Weyl type theorem and hypercyclic operators

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Abstract

Using a variant of the essential approximate point spectrum, we give the necessary and sufficient conditions for \( T \) for which the a-Browder’s theorem or the a-Weyl’s theorem holds. Also, the relation between hypercyclic operators (or supercyclic operators) and the operators which satisfy Weyl type theorem is discussed.

Keywords: a-Weyl’s theorem; a-Browder’s theorem; Hypercyclic operator; Supercyclic operator

1. Introduction

Throughout this note let \( B(H) \) (\( K(H) \)) denote the algebra of bounded linear operators (compact operators) acting on a complex, separable, infinite dimensional Hilbert space \( H \). If \( T \in B(H) \), write \( N(T) \) and \( R(T) \) for the null space and the range of \( T \); \( \sigma(T) \) for the spectrum of \( T \); \( \pi_{00}(T) = \pi_{0}(T) \cap \text{iso} \sigma(T) \), where \( \pi_{0}(T) = \{ \lambda \in \mathbb{C}: 0 < \dim N(T - \lambda I) < \infty \} \) are the eigenvalues of finite multiplicity. An operator \( T \in B(H) \) is called upper semi-Fredholm if it has closed range with finite dimensional null space and if \( R(T) \) has finite co-dimension, \( T \in B(H) \) is called a lower semi-Fredholm operator. We call \( T \in B(H) \) Fredholm if it has closed range with finite dimensional null space and its range is of finite co-dimension. The index of a Fredholm operator \( T \in B(H) \) if given by

\[
\text{ind}(T) = \dim N(T) - \dim R(T) \perp (= \dim N(T) - \dim N(T^*))
\]
An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. And $T \in B(H)$ is called Browder if it is Fredholm “of finite ascent and descent”: equivalently [3, Theorem 7.9.3] if $T$ is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, the Browder spectrum $\sigma_b(T)$, the upper semi-Fredholm spectrum and the lower semi-Fredholm spectrum of $T \in B(H)$ are defined by (cf. [3,5])

$$
\sigma_e(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not Fredholm} \},
$$

$$
\sigma_w(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not Weyl} \},
$$

$$
\sigma_b(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not Browder} \},
$$

$$
\sigma_{SF_+}(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not upper semi-Fredholm} \},
$$

$$
\sigma_{SF_-}(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not lower semi-Fredholm} \}.
$$

We say that the Weyl’s theorem holds for $T \in B(H)$ if there is equality

$$
\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).
$$

Harte and Lee [4] have discussed a variant of the Weyl’s theorem: “the Browder’s theorem holds” for $T$ if

$$
\sigma(T) = \sigma_w(T) \cup \pi_{00}(T).
$$

What is missing is the disjointness between the Weyl spectrum and the isolated eigenvalues of finite multiplicity: equivalently

$$
\sigma_w(T) = \sigma_b(T).
$$

Rakočević [10] has looked at variants of “Weyl’s theorem” and “Browder’s theorem” in which the spectrum is replaced by the approximate point spectrum: “the a-Weyl’s theorem holds” for $T$

$$
\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),
$$

where we write $\sigma_a(T)$ for the approximate point spectrum of $T$, $\pi_{00}^a(T) = \pi_0(T) \cap \text{iso } \sigma_a(T)$ and

$$
\sigma_{ea}(T) = \bigcap \{ \sigma_a(T + K): K \in K(H) \}.
$$

It is well known that $\sigma_{ea}(T)$ coincides with

$$
\sigma_{ea}(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \notin SF_-(H) \},
$$

where $SF_-(H) = \{ T \in B(H): T \text{ is upper semi-Fredholm of ind}(T) \leq 0 \}$. Finally, “the a-Browder’s theorem holds” for $T$

$$
\sigma_{ea}(T) = \sigma_ab(T),
$$

where $\sigma_{ab}(T) = \bigcap \{ \sigma_a(T + K): K \in K(H) \cap \text{comm}(T) \}$. It is well known that $\lambda \notin \sigma_{ab}(T)$ if and only if $T - \lambda I$ is upper semi-Fredholm and $T - \lambda I$ has finite ascent. We call $\sigma_{ea}(T)$ and $\sigma_{ab}(T)$ the essential approximate point spectrum and the Browder essential approximate point spectrum, respectively.

