \setminus I(Q)$. Then $(P \setminus \{p\}) \times (Q \setminus \{q\})$ is dense in X and as X is submaximal, it follows from Lemma 2.1 that $(P \times \{q\}) \cup (\{p\} \times Q)$ is discrete. Hence P and Q are discrete, and hence finite as they are feebly compact. On the other hand, if $P = I(P)$, then P is discrete and finite. Hence either P or Q is finite and discrete.

Now suppose there exists $i_0 \in I$ such that X_{i_0} is not finite and discrete. By the preceding argument $\prod\{X_i: i \in I \setminus \{i_0\}\}$ is finite and discrete. Hence all but finitely many of its factors must be one-point spaces. This means that X is homeomorphic to a free union of finitely many copies of X_{i_0}, and so X_{i_0} is homeomorphic to an open-and-closed subspace of X. It follows from Corollary 2.3 that X_{i_0} is maximal feebly compact. □

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