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Simple Lie algebras of small characteristic VI. Completion of the classification

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ABSTRACT

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$. It is proved in this paper that if the p-envelope of $\operatorname{ad} L$ in $\operatorname{Der} L$ contains a torus of maximal dimension whose centralizer in $\operatorname{ad} L$ acts nontriangulably on L, then $p = 5$ and L is isomorphic to one of the Melikian algebras $\mathcal{M}(m, n)$. In conjunction with [A. Premet, H. Strade, Simple Lie algebras of small characteristic V. The non-Melikian case, J. Algebra 314 (2007) 664–692, Theorem 1.2], this implies that, up to isomorphism, any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is either classical or a filtered Lie algebra of Cartan type or a Melikian algebra of characteristic 5. This result finally settles the classification problem for finite-dimensional simple Lie algebras over algebraically closed fields of characteristic $\neq 2, 3$.

1. Introduction

This paper concludes the series [P-St 97,P-St 99,P-St 01,P-St 04,P-St 05]. Its goal is to finish the proof of the following theorem which was announced in [St 04] and [P-St 06]:

Theorem 1.1 *(Classification Theorem). Any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is of classical, Cartan or Melikian type.*

For $p > 7$, the finite-dimensional simple Lie algebras were classified by the second author in the series of papers [St 89,St 91,St 92,St 93,St 94,St 98]. It should be mentioned that the Classification

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Theory was inspired by the ground-breaking work of Block and Wilson [B-W 82,B-W 88] who handled the so-called restricted case (also for $p > 7$).

In what follows, F will denote an algebraically closed field of characteristic $p > 3$, and L will always stand for a finite-dimensional simple Lie algebra over F. As usual, we identify L with the subalgebra $\operatorname{ad} L$ of the derivation algebra $\operatorname{Der} L$ and denote by L_p the semisimple p-envelope of L (it coincides with the p-closure of $\operatorname{ad} L$ in the restricted Lie algebra $\operatorname{Der} L$). Given a torus T of maximal dimension in L_p we let H stand for the centralizer of T in L; that is,

$$H := \mathfrak{c}_L(T) = \{x \in L \mid [t, x] = 0 \ \forall t \in T\}.$$

Let $\Gamma(L, T)$ be the set of roots of L relative to T; that is, the set of all *nonzero* linear functions $\gamma \in T^*$ for which the subspace $L_\gamma := \{x \in L \mid [t, x] = \gamma(t)x \ \forall t \in T\}$ is nonzero. Then H is a nilpotent subalgebra of L (possibly zero) and L decomposes as $L = H \oplus \bigoplus_{\gamma \in \Gamma(L,T)} L_\gamma$. By [P-St 04, Corollary 3.7] any root γ in $\Gamma(L, T)$ is either *solvable* or *classical* or *Witt* or *Hamiltonian*. Accordingly, the semisimple quotient $L[\gamma] = L(\gamma)/\operatorname{rad} L(\gamma)$ of the 1-section $L(\gamma) := H \oplus \bigoplus_{i \in \mathbb{F}_p^\times} L_{i\gamma}$ is either (0) or $\mathfrak{sl}(2)$ or the Witt algebra $W(1; \underline{1})$ or contains an isomorphic copy of the Hamiltonian algebra $H(2; \underline{1})^{(2)}$ as an ideal of codimension $\leqslant 1$. For $\alpha, \beta \in \Gamma(L, T)$ we denote by $L(\alpha, \beta)$ the 2-section $\sum_{i, j \in \mathbb{F}_p} L_{i\alpha + j\beta}$, where $L_0 = H$ by convention.

We say that T is *standard* if $H^{(1)}$ consists of nilpotent derivations of L and *nonstandard* otherwise. In [P-St 04] and [P-St 05], it was shown that if all tori of maximal dimension in L_p are standard, then L is either classical or a filtered Lie algebra of Cartan type. On the other hand, the main results of [P 94] imply that if L_p contains a nonstandard torus of maximal dimension, say T', then there are $\alpha, \beta \in \Gamma(L, T')$ such that the factor algebra $L(\alpha, \beta)/\operatorname{rad} L(\alpha, \beta)$ is isomorphic to the restricted Melikian algebra $\mathcal{M}(1, 1)$. In particular, $p = 5$ in this case.

The main result of the present paper is the following:

Theorem 1.2. *If the semisimple p-envelope of L contains nonstandard tori of maximal dimension, then L is isomorphic to one of the Melikian algebras $\mathcal{M}(m, n)$, where $(m, n) \in \mathbb{N}^2$.*

Together with the main results of [P-St 04] and [P-St 05] Theorem 1.2 implies the Classification Theorem. In view of [St 04, Corollary 7.2.3] we also obtain:

Corollary 1.3. *Any finite-dimensional restricted simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is, up to isomorphism, either one of $W(n; \underline{1})$, $n \geqslant 1$, $S(n; \underline{1})^{(1)}$, $n \geqslant 3$, $H(2r; \underline{1})^{(2)}$, $r \geqslant 1$, $K(2r + 1; \underline{1})^{(1)}$, $r \geqslant 1$, $\mathcal{M}(1, 1)$, or has the form $(\operatorname{Lie} G)^{(1)}$, where G is a simple algebraic F-group of adjoint type.*

For the reader's convenience, we now give a brief overview of the proof of Theorem 1.2. Since our goal is to show that $L \cong \mathcal{M}(m, n)$, we need to produce a subalgebra $L_{(0)}$ of codimension 5 in L. As in the previous two papers of the series, local analysis is vital here. All possible types of 2-sections in simple Lie algebras are described in [P-St 04, Section 4]. The list of 2-sections is long, but a thorough investigation shows that most of them cannot occur in our situation. We prove in Section 5 that if T is a nonstandard torus of maximal dimension in L_p and $\alpha, \beta \in \Gamma(L, T)$ are $\mathbb{F}_p$-independent, then $\operatorname{rad} L(\alpha, \beta) \subset T$ and either $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ of $L[\alpha, \beta]^{(1)} \cong H(2; (2, 1))^{(2)}$; see Theorem 5.8. In particular, this implies that all root spaces of L with respect to T are 5-dimensional. This intermediate result is crucial for the rest of the paper. In order to prove it we have to refine our earlier description of 2-sections with core of type $H(2; (2, 1))^{(2)}$; see Theorem 3.6(5). The proof of Theorem 3.6(5) relies heavily on a classification of certain toral derivations of $H(2; (2, 1))$. The latter is obtained in Section 2, the longest section of the paper.

In Section 6, we show the restricted Melikian algebra $\mathcal{M}(1, 1)$ has no nontrivial central extensions and describe the p-characters of irreducible $\mathcal{M}(1, 1)$-modules of dimension $\leqslant 125$. This gives us important new information on the p-mapping of L_p; see Section 7. To proceed further we need a sufficiently generic nonstandard torus of maximal dimension in L_p. We show in Section 9 that there is a nonstandard torus T of maximal dimension in L_p for which $H^3 = [\mathfrak{c}_L(T), [\mathfrak{c}_L(T), \mathfrak{c}_L(T)]]$ contains no nonzero toral elements. We then use the new information on the p-mapping of L_p to construct for every $\alpha \in \Gamma(L,T)$ a subalgebra $Q(\alpha) \subset L(\alpha)$ such that $L(\alpha) = H \oplus Q(\alpha)$, and set $L_{(0)} := \sum_{\alpha\in\Gamma(L,T)} Q(\alpha)$. By construction, $L_{(0)}$ is a subspace of L. In order to show that it is a subalgebra, we need to check that $[Q(\alpha), Q(\beta)] \subset Q(\alpha) \oplus \sum_{i\in\mathbb{F}_p} Q(\beta + i\alpha)$ for all $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma(L,T)$. This is carried out in Section 10. The rest of the proof is routine.

All Lie algebras in this paper are assumed to be finite-dimensional. We adopt the notation introduced in [P-St 97,P-St 99,P-St 01,P-St 04] with the following two exceptions: the divided power algebra $A(m;\underline{n})$ is denoted here by $\mathcal{O}(m;\underline{n})$, and the Melikian algebra $\mathfrak{g}(m,n)$ by $\mathcal{M}(m,n)$. Given a Lie subalgebra M of L, we write M_p for the p-envelope of M in L_p.

2. Toral elements and one-sections in $H(2;(2,1))$

The Lie algebra $H(2;(2,1))$ will appear quite frequently in what follows, and to deal with it we need some refinements of [B-W 88, (10.1.1)], [St 91, (VI.4)] and [P-St 04, Proposition 2.1]. Set $S := H(2;(2,1))^{(2)}$, $G := H(2;(2,1))$, and denote by $S_{(i)}$ (resp., $G_{(i)}$) the ith component of the standard filtration of S (resp., G). Recall that $S_p = H(2;(2,1))^{(2)} \oplus FD_1^p$; see [St 04, Theorem 7.2.2(5)], for instance. By [B-W 88, Proposition 2.1.8(viii)], $G = V \oplus S$ where

$$V = FD_H\big(x_1^{(p^2)}\big) \oplus FD_H\big(x_2^{(p)}\big) \oplus D_H\big(x_1^{(p^2-1)}x_2^{(p-1)}\big).$$

Note that V is a Lie subalgebra of G, and in $\operatorname{Der} S$ we have $V^{[p]} = V^3 = 0$. We denote by $\mathcal{G}$ the p-envelope of G in $\operatorname{Der} S$. As $V^{[p]} = 0$, it follows from Jacobson's formula [St 04, p. 17] that $\mathcal{G} = V \oplus S_p$. We remind the reader that G is a Lie subalgebra of the Hamiltonian algebra $H(2) = \operatorname{span}\{D_H(f) \mid f \in \mathcal{O}(2)\}$ and

$$\big[D_H(f), D_H(g)\big] = D_H\big(D_1(f)D_2(g) - D_2(f)D_1(g)\big) \quad \big(\forall f, g \in \mathcal{O}(2)\big).$$

Furthermore, $D_H(f) = D_H(g)$ if and only if $f - g \in F$.

