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Inequalities and Existence Theorems in the Theory of Matrices

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I. INTRODUCTION

While the study of matrix inequalities has an extended history which reaches back to Hadamard's discovery [1] of the determinantal inequality which now bears his name, the discussion of existence problems to which inequalities give rise is of much more recent date. This subject, which is largely due to or inspired by the work of A. Horn, is less than ten years old; but as it already embraces a distinctive body of results, it seems appropriate to survey the present state of our knowledge and, in particular, to formulate some of the unsolved problems.

Before embarking on a detailed examination of the material, however, we shall indicate in broad outline the type of question with which we shall be mainly concerned. Let $\mathfrak{S}$ be a system of inequalities[1] in the indeterminates $x_1, \cdots, x_m$. Denote by $\phi_1, \cdots, \phi_m$ real-valued functions defined on the class $\mathscr{M}_n$ of complex $n \times n$ matrices, and let $\mathfrak{M}$ be a subclass of $\mathscr{M}_n$. Suppose now that, for every matrix $A \in \mathfrak{M}$, the system $\mathfrak{S}$ becomes validw hen we make the substitution

$$x_1 = \phi_1(A), \cdots, x_m = \phi_m(A). \tag{1.1}$$

A knowledge of this system of matrix inequalities prompts at once the following question: given real numbers $c_1, \cdots, c_m$, to find conditions for the existence of a matrix $A \in \mathfrak{M}$ such that

$$\phi_1(A) = c_1, \cdots, \phi_m(A) = c_m .$$

In view of our assumption, it is plain that a *necessary* condition is the validity of $\mathfrak{S}$ for $x_1 = c_1, \cdots, x_m = c_m$. If this condition is also *sufficient*, then we may say that the system $\mathfrak{S}$ with (1.1) is *essential for the class* $\mathfrak{M}$. The assertion of sufficiency constitutes, of course, an existence theorem in matrix theory.

[1] This system may contain both strict and wide inequalities.

Most of the problems discussed below are concerned with the determination of essential systems of inequalities for specified classes of matrices. As a trivial illustration, let us consider Hadamard's inequality: for any hermitian, positive definite, nondiagonal $n \times n$ matrix $H = (h_{rs})$, we have

$$h_{11} \cdots h_{nn} > \det H > 0. \tag{1.2}$$

Now let $a_1, \cdots, a_n, d$ be real numbers. What conditions guarantee the existence of a hermitian, positive definite, nondiagonal $n \times n$ matrix $H = (h_{rs})$ such that

$$h_{11} = a_1, \cdots, h_{nn} = a_n, \det H = d?$$

In view of (1.2), the inequalities

$$a_1 \cdots a_n > d > 0$$

are necessary, and it is an easy exercise to verify that they are also sufficient. Thus the inequalities (1.2) are essential for the class of matrices specified above, and we have here a particularly simple existence theorem of a type which we shall encounter repeatedly in the sequel.

Unless the contrary is stated, it will be understood throughout the discussion that all matrices have real or complex elements and are of type $n \times n$. The transpose of A will be denoted by A^T and the transposed conjugate by A^*. The unit matrix of order n will be denoted by I_n. Again $[A]_k$ will stand for the submatrix of A formed by the elements in the first k rows and the first k columns.

For any real numbers $x_1, \cdots, x_n$, we shall denote by $\bar{x}_1, \cdots, \bar{x}_n$ these numbers arranged in non-ascending order of magnitude. If x_k, y_k $(1 \leqslant k \leqslant n)$ are real numbers and if

$$\bar{x}_1 + \cdots + \bar{x}_k \leqslant \bar{y}_1 + \cdots + \bar{y}_k \qquad (1 \leqslant k \leqslant n), \tag{1.3}$$

then we shall write $(x_1, \cdots, x_n) \prec\prec (y_1, \cdots, y_n)$. If, in addition, there is equality in (1.3) for $k = n$, we shall write $(x_1, \cdots, x_n) \prec (y_1, \cdots, y_n)$. This latter notation is due to Hardy, Littlewood, and Pólya [2, p. 45].

II. Real Symmetric Matrices

We begin by considering problems in which we seek to prescribe certain quantities connected with real symmetric matrices. Some of the results below can be obviously extended to hermitian matrices, and we shall make occasional use of this fact.

Our first result involves the diagonal elements and characteristic roots.

THEOREM 1. *Let a_k, ω_k $(1 \leqslant k \leqslant n)$ be real numbers. Then the relation*

$$(a_1, \cdots, a_n) \prec (\omega_1, \cdots, \omega_n) \tag{2.1}$$

is necessary and sufficient for the existence of a real symmetric matrix with diagonal elements $a_1, \cdots, a_n$ and characteristic roots $\omega_1, \cdots, \omega_n$.[2]

The necessity of (2.1), which follows for example easily from Lemma 1 below, is implicit in the work of Fan [3] and possibly already in that of Schur [4]. The much harder proof of the sufficiency was first given by Horn [5], and to indicate Horn's argument we need to refer to doubly-stochastic matrices. A matrix is called *doubly-stochastic* if its elements are nonnegative and the sum of the elements in each row and in each column is equal to 1. The study of these matrices was initiated about forty years ago by Schur [4], and a fundamental result, discovered by Hardy, Littlewood, and Pólya [2, Theorem 46] may be stated as follows.