For $x \in H$, the orbit of $x$ under $T$ is the set of images of $x$ under successive iterates of $T$:

$$
\text{Orb}(T, x) = \{ x, Tx, T^2x, \ldots \}.
$$
A vector \( x \in H \) is supercyclic if the set of scalar multiples of \( \text{Orb}(T, x) \) is dense in \( H \), and \( x \) is hypercyclic if \( \text{Orb}(T, x) \) is dense. A hypercyclic operator is one that has a hypercyclic vector. We similarly define the notion of supercyclic operator. We denote by \( HC(H) \) (\( SC(H) \)) the set of all hypercyclic (supercyclic) operators in \( B(H) \) and \( \overline{HC(H)} \) (\( \overline{SC(H)} \)) the norm-closure of the class \( HC(H) \) (\( SC(H) \)). Supercyclic operators were introduced by Hilden and Wallen in 1974 [7]. Many fundamental results regarding the theory of hypercyclic and supercyclic operators were established by C. Kitai in her thesis [8].

In Section 2, we give the necessary and sufficient conditions for \( T \in B(H) \) for which the a-Browder’s theorem or a-Weyl’s theorem holds. In Section 3, we study the relations between hypercyclic operators (supercyclic operators) and the operators for which Weyl type theorem holds.

2. Weyl type theorem for operator \( T \)

We turn to a variant of the essential approximate point spectrum. \( T \in B(H) \) is called a generalized upper semi-Fredholm operator if there exists \( T \)-invariant subspaces \( M \) and \( N \) such that \( H = M \oplus N \) and \( T|_M \in SF_+^0(M) \), \( T|_N \) is quasinilpotent. Clearly, if \( T \) is generalized upper semi-Fredholm, there exists \( \epsilon > 0 \) such that \( T - \lambda I \in SF_+^0(H) \) and \( N(T - \lambda I) \subseteq \bigcap_{n=1}^\infty R[(T - \lambda I)^n] \) if \( 0 < |\lambda| < \epsilon \). Clearly, if \( \lambda \in \sigma_a(T), T - \lambda I \) is generalized upper semi-Fredholm.

The new spectrum set is defined as follows. Let

\[
\rho_1(T) = \{ \lambda \in \mathbb{C} : \text{there exists } \epsilon > 0 \text{ such that } T - \mu I \text{ is generalized upper semi-Fredholm if } 0 < |\mu - \lambda| < \epsilon \}
\]

and let \( \sigma_1(T) = \mathbb{C} \setminus \rho_1(T) \). Then

\[
\sigma_1(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_{ab}(T) \subseteq \sigma_a(T).
\]

\( T \) is called approximate isoloid (a-isoloid) (or isoloid) if \( \lambda \in \sigma_a(T) \Rightarrow N(T - \lambda I) \neq \{0\} \) and \( T \) is called finite approximate isoloid (f-a-isoloid) (or finite isoloid, f-isoloid) operator if the isolated points of approximate point spectrum (of the spectrum) are all eigenvalues of finite multiplicity. Clearly, f-a-isoloid implies a-isoloid and finite isoloid, but the converse is not true.

**Theorem 2.1.** a-Browder’s theorem holds for \( T \) \( \Leftrightarrow \) acc \( \sigma_a(T) = \sigma_1(T) \cup \text{acc}[\sigma_a(T)] \).

**Proof.** Suppose that acc \( \sigma_a(T) = \sigma_1(T) \cup \text{acc}[\sigma_a(T)] \). We only need to prove that \( \sigma_{ab}(T) \subseteq \sigma_{ea}(T) \). Let \( \lambda_0 \notin \sigma_{ea}(T) \), then \( \lambda_0 \notin \sigma_1(T) \). Also \( \lambda_0 \notin \text{acc}[\sigma_a(T)] \). In fact, using the punctured neighborhood theorem, there exists \( \epsilon > 0 \) such that \( T - \lambda I \in SF_+^0(H) \) and \( N(T - \lambda I) \subseteq \bigcap_{n=1}^\infty R[(T - \lambda I)^n] \) if \( 0 < |\lambda - \lambda_0| < \epsilon \). If \( \lambda_0 \in \text{acc}[\sigma_a(T)] \), there exists \( \lambda_1 \) such that \( 0 < |\lambda - \lambda_0| < \epsilon \) and \( \lambda_1 \in \text{iso} \sigma_a(T) \). Then \( T \) has the single valued extension property in \( \lambda_1 \). Theorem 11 in [1] asserts that \( T - \lambda_1 I \) has finite ascent. It induces that \( N(T - \lambda_1 I) = N(T - \lambda_1 I) \cap \bigcap_{n=1}^\infty R[(T - \lambda_1 I)^n] = \{0\} \) [11, Lemma 3.4], which means that \( T - \lambda_1 I \) is bounded from below. It is in contradiction to the fact that \( \lambda_1 \in \sigma_a(T) \). Therefore \( \lambda_0 \notin \sigma_1(T) \cup \text{acc}[\sigma_a(T)] \), which means that \( \lambda_0 \in \text{iso} \sigma_a(T) \cup \rho_a(T) \), where \( \rho_a(T) = \mathbb{C} \setminus \sigma_a(T) \). Then \( T - \lambda_0 I \) has finite ascent [1, Theorem 11]. It induces that \( \lambda_0 \) is not in \( \sigma_{ab}(T) \). Then \( \sigma_{ea}(T) = \sigma_{ab}(T) \), which means that the a-Browder’s theorem holds for \( T \).