Lemma 2.1. *Every toral element t of S_p contained in $S \setminus S_{(0)}$ is conjugate under the automorphism group of S to an element*

$$t_\mu = D_H\big(x_1 + \mu x_1^{(p)} + \big(x_1 + \mu x_1^{(p)}\big) r x_2^{(p-1)}\big), \quad r = 1 + \mu x_1^{(p-1)},$$

where $\mu \in \{0,1\}$. Each such element is toral.

Proof. (a) Write $t = aD_1 + bD_2 + w$ with $a, b \in F$ and $w \in S_{(0)}$. By our assumption, t is a toral element of S_p; that is, $t^{[p]} = t$. Since $(aD_1 + bD_2)^{[p]} = a^p D_1^p$ and $w^{[p]} \in S_{(0)}$, Jacobson's formula yields $a = 0$. Since $t \notin S_{(0)}$, it must be that $b \neq 0$. There exists a special automorphism σ of the divided power algebra $\mathcal{O}(2;(2,1))$ such that $\sigma(x_1) = b^{-1}x_1$ and $\sigma(x_2) = bx_2$. It induces an automorphism Φ_σ of the Lie algebra S via $\Phi_\sigma(E) = \sigma \circ E \circ \sigma^{-1}$ for all $E \in S$; see [St 04, Theorem 7.3.6]. After adjusting t by Φ_σ it can be assumed that $b = 1$. The description of $\operatorname{Aut} S$ given in [St 04, Theorems 7.3.5 and 7.3.2] implies that for any $\lambda \in F$ and any pair of nonnegative integers (m,n) such that either $(m,n) = (p^2, 0)$ or $m + n \geqslant 3$, $m < p^2$, $n < p$ and $(m,n) \neq (p,1)$ there exists $\sigma_{m,n,\lambda} \in \operatorname{Aut} S$ with

$$\sigma_{m,n,\lambda}(u) \equiv u + \lambda\big[D_H\big(x_1^{(m)}x_2^{(n)}\big), u\big] \pmod{S_{(i+m+n-1)}} \quad (\forall u \in S_{(i)}).$$

Because

$$\big[D_2, D_H\big(x_1^{(m)}x_2^{(n)}\big)\big] = D_H\big(x_1^{(m)}x_2^{(n-1)}\big) \quad (1 \leqslant n \leqslant p-1),$$

it is not hard to see that there is $g \in \operatorname{Aut} S$ such that $g(t) = D_H(x_1 + \mu x_1^{(p)}) + D_H(f x_2^{(p-1)})$ for some $\mu \in F$ and $f = \sum_{i=1}^{p^2-1} \lambda_i x_1^{(i)}$ with $\lambda_i \in F$. If $\mu \neq 0$, then there exists $\alpha \in F$ with $\alpha^{p-1}\mu = 1$ and a special automorphism σ' of the divided power algebra $\mathcal{O}(2;(2,1))$ for which $\sigma'(x_1) = \alpha x_1$ and $\sigma'(x_2) = x_2$. It gives rise to an automorphism $\Phi_{\sigma'}$ of the Lie algebra S such that $\Phi_{\sigma'}(D_H(x_1^{(r)} x_2^{(s)})) = \alpha^{r-1} D_H(x_1^{(r)} x_2^{(s)})$ for all admissible r and s; see [St 04, Theorem 7.3.6]. Adjusting t by $\Phi_{\sigma'}$ we may assume without loss that $\mu \in \{0,1\}$.

Put $r = D_1(x_1 + \mu x_1^{(p)}) = 1 + \mu x_1^{(p-1)}$, $f' := D_1(f)$, and assume from now on that $t = D_H(x_1 + \mu x_1^{(p)}) + D_H(f x_2^{(p-1)})$.

(b) As $(\operatorname{ad} D_H(f x_2^{(p-1)}))(\operatorname{ad} D_H(x_1 + \mu x_1^{(p)}))^k (D_H(f x_2^{(p-1)})) = 0$ for $0 \leqslant k \leqslant p-3$, $D_H(x_1 + \mu x_1^{(p)})^{[p]} = D_H(f x_2^{(p-1)})^{[p]} = 0$, and

$$\big[D_H\big(x_1 + \mu x_1^{(p)}\big), D_H\big(r^i f x_2^{(j)}\big)\big] = D_H\big(r^{i+1} f x_2^{(j-1)}\big) \quad (1 \leqslant i, j \leqslant p-1),$$

Jacobson's formula yields

$$\begin{aligned}
t^{[p]} &= \big(\operatorname{ad} D_H\big(x_1 + \mu x_1^{(p)}\big)\big)^{p-1}\big(D_H\big(f x_2^{(p-1)}\big)\big) \\
&\quad + \frac{1}{2}\big[D_H\big(f x_2^{(p-1)}\big), \big(\operatorname{ad} D_H\big(x_1 + \mu x_1^{(p)}\big)\big)^{p-2}\big(D_H\big(f x_2^{(p-1)}\big)\big)\big] \\
&= D_H\big(r^{p-1} f\big) + \frac{1}{2}\big[D_H\big(f x_2^{(p-1)}\big), D_H\big(r^{p-2} f x_2\big)\big] \\
&= D_H\big(r^{p-1} f\big) + \frac{1}{2} D_H\big(f' r^{p-2} f x_2^{(p-1)}\big) - \frac{1}{2}\binom{p-1}{1} D_H\big(f D_1\big(r^{p-2} f\big) x_2^{(p-1)}\big) \\
&= D_H\big(r^{p-1} f\big) + D_H\big(f f' r^{p-2} x_2^{(p-1)}\big) - \mu D_H\big(r^{p-3} x_1^{(p-2)} f^2 x_2^{(p-1)}\big).
\end{aligned}$$

As $r^{p-1} = r^{-1}$, the RHS equals t if and only if $f = (x_1 + \mu x_1^{(p)})r$, as claimed. $\square$

Denote by $\mathcal{O}(2;(2,1))_{(k)}[x_1]$ the subalgebra of $\mathcal{O}(2;(2,1))$ spanned by all $x_1^{(i)}$ with $k \leqslant i < p^2$ and let $\mathcal{O}(2;(2,1))[x_1] := \mathcal{O}(2;(2,1))_{(0)}[x_1]$. For $u \in \mathcal{O}(2;(2,1))[x_1]$ put $u' := D_1(u)$ and set $\tilde{r} := x_1 + \mu x_1^{(p)}$, so that $t_\mu = D_H(\tilde{r} + r\tilde{r} x_2^{(p-1)})$. Note that $\tilde{r}' = r$.

Lemma 2.2. *Let t_μ be as in Lemma* 2.1 *and put $C_\mu := \mathfrak{c}_{\mathcal{G}}(t_\mu)$.*

(i) *The Lie algebra C_μ has an abelian ideal C'_μ of codimension* 2 *spanned by all $D_H(u + u'\tilde{r} x_2^{(p-1)})$ with $u \in \mathcal{O}(2;(2,1))[x_1]$ and by $D_H(x_1^{(p^2)})$. Furthermore, $C_\mu = F n_\mu \oplus F h_\mu \oplus C'_\mu$, where $n_\mu = D_1^p + \mu D_H(x_2^{(p)})$ and $h_\mu = D_H(r^{-1} x_2 - x_2^{(p)})$.*

(ii) *Given $a \in F$ and $v \in \mathcal{O}(2;(2,1))[x_1]$ put*

$$\varphi_a(v) := \sum_{i=0}^{p-1} a^i D_H\big(r^{-i} v x_2^{(i)}\big) + a^{p-1} D_H\big(\tilde{r} v' x_2^{(p-1)}\big).$$

Then for every $k \in \mathbb{F}_p^\times$ the k-eigenspace of $\operatorname{ad} t_\mu$ has dimension p^2 and is spanned by all $\varphi_k(u)$ with $u \in \mathcal{O}(2;(2,1))[x_1]$.

(iii) *In $\mathcal{G}$ we have $h_\mu^{[p]} = -\mu h_\mu - n_\mu$ and $n_\mu^{[p]} = 0$.*

(iv) *If $\mu = 0$, then C_μ is nilpotent and $F t_\mu$ is a maximal torus in $\mathcal{G}$.*

Proof. (i) It is straightforward to see that C'_μ is abelian and $t_\mu \in C'_\mu$. Also,

$$\left[D_1^p, t_\mu\right] = \mu D_H\left(r x_2^{(p-1)}\right) = -\mu\left[D_H\left(x_2^{(p)}\right), t_\mu\right],$$

implying $n_\mu \in C_\mu$. For all $u \in \langle x_1^{(i)} | 0 \leqslant i \leqslant p^2\rangle$ we have

$$\begin{aligned}\left[D_H\left(r^{-1}x_2\right), D_H\left(u + u'\tilde{r}x_2^{(p-1)}\right)\right] &= -D_H\left(r^{-1}u'\right) + D_H\left(r^{-2}\left(r'u'\tilde{r} - (u'\tilde{r})'r\right)x_2^{(p-1)}\right),\\ \left[D_H\left(x_2^{(p)}\right), D_H\left(u + u'\tilde{r}x_2^{(p-1)}\right)\right] &= -D_H\left(u'x_2^{(p-1)}\right).\end{aligned}$$

As a consequence,

$$\left[h_\mu, D_H\left(u + u'\tilde{r}x_2^{(p-1)}\right)\right] = -D_H\left(r^{-1}u' + \left(r^{-1}u'\right)'\tilde{r}x_2^{(p-1)}\right) \tag{2.1}$$

for all $u \in \mathcal{O}(2; (2,1))[x_1]$. Putting $u = \tilde{r}$ gives $h_\mu \in C_\mu$.