LEMMA 1. *Let x, y be two real vectors with n components each. Then there exists a doubly-stochastic matrix D such that $x = yD$ if and only if $x \prec y$.*

Next, let $A = (a_{rs})$ be called *orthostochastic* if there exists an orthogonal matrix $U = (u_{rs})$ such that $a_{rs} = u_{rs}^2$ $(1 \leqslant r, s \leqslant n)$. The set of orthostochastic matrices is contained properly in the set of doubly-stochastic matrices, and the main step in Horn's proof of the sufficiency of (2.1) is the proof of the following refinement of Lemma 1.

LEMMA 2. *Lemma 1 remains valid if the term "doubly-stochastic" is replaced by "orthostochastic."*

The proof of this result is quite complicated, but once it has been given, the sufficiency of (2.1) follows very easily. For, by (2.1) and Lemma 2, there exists an orthogonal matrix $U = (u_{rs})$ such that

$$(a_1, \cdots, a_n) = (\omega_1, \cdots, \omega_n) A,$$

where $A = (a_{rs})$ and $a_{rs} = u_{rs}^2$ $(1 \leqslant r, s \leqslant n)$. Write $\Omega = \operatorname{diag}(\omega_1, \cdots, \omega_n)$. Then $U^T \Omega U$ is a real symmetric matrix with diagonal elements $a_1, \cdots, a_n$ and characteristic roots $\omega_1, \cdots, \omega_n$.

There is an alternative and somewhat easier approach to Theorem 1. This depends on the following two lemmas, the proofs of which involve fairly standard ideas.

[2] It is plainly irrelevant whether we insist on a definite order of the diagonal elements.

LEMMA 3. *If* $(x_1, \cdots, x_n) \prec (y_1, \cdots, y_n)$, *then there exist numbers* $z_1, \cdots, z_{n-1}$ *which separate* $y_1, \cdots, y_n$[3] *and satisfy the relation*

$$(x_1, \cdots, x_{n-1}) \prec (z_1, \cdots, z_{n-1}).$$

LEMMA 4. *Let* S *be a real symmetric* $(n-1) \times (n-1)$ *matrix whose characteristic roots separate the numbers* $\omega_1, \cdots, \omega_n$. *Then, by bordering* S *with an additional row and column, it is possible to obtain a real symmetric* $n \times n$ *matrix with characteristic roots* $\omega_1, \cdots, \omega_n$.

Both these lemmas were derived, in essence, in [6]. Moreover, a result substantially equivalent to Lemma 4 was proved independently by Fan and Pall [7].

The sufficiency of the condition (2.1) can now be readily established. By (2.1) and Lemma 3, there exist numbers $\alpha_1, \cdots, \alpha_{n-1}$ which separate $\omega_1, \cdots, \omega_n$ and such that

$$(a_1, \cdots, a_{n-1}) \prec (\alpha_1, \cdots, \alpha_{n-1}). \tag{2.2}$$

Assuming that our assertion is valid for $n-1$, we infer from (2.2) the existence of a real symmetric $(n-1) \times (n-1)$ matrix S with characteristic roots $\alpha_1, \cdots, \alpha_{n-1}$ and diagonal elements $a_1, \cdots, a_{n-1}$. Hence we can border S in such a way as to obtain a real symmetric $n \times n$ matrix A with characteristic roots $\omega_1, \cdots, \omega_n$. Moreover, in view of (2.1), the last element on the diagonal of A is equal to

$$\omega_1 + \cdots + \omega_n - (a_1 + \cdots + a_{n-1}) = a_n .$$

This completes the induction proof of the sufficiency of (2.1).

It may be of interest to note that the more interesting half of Lemma 2 can now be inferred as an immediate corollary. For if

$$(x_1, \cdots, x_n) \prec (y_1, \cdots, y_n),$$

then, by Theorem 1, there exists a real symmetric matrix A with diagonal elements x_k and characteristic roots y_k $(1 \leqslant k \leqslant n)$. Denoting by $U = (u_{rs})$ an orthogonal matrix such that

$$A = U^T \cdot \operatorname{diag}(y_1, \cdots, y_n) \cdot U,$$

we have

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n)\, \Omega,$$

where $\Omega = (u_{rs}^2)$ is an orthostochastic matrix.

[3] This means that $y_1 \geqslant z_1 \geqslant y_2 \geqslant z_2 \geqslant \cdots \geqslant z_{n-1} \geqslant y_n$.

Next, let us call the characteristic roots of $\frac{1}{2}(A + A^*)$ the *associated roots* of A. A result in some respects analogous to Theorem 1 is given by

THEOREM 2. *Let $\omega_1, \cdots, \omega_n$ be complex and $\alpha_1, \cdots, \alpha_n$ real numbers. Then the relation*

$$(\Re\omega_1, \cdots, \Re\omega_n) \prec (\alpha_1, \cdots, \alpha_n) \tag{2.3}$$

is necessary and sufficient for the existence of a complex matrix with characteristic roots $\omega_1, \cdots, \omega_n$ and associated roots $\alpha_1, \cdots, \alpha_n$.

The necessity of (2.3) was observed by Fan [8] as an immediate consequence of the necessity part of Theorem 1 (for hermitian matrices). The sufficiency can also be deduced readily from Theorem 1; details will be found in [6].

There are a number of variants of the problem dealt with by Theorem 1. In some ways it is more natural to seek to prescribe not the characteristic roots but the characteristic polynomial. We may therefore ask: what conditions are necessary and sufficient for the existence of a real symmetric matrix with prescribed diagonal elements and characteristic polynomial? This qeustion was discussed in [9] and an algorithm was developed for deciding in each case whether there exists a matrix with the stated properties.