Conversely, suppose that the a-Browder’s theorem holds for \( T \). If \( \lambda_0 \notin \sigma_1(T) \cup \text{acc}[\sigma_a(T)] \), then there exists \( \epsilon > 0 \) such that \( T - \lambda I \) is generalized upper semi-Fredholm if \( 0 < |\lambda - \lambda_0| < \epsilon \).
For any \( \lambda \), there exists \( \epsilon' \) such that \( T - \lambda' I \in SF_{+}(H) \) and \( N(T - \lambda' I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda' I)^{n}] \) if \( 0 < |\lambda' - \lambda| < \epsilon' \). Since the a-Browder’s theorem holds for \( T \), it follows that \( T - \lambda' I \) has finite ascent. Then \( N(T - \lambda' I) = N(T - \lambda' I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda' I)^{n}] = \{0\} \) [11, Lemma 3.4], which means that \( T - \lambda' I \) is bounded from below if \( 0 < |\lambda' - \lambda| < \epsilon \). Therefore \( \lambda \in \sigma_{a}(T) \cup \rho_{a}(T) \). By Theorem 2.1, we only need to prove that \( acc_{a}(T) \cup \sigma_{a}(T) \). Thus \( acc_{a}(T) \subseteq \rho_{a}(T) \). Since \( \lambda_{0} \notin acc_{a}(T) \), it follows that \( \lambda_{0} \notin acc_{a}(T) \) and \( \lambda_{0} \notin acc_{a}(T) \cup \rho_{a}(T) \). If \( \lambda_{0} \notin acc_{a}(T) \), then \( \lambda_{0} \notin acc_{a}(T) \cup \rho_{a}(T) \). For the converse, since \( acc_{a}(T) \subseteq \sigma_{a}(T) \cup \rho_{a}(T) \), we know that \( acc_{a}(T) \subseteq \sigma_{a}(T) \cup \rho_{a}(T) \). Thus \( acc_{a}(T) = \sigma_{a}(T) \cup \rho_{a}(T) \). □

For the a-Weyl’s theorem, we have

**Theorem 2.2.** \( T \in B(H) \) is f-a-isoloid and the a-Weyl’s theorem holds for \( T \iff \sigma_{ab}(T) = \sigma_{a}(T) \cup acc[\sigma_{a}(T)] \).

**Proof.** Suppose \( T \) is f-a-isoloid and the a-Weyl’s theorem holds for \( T \). Since a-Weyl’s theorem implies a-Browder’s theorem, it follows that \( acc_{a}(T) = \sigma_{a}(T) \cup acc[\sigma_{a}(T)] \). Thus from Theorem 2.1, we only need to prove that \( acc_{a}(T) = \sigma_{ab}(T) \). \( acc_{a}(T) \subseteq \sigma_{ab}(T) \) is clear. If \( \lambda_{0} \notin acc_{a}(T) \), then \( \lambda_{0} \notin acc_{a}(T) \cup \rho_{a}(T) \). Since \( T \) is f-a-isoloid, it follow that \( \lambda_{0} \notin acc_{a}(T) \cup \rho_{a}(T) \). The fact that a-Weyl’s theorem holds for \( T \) tells us that \( T - \lambda_{0} I \in SF_{+}(H) \) with finite ascent, which means that \( \lambda_{0} \notin \sigma_{ab}(T) \).

Conversely, suppose \( \sigma_{ab}(T) = \sigma_{a}(T) \cup acc[\sigma_{a}(T)] \). Since \( acc_{a}(T) \subseteq \sigma_{ab}(T) = \sigma_{a}(T) \cup acc[\sigma_{a}(T)] \), from the proof of Theorem 2.1, we know that a-Browder’s theorem holds for \( T \), that is \( \sigma_{a}(T) \subseteq N(T - \lambda_{0} I) \subseteq \rho_{SF}(T) \). Let \( \lambda_{0} \in \rho_{SF}(T) \), then \( \lambda_{0} \notin acc_{a}(T) \cup \rho_{SF}(T) \). Thus \( \lambda_{0} \notin \sigma_{ab}(T) \), which means that \( \lambda_{0} \notin acc_{a}(T) \cup \rho_{SF}(T) \). Therefore a-Weyl’s theorem holds for \( T \). Using the fact that \( acc_{a}(T) = \sigma_{a}(T) \), we can prove that \( T \) is f-a-isoloid. □

3. Hypercyclic operators and supercyclic operators

The following lemmas give the essential facts for hypercyclic operators and supercyclic operators that we will need to prove the main theorem [6, Theorems 2.1 and 3.3].