(ii) We claim that for all $u \in \langle x_1^{(i)} \mid 1 \leqslant i \leqslant p^2\rangle$ and all $k \in \mathbb{F}_p^\times$ the following relations hold:

$$\left[D_H\left(u + u'\tilde{r}x_2^{(p-1)}\right), \varphi_k(v)\right] = k\varphi_k\left(r^{-1}u'v\right), \tag{2.2}$$

$$\left[D_H\left(r^{-1}x_2 - x_2^{(p)}\right), \varphi_k(v)\right] = \left[h_\mu, \varphi_k(v)\right] = -\varphi_k\left(r^{-1}v'\right). \tag{2.3}$$

Indeed, since $k^{p-1} = 1$, $r^p = 1$, and $x_2^{(p-2)} \cdot x_2^{(k)} = 0$ for $2 \leqslant k \leqslant p-1$, the LHS of (2.2) equals $D_H(w)$, where

$$\begin{aligned}w &= D_1\left(u + u'\tilde{r}x_2^{(p-1)}\right) \cdot D_2\left(\varphi_k(v)\right) - D_2\left(u + u'\tilde{r}x_2^{(p-1)}\right) \cdot D_1\left(\varphi_k(v)\right)\\ &= \left(u' + u''\tilde{r}x_2^{(p-1)} + u'rx_2^{(p-1)}\right) \cdot \left(\sum_{i=1}^{p-1} k^i r^{-i} v x_2^{(i-1)} + \tilde{r}v'x_2^{(p-2)}\right) - u'\tilde{r}x_2^{(p-2)} \cdot \left(v' + k\left(r^{-1}v\right)'x_2\right)\\ &= u'\left(\sum_{i=1}^{p-1} k^i r^{-i} v x_2^{(i-1)}\right) + u'\tilde{r}v'x_2^{(p-2)}\\ &\quad + ku''\tilde{r}r^{-1}vx_2^{(p-1)} + ku'vx_2^{(p-1)} - u'\tilde{r}v'x_2^{(p-2)} + ku'\tilde{r}\left(r^{-1}v\right)'x_2^{(p-1)}\\ &= k\sum_{i=0}^{p-2} k^i r^{-i}\left(r^{-1}u'v\right)x_2^{(i)} + k\left(u''\tilde{r}r^{-1}v + u'v + u'\tilde{r}\left(r^{-1}v\right)'\right)x_2^{(p-1)}\\ &= k\sum_{i=0}^{p-1} k^i r^{-i}\left(r^{-1}u'v\right)x_2^{(i)} + k\tilde{r}\left(r^{-1}u'v\right)'x_2^{(p-1)}.\end{aligned}$$

But then $D_H(w) = k\varphi_k(r^{-1}u'v)$ and (2.2) follows. Since

$$\left(-r^{-1}v'\right)' = r^{-2}r'v' - r^{-1}v'',$$

the LHS of (2.3) equals $D_H(y)$, where

$$\begin{aligned}y &= \left(r^{-1}\right)'x_2 \cdot \left(\sum_{i=1}^{p-1} k^i r^{-i} v x_2^{(i-1)} + \tilde{r}v'x_2^{(p-2)}\right)\\ &\quad - r^{-1} \cdot \left(\sum_{i=1}^{p-1} k^i i\left(r^{-1}\right)^{i-1}\left(-r^{-2}r'\right)vx_2^{(i)} + \sum_{i=0}^{p-1} k^i r^{-i} v' x_2^{(i)}\right) - r^{-1} \cdot (\tilde{r}v')'x_2^{(p-1)} + x_2^{(p-1)}v'\end{aligned}$$

$$= -r^{-2}r' \cdot \left(\sum_{i=1}^{p-1} k^i i r^{-i} v x_2^{(i)} - \tilde{r} v' x_2^{(p-1)}\right)$$

$$+ r^{-2}r' \cdot \left(\sum_{i=1}^{p-1} k^i i r^{-i} v x_2^{(i)}\right) + \sum_{i=0}^{p-1} k^i r^{-i}\left(-r^{-1}v'\right)x_2^{(i)} - r^{-1} \cdot (rv' + \tilde{r}v'')x_2^{(p-1)} + x_2^{(p-1)}v'$$

$$= r^{-2}r'\tilde{r}v'x_2^{(p-1)} + \sum_{i=0}^{p-1} k^i r^{-i}\left(-r^{-1}v'\right)x_2^{(i)} - r^{-1}\tilde{r}v''x_2^{(p-1)}$$

$$= \sum_{i=0}^{p-1} k^i r^{-i}\left(-r^{-1}v'\right)x_2^{(i)} + \left(\tilde{r}r^{-2}r'v' - \tilde{r}r^{-1}v''\right)x_2^{(p-1)}$$

$$= \sum_{i=0}^{p-1} k^i r^{-i}\left(-r^{-1}v'\right)x_2^{(i)} + \tilde{r}\left(-r^{-1}v'\right)'x_2^{(p-1)}.$$

This shows that $D_H(y) = D_H(-r^{-1}v')$, proving (2.3).

Setting $u = \tilde{r}$ in (2.2) now gives $[t_\mu, \varphi_k(v)] = k\varphi_k(v)$. Since $\varphi_k(v) \neq 0$ for all nonzero $v \in \mathcal{O}(2;(2,1))[x_1]$, comparing dimensions yields that C_μ is spanned by h_μ, n_μ and C'_μ and that for every $k \in \mathbb{F}_p^\times$ the k-eigenspace of $\operatorname{ad} t_\mu$ has dimension p^2 and is spanned by all $\varphi_k(v)$ with $v \in \mathcal{O}(2;(2,1))[x_1]$.

(iii) Clearly, $n_\mu^{[p]} = D_1^{p^2} - \mu^p (x_2^{(p-1)}D_1)^p = 0$. Next observe that

$$[h_\mu, n_\mu] = \left[D_H\left(r^{-1}x_2 - x_2^{(p)}\right), D_1^p + \mu D_H\left(x_2^{(p)}\right)\right] = \mu D_H\left(\left(r^{-1}\right)' x_2 \cdot x_2^{(p-1)}\right) = 0.$$

We claim that $h_\mu^{[p]} + \mu h_\mu + n_\mu = 0$. If $\mu = 0$, then $h_\mu = D_1 - x_2^{(p-1)}D_1$ and $n_\mu = D_1^p$; hence, our claim is true in this case. Assume now that $\mu \neq 0$ and set $q := h_\mu + \mu^{-1}n_\mu$. Since our remarks at the beginning of this part imply that $q^{[p]} = (h_\mu + \mu^{-1}n_\mu)^{[p]} = h_\mu^{[p]}$, we are reduced to showing that $q^{[p]} + \mu q = 0$. As $[D_H(x_1^{(p-1)}x_2), (\operatorname{ad} D_1)^i(D_H(x_1^{(p-1)}x_2))] = 0$ for all $i \leqslant p-2$, we see that

$$q^{[p]} = \left(\mu^{-1}D_1^p - D_1 - \mu D_H\left(x_1^{(p-1)}x_2\right)\right)^{[p]} = \left(-D_1 - \mu D_H\left(x_1^{(p-1)}x_2\right)\right)^{[p]}$$

$$= -D_1^p - (\operatorname{ad} D_1)^{p-1}\left(\mu D_H\left(x_1^{(p-1)}x_2\right)\right) - \frac{1}{2}\left[\mu D_H\left(x_1^{(p-1)}x_2\right), (\operatorname{ad} D_1)^{p-2}\left(\mu D_H\left(x_1^{(p-1)}x_2\right)\right)\right]$$

$$= -D_1^p + \mu D_1 + \mu^2 D_H\left(x_1^{(p-1)}x_2\right) = -\mu q,$$

and our claim follows.

(iv) Now suppose $\mu = 0$. Then $t_\mu = D_H(x_1(1 + x_2^{(p-1)}))$, $h_\mu = D_H(x_2 - x_2^{(p)}) = (x_2^{(p-1)} - 1)D_1$ and $n_\mu = D_1^p$. Set $C := C_0$ and $C_{(0)} := C \cap G_{(0)}$. By Lemma 2.2(i), which we have already proved, C is spanned by D_1^p, $(x_2^{(p-1)} - 1)D_1$ and by all $D_H(x_1^{(k+1)} + x_1^{(k)}\tilde{r}x_2^{(p-1)})$ with $0 \leqslant k \leqslant p^2 - 1$. As a consequence, $C = FD_1^p \oplus F(x_2^{(p-1)} - 1)D_1 \oplus Ft_\mu \oplus C_{(0)}$. As $G_{(0)}$ is a restricted subalgebra of $\mathcal{G}$, so is $C_{(0)}$. From this it is immediate that $C_{(0)}$ is a p-nilpotent subalgebra of $\mathcal{G}$. Note that $C \cap S = Ft_\mu \oplus C_{(0)}$ is an ideal of C. Since $((x_2^{(p-1)} - 1)D_1)^{[p]} = -D_1^p$ and $(D_1^p)^{[p]} = 0$ (as derivations of S), Jacobson's formula implies that $C^{[p]} \subset FD_1^p \oplus Ft_\mu \oplus C_{(0)}$ and $C^{[p]^2} \subset Ft_\mu \oplus C_{(0)}$. Since $C_{(0)}$ is p-nilpotent and $[t_\mu, C] = 0$, it follows that $C^{[p]^e} = Ft_\mu$ for all $e \gg 0$. Hence C is a restricted nilpotent subalgebra of $\mathcal{G}$ and Ft_μ is the unique maximal torus of C. $\square$

If u belongs to the linear span of all $x_1^{(i)}$ with $2 \leqslant i \leqslant p^2$, then $r^{-1}u' \in \mathcal{O}(2;(2,1))_{(1)}$, forcing $(r^{-1}u')^p = 0$. For $k \in \mathbb{F}_p^\times$ we write S_k for the k-eigenspace of $\operatorname{ad} t_\mu$. In view of (2.2) we have that $(\operatorname{ad} D_H(u + \tilde{r}u'x_2^{(p-1)}))^p(S_k) = (0)$ for all $k \in \mathbb{F}_p^\times$. Since

$$\left(\operatorname{ad} D_H\left(u + \tilde{r}u'x_2^{(p-1)}\right)\right)^p(C_\mu) \subset \left(\operatorname{ad} D_H\left(u + \tilde{r}u'x_2^{(p-1)}\right)\right)^{p-1}\left(C'_\mu\right) \subset \left(C'_\mu\right)^{(1)} = (0)$$

by Lemma 2.2(i), it follows that $(\operatorname{ad} D_H(u + \tilde{r}u'x_2^{(p-1)}))^p = 0$. Therefore, for all u as above and $c \in F$ the exponential $\exp(c \operatorname{ad} D_H(u + u'\tilde{r}x_2^{(p-1)}))$ is well defined as a linear operator on S.