Theorem 1 also suggests the analogous problem for normal matrices. Now, if A is a normal matrix with diagonal elements a_k and characteristic roots ω_k $(1 \leqslant k \leqslant n)$ and if z is any complex number, then $B = \frac{1}{2}(zA + z^*A^*)$ is a hermitian matrix with diagonal elements $\Re(a_k z)$ and characteristic roots $\Re(\omega_k z)$ $(1 \leqslant k \leqslant n)$. Hence, by the necessity part of Theorem 1 (for hermitian matrices), we infer that

$$(\Re(a_1 z), \cdots, \Re(a_n z)) \prec (\Re(\omega_1 z), \cdots, \Re(\omega_n z)).$$

Suppose, next, that this condition is valid for certain complex numbers a_k, ω_k $(1 \leqslant k \leqslant n)$ and *every* complex number z. Does it follow that there exists a normal matrix with diagonal elements a_k and characteristic roots ω_k $(1 \leqslant k \leqslant n)$? This is one of the most tantalizing unsolved problems in the subject.

Returning now to real symmetric matrices, we note a further problem discussed by Horn, namely, the existence of positive definite matrices with prescribed characteristic roots and prescribed determinants of a nest of principal submatrices. Horn [10] proved the following result.

THEOREM 3. *Let r_k, ω_k $(1 \leqslant k \leqslant n)$ be positive numbers. Then the relation*

$$(\log r_1, \cdots, \log r_n) \prec (\log \omega_1, \cdots, \log \omega_n)$$

is necessary and sufficient for the existence of a real, symmetric, positive definite matrix S with characteristic roots $\omega_1, \cdots, \omega_n$ *and such that*

$$\det [S]_k = r_1 \cdots r_k \quad (1 \leqslant k \leqslant n).$$

The necessity of the stated condition is an easy consequence of Weyl's inequality (see Section III below). The sufficiency can be established by induction and the use of Lemmas 3 and 4 (cf. [11]).

Finally, we mention two open problems. (i) What are the conditions for the existence of a real symmetric matrix with prescribed diagonal elements, rank, and signature? (ii) Denote by $\pi_k(A)$ the product of all principal k-rowed minors of A. Given positive numbers $p_1, \cdots, p_n$, under what conditions does there exist a positive definite hermitian matrix H with $\pi_k(H) = p_k$ $(1 \leqslant k \leqslant n)$? In view of a result of Szász ([12], cf. [13]), the inequalities

$$p_1 \geqslant p_2^{1/\binom{n-1}{1}} \geqslant p_3^{1/\binom{n-1}{2}} \geqslant \cdots \geqslant p_n$$

are certainly necessary.

III. Singular Values of Complex Matrices

If A is any complex matrix, then A^*A is hermitian and either positive definite or positive semidefinite. Its characteristic roots are therefore real and nonnegative, and their nonnegative square roots are called the *singular values* of A.

It was noted by Weyl [14], and independently by Turnbull and Aitken [15, p. 110], that if the characteristic roots ω_k and singular values σ_k $(1 \leqslant k \leqslant n)$ of a matrix A are numbered so that

$$|\omega_1| \geqslant \cdots \geqslant |\omega_n|, \qquad \sigma_1 \geqslant \cdots \geqslant \sigma_n, \tag{3.1}$$

then

$$\left.\begin{aligned} &|\omega_1 \cdots \omega_k| \leqslant \sigma_1 \cdots \sigma_k \qquad (1 \leqslant k < n) \\ &|\omega_1 \cdots \omega_n| = \sigma_1 \cdots \sigma_n \end{aligned}\right\} \tag{3.2}$$

The proof of these relations is not difficult. For $k = 1$, the assertion is almost self-evident, and for $k = r$ it follows if we apply the inequality of case $k = 1$ to the rth compound of A.

Horn considered the converse problem to which these inequalities give rise. His conclusion [10], together with Weyl's, may be summarized in the following theorem, which contains one of the earliest results identifying a system of essential inequalities.

THEOREM 4. *Let $\omega_1, \cdots, \omega_n$ be complex numbers, let $\sigma_1, \cdots, \sigma_n$ be real non-negative numbers, and suppose that (3.1) is satisfied. Then the relations (3.2) are necessary and sufficient for the existence of a complex matrix with characteristic roots ω_k and singular values σ_k $(1 \leqslant k \leqslant n)$.*

We indicate briefly the proof of the sufficiency of (3.2) and confine ourselves, for simplicity, to the case when all ω's and σ's are nonzero. By (3.2) and Theorem 3, there exists a real symmetric matrix S with characteristic roots $\sigma_1^2, \cdots, \sigma_n^2$ and such that

$$| \omega_1 \cdots \omega_k |^2 = \det [S]_k \qquad (1 \leqslant k \leqslant n).$$

Since S is positive definite, it is not difficult to verify that there exists an upper triangular matrix P with diagonal elements $| \omega_1 |, \cdots, | \omega_n |$ such that $S = P^T P$. Writing now

$$\Omega = \operatorname{diag} (\omega_1/| \omega_1 |, \cdots, \omega_n/| \omega_n |),$$

we see that the matrix ΩP has clearly the requisite properties. For the case when some ω's and σ's vanish, we refer to Horn's paper [10].

Theorem 4 may be interpreted in a different way. A complex matrix A possesses a polar factorization $A = HU$, where U is unitary and H is a hermitian matrix whose characteristic roots are identical with the singular values of A. Moreover, H is unique for every A, while U is unique when A is nonsingular. Theorem 4 therefore furnishes us with complete information about the relations between the characteristic roots of a matrix A and the characteristic roots of its "hermitian part" H. The analogous problem which consists in establishing the complete set of relations between the characteristic roots of a nonsingular matrix A and those of its "unitary part" U is a good deal harder. However, a solution has been given by Horn and Steinberg [16].