**Lemma 3.1.** \( \overline{HC/(H)} \) is the class of all those operators \( T \in B(H) \) satisfying the conditions:

1. \( \sigma_{a}(T) \cup \partial D \) is connected;
2. \( \partial(T) \setminus \sigma_{b}(T) = \emptyset \);
3. \( \text{ind}(T - \lambda I) \geq 0 \) for every \( \lambda \in \rho_{SF}(T) \), where \( \rho_{SF}(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \) is semi-Fredholm\}.

**Lemma 3.2.** \( \overline{SC/(H)} \) is the set of all those \( T \in B(H) \) satisfying the conditions:

1. \( \partial(T) \setminus \sigma_{b}(T) = \emptyset \) and \( \partial(D) \) is connected (for some \( r \geq 0 \));
2. \( \sigma_{a}(T) \cup \partial(D) \) is connected (for some \( r \geq 0 \));
3. either \( \sigma(T) \setminus \sigma_{a}(T) = \emptyset \) or \( \sigma(T) \setminus \sigma_{b}(T) = \{a\} \) for some \( a \neq 0 \);
4. \( \text{ind}(T - \lambda I) \geq 0 \) for every \( \lambda \in \rho_{SF}(T) \).

Let \( H(T) \) be the class of complex-valued functions which are analytic in a neighbourhood of \( \sigma(T) \) and are not constant on any neighbourhood of any component of \( \sigma(T) \). Our results are:
Theorem 3.3. Suppose $\sigma(T) = \sigma_1(T)$ is connected. If $f \in H(T)$ such that $|f(\lambda)| = 1$ for some $\lambda_0 \in \sigma(T)$, then $f(T) \in \overline{HC(H)}$.

Proof. Using the fact that $\sigma(T) \cap \sigma_1(T) = \emptyset$ and $\sigma(T) = \sigma_1(T)$, we know that $\sigma(T) = \emptyset$. Since each of $\sigma(T)$ and $\sigma_0(T)$ obeys the two way spectral mapping theorem, it follows that $\sigma(f(T)) \subseteq f(\sigma(T)) \subseteq f(\sigma_0(T))$. Thus $f(T)$ is semi-Fredholm operator. For every $\lambda \in \rho_{SF}(T)$ and $\lambda I \in \sigma_0(T)$, we know that $T - \lambda I$ is invertible. Then $\text{ind}(T - \lambda I) = 0$. This proof tells us that for every $\lambda \in \rho_{SF}(T)$, $\text{ind}(T - \lambda I) \geq 0$, then $\text{ind}(f(T) - \lambda I) \geq 0$ if $f(T) - \mu I$ is semi-Fredholm operator. Thus Theorem 5 in [4] shows that $\sigma_w(f(T)) = f(\sigma(T))$. Since $\sigma(T) = \sigma_1(T)$ and $\sigma_1(T) \subseteq \sigma_0(T)$, it follows that $\sigma_w(T) = \sigma_1(T) = \sigma(T)$ is connected. Then $\sigma_w(f(T)) = f(\sigma(T))$ is connected. We have $|f(\lambda)| = 1$ for some $\lambda \in \sigma(T)$, hence $f(\lambda) \in \partial D$ and $f(\lambda) \in f(\sigma(T)) = f(\sigma(T)) = f(\sigma(T))$, thus $f(\lambda) \in \sigma_w(f(T)) \cap \partial D$. Since $\sigma_w(f(T))$ and $\partial D$ are connected, $\sigma_w(f(T)) \cup \partial D$ is connected. Using Lemma 3.1, $f(T) \in \overline{HC(H)}$. □

Remark. We also can prove the following facts:

1. In the proof of Theorem 3.3, we can see that if $\sigma(T) = \sigma_1(T)$ is connected, then for every $f \in H(T)$, there exists $c \neq 0$ such that $cf(T) \in \overline{HC(H)}$.

2. In fact, for every $f \in H(T)$, there exist $\lambda_0 \in \sigma(T)$ such that $f(\lambda_0) \neq 0$. We can take $c = 1/f(\lambda_0)$, then $|cf(\lambda_0)| = 1$. Thus $cf(T) \in \overline{HC(H)}$.

3. If $f(T)$ is $p$-hyponormal or $M$-hyponormal operator and if $\sigma(T)$ is connected, then for every $f \in H(T)$, there exists $c \neq 0$ such that $cf(T) \in \overline{HC(H)}$.