Lemma 2.3. *Suppose $\mu \neq 0$ and let $Z(t_\mu)$ denote the stabilizer of t_μ in* $\operatorname{Aut} S$*.*

(i) $\exp(c \operatorname{ad} D_H(x_1^{(m)} + x_1^{(m-1)}\tilde{r}x_2^{(p-1)})) \in Z(t_\mu)$ *for all* $3 \leqslant m \leqslant p^2$.
(ii) *For every $h \in G \cap C_\mu$ with $h \notin C'_\mu$ there exist $z \in Z(t_\mu)$ and $a \in F^\times$ such that $z(h) = ah_\mu + bt_\mu + sD_H(x_1^{(p^2)})$ for some $b, s \in F$.*
(iii) *If $h \in (G \cap C_\mu) \setminus C'_\mu$, then for every $k \in \mathbb{F}_p^\times$ there is $v_k \in 1 + \mathcal{O}(2;(2,1))_{(1)}[x_1]$ such that $\varphi_k(v_k)$ is an eigenvector for* $\operatorname{ad} h$ *and $\varphi_k(v_k)^{[p]}$ is a nonzero p-semisimple element of $\mathcal{G}$.*
(iv) *For every $h \in (G \cap C_\mu) \setminus C'_\mu$ there exists a nonzero $x \in \mathfrak{c}_S(t_\mu)$ such that* $\operatorname{ad} x$ *is not nilpotent and $[h, x] = \lambda x$ for some nonzero $\lambda \in F$.*

Proof. (a) For $1 \leqslant m \leqslant p^2$ set $\mathcal{D}_m := \operatorname{ad} D_H(x_1^{(m)} + x_1^{(m-1)}\tilde{r}x_2^{(p-1)})$. As $(\operatorname{ad} \mathcal{D}_m)^p = 0$ for $m \geqslant 3$, in order to prove (i) it suffices to show that

$$\sum_{i=1}^{p-1} \frac{1}{i!(p-i)!}\left[\mathcal{D}_m^i(y_1), \mathcal{D}_m^{p-i}(y_2)\right] = 0 \quad (\forall y_1, y_2 \in S,\ \forall m \geqslant 3). \tag{2.4}$$

It follows from Lemma 2.2(i) that $\mathcal{D}_m^2(C_\mu) \subset (C'_\mu)^{(1)} = (0)$. Therefore, we just need to show that (2.4) holds for all $y_1 = \varphi_k(v_1)$ and $y_2 = \varphi_l(v_2)$, where $k, l \in \mathbb{F}_p^\times$ and $v_1, v_2 \in \mathcal{O}(2;(2,1))[x_1]$.

For $3 \leqslant m \leqslant p$ we have $(r^{-1}x_1^{(m-1)})^{(p+1)/2} = 0$, since $\mathcal{O}(2;(1,1))$ is a subalgebra of $\mathcal{O}(2;(2,1))$ and $\frac{(m-1)(p+1)}{2} > p$. In light of (2.2) this gives $(\operatorname{ad} \mathcal{D}_m)^{(p+1)/2}(\varphi_i(v)) = 0$ for all $i \in \mathbb{F}_p^\times$ and $v \in \mathcal{O}(2;(2,1))[x_1]$. Hence (2.4) holds for $m \leqslant p$.

If $m \geqslant p+2$, then (2.2) yields that $\mathcal{D}_m^i(\varphi_k(v_1)) = \varphi_k(w_1)$ and $\mathcal{D}_m^{p-i}(\varphi_l(v_2)) = \varphi_l(w_2)$ for some $w_1 \in \mathcal{O}(2;(2,1))_{(i(p+1))}[x_1]$ and $w_2 \in \mathcal{O}(2;(2,1))_{((p-i)(p+1))}[x_1]$. As $[\varphi_k(w_1), \varphi_l(w_2)] = 0$ in this case, we deduce that (2.4) holds for $m \geqslant p+2$. As $\mathcal{O}(2;(2,1))_{(p^2)}[x_1] = 0$, this argument also shows that (2.4) holds if $m = p+1$ and either v_1 or v_2 belongs to $\mathcal{O}(2;(2,1))_{(1)}[x_1]$.

Thus, in order to prove (i) it suffices to show that (2.4) holds for $m = p+1$ and $v_1 = v_2 = 1$. Suppose the contrary and set

$$Y := \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!}\left[\mathcal{D}_{p+1}^i\left(\varphi_k(1)\right), \mathcal{D}_{p+1}^{p-i}\left(\varphi_l(1)\right)\right].$$

Arguing as in the preceding paragraph we now observe that Y is a nonzero multiple of either $\varphi_{k+l}(x_1^{(p^2-1)})$ (if $k + l \neq 0$) or $D_H(x_1^{(p^2-1)}(1 - x_2^{(p-1)}))$ (if $k + l = 0$). In any event, $(\operatorname{ad} n_\mu)^{p-1}(Y) \neq 0$.

Set $N_\mu := \operatorname{ad} n_\mu$. We know from the proof of Lemma 2.2 that $[N_\mu, \mathcal{D}_{p+1}] = \mathcal{D}_1$, $[\mathcal{D}_1, \mathcal{D}_{p+1}] = 0$ and $N_\mu(\varphi_i(1)) = 0$ for all $i \in \mathbb{F}_p^\times$. From this it follows that

$$N_\mu^{p-1}(Y) = \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!}\left(\sum_{j=0}^{p-1} (-1)^j\left[N_\mu^j\left(\mathcal{D}_{p+1}^i\left(\varphi_k(1)\right)\right), N_\mu^{p-1-j}\left(\mathcal{D}_{p+1}^{p-i}\left(\varphi_l(1)\right)\right)\right]\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{p-1}(-1)^i\big[\mathcal{D}_1^i\big(\varphi_k(1)\big), \big(\mathcal{D}_1^{p-i-1}\mathcal{D}_{p+1}\big)\big(\varphi_l(1)\big)\big] \\
&\quad + \sum_{i=1}^{p-1}(-1)^{i-1}\big[\big(\mathcal{D}_1^{i-1}\mathcal{D}_{p+1}\big)\big(\varphi_k(1)\big), \mathcal{D}_1^{p-i}\big(\varphi_l(1)\big)\big] \\
&= \mathcal{D}_1^{p-1}\big(\big[\varphi_k(1), \mathcal{D}_{p+1}\big(\varphi_l(1)\big)\big]\big) - \big[\varphi_k(1), \big(\mathcal{D}_1^{p-1}\mathcal{D}_{p+1}\big)\big(\varphi_l(1)\big)\big] \\
&\quad + \mathcal{D}_1^{p-1}\big(\big[\mathcal{D}_{p+1}\big(\varphi_k(1)\big), \big(\varphi_l(1)\big)\big]\big) - \big[\big(\mathcal{D}_1^{p-1}\mathcal{D}_{p+1}\big)\big(\varphi_k(1)\big), \varphi_l(1)\big] \\
&= \big(\mathcal{D}_1^{p-1}\mathcal{D}_{p+1}\big)\big(\big[\varphi_k(1), \varphi_l(1)\big]\big) - l\big[\varphi_k(1), \varphi_l\big(x_1^{(p)}\big)\big] - k\big[\varphi_k\big(x_1^{(p)}\big), \varphi_l(1)\big]
\end{aligned}
$$

(we used (2.2) and the equalities $r^p = 1$, $k^p = k$ and $l^p = l$). On the other hand, comparing components of x_2-degree 0 and 1 one observes that

$$
\big[\varphi_k(u), \varphi_l(v)\big] = \begin{cases} \varphi_{k+l}((lu'v - kuv')r^{-1}) & \text{if } k+l \neq 0, \\ kD_H((u'v - uv') + (u'v - uv')'\tilde{r}x_2^{p-1}) & \text{if } k+l = 0 \end{cases}
$$

for all $u, v \in \mathcal{O}(2;(2,1))[x_1]$. But then $l[\varphi_k(1), \varphi_l(x_1^{(p)})] + k[\varphi_k(x_1^{(p)}), \varphi_l(1)] = 0$ and $[\varphi_k(1), \varphi_l(1)] = 0$, forcing $N_\mu^{p-1}(Y) = 0$, a contradiction. Statement (i) follows.

(b) Observe that $C_\mu \cap G = C'_\mu \oplus Fh_\mu$. If $h \in C_\mu \cap G$ and $h \notin C'_\mu$, then Lemma 2.2(i) implies that there are $a \in F^\times$, $b, s \in F$ such that $h = ah_\mu + bt_\mu + sD_H(x_1^{(p^2)}) + \sum_{i=2}^{p^2-1} a_i D_H(x_1^{(i)} + x_1^{(i-1)}\tilde{r}x_2^{(p-1)})$ for some $a_i \in F$. Since C'_μ is abelian, r is invertible, and

$$
(\exp a_i\mathcal{D}_m)(h_\mu) = h_\mu + a_i D_H\big(r^{-1}\big(x_1^{(m-1)} + x_1^{(m-2)}\tilde{r}x_2^{(p-1)}\big)\big) \quad \big(3 \leqslant m \leqslant p^2\big)
$$

by (2.1), we can clear the a_i's by applying suitable automorphisms from $Z(t_\mu)$. This proves statement (ii).