In view of Theorems 1 and 4 it is natural to ask the following question: what are the conditions for the existence of a (complex) matrix with prescribed diagonal elements and singular values? This question has not yet been discussed in the literature, but it is unlikely to prove excessively difficult.

IV. STOCHASTIC AND DOUBLY-STOCHASTIC MATRICES

A matrix with nonnegative elements each of whose row-sums is equal to 1 is called *stochastic*. Stochastic matrices play an important part in the theory of Markov chains and have been the subject of a good deal of research (see e.g. [17], p. 82 ff.).

The problem of simultaneous characterization of all characteristic roots of a stochastic $n \times n$ matrix is plainly very difficult, and the published work on the subject is confined to the study of real roots. Suleĭmanova [18] introduced a geometric technique for treating the question. In a series of three papers, H. Perfect [19-21], partly relying on this technique and partly using a more powerful method based on a theorem of Brauer [22, Theorem 27], obtained various sufficient conditions for a set of real numbers to be the set of characteristic roots of a stochastic matrix. It is not possible within the scope of the present survey to give a comprehensive account of this work and we shall content ourselves, in the main, with quoting the most interesting result.

A set of real numbers will be called *balanced* if the sum of the moduli of the negative numbers in the set (if there are such) does not exceed the greatest number in the set. One of Perfect's results [20], obtained by an actual construction of the requisite matrix, may be stated thus.

THEOREM 5. *Let $\lambda_k (1 \leqslant k \leqslant n)$ be real numbers. If*

$$\lambda_1 = 1, |\lambda_2| \leqslant 1, \cdots, |\lambda_n| \leqslant 1 \tag{4.1}$$

and if the set of numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ can be partitioned into disjoint balanced sets, then there exists a stochastic $n \times n$ matrix with $\lambda_1, \lambda_2, \cdots, \lambda_n$ as its characteristic roots.

On the other hand, it is plain that (4.1) and the inequality

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n \geqslant 0$$

are necessary for the existence of such a matrix. For all characteristic roots of a stochastic matrix lie on the closed unit disk, 1 is always a characteristic root, and the trace of the matrix must be nonnegative.

There is thus still a considerable gap between the known necessary and sufficient conditions, and this gap has not been closed for $n \geqslant 5$. For $n \leqslant 4$, however, the necessary conditions are easily seen to be also sufficient.

Perfect also considered the analogous problem for stochastic matrices with strictly positive elements, and, furthermore, she raised the question of characterizing the diagonal elements of stochastic $n \times n$ matrices whose characteristic roots are given real numbers. For $n > 3$, only incomplete results are available so far.

Let us next turn to the problem of complex characteristic roots of stochastic matrices. Even the question involving a single root is difficult, though tractable. This question is concerned with the determination of the region $\mathfrak{S}_n$ of the complex plane specified by the requirement that $z \in \mathfrak{S}_n$ if and only if z is a characteristic root of some stochastic $n \times n$ matrix. Karpelevich [23] gave a complete description of the region $\mathfrak{S}_n$, but as both the statement and the proof of his result are very complicated, we refer the reader to the original paper and, following Dmitriev and Dynkin [24], merely determine the points of $\mathfrak{S}_n$ on the unit circle.

If ω is a complex number of unit modulus which is a characteristic root of a stochastic matrix $S = (s_{kj})$, then there exist complex numbers $z_1, \cdots, z_n$, not all zero, such that

$$\omega z_k = \sum_{j=1}^{n} s_{kj} z_j \qquad (1 \leqslant k \leqslant n). \tag{4.2}$$

The convex hull Q of the points $z_1, \cdots, z_n$ is a polygon with at most n vertices and it does not consist of the origin only. Since S is stochastic, equations (4.2) show that $\omega z_k \in Q$ $(1 \leqslant k \leqslant n)$ and hence that $\omega Q \subseteq Q$. If $\zeta \neq 0$ is a point of Q at a maximum distance from the origin, then ζ is a vertex of Q and also $\omega\zeta \in Q$. But $|\omega\zeta| = |\zeta|$, and therefore $\omega\zeta$ is again a vertex of Q. By repetition of the same argument, we infer that $\omega^k\zeta$ is a vertex of Q for every nonnegative integer k. Hence at most n among the numbers $\omega^k\zeta$ $(k = 0, 1, 2, \cdots)$ are distinct, and so ω is one of the numbers

$$e^{2\pi i a/b} \qquad (0 < a \leqslant b \leqslant n). \tag{4.3}$$

Conversely, it is at once obvious that a number of the form (4.3) belongs to $\mathfrak{S}_n$. The numbers (4.3) therefore specify precisely the points of $\mathfrak{S}_n$ on the unit circle, and it can be shown (see [23]) that the region $\mathfrak{S}_n$ is bounded by certain arcs joining these points in cyclic order.

The study of stochastic matrices has many features in common with the study of the wider class of nonnegative matrices (cf. [17, Chap. XIII]), and here again it is easy to point to open problems. Thus, for example, the conditions for the existence of a nonnegative matrix with prescribed real characteristic roots and diagonal elements are not known. Perfect [25] obtained partial answers to the following question: what conditions are necessary and sufficient for the existence of a nonnegative matrix with prescribed real characteristic roots and one prescribed diagonal element?