If $T^*$ is hyponormal or $M$-hyponormal operator, we know that $\sigma(T) = \sigma_0(T)$ and $T - \lambda I$ has finite descent for each $\lambda \in \rho_{SF}(T)$. Then we can get that $\sigma(T) = \sigma_1(T)$.

Corollary 3.4. If $T \in B(H)$ is $f$-isoloid and the Weyl’s theorem holds for $T$ (or $T$ is $f$-a-isoloid and the $a$-Weyl’s theorem holds for $T$), then

1. $T \in \overline{HC(H)} \Leftrightarrow \sigma(T) = \sigma_1(T)$ and $\sigma(T) \cup \partial D$ is connected;
2. $T \in \overline{SC(H)} \Leftrightarrow \sigma(T) = \sigma_1(T)$ or $\sigma(T) = \sigma_1(T) \cup \{\lambda\}$, where $\lambda \neq 0$ and $T - \lambda I$ is Browder, and $\sigma(T) \cup \partial \{rD\}$ is connected for some $r \geq 0$.

Proof. Suppose $T \in \overline{HC(H)}$. Let $\lambda_0 \notin \sigma_1(T)$. Then there exists $\epsilon > 0$ such that $T - \lambda I$ is generalized upper semi-Fredholm. For every $\lambda$, there exists $\epsilon'$ such that $T - \lambda' I \in SF^+_+(H)$ and $N(T - \lambda I) \subseteq \{ \lambda \in \rho_{SF}(T) \cup \{\lambda\} : 0 < |\lambda' - \lambda| < \epsilon' \}$. Since $T \in \overline{HC(H)}$, it induces that $\text{ind}(T - \lambda I) \geq 0$ (Lemma 3.1). Then $T - \lambda I$ is Weyl if $0 < |\lambda' - \lambda| < \epsilon$. Since the Weyl’s theorem holds for $T$, then $T - \lambda I$ is Browder and hence $T - \lambda I$ is invertible if $0 < |\lambda' - \lambda| < \epsilon$. It implies $\lambda \in \text{iso} \sigma(T) \cup \rho(T)$, where $\rho(T) = C \setminus \sigma(T)$. We claim that $\lambda \notin \text{iso} \sigma(T)$. If not, since $T$ is finite isoloid and the Weyl’s theorem holds for $T$, it follows that $\lambda \in \pi_{00}(T) = \sigma(T) \setminus \sigma_0(T)$. Then $T - \lambda I$ is Browder. It is in contradiction to the fact that $T \in \overline{HC(H)}$ (Lemma 3.1). Thus $\lambda \notin \sigma(T)$. It induces that $\lambda_0 \in \text{iso} \sigma(T) \cup \rho(T)$. Using the same way, we prove that $T - \lambda_0 I$ is in-
invertible, which means that $\lambda_0 \notin \sigma(T)$. Now we have that $\sigma(T) = \sigma_1(T)$. Since $\sigma_1(T) \subseteq \sigma_w(T)$, we get $\sigma_w(T) = \sigma(T)$. The condition $T \in H\mathcal{C}(\mathcal{H})$ tells us that $\sigma_w(T) \cup \partial D$ is connected, then $\sigma(T) \cup \partial D$ is connected.

Conversely, suppose that $\sigma(T) = \sigma_1(T)$ and $\sigma(T) \cup \partial D$ is connected. Since $\sigma_w(T) = \sigma(T)$, it follows that $\sigma_w(T) \cup \partial D$ is connected. The proof of Theorem 3.3 tells us that $T \in H\mathcal{C}(\mathcal{H})$.

Similarly, we can prove (2) \qed

If $T \in H\mathcal{C}(\mathcal{H})$ or $T \in \mathcal{S}\mathcal{C}(\mathcal{H})$, we have

**Corollary 3.5.** Suppose $T \in H\mathcal{C}(\mathcal{H})$, then the following statements are equivalent:

1. $\sigma(T) = \sigma_1(T)$;
2. $T$ is f-a-isoloid and the a-Weyl’s theorem holds for $T$;
3. $f(T)$ is f-a-isoloid and the a-Weyl’s theorem holds for $f(T)$ for any $f \in H(T)$;
4. $T$ is f-isoloid and the Weyl’s theorem holds for $T$;
5. $f(T)$ is f-isoloid and the Weyl’s theorem holds for $f(T)$ for any $f \in H(T)$.

**Proof.** (1) $\Rightarrow$ (2). Since $\sigma(T) = \sigma_1(T)$ and $\sigma_1(T) \cup \text{acc}[	ext{iso } \sigma_a(T)] \subseteq \sigma_{ab}(T) \subseteq \sigma(T)$, it follows that $\sigma_{ab}(T) = \sigma_1(T) \cup \text{acc}[	ext{iso } \sigma_a(T)]$. Then from Theorem 2.2, we know $T$ is f-a-isoloid and the a-Weyl’s theorem holds for $T$.