In dealing with (iii) we may assume that $h = h_\mu + sD_H(x_1^{(p^2)})$ where $s \in F$. In view of (2.3) we need to find $v_k = 1 + b_1x_1^{(1)} + b_2x_1^{(2)} + \cdots + b_{p^2-1}x_1^{(p^2-1)}$ and $\eta_k \in F$ satisfying the condition

$$
\begin{aligned}
\eta_k\varphi_k(v_k) &= \big[h_\mu + sD_H\big(x_1^{(p^2)}\big), \varphi_k(v_k)\big] \\
&= -\varphi_k\big(r^{-1}v_k'\big) + sD_H\left(x_1^{(p^2-1)} \cdot \left(\sum_{i=0}^{p-1} k^i r^{-i} v_k x_2^{(i-1)} + k^{p-1}\tilde{r}v_k' x_2^{(p-2)}\right)\right) \\
&= -\varphi_k\big(r^{-1}v_k'\big) + sk\varphi_k\big(x_1^{(p^2-1)}v_k\big).
\end{aligned}
$$

This holds if and only if

$$
-b_1 - b_2x_1 - \cdots - b_{p^2-1}x_1^{(p^2-2)} + skx_1^{(p^2-1)} = \eta_k r\big(1 + b_1x_1^{(1)} + \cdots + b_{p^2-1}x_1^{(p^2-1)}\big).
$$

Set $b_0 := 1$. Because

$$
\begin{aligned}
r\left(1 + \sum_{i=1}^{p^2-1} b_i x_1^{(i)}\right) &= \left(1 + \sum_{i=1}^{p^2-1} b_i x_1^{(i)}\right) + \mu\left(x_1^{(p-1)} + \sum_{i=1}^{p-1} b_{ip}x_1^{(ip+p-1)}\right) \\
&= \left(1 + \sum_{i=1}^{p^2-1} b_i x_1^{(i)}\right) + \mu\sum_{i=0}^{p-1} b_{ip}x_1^{(ip+p-1)}
\end{aligned}
$$

by Lucas' theorem, this leads to the system of equations

$$\begin{aligned} b_0 &= 1; \\ b_i &= -\eta_k b_{i-1}, && 1 \leqslant i \leqslant p^2-1,\ i \notin p\mathbb{Z}; \\ b_{ip} &= -\eta_k(b_{ip-1} + \mu b_{i-1}), && 1 \leqslant i \leqslant p-1; \\ \eta_k b_{p^2-1} &= sk. \end{aligned}$$

Arguing recursively, one observes that there is a bijection between the solutions to this system and the roots of a polynomial of the form $X^{p^2} + \sum_{i=1}^{p^2-1} \lambda_i X^i - sk$, where $\lambda_i \in F$. Since F is algebraically closed, it follows that our eigenvalue problem has at least one solution.

(c) In view of our discussion in part (b), $\varphi_k(v_k) \equiv D_H(x_2) + b_1 D_H(x_1) \pmod{S_{(0)}}$. Since $D_H(x_2) = -D_1$ and $S_{(0)}$ is a restricted subalgebra of $\mathcal{G}$, Jacobson's formula shows that $\varphi_k(v_k)^{[p]} = -D_1^p + w_k$ for some $w_k \in S$. In particular, $\varphi_k(v_k)^{[p]} \neq 0$. Note that $\varphi_k(v_k)^{[p]} \in C_\mu \cap S_p \cap \ker \operatorname{ad} h$. Now, using (2.1) it is easy to observe that $C'_\mu \cap \ker \operatorname{ad} h = F t_\mu$, whilst from Lemma 2.2 it is immediate that $C_\mu \cap S_p = F(\mu h_\mu + n_\mu)$. Lemma 2.2 also implies that $\mu h_\mu + n_\mu = -h_\mu^{[p]}$ and $h_\mu^{[p]^2} = -\mu^p h_\mu^{[p]}$.

Let h_s denote the p-semisimple part of h in $\mathcal{G}$, an element of $C_\mu \cap \ker \operatorname{ad} h \cap S_p$. Since the above discussion shows that $C_\mu \cap S_p \cap \ker \operatorname{ad} h$ has dimension $\leqslant 2$, in order to finish the proof of (iii) we need to show that t_μ and h_s are linearly independent.

Suppose the contrary. Then $\operatorname{ad} h$ acts nilpotently on C'_μ. Recall that $h \in h_\mu + C'_\mu$ and C'_μ is abelian. So $\operatorname{ad} h_\mu$ acts on C'_μ nilpotently, too. Since $\mu \neq 0$, our earlier remarks and Lemma 2.2(iii) now show that $\operatorname{ad}(h_\mu^{[p]}) = -\mu \operatorname{ad} h_\mu - \operatorname{ad} n_\mu$ acts trivially on C'_μ. Since this violates (2.1), we reach a contradiction. Statement (iii) follows.

(d) In proving (iv) we may assume that $h = h_\mu + sD_H(x_1^{(p^2)})$; see part (b). We claim that there exist $u = x_1 + c_1 x_1^{(2)} + \cdots + c_{p^2-2} x_1^{(p^2-1)}$ and $\lambda \in F^\times$ such that

$$\big[h, D_H\big(u + u'\tilde{r}x_2^{(p-1)}\big)\big] = \lambda D_H\big(u + u'\tilde{r}x_2^{(p-1)}\big).$$

Since C'_μ is abelian, it follows from (2.1) that

$$\big[h, D_H\big(u + u'\tilde{r}x_2^{(p-1)}\big)\big] = \big[h_\mu, D_H\big(u + u'\tilde{r}x_2^{(p-1)}\big)\big] = -D_H\big(r^{-1}u' + \big(r^{-1}u'\big)'\tilde{r}x_2^{(p-1)}\big).$$

Thus, we seek u such that $r^{-1}u' = a - \lambda u$ for some $a \in F$. Since $r^{-1} = 1 - \mu x_1^{(p-1)}$, this entails that $a = 1$, $c_1 = -\lambda$, and

$$\big(1 - \mu x_1^{(p-1)}\big)\left(1 + \sum_{i=1}^{p^2-2} c_i x_1^{(i)}\right) = 1 + c_1\left(x_1 + \sum_{i=1}^{p^2-2} c_i x_1^{(i+1)}\right). \tag{2.5}$$

Since $x_1^{(p-1)} \cdot (1 + \sum_{i=1}^{p^2-2} c_i x^{(i)}) = (x_1^{(p-1)} + \sum_{i=1}^{p-1} c_{ip} x_1^{(ip+p-1)})$ by Lucas' theorem, we see that $c_{i+1} = c_1 c_i$ if $p \nmid (i+2)$. Induction on k shows that $c_{kp+p-1} = c_1^k (c_1^{p-1} + \mu)^{k+1}$ for $0 \leqslant k \leqslant p-1$. As $c_{p^2-1} = 0$, this yields $c_1^{p-1}(c_1^{p-1} + \mu)^p = 0$. As $c_1 = -\lambda \neq 0$, we see that c_1 must satisfy the equation $X^{p-1} + \mu = 0$. Conversely, any root of this equation gives rise to a solution of (2.5) with $\lambda = -c_1 \neq 0$ (recall that $\mu \neq 0$ by our assumption). The claim follows.

We now set $x := D_H(u + u'\tilde{r}x_2^{(p-1)})$, where u is as above. Clearly, $x \in S$. Since $r^{-1}u' - 1 \in \mathcal{O}_{(1)}(2;(2,1))$, it follows from (2.2) that $(\operatorname{ad} x)^p(\varphi_k(v)) = k^p \varphi((r^{-1}u')^p v) = k\varphi_k(v)$ for all $v \in \mathcal{O}(2;(2,1))[x_1]$ and all $k \in \mathbb{F}_p$. This implies that $\operatorname{ad} x$ is not nilpotent, completing the proof. $\square$

We now let $\mathfrak{t}$ be a 2-dimensional torus in $\mathcal{G}$.

Lemma 2.4. *There exist nonzero $u_1, u_2 \in S$ such that $\mathfrak{t} = F(D_1^p + u_1) \oplus Fu_2$.*

Proof. Since $V^{[p]} = 0$, the restricted Lie algebra $\mathcal{G}/S_p$ is p-nilpotent. As $\mathfrak{t}$ is a torus, it must be that $\mathfrak{t} \subset S_p$. Then $\mathfrak{t} \cap S \neq (0)$, for $\dim \mathfrak{t} = 2$.

Suppose $\mathfrak{t} \subset S$. Since $S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$ and $S_{(-1)}/S_{(0)}$ is a 2-dimensional irreducible module over $S_{(0)}/S_{(1)}$, every nonzero element of $\mathfrak{t} \cap S_{(0)}$ acts invertibly on $S_{(-1)}/S_{(0)}$. Therefore, $\mathfrak{t} \cap S_{(0)} \neq (0)$ would force $\mathfrak{t} \subset S_{(0)}$, which is false because $S_{(0)}$ has toral rank 1 in S. On the other hand, if $\mathfrak{t} \cap S_{(0)} = (0)$ (and still $\mathfrak{t} \subset S$), then $\mathfrak{t}$ would contain an element of the form $D_1 + u$ with $u \in S_{(0)}$. But this would yield $D_1^p \in \mathfrak{t} + S = S$, as $S_{(0)}$ is a restricted subalgebra of S_p. Therefore, $\mathfrak{t} \not\subset S$. Since D_1 is nilpotent and S has codimension 1 in S_p, our statement follows immediately. □

Lemma 2.5. *Let $\mathfrak{h} = \mathfrak{c}_S(\mathfrak{t})$ and let $\alpha \in \Gamma(S, \mathfrak{t})$.*

(1) *If α vanishes on $\mathfrak{h}$, then $G(\alpha)$ is solvable.*
(2) *If α does not vanish on $\mathfrak{h}$, then $G(\alpha) \cong H(2; \underline{1})$.*
(3) $\dim G_\gamma = p + \delta_{\gamma,0}$ *for all $\gamma \in \Gamma(G, \mathfrak{t}) \cup \{0\}$.*
(4) $\Gamma(S, \mathfrak{t}) \cup \{0\}$ *is a two-dimensional vector space over $\mathbb{F}_p$.*

Proof. Note that $\mathfrak{c}_{S_p}(\mathfrak{t}) = \mathfrak{t} + \mathfrak{h}$ and $\mathfrak{t}$ is a standard torus of maximal dimension in S_p. Therefore, the results of [B-W 88, (10.1.1)] and [St 91, (VI)] apply to $\mathfrak{t}$.