We next turn to a subclass of stochastic matrices, the doubly-stochastic matrices defined in Section II. Surprisingly enough our information about spectral properties is in this case even more meager than in the case of stochastic matrices. Let Π_k denote the closed k-gon whose vertices are the kth roots of unity. It was observed by J. F. C. Kingman (cf. [26]) that if a complex number belongs to the union of $\Pi_2, \Pi_3, \cdots, \Pi_n$, then it is a characteristic root of some doubly-stochastic $n \times n$ matrix; but beyond this it is difficult to point to any results in the field. In the remainder of the present section we shall therefore deal with a different problem—the characterization of diagonal elements.

Horn [5], making an ingenious use of the theorem of Hardy, Littlewood, and Pólya (Lemma 1), derived the following result (alternative proofs of which will be found in [27] and [28]).

THEOREM 6. *Let* a_k $(1 \leqslant k \leqslant n)$ *be real numbers. Then the inequalities*

$$0 \leqslant a_k \leqslant 1 \qquad (1 \leqslant k \leqslant n) \tag{4.4}$$

and

$$\sum_{k=1}^{n} a_k \leqslant n - 2 + 2 \min_{1 \leqslant j \leqslant n} a_j$$

are necessary and sufficient for the existence of a doubly-stochastic matrix with diagonal elements a_k $(1 \leqslant k \leqslant n)$.

Now, by a well-known result of G. Birkhoff [29], every doubly-stochastic matrix can be represented as a convex combination of permutation matrices. We define an *even* doubly-stochastic matrix as a matrix which can be represented as a convex combination of permutation matrices corresponding to even permutations. No characterization of even doubly-stochastic matrices is, as yet, known, but it is not difficult to describe the sets of their diagonal elements by proving the following result [30].

THEOREM 7. *Let* a_k $(1 \leqslant k \leqslant n)$ *be real numbers. Then the inequalities (4.4) and*

$$\sum_{k=1}^{n} a_k \leqslant n - 3 + 3 \min_{1 \leqslant j \leqslant n} a_j$$

are necessary and sufficient for the existence of an even doubly-stochastic matrix with diagonal elements a_k $(1 \leqslant k \leqslant n)$.

The proof is based, in essence, on a fairly straightforward application of the following useful principle [31, pp. 23-24].

LEMMA 5. *Let* $u, u_1, \cdots, u_m$ *be vectors in euclidean space* E^n *with inner product* $(,)$. *Then* u *belongs to the convex hull of* $u_1, \cdots, u_m$ *if and only if, for every vector* $v \in E^n$,

$$(u, v) \leqslant \max_{1 \leqslant k \leqslant m} (u_k, v).$$

Again, it is natural to inquire whether there exists an analogue of Theorem 6 for infinite doubly-stochastic matrices. This is, indeed, the case and it can be shown that the real numbers a_k $(k \geqslant 1)$ are the diagonal elements of some infinite doubly-stochastic matrix if and only if $0 \leqslant a_k \leqslant 1$ $(k \geqslant 1)$ and

$$2(1 - \inf_{j \geqslant 1} a_j) \leqslant \sum_{k=1}^{\infty} (1 - a_k) \leqslant \infty.$$

This is a special case of the following result established in [28].

THEOREM 8. *Let a_k, ρ_k, σ_k ($k \geqslant 1$) be real numbers. Then there exists an infinite matrix with nonnegative elements, with diagonal elements a_1, a_2, $\cdots$, with row-sums ρ_1, ρ_2, $\cdots$, and with column-sums σ_1, σ_2, $\cdots$ if and only if*

$$0 \leqslant a_k \leqslant \min(\rho_k, \sigma_k) \qquad (k \geqslant 1),$$

$$\sup_{j \geqslant 1} (\rho_j + \sigma_j - 2a_j) \leqslant \sum_{k=1}^{\infty} (\rho_k - a_k) = \sum_{k=1}^{\infty} (\sigma_k - a_k) \leqslant \infty.$$

The proof consists in a stepwise construction of the requisite matrix. Horn [32] extended the scope of the above result by dispensing with the requirement that all row-sums and column-sums should be finite.

V. ORTHOGONAL, UNITARY, AND NORMAL MATRICES

The problem of characterizing the diagonal elements of orthogonal or related matrices was considered by Horn in the important paper [5] to which we have already had occasion to refer. Making use of Lemma 2, Horn obtained the following result.

THEOREM 9. *The real numbers a_k ($1 \leqslant k \leqslant n$) are the diagonal elements of a proper orthogonal matrix*[4] *if and only if the vector $(a_1, \cdots, a_n)$ lies in the convex hull of those points of the form $(\pm 1, \cdots, \pm 1)$ which have an even number of negative coordinates.*

Let us call $(a_{11}, \cdots, a_{nn})$ the *diagonal vector* of the matrix $A = (a_{rs})$. The theorem then implies, in particular, that the set of diagonal vectors of proper orthogonal matrices is a convex set. The reasons for this remarkable fact are far from clear, and a direct proof would be of the greatest interest.

Theorem 9 cannot be used in the form given above to test whether given real numbers a_1, $\cdots$, a_n can be the diagonal elements of a proper orthogonal $n \times n$ matrix. However, with the help of Lemma 5, it is not difficult to deduce that this is the case if and only if $|a_k| \leqslant 1$ ($1 \leqslant k \leqslant n$) and

$$\sum_{k=1}^{n} |a_k| \leqslant n - 2 + 2\delta \min_{1 \leqslant j \leqslant n} |a_j|,$$

where δ is 1 if the number of negative a's is even and 0 otherwise (see [33]).

Horn also solved the analogous problem for orthogonal and for unitary matrices by proving

[4] An orthogonal matrix is said to be *proper* if its determinant is equal to $+1$.