(2) $\Rightarrow$ (1). See Corollary 3.4.

(2) $\Rightarrow$ (3). Since $T \in H\mathcal{C}(\mathcal{H})$, it induces that for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{SF_+}(T)$, ind$(T - \lambda I) \geq 0$. Theorem 2.2 in [2] tells us that the a-Weyl’s theorem holds for $f(T)$ for every $f \in H(T)$. In the following, we will prove $f(T)$ is f-a-isoloid. Let $\mu_0 \in \text{iso } \sigma(f(T))$ and let $f(T) - \mu_0 I = a(T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T)$, where $a \neq 0$, $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Without loss of generality, we suppose that $\lambda_j \in \text{iso } \sigma_a(T)$. Since $T$ is f-a-isoloid, it follows that $0 < \dim N[(T - \lambda_i)^{n_1}] < \infty$ for every $\lambda_i$. We know that $N(f(T) - \mu_0 I) = N[(T - \lambda_1 I)^{n_1} \oplus N[(T - \lambda_2 I)^{n_2} \oplus \cdots \oplus N((T - \lambda_k I)^{n_k}]]$, so $0 < \dim N(f(T) - \mu_0 I) < \infty$, which means that $f(T)$ is f-a-isoloid.

(3) $\Rightarrow$ (2). It is clear.

Since $T \in H\mathcal{C}(\mathcal{H})$, we can prove that $\sigma(T) = \sigma_a(T), \sigma_w(T) = \sigma_{ea}(T)$ and $\pi_{00}(T) = \pi_{00}(T)$. Also for any $f \in H(T)$, we have that $\sigma(f(T)) = \sigma_a(f(T)), \sigma_{ea}(f(T)) = \sigma_w(f(T))$ and $\pi_{00}(f(T)) = \pi_{00}(f(T))$. Then (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5). \qed

**Corollary 3.6.** Suppose $T \in \mathcal{S}\mathcal{C}(\mathcal{H})$, then the following statements are equivalent:

1. $\sigma(T) = \sigma_1(T)$ or $\sigma(T) = \sigma_1(T) \cup \{\lambda\}$, where $\lambda \neq 0$ and $T - \lambda I$ is Browder;
2. $T$ is f-a-isoloid and the a-Weyl’s theorem holds for $T$;
3. $f(T)$ is f-a-isoloid and the a-Weyl’s theorem holds for $f(T)$ for any $f \in H(T)$;
4. $T$ is f-isoloid and the Weyl’s theorem holds for $T$;
5. $f(T)$ is f-isoloid and the Weyl’s theorem holds for $f(T)$ for any $f \in H(T)$.

If $\sigma_{SF_-}(T) = \sigma_1(T)$, we claim that ind$(T - \lambda I) \leq 0$ for any $\lambda \in \rho_{SF}(T)$. In fact, if there exists $\lambda_0 \in \rho_{SF}(T)$ such that ind$(T - \lambda_0 I) > 0$, then $T - \lambda_0 I$ is lower semi-Fredholm. Thus $\lambda_0 \notin \sigma_1(T)$ because $\sigma_{SF_-}(T) = \sigma_1(T)$. By the definition of $\sigma_1(T)$, for any sufficiently small $\epsilon > 0$, there exists $\lambda$ such that $0 < |\lambda - \lambda_0| < \epsilon$ and $T - \lambda I \in SF_{-}(\mathcal{H})$. Using the punctured neighborhood theorem, we know that ind$(T - \lambda_0 I) = \text{ind}(T - \lambda I)$ if $|\lambda - \lambda_0| > 0$ is sufficiently
small. Then \( \text{ind}(T - \lambda_0 I) \leq 0 \). It is in contradiction to the fact that \( \text{ind}(T - \lambda_0 I) > 0 \). Thus \( \text{ind}(T - \lambda I) \leq 0 \) for any \( \lambda \in \rho_{SF}(T) \). So it induces that \( \text{ind}(T - \lambda I) \text{ind}(T - \mu I) \geq 0 \) for each pair \( \lambda, \mu \in \mathbb{C}\setminus\sigma_{SF_+}(T) \).

**Theorem 3.7.** Suppose \( \sigma_{SF_-}(T) = \sigma_1(T) \). If \( \text{iso}\sigma(T) = \emptyset \) and \( \sigma_w(T) \) is connected, then for each \( f \in H(T) \), there exists \( c \neq 0 \) such that \( cf(T^*) \in \overline{HC(H)} \).