If α does not vanish on $\mathfrak{h}$, then $G(\alpha) \cong H(2; \underline{1})$ by [P-St 04, Proposition 2.1(2)]. Suppose $\alpha(\mathfrak{h}) = 0$. As $\mathfrak{t}$ is a maximal torus of S_p, we have that $\alpha(L_{i\alpha}^{[p]}) = 0$ for all $i \in \mathbb{F}_p^\times$. Then $S(\alpha)$ is nilpotent due to the Engel–Jacobson theorem. As G/S is nilpotent too, we conclude that $G(\alpha)$ is solvable.

By [B-W 88, (10.1.1(e))], there is a 2-dimensional torus $\mathfrak{t}'$ in S_p such that all roots in $\Gamma(S, \mathfrak{t}')$ are proper. Then [St 91, (VI.2(2))] applies showing that all root spaces of G with respect to $\mathfrak{t}'$ are p-dimensional and $\dim \mathfrak{c}_G(\mathfrak{t}') = p + 1$. By [P 89], all root spaces of G with respect to $\mathfrak{t}$ must have the same property, and $\dim \mathfrak{c}_G(\mathfrak{t}) = p + 1$ (see also [P-St 99, Corollary 2.11]). As $\dim S = p^3 - 2$ and $\dim S_\gamma \leqslant p$ for all $\gamma \in \Gamma(S, \mathfrak{t})$, we derive that $|\Gamma(S, \mathfrak{t})| = p^2 - 1$. As a consequence, the set $\Gamma(S, \mathfrak{t}) \cup \{0\}$ is 2-dimensional vector space over $\mathbb{F}_p$. This completes the proof. □

Lemma 2.6. *Under the above assumptions on $\mathfrak{t}$ and S the following hold*:

(1) *If $TR(\mathfrak{h}, S) = 2$, then all roots in $\Gamma(S, \mathfrak{t})$ are Hamiltonian improper.*
(2) *If $TR(\mathfrak{h}, S) = 1$, then $\Gamma(S, \mathfrak{t})$ contains a solvable root.*
(3) *Suppose that $TR(\mathfrak{h}, S) = 1$ and $\mathfrak{h}_p \cap S_{(0)}$ contains a nonnilpotent element. Then for any solvable $\alpha \in \Gamma(S, \mathfrak{t})$ the 1-section $G(\alpha)$ is nilpotent.*

Proof. Suppose $TR(\mathfrak{h}, S) = 2$. Then no root in $\Gamma(S, \mathfrak{t})$ vanishes on $\mathfrak{h}$; hence, all roots in $\Gamma(S, \mathfrak{t})$ are Hamiltonian by Proposition 2.5(2). If $\mathfrak{h} \cap S_{(0)}$ contains a nonnilpotent element, x say, then the image of x in $S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$ acts invertibly on $S_{(-1)}/S_{(0)}$. As $\mathfrak{h}$ is nilpotent, this would force $\mathfrak{h} \subset S_{(0)}$, and hence $TR(\mathfrak{h}, S) = 1$, a contradiction. Consequently, $\mathfrak{t} \cap S_{(0)} = (0)$. By [B-W 88, (10.1.1(d))] (see the proof on pp. 232–233), every Hamiltonian root is then improper.

Now suppose $TR(\mathfrak{h}, S) = 1$. Then the unique maximal torus of $\mathfrak{h}_p$ is spanned by a toral element, hence it follows from Lemma 2.5(4) that there is a root in $\Gamma(S, \mathfrak{t})$ which vanishes on $\mathfrak{h}$. Every such root is solvable by Proposition 2.5(1).

Finally, suppose that $TR(\mathfrak{h}, S) = 1$ and $\mathfrak{h}_p \cap S_{(0)}$ contains a nonnilpotent element. Since $S_{(0)}$ is a restricted subalgebra of S_p, we then have $S_{(0)} \cap \mathfrak{h}_p \cap \mathfrak{t} \neq (0)$. Since $\mathfrak{t} \cap S = Fu_2$ for some nonzero $u_2 \in S$ (see Lemma 2.4), it must be that $u_2 \in S_{(0)}$ and $u_2^{[p]} \in Fu_2$.

If $\alpha \in \Gamma(S, \mathfrak{t})$ is solvable, then $\alpha(\mathfrak{h}) = 0$ by Lemma 2.5(2). As explained in the proof of Lemma 2.5 the Lie algebra $S(\alpha)$ is nilpotent. There exists an element $t \in F^\times u_2$ with $t^{[p]} = t$ such that $G(\alpha) =$

$\mathfrak{c}_G(t)$. Set $W := \{v - (\operatorname{ad} t)^{p-1}(v) \mid v \in V\}$. By construction, $W \subset \mathfrak{c}_G(t)$ and $G = W \oplus S$. Since $V \subset G_{(1)}$ and $t \in S_{(0)}$, we have the inclusion $W \subset G_{(1)}$. In particular, all elements of W act nilpotently on $\mathfrak{c}_G(t)$.

Since $S(\alpha)$ is a nilpotent ideal of $G(\alpha)$, the set $(\operatorname{ad}_{G(\alpha)} S(\alpha)) \cup (\operatorname{ad}_{G(\alpha)} W)$ is weakly closed and consists of nilpotent endomorphisms. Since $G(\alpha) = W \oplus S(\alpha)$, the Engel–Jacobson theorem now shows that $G(\alpha)$ is nilpotent. $\square$

Lemma 2.7. *If $t \in S_p$ is a toral element not contained in S, then t is conjugate to $D_1^p + D_1 + D_H(x_1x_2)$ under the automorphism group of S.*

Proof. By our assumption, $t = aD_1^p + w$ for some $a \in F^\times$ and $w \in S$. Choose $\alpha \in F$ satisfying $\alpha^p = a$ and let σ_α denote the automorphism of S which sends $D_H(x_1^{(i)}x_2^{(j)})$ to $\alpha^{i-1}D_H(x_1^{(i)}x_2^{(j)})$; see [St 04, Theorem 7.3.6]. Then $\sigma_\alpha(t) = -aD_H(\alpha^{-1}x_2)^p + w'$ for some $w' \in S$. Hence we may assume that $a = 1$. The description of $\operatorname{Aut} S$ given in [St 04, Theorems 7.3.5 and 7.3.2] shows that for any pair of nonnegative integers $(m,n) \neq (p,1)$ such that either $p \leqslant m < p^2$ and $n < p$ or $(m,n) = (p^2, 0)$ and any $\lambda \in F$ there is $\sigma_{m,n,\lambda} \in \operatorname{Aut} S$ such that $\sigma_{m,n,\lambda}(u) \equiv u + \lambda[D_H(x_1^{(m)}x_2^{(n)}), u] \pmod{S_{i+(m+n-1)}}$ for all $u \in S_{(i)}$ Using Jacobson's formula (with $u = D_1$) it is not hard to observe that

$$\sigma_{m,n,\lambda}\big(D_1^p\big) \equiv D_1^p - \lambda D_H\big(x_1^{(m-p)}x_2^{(n)}\big) \pmod{S_{(m+n-p-1)}}.$$

This implies that there exists $g \in \operatorname{Aut} S$ such that $g(t) = D_1^p + bD_1 + D_H(x_1^{(p^2-p)}\psi)$ for some $\psi \in F[x_1,x_2] \subset \mathcal{O}(2;(1,1))$ with $\psi(0) = 0$. Write $\psi = \sum_{i=0}^{p-1} \psi_i x_1^{(i)}$ with $\psi_i \in F[x_2]$, where $\psi_0(0) = 0$. The element $g(t)$ being toral, it must be that $b = 1$. Note that $(\operatorname{ad} D_H(x_1^{(p^2-p)}\psi))(\operatorname{ad}(D_1^p + D_1))^i(D_H(x_1^{(p^2-p)}\psi)) = 0$ for $0 \leqslant i \leqslant p-3$ and

$$\big(\operatorname{ad} D_H\big(x_1^{(p^2-p)}\psi\big)\big)\big(\operatorname{ad}\big(D_1^p + D_1\big)\big)^{p-2}\big(D_H\big(x_1^{(p^2-p)}\psi\big)\big) = \big[D_H\big(x_1^{(p^2-p)}\psi\big), D_H\big(x_1^{(p)}\psi\big)\big].$$

Because

$$\big(\operatorname{ad} D_1^p + \operatorname{ad} D_1\big)^{p-1} = \sum_{i=0}^{p-1}(-1)^i(\operatorname{ad} D_1)^{pi}(\operatorname{ad} D_1)^{p-i-1} = \sum_{i=1}^{p}(-1)^{i-1}(\operatorname{ad} D_1)^{i(p-1)}$$

and $D_1^p(\psi) = 0$, Jacobson's formula yields that

$$\begin{aligned} g(t)^{[p]} &= \big(D_1^p + D_1\big)^{[p]} + \big(\operatorname{ad}\big(D_1^p + D_1\big)\big)^{p-1}\big(D_H\big(x_1^{(p^2-p)}\psi\big)\big) + \frac{1}{2}\big[D_H\big(x_1^{(p^2-p)}\psi\big), D_H\big(x_1^{(p)}\psi\big)\big] \\ &= D_1^p + D_H(\psi) - D_H\big(x_1^{(p-1)}\psi\big) + \sum_{i \geqslant p} D_H\big(x_1^{(i)}q_i\big) \end{aligned}$$

for some $q_i \in F[x_2]$. As the RHS equals $D_1^p - D_H(x_2) + D_H(x_1^{(p^2-p)}\psi)$ and $x_1^{(p-1)}\psi = x_1^{(p-1)}\psi_0$, we derive that $\psi_0 = -x_2$, $\psi_i = 0$ for $1 \leqslant i \leqslant p-2$, and $\psi_{p-1} = \psi_0$. In other words, $\psi = -(1 + x_1^{(p-1)})x_2$ and

$$g(t) = \big(D_1^p + D_1\big) - D_H\big(x_1^{(p^2-p)}x_2\big) - D_H\big(x_1^{(p^2-1)}x_2\big).$$