THEOREM 10. *The real (complex) numbers $a_1, \cdots, a_n$ are the diagonal elements of an orthogonal (unitary) matrix if and only if $|a_1|, \cdots, |a_n|$ are the diagonal elements of a doubly-stochastic matrix, i.e. (in view of Theorem 6), if and only if $|a_k| \leqslant 1$ $(1 \leqslant k \leqslant n)$ and*

$$\sum_{k=1}^{n} |a_k| \leqslant n - 2 + 2 \min_{1 \leqslant j \leqslant n} |a_j|.$$

The proof of the theorem for orthogonal matrices occupies only a few lines, but unitary matrices are quite troublesome and a simplified proof for that case would be very welcome. The connection between diagonal vectors of doubly-stochastic matrices and those of orthogonal (or unitary) matrices is still obscure, and further clarification is needed here.

We may seek to prescribe not merely the diagonal elements but also the determinant of a unitary matrix. In that case the question becomes, presumably, somewhat harder. Another variant of the problem solved by Theorem 10 runs as follows. Let P be a real symmetric, positive definite matrix; when is it possible to find a complex matrix X which satisfies the equation $X^*X = P$ and has prescribed diagonal elements?

Again, we can formulate several problems involving the "completion" of matrices. For example, let $1 \leqslant r, s < n$ and let A be a given complex $r \times s$ matrix. When does there exist a unitary $n \times n$ matrix of which A is the submatrix standing in the left-hand corner? The solution of this particular problem is quite easy: a matrix of the requisite type exists if and only if $I_r - AA^*$ is positive definite or positive semidefinite and of rank not exceeding $n - s$. However, if the term "unitary" is replaced by "normal", we arrive at a difficult unsolved problem. For some relevant observations, see a note by Drazin [34].

A slightly different type of completion has been discussed by Fan and Pall [7]. Let A, B be complex square matrices of order n, m respectively, and suppose that $n > m$. In the terminology of Fan and Pall, B is *imbeddable* in A if there exists a unitary $n \times n$ matrix U such that B is a principal submatrix of U^*AU. If A, B are hermitian matrices with characteristic roots $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ and $\beta_1 \geqslant \cdots \geqslant \beta_m$ respectively, then B is imbeddable in A if and only if

$$\alpha_{n-m+k} \leqslant \beta_k \leqslant \alpha_k \qquad (1 \leqslant k \leqslant m).$$

The even more interesting analogous problem for two normal matrices is as yet unsolved except for the cases $m = 1$ and $m = n - 1$.

Another problem that may be mentioned is concerned with the existence of a unitary, or a normal, matrix whose real part[5] is prescribed. Or, again,

[5] By the real part of the matrix (x_{rs}) we understand the matrix $(\Re x_{rs})$.

we may wish to prescribe the elements on and above the main diagonal of a matrix. For the case of a normal matrix this problem is, of course, trivial if the given elements on the main diagonal are real, but in other cases it is probably quite difficult.

VI. Matrices over an Arbitrary Field

In Section II we considered the problem of prescribing the diagonal elements and also the characteristic roots (or the characteristic polynomial) of a real symmetric matrix. If the requirement of symmetry is abandoned, the problem becomes, of course, much easier and in that case it is possible to operate with an arbitrary field. Indeed, Farahat and Ledermann [35] noted the following result.

THEOREM 11. *Let $\chi(x)$ be a monic polynomial of degree n over a field F, and let $a_1, \cdots, a_{n-1} \in F$. Then there exists a matrix over F with characteristic polynomial $\chi(x)$ and having $a_1, \cdots, a_{n-1}$ as elements on the main diagonal.*

To prove this assertion, let $\chi(x) = x^n + c_1x^{n-1} + \cdots + c_n$, and define a_n by the equation $a_1 + \cdots + a_{n-1} + a_n = -c_1$. Clearly $\chi(x)$ can be written in the form

$$\chi(x) = \phi_n(x) - e_{n-1}\phi_{n-1}(x) - \cdots - e_0\phi_0(x), \tag{6.1}$$

where

$$\phi_0(x) = 1, \qquad \phi_k(x) = (x - a_1) \cdots (x - a_k) \qquad (1 \leqslant k \leqslant n).$$

Equating the coefficients of x^{n-1} on the two sides of (6.1), we see that $e_{n-1} = 0$. It follows that the matrix

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ e_0 & e_1 & e_2 & \cdots & e_{n-2} & a_n \end{pmatrix}$$

has the requisite properties.

The conclusion just established is still weak, for when the characteristic polynomial is given, it may be possible to prescribe not merely the first $n - 1$ diagonal elements but the entire $(n - 1) \times (n - 1)$ submatrix in the top left-hand corner. In the paper referred to, Farahat and Ledermann obtained the following result.

THEOREM 12. *Let B be a nonderogatory*[6] $(n-1) \times (n-1)$ *matrix over F, and let $\chi(x)$ be a monic polynomial of degree n over F. Then there exists an $n \times n$ matrix A over F with characteristic polynomial $\chi(x)$ and such that $[A]_{n-1} = B$.*

It is not difficult to see that Theorem 11 can be derived as a simple consequence of Theorem 12.

We conclude with a question: if $\chi(x)$ is an irreducible monic polynomial over F, what are the conditions for the existence of a symmetric matrix over F which has $\chi(x)$ as its characteristic polynomial?

VII. INCIDENCE MATRICES

Let $S_1, \cdots, S_m$ be subsets of a set consisting of the elements $x_1, \cdots, x_n$. Defining a_{ik} as 1 or 0 according as $x_k \in S_i$ or $x_k \notin S_i$, we obtain an $m \times n$ matrix $A = (a_{ik})$ which is called the *incidence matrix* of the sets $S_1, \cdots, S_m$ and elements $x_1, \cdots, x_n$. Here we shall find it convenient to refer to any rectangular matrix of zeros and ones as an incidence matrix.