**Proof.** Using the fact that \( \text{ind}(T - \lambda I) \text{ind}(T - \mu I) \geq 0 \) for each pair \( \lambda, \mu \in \mathbb{C}\setminus\sigma_{SF_+}(T) \) and using [4, Theorem 5], we know that \( f(\sigma_w(T)) = \sigma_w(f(T)) \) for any \( f \in H(T) \). Since \( \sigma_w(T) \) is connected and \( \text{iso}\sigma(T) = \emptyset \), it follows that \( \sigma_w(f(T)) = \sigma_w(T) \) is connected and \( \text{iso}\sigma(f(T)) = \emptyset \). Then \( \sigma_w(f(T^*)) = \sigma_w(T^*) \) is connected and \( \text{iso}\sigma(f(T^*)) = \emptyset \).

Since \( \text{ind}(T - \lambda I) \leq 0 \) for each \( \lambda \in \rho_{SF}(T) \). Using Lemma 3.1 and the statement in Remark, we can say that for each \( f \in H(T) \), there exists \( c \neq 0 \) such that \( cf(T^*) \in \overline{HC(H)} \). \( \square \)

If \( \sigma_{SF_-}(T) = \sigma_1(T) \), \( T \) must be \( f \)-a-isoloid. In fact, if \( \lambda_0 \in \text{iso}\sigma_a(T) \), then \( \lambda_0 \notin \sigma_1(T) \), hence \( \lambda_0 \notin \sigma_{SF_-}(T) \), that is \( T - \lambda_0 I \) is lower semi-Fredholm. But since \( T \in \text{iso}\sigma_a(T) \), it means that \( T \) has the single valued extension property at \( \lambda_0 \). Theorem 11 in [1] tells us that \( T - \lambda_0 I \) has finite ascent. Thus \( \dim N(T - \lambda_0 I) \) is finite, which means that \( T - \lambda_0 I \) is Fredholm. If \( N(T - \lambda_0 I) = \{0\} \), \( T - \lambda_0 I \) is bounded from below. It is in contradiction to the fact that \( \lambda_0 \in \text{iso}\sigma_a(T) \). Then \( 0 < \dim N(T - \lambda_0 I) < \infty \), which means that \( T \) is \( f \)-a-isoloid.

**Corollary 3.8.** Suppose \( \sigma_{SF_-}(T) = \sigma_1(T) \) and \( \sigma_w(T) \) is connected. If \( \text{iso}\sigma(T) = \emptyset \) and if the \( a \)-Weyl’s theorem holds for \( T \), then the following statements are true:

1. For each \( f \in H(T) \), there exists \( c \neq 0 \) such that \( cf(T^*) \in \overline{HC(H)} \);
2. \( \text{ind}(T - \lambda I) \text{ind}(T - \mu I) \geq 0 \) for each pair \( \lambda, \mu \in \mathbb{C}\setminus\sigma_{SF_+}(T) \);
3. The \( a \)-Weyl’s theorem holds for \( f(T) \) for each \( f \in H(T) \);
4. The \( a \)-Browder’s theorem holds for \( f(T) \) for each \( f \in H(T) \);
5. The Weyl’s theorem holds for \( f(T) \) for each \( f \in H(T) \);
6. The Browder’s theorem holds for \( f(T) \) for each \( f \in H(T) \);
7. \( f(\sigma_1(T)) = \sigma_1(f(T)) \) for each \( f \in H(T) \).

**Proof.** (1) and (2). See Theorem 3.7 and remark before Theorem 3.7.

(3). Since \( T \) is \( f \)-a-isoloid, [2, Theorem 2.2] asserts that the result is true.

(4)–(6). Since \( a \)-Weyl’s theorem implies \( f \)-Weyl’s theorem, \( a \)-Browder’s theorem and Browder’s theorem, the result is true.