Next we show that this element is toral. Note that

$$\big(D_1^p + D_1\big) - D_H\big(x_1^{(p^2-p)}x_2\big) - D_H\big(x_1^{(p^2-1)}x_2\big) = \big(D_1^p + D_1\big) - \big[D_1^p + D_1, D_H\big(x_1^{(p^2)}x_2\big)\big]$$

and $\binom{p^2-1}{p} - \binom{p^2-1}{p-1} = \binom{p-1}{1} - 1 = -2$ by Lucas' theorem. Then

$$\begin{aligned}\bigl[D_H\bigl(x_1^{(p^2-p)}\bigl(1+x_1^{(p-1)}\bigr)x_2\bigr), D_H\bigl(x_1^{(p)}\bigl(1+x_1^{(p-1)}\bigr)x_2\bigr)\bigr] &= \bigl[D_H\bigl(x_1^{(p^2-p)}x_2\bigr), D_H\bigl(x_1^{(p)}x_2\bigr)\bigr]\\ &= -2D_H\bigl(x_1^{(p^2-1)}x_2\bigr).\end{aligned}$$

In view of the earlier computations this gives

$$\begin{aligned}&\bigl(D_1^p + D_1 - D_H\bigl(x_1^{(p^2-p)}x_2\bigr) - D_H\bigl(x_1^{(p^2-1)}x_2\bigr)\bigr)^{[p]}\\ &\quad = D_1^p - \bigl(\mathrm{ad}\bigl(D_1^p + D_1\bigr)\bigr)^p\bigl(D_H\bigl(x_1^{(p^2)}x_2\bigr)\bigr) - D_H\bigl(x_1^{(p^2-p)}x_2\bigr)\\ &\quad = D_1^p - D_H(x_2) - D_H\bigl(x_1^{(p^2-p)}x_2\bigr) - D_H\bigl(x_1^{(p^2-1)}x_2\bigr).\end{aligned}$$

So the element $D_1^p + D_1 - D_H((x_1^{(p^2-p)} + x_1^{(p^2-1)})x_2)$ is indeed toral.

As a result, all toral elements in $S_p \setminus S$ are conjugate under $\operatorname{Aut} S$. To finish the proof it remains to note that the element $D_1^p + D_1 + D_H(x_1x_2) \in S_p \setminus S$ is toral. $\square$

3. Two-sections in simple Lie algebras

In this section our standing hypothesis is that L is a finite-dimensional simple Lie algebra and T is a torus of maximal dimension in the semisimple p-envelope L_p of L. Given $\alpha_1, \dots, \alpha_s \in \Gamma(L,T)$ we denote by $\mathrm{rad}_T\, L(\alpha_1, \dots, \alpha_s)$ the maximal T-*invariant* solvable ideal of the s-section $L(\alpha_1, \dots, \alpha_s)$ and put

$$L[\alpha_1, \dots, \alpha_s] := L(\alpha_1, \dots, \alpha_s)/\,\mathrm{rad}_T\, L(\alpha_1, \dots, \alpha_s). \tag{3.1}$$

We let $\widetilde{S} = \widetilde{S}(\alpha_1, \dots, \alpha_s)$ be the T-socle of $L[\alpha_1, \dots, \alpha_s]$, the sum of all minimal T-stable ideals of the Lie algebra $L[\alpha_1, \dots, \alpha_s]$. Then $\widetilde{S} = \bigoplus_{i=1}^r \widetilde{S}_i$, where each $\widetilde{S}_i$ is a *minimal* T-stable ideal of $L[\alpha_1, \dots, \alpha_s]$. It is immediate from the definition that both T and $L(\alpha_1, \dots, \alpha_s)_p$ act on $L[\alpha_1, \dots, \alpha_s]$ as derivations and preserve $\widetilde{S}$. Thus, there is a natural restricted Lie algebra homomorphism $T + L(\alpha_1, \dots, \alpha_s)_p \to \operatorname{Der} \widetilde{S}$ which will be denoted by $\Psi_{\alpha_1,\dots,\alpha_s}$. Note that $L(\alpha_1, \dots, \alpha_s) \cap \ker \Psi_{\alpha_1,\dots,\alpha_s} = \mathrm{rad}_T\, L(\alpha_1, \dots, \alpha_s)$ and, moreover, the image of $\Psi_{\alpha_1,\dots,\alpha_s}$ can be identified with a semisimple restricted Lie subalgebra of $\operatorname{Der} \widetilde{S}$ containing $L[\alpha_1, \dots, \alpha_s]$ as an ideal.

We often regard the linear functions on T as functions on the nilpotent restricted Lie algebra $\mathfrak{c}_{L_p}(T)$ by using the rule $\gamma(x) := (\gamma(x^{[p]^e}))^{p^{-e}}$ for all $x \in \mathfrak{c}_{L_p}(T)$, where $e \gg 0$ (this makes sense because T coincides with the set of all p-semisimple elements of $\mathfrak{c}_{L_p}(T)$).

Let $\mathrm{nil}\, H_p$ denote the maximal p-nilpotent ideal of the restricted Lie algebra H_p. According to [P-St 04, Corollary 3.9], the inclusion $H^4 \subset \mathrm{nil}\, H_p$ holds and all roots in $\Gamma(L,T)$ are linear functions on H.

Lemma 3.1. *If $\delta \in \Gamma(L,T)$ has the property that $\delta(H) \neq 0$, then $\delta([L_\delta, L_{-\delta}]^2) = 0$ and $[L_\delta, L_{-\delta}]^3 \subset \mathrm{nil}\, H_p$.*

Proof. This is immediate from [P-St 04, Proposition 3.4]. $\square$

Proposition 3.2. *Let $\mathfrak{t}$ be a torus in L_p whose centralizer in L is nilpotent, and assume further that $\mathfrak{t}$ contains the all p-semisimple elements of the p-envelope of $\mathfrak{c}_L(\mathfrak{t})$ in L_p. Let $\eta \in \Gamma(L,\mathfrak{t})$ be such that $L(\eta)$ is nonsolvable and denote by $S(\eta)$ the socle of the semisimple Lie algebra $L(\eta)/\,\mathrm{rad}\, L(\eta)$. Then the following hold*:

(1) *the radical* $\mathrm{rad}\, L(\eta)$ *is* $\mathfrak{t}$*-stable*;
(2) *the socle $S(\eta)$ is a simple Lie algebra invariant under the action of* $\mathfrak{t}$;
(3) *the centralizer $\mathfrak{c}_S(\mathfrak{t})$ is a Cartan subalgebra of toral rank 1 in S.*

Proof. The torus $\mathfrak{t}$ satisfies the conditions of [P-St 04, Theorem 3.6]. Moreover, our first statement is nothing but [P-St 04, Theorem 3.6(1)]. The last two statements are immediate consequences of [P-St 04, Theorem 3.6(3)] and [P-St 04, Theorem 3.6(4)]. □

Theorem 3.3. *For every $\gamma \in \Gamma(L,T)$ the radical* $\operatorname{rad} L(\gamma)$ *is T-stable and either $L[\gamma]$ is one of* (0), $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$, $H(2;\underline{1})^{(1)}$ *or $p=5$, L_p possesses nonstandard tori of maximal dimension, and $L[\gamma] \cong H(2;\underline{1})^{(2)} \oplus F(1+x_1)^4\partial_2$. If γ is nonsolvable, then the derived subalgebra $L[\gamma]^{(1)}$ is simple.*

Proof. This is immediate from [P-St 04, Corollary 3.7]. □

Lemma 3.4. *Let $\mathfrak{g} = H(2;\underline{1})^{(2)} \oplus F(1+x_1)^{p-1}\partial_2$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then either $\mathfrak{h}$ is abelian or $\mathfrak{h}^3$ contains a nonzero toral element of $\mathfrak{g}$.*

Proof. We regard $\mathfrak{g}$ as a restricted Lie subalgebra of $\widetilde{\mathfrak{g}} := H(2;\underline{1})$. Recall that $\widetilde{\mathfrak{g}} = H(2;\underline{1})^{(2)} \oplus FD_H(x_1^{(p)}) \oplus FD_H(x_2^{(p)}) \oplus FD_H(x_1^{(p-1)}x_2^{(p-1)})$. Since $\widetilde{\mathfrak{g}}^{[p]} \subset H(2;\underline{1})^{(2)}$ by Jacobson's formula, $\mathfrak{h}$ coincides with $\mathfrak{c}_{\mathfrak{g}}(y)$ for some nonzero toral element $y \in H(2;\underline{1})^{(2)}$. By a result of Demuškin, there is $\sigma \in \operatorname{Aut} H(2;\underline{1})^{(2)}$ such that either $\sigma(y) = D_H((1+x_1)x_2)$ or $\sigma(y)$ is a nonzero multiple of $D_H(x_1x_2)$; see [St 04, Theorem 7.5.8]. In the latter case, there exist $a,b \in F$ such that $\sigma(\mathfrak{h})$ is contained in the span of $aD_H(x_1^{(p)}) + bD_H(x_2^{(p)})$ and all $D_H(x_1^{(i)}x_2^{(i)})$ with $1 \leqslant i \leqslant p-1$, hence is abelian. Then $\mathfrak{h}$ is abelian, too. So assume we are in the former case. Then there are $a,b,c \in F$ such that $\sigma(\mathfrak{h})$ coincides with the span of all $D_H((1+x_1)^i x_2^{(i)})$ with $1 \leqslant i \leqslant p-2$ and $z := a(1+x_1)^{p-1}D_2 + bD_H(x_2^{(p)}) + cD_H((1+x_1)^{p-1}x_2^{(p-1)})$. If $a=0$, then it is easy to check that $\sigma(\mathfrak{h})$ is abelian, whilst if $a \neq 0$, then $(\operatorname{ad} z)^2(D_H((1+x_1)^3x_2^{(3)}))$ is a nonzero multiple of $\sigma(y)$. This completes the proof. □