Since every incidence matrix is plainly the incidence matrix of a system of sets and elements, it is not surprising that incidence matrices should prove a powerful tool in combinatorial investigations and that they should figure prominently in the discussion of topics such as Latin squares, systems of distinct representatives, finite projective planes, and incomplete balanced block designs. We cannot here enter into details and refer the interested reader to the useful surveys of Ryser [36] and Hoffman [37]. Our present object is merely to touch on a few questions which pertain to matrix theory proper.

The first and most natural problem is concerned with the existence of incidence matrices having prescribed row-sums and column-sums. The following result was found independently by Gale [38] and by Ryser [39] (cf. also [40]).

THEOREM 13. *Let $r_1, \cdots, r_m, c_1, \cdots, c_n$ be nonnegative integers and, for $1 \leqslant k \leqslant n$, define u_k as the number of integers among $r_1, \cdots, r_m$ which are not smaller than k. Then the condition*

$$(c_1, \cdots, c_n) \prec (u_1, \cdots, u_n)$$

is necessary and sufficient for the existence of an $m \times n$ incidence matrix with row-sums $r_1, \cdots, r_m$ and column-sums $c_1, \cdots, c_n$.

Gale also gave a simple algorithm for constructing an incidence matrix

[6] A matrix is said to be *nonderogatory* if its minimum polynomial and its characteristic polynomial are identical.

with given row-sums and column-sums. It may be mentioned in passing that the evaluation of the number of such matrices is a completely unsolved problem.

In addition to row-sums and column-sums we may wish to prescribe other quantities, such as the trace of the matrix[7]; the conditions for the existence of a matrix satisfying these, or similar, requirements are not fully known. Another unsolved and difficult problem is concerned with prescribing the determinant of an $n \times n$ incidence matrix and, in particular, with evaluating the maximum of det A as A ranges over the set of such matrices. A number of results involving determinants will be found in a paper by Ryser [41].

It is also easy to pose problems relating to infinite matrices. For example, is there an infinite analogue of Theorem 13? More precisely, if $\{r_k\}$, $\{c_k\}$ are sequences of nonnegative integers, under what conditions does there exist an infinite incidence matrix with row sums $r_1, r_2, \cdots$ and column sums $c_1, c_2, \cdots$?

A somewhat different problem may be stated as follows. Let $\mathfrak{P}$ be a set of ordered pairs (i, k) of integers such that $1 \leqslant i, k \leqslant n$. What conditions must be satisfied by $\mathfrak{P}$ in order that there should exist a doubly-stochastic matrix (d_{ik}) such that $d_{ik} > 0$ or $d_{ik} = 0$ according as $(i, k) \in \mathfrak{P}$ or $(i, k) \notin \mathfrak{P}$? There is, of course, an analogous problem for infinite doubly-stochastc matrices.

VIII. Problems Involving More than one Matrix

So far the discussion has been concerned with quantities associated with a single matrix, but it is plainly feasible to propound analogous problems for several matrices. The mention of this topic has been deferred till the end since the problems arising here are exceptionally difficult: although a good many inequalities linking two or more matrices have been discovered, not a single significant case is known where it is possible to assert that a system of inequalities is essential.

Let us begin by considering two hermitian matrices A, B. Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\sigma_k\}$ denote the sets of characteristic roots of A, B, $A + B$ respectively, and suppose that the numbers in each set are arranged in nonascending order of magnitude. It is then quite easy to infer the relation

$$(\sigma_1, \cdots, \sigma_n) \prec (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n). \tag{8.1}$$

Wielandt [42] obtained a generalization of the Fischer-Courant minimax

[7] By the trace of an $m \times n$ matrix (a_{ik}) we mean the value of $a_{11} + \cdots + a_{pp}$, where $p = \min(m, n)$.

principle (see [43], p. 181) and from it deduced a set of inequalities more extensive than (8.1). His result may be stated as follows.

THEOREM 14. *With the notation as above, we have*

$$\sigma_{i_1} + \cdots + \sigma_{i_k} \leqslant \alpha_{i_1} + \cdots + \alpha_{i_k} + \beta_1 + \cdots + \beta_k$$

whenever $1 \leqslant k \leqslant n$ *and* $1 \leqslant i_1 < \cdots < i_k \leqslant n$.

These inequalities (which reduce to an equality for $k = n$) are deduced from a minimax characterization for the sum $\alpha_{i_1} + \cdots + \alpha_{i_k}$. An analogous characterization for the more general sum $c_1\alpha_1 + \cdots + c_n\alpha_n$, where the c's are given real numbers, is not known.

Lidskii [44] found a result substantially equivalent to Theorem 14. We may also mention much older inequalities due to Weyl [45] which are not contained in Theorem 14. Amir-Moéz [46] obtained a more comprehensive set of inequalities which contain those of Weyl and Wielandt as special cases, and still further inequalities were discovered quite recently by Horn [47]. Horn also considered the following converse problem: given real numbers α_k, β_k, σ_k $(1 \leqslant k \leqslant n)$, what conditions are necessary and sufficient for the existence of hermitian matrices A, B so that $\{\alpha_k\}$, $\{\beta_k\}$, $\{\sigma_k\}$ are the characteristic roots of A, B, $A + B$ respectively? A definitive answer was obtained only for $n \leqslant 4$; and the general, very interesting, problem still awaits a solution. Both the results and arguments in Horn's paper [47] are complicated, and we refer the reader to the original treatment.