(7) Let \( \mu_0 \in f(\sigma_1(T)) \) and let \( \lambda_0 = f(\lambda_0) \), where \( \lambda_0 \in \sigma_1(T) \). If \( \mu_0 \notin \sigma_1(f(T)) \), there exists \( \delta > 0 \) such that \( f(T) - \mu I \) is generalized upper semi-Fredholm \( 0 < |\mu - \mu_0| < \delta \). For each \( \mu \), there exists \( \delta' > 0 \) such that \( f(T) - \mu' I \in SF_{\infty}(H) \) and \( N(f(T) - \mu' I) \subseteq \bigcap_{n=1}^{\infty} R[(f(T) - \mu' I)^n] \) if \( 0 < |\mu' - \mu| < \delta' \). Since \( a \)-Browder’s theorem holds for \( f(T) \), it follows that \( f(T) - \mu' I \) has finite ascent. Then \( N(f(T) - \mu' I) = N(f(T) - \mu' I) \cap \bigcap_{n=1}^{\infty} R[(f(T) - \mu' I)^n] = \{0\} \) [11, Lemma 3.4], which means that \( \mu \in \text{iso}\sigma_a(f(T)) \cup \rho_a(f(T)) \). For \( \lambda_0 \), there exists \( \epsilon > 0 \) such that \( f(\lambda) \in \text{iso}\sigma_a(f(T)) \cup \rho_a(f(T)) \) if \( 0 < |\lambda - \lambda_0| < \epsilon \). Then \( \lambda \in \text{iso}\sigma_a(T) \cup \rho_a(T) \). Thus \( \lambda \notin \sigma_1(T) \). Since \( \sigma_1(T) = \sigma_{SF_+}(T), T - \lambda_0 I \) is Fredholm with the index \( \text{ind}(T - \lambda_0 I) \leq 0 \). Now we have proved that for \( \lambda_0 \), there exists \( \epsilon > 0 \) such that \( T - \lambda_0 I \) is generalized upper semi-Fredholm. Then \( \lambda_0 \notin \sigma_1(T) \). It is in contradiction to the fact that \( \lambda_0 \in \sigma_1(T) \). Thus \( f(\sigma_1(T)) \subseteq \sigma_1(f(T)) \) for any \( f \in H(T) \).
For the converse, let \( \mu_0 \notin f(\sigma_1(T)) \) and let \( f(T) - \mu_0 I = a(T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T) \), where \( \lambda_i \neq \lambda_j, a \neq 0 \) and \( g(T) \) is invertible. Then \( \lambda_i \notin \sigma_1(T) \). Since \( \sigma_{SF_+}(T) = \sigma_1(T) \), we know that \( T - \lambda_i I \) is Fredholm and \( \text{ind}(T - \lambda_i I) \leq 0 \) for each \( i = 1, 2, \ldots, k \). Then \( f(T) - \mu_0 I \) is Fredholm and \( \text{ind}(f(T) - \mu_0 I) \leq 0 \), which means that \( \mu_0 \notin \sigma_1(f(T)) \). Thus \( \sigma_1(f(T)) \subseteq f(\sigma_1(T)) \) for any \( f \in H(T) \). \( \square \)

K.K. Oberai [9] has examples which show that the Weyl’s theorem for \( T \) is not sufficient for the Weyl’s theorem for \( T + F \) with finite rank \( F \). So does the a-Weyl’s theorem. But if \( T \in HC(H) \) or \( T \in SC(H) \), we have:

**Theorem 3.9.** Suppose that \( T \in SC(H) \) or \( T \in HC(H) \). If \( T \) is f-a-isoloid and the a-Weyl’s theorem holds for \( T \), then for every compact operator \( F \) commuting with \( T \), \( T + F \) is f-a-isoloid and the a-Weyl’s theorem holds for \( T + F \).

**Proof.** Suppose \( T \in SC(H) \). From Theorem 2.2, we need to prove that \( \sigma_{ab}(T + F) \subseteq \sigma_1(T + F) \cup \text{acc}[\sigma_0(T + F)] \). Let \( \lambda_0 \notin \sigma_1(T + F) \cup \text{acc}[\sigma_0(T + F)] \). Then there exists \( \epsilon > 0 \) such that \( T + F - \lambda I \) is generalized upper semi-Fredholm if \( 0 < |\lambda - \lambda_0| < \epsilon \). For any \( \lambda \), there exists \( \epsilon' > 0 \) such that \( T + F - \lambda' I \in SF_{+}^{\epsilon}(H) \) if \( 0 < |\lambda' - \lambda| < \epsilon' \). Then \( T - \lambda' I \in SF_{+}^{\epsilon}(H) \). Since \( T \in SC(H) \), \( T - \lambda' I \) is Weyl. Then \( T - \lambda' I \) is Browder because the Weyl’s theorem holds for \( T \). But since \( \sigma(T) \setminus \sigma_0(T) \) contains at most one point \( \alpha \neq 0 \), it follows that \( \lambda \in \sigma(T) \cup \rho(T) \). The fact that \( T \) is f-a-isoloid and the a-Weyl’s theorem holds for \( T \) implies that \( T - \lambda I \) is Browder. Once again, we get that \( \lambda_0 \in \text{iso } \sigma(T) \cup \rho(T) \). Therefore \( T - \lambda_0 I \) is Browder. Then \( T + F - \lambda_0 I \) is Browder, which means that \( \lambda_0 \notin \sigma_{ab}(T + F) \). Thus \( T + F \) is f-a-isoloid and the a-Weyl’s theorem holds for \( T + F \). \( \square \)

**References**