Next, we turn to the analogous problem for normal matrices: given complex numbers α_k, β_k, σ_k $(1 \leqslant k \leqslant n)$, what conditions are necessary and sufficient for the existence of normal matrices A, B such that $\{\alpha_k\}$, $\{\beta_k\}$, $\{\sigma_k\}$ are the characteristic roots of A, B, $A + B$ respectively? There is no doubt that this problem is extremely hard. However, Wielandt [48] discussed successfully the following more tractable variant: given complex numbers $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \sigma$, what conditions are necessary and sufficient for the existence of normal matrices A, B with characteristic roots $\{\alpha_k\}$, $\{\beta_k\}$ respectively and such that σ is a characteristic root of $A + B$? Wielandt's solution is given not by means of a system of inequalities but in geometric terms, and it does not obviously lead to an effective procedure for deciding whether a set of given numbers possesses the requisite properties.

Finally, we consider general complex matrices. Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\sigma_k\}$ denote the singular values of A, B, $A + B$ respectively, each set being arranged in non-ascending order of magnitude. Two noteworthy systems of inequalities, both due to Fan [49] are known:

$$(\sigma_1, \cdots, \sigma_n) \prec\prec (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n),$$

$$\sigma_{r+s+1} \leqslant \alpha_{r+1} + \beta_{s+1} \qquad (r \geqslant 0, s \geqslant 0, r + s < n).$$

Further, let $\{\pi_k\}$ be the set of singular values, again arranged in nonascending order, of AB. Fan, in the same paper, obtained the relations

$$\pi_{r+s+1} \leqslant \alpha_{r+1}\beta_{s+1} \qquad (r \geqslant 0, s \geqslant 0, r + s < n).$$

A different set of inequalities was found by Horn [50] who showed that

$$\pi_1 \cdots \pi_k \leqslant \alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_k \qquad (1 \leqslant k \leqslant n) \tag{8.2}$$

(with, of course, equality for $k = n$). None of these relations is difficult to establish. For instance, Horn's inequalities (8.2) follow by a combination of the polar factorization theorem and the use of compound matrices. An alternative proof, depending on quite different ideas, has been given by de Bruijn [51], who also derived a far-reaching generalization of (8.2).

All these inequalities suggest the obvious question: given real numbers $\{\alpha_k\}$, $\{\beta_k\}$, $\{\sigma_k\}$, $\{\pi_k\}$, what are the necessary and sufficient conditions that they should be the singular values of (complex) matrices $A, B, A + B, AB$ respectively? But beyond the obvious necessary conditions which follow from the stated inequalities, nothing at all is known about the solution of this problem.

Note Added in Proof

(i) When writing my survey, I overlooked two papers bearing on the problem proposed at the end of Section VI. A. P. Šapiro (Characteristic polynomials of third order rational symmetric matrices, *Dokl. Akad. Nauk SSSR* **119**, 890-892 (1958) (Russian)) has shown that a monic irreducible cubic polynomial with rational coefficients and real zeros is necessarily the characteristic polynomial of a symmetric matrix with rational elements. F. Krakowski (Eigenwerte und Minimalpolynome symmetrischer Matrizen in kommutativen Körpern, *Commun. Math. Helv.* **23**, 224-240 (1958)) derived a large number of results some of which we briefly indicate. If a field F is formally real, then the characteristic roots of symmetric matrices over F are precisely the elements which define totally real algebraic extensions over F. If F is not formally real, then every algebraic element over F is a characteristic root of a symmetric matrix over F. In particular, an algebraic number is a characteristic root of a symmetric matrix with rational elements if and only if all its conjugates are real. Similar results are established for minimum polynomials and for skew-symmetric and orthogonal matrices. However, the method used does not allow us to infer the order of the matrices whose existence is asserted.

(ii) In Section VII we mentioned the problem of estimating the maximum of det A, say $f(n)$, as A ranges over the set of $n \times n$ incidence matrices. A recent paper by E. Ehlich (Determinanten-abschätzungen für binäre Matrizen, *Math. Z.* **82**, 123-132 (1964)) discusses this and similar questions. Interesting results for $f(n)$ and related functions have been obtained by J. H. E. Cohn (On the value of determinants, *Proc. Am. Math. Soc.* **14**, 581-588 (1963)) who showed, in particular, that $\log f(n)/n \log n \to 1$ as $n \to 1$.

(iii) A very remarkable existence theorem concerning positive definite matrices has been just proved by M. Fiedler (Relations between the diagonal elements of two mutually inverse positive definite matrices, *Czech. Math. J.* 14, 39-51 (1964)). Fiedler's result states that the $2n$ real numbers $a_{11}, \cdots, a_{nn}, b_{11}, \cdots, b_{nn}$ can serve as the diagonal elements of a real symmetric positive definite $n \times n$ matrix $A = (a_{rs})$ and its inverse $A^{-t} = (b_{rs})$ if and only if the inequalities

$$a_{rr} > 0, \qquad b_{rr} > 0, \qquad a_{rr}b_{rr} \geqslant 1,$$

$$\sqrt{(a_{rr}b_{rr})} - 1 \leqslant \sum_{\substack{1 \leqslant s \leqslant n \\ s \neq r}} (\sqrt{(a_{ss}b_{ss})} - 1)$$

are valid for $1 \leqslant r \leqslant n$. This result leads to the determination of the essential system of inequalities between the lengths of $2n$ vectors which constitute a biorthogonal system in unitary n-dimensional space.

Fiedler's paper, as well as the two papers mentioned in the preceding paragraph, appeared after the present survey had been submitted for publication.

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