Next we recall our results on 2-sections of L with respect to T. Let $\alpha,\beta \in \Gamma(L,T)$ be such that $L(\alpha,\beta)$ is nonsolvable. As explained in [P-St 04, p. 793], the T-socle $\widetilde{S} = \widetilde{S}(\alpha,\beta)$ is either a unique minimal ideal of $L[\alpha,\beta]$ or $\widetilde{S} = \widetilde{S}_1 \oplus \widetilde{S}_2$, where $TR(\widetilde{S}_i) = 1$ for $i = 1,2$ and each $\widetilde{S}_i$ is T-stable. Moreover, in the latter case the following holds:

Theorem 3.5. *(Cf. [P-St 04, Theorem 4.1].) If $\widetilde{S} = \widetilde{S}_1 \oplus \widetilde{S}_2$, then there exist $\delta_1,\delta_2 \in \Gamma(L,T)$ such that*

$$L[\delta_1]^{(1)} \oplus L[\delta_2]^{(1)} \subset L[\alpha,\beta] \subset L[\delta_1] \oplus L[\delta_2].$$

When the T-socle $\widetilde{S}$ is a minimal ideal of $L[\alpha,\beta]$, we have two possibilities: either $TR(\widetilde{S}) = 2$ or $TR(\widetilde{S}) = 1$.

Theorem 3.6. *Suppose $\widetilde{S}$ is the unique minimal ideal of $L(\alpha,\beta)$ and $TR(\widetilde{S}) = 2$. Then $\widetilde{S}$ is simple, $\Psi_{\alpha,\beta}(L_\gamma) \subset \widetilde{S}$ for all $\gamma \in \Gamma(L,T)$, and one of the following holds:*

(1) *$\widetilde{S}$ is one of $W(2;\underline{1})$, $S(3;\underline{1})^{(1)}$, $H(4;\underline{1})^{(1)}$, $K(3;\underline{1})^{(1)}$ and $L[\alpha,\beta] = \widetilde{S}$;*
(2) *$\widetilde{S}$ is one of $W(1;\underline{2})$, $H(2;\underline{1};\Phi(\tau))^{(1)}$, $H(2;\underline{1};\Delta)$ and*

$$L[\alpha,\beta] = \widetilde{S} + \Psi_{\alpha,\beta}(T) \cap L[\alpha,\beta];$$

(3) *$\widetilde{S} \cong \mathcal{M}(1,1)$ and $L[\alpha,\beta] = \widetilde{S}$;*
(4) *$\widetilde{S}$ is a classical Lie algebra of type A_2, B_2 or G_2 and $L[\alpha,\beta] = \widetilde{S}$;*
(5) *$\widetilde{S} = H(2;(2,1))^{(2)}$ and $\Psi_{\alpha,\beta}(T) \subset \widetilde{S}_p$. Moreover,*

$$H\bigl(2;(2,1)\bigr)^{(2)} \subset L[\alpha,\beta] \subset H\bigl(2;(2,1)\bigr)^{(2)} \oplus FD_H\bigl(x_1^{(p^2)}\bigr) \oplus FD_H\bigl(x_1^{(p^2-1)}x_2^{(p-1)}\bigr).$$

In cases (1), (3), (4) *the Lie algebra $L[\alpha,\beta]$ is simple, and $L[\alpha,\beta]^{(1)}$ is simple in all cases.*

Proof. If $\widetilde{S}$ is not isomorphic to $H(2;(2,1))^{(2)}$, then the statement follows immediately from [P-St 04, Theorem 4.2]. So assume $\widetilde{S} \cong H(2;(2,1))^{(2)}$. Then [P-St 04, Theorem 4.2] says that $L[\alpha,\beta] \subset \mathcal{G}$ where $\mathcal{G}$ is the p-envelope of $G = H(2;(2,1))$ in $\operatorname{Der}\widetilde{S}$. Recall that $\Psi_{\alpha,\beta}: T + L(\alpha,\beta)_p \to \operatorname{Der}\widetilde{S}$ is a restricted Lie algebra homomorphism. Hence $\widetilde{S}_p$ lies in the image of $\Psi_{\alpha,\beta}$. In the present case, $\operatorname{Der}\widetilde{S} = \mathcal{G} \oplus F(x_1D_1 + x_2D_2)$; see [B-W 88, Proposition 2.1.8(vii)] for instance. If $\Psi_{\alpha,\beta}(T) \not\subset \mathcal{G}$, then there is a surjective restricted Lie algebra homomorphism $\Psi_{\alpha,\beta}(T + L(\alpha,\beta)_p) \twoheadrightarrow F(x_1D_1 + x_2D_2)$ whose kernel contains $\widetilde{S}_p$. But then [St-F, Lemma 2.4.4(2)] yields that the restricted Lie algebra $\Psi_{\alpha,\beta}(T + L(\alpha,\beta)_p)$ contains 3-dimensional tori, a contradiction. Consequently, $\Psi_{\alpha,\beta}(T + L(\alpha,\beta)_p) \subset \mathcal{G}$, forcing $\Psi_{\alpha,\beta}(T) \subset \mathcal{G}^{[p]} \subset \widetilde{S}_p$.

Let $\mathfrak{t}'$ be an optimal 2-dimensional torus in $\widetilde{S}_p$. By [B-W 88, Lemma 1.7.2(b)], there is a torus T' of maximal dimension in $T + L(\alpha,\beta)_p$ such that $\Psi_{\alpha,\beta}(T') = \mathfrak{t}'$. Let H' denote the centralizer of T' in L. Note that $L(\alpha,\beta) = L(\alpha',\beta')$ for some $\alpha',\beta' \in \Gamma(L,T')$ (this follows from the main result of [P 89] and [P-St 99, Corollary 2.10]). Each $i\alpha' + j\beta'$ with $i,j \in \mathbb{F}_p$ can be viewed as a linear function of $\mathfrak{t}'$.

Since $\mathfrak{t}'$ is optimal, $\mathfrak{t}' \cap \widetilde{S} = \mathfrak{t}' \cap \widetilde{S}_{(0)}$ is spanned by a nonzero toral element, t_2 say; see [St 92, (VI.1)]. Since $\Gamma(\widetilde{S},\mathfrak{t}') \cup \{0\}$ is a 2-dimensional vector space over $\mathbb{F}_p$, by Lemma 2.5(4), there is $\delta' \in \Gamma(L(\alpha,\beta),T')$ such that $\delta'(t_2) = 0$. Since, then, δ' also vanishes on $\mathfrak{c}_{\widetilde{S}}(\mathfrak{t}')$, the Engel–Jacobson theorem yields that $\widetilde{S}(\delta')$ is nilpotent. Since $\mathcal{G}/\widetilde{S}$ is solvable, $\mathcal{G}(\delta')$ must be, also. But then $L(\delta')$ is solvable, too. As explained in [St 92, (VI.4)] the union $\bigcup_{i\in\mathbb{F}_p^\times} \widetilde{S}_{i\delta'}$ contains a nonnilpotent element of $\mathcal{G}$. Hence $\bigcup_{i\in\mathbb{F}_p^\times} L_{i\delta'}$ contains a nonnilpotent element of L_p. Since $L_{i\delta'} \subset \operatorname{rad} L(\delta')$ for all $i \in \mathbb{F}_p^\times$, it follows from [P-St 04, Proposition 3.8] that δ' vanishes on H'.

Recall that $\widetilde{S}_p = FD_1^p \oplus \widetilde{S}$ and $\mathcal{G} = S_p \oplus V$, where V is the F-span of $D_H(x_1^{(p^2)})$, $D_H(x_2^{(p)})$ and $D_H(x_1^{(p^2-1)}x_2^{(p-1)})$. Hence $\mathcal{G}^3 \subset \widetilde{S}$. Pick a toral element $t_1 \in \mathfrak{t}' \setminus \widetilde{S}$ (such an element exists by Lemma 2.4). By Lemma 2.7, we may assume that $t_1 = D_1^p + D_1 + D_H(x_1x_2)$ (one should keep in mind here that $\widetilde{S}_{(0)}$ is invariant under all automorphisms of S; see [St 04, Theorem 4.2.6]). Set $V' := (\mathrm{Id} - (\operatorname{ad} t_2)^{p-1})(\mathrm{Id} - (\operatorname{ad} t_1)^{p-1})(V)$. Then

$$\mathfrak{c}_{\widetilde{S}}(\mathfrak{t}') \subset \Psi_{\alpha,\beta}(H') \subset \mathfrak{c}_{\mathcal{G}}(\mathfrak{t}') = \mathfrak{c}_{\widetilde{S}_p}(\mathfrak{t}') \oplus V', \qquad \mathfrak{c}_{\mathcal{G}}(\mathfrak{t}')^3 \subset \mathfrak{c}_{\widetilde{S}}(\mathfrak{t}') \subset \Psi_{\alpha,\beta}(H').$$

The elements $(\mathrm{Id} - (\operatorname{ad} t_1)^{p-1})(D_H(x_1^{(p^2)}))$ and $(\mathrm{Id} - (\operatorname{ad} t_1)^{p-1})(D_H(x_1^{(p^2-1)}x_2^{(p-1)}))$ lie in $G_{(p-2)} \subset G_{(1)}$ whereas $[t_1, D_H(x_2^{(p)})] = 0$. Consequently, $(\mathrm{Id} - (\operatorname{ad} t_1)^{p-1})(V) \subset G_{(1)}$. As $\operatorname{ad} t_2$ preserves $G_{(1)}$ we get $V' \subset G_{(1)}$.

We claim that $L[\alpha,\beta] \subset G$. Indeed, suppose the contrary. Recall that $G = \widetilde{S} \oplus V' \subsetneq L[\alpha,\beta] + V'$ and $\mathcal{G} = \widetilde{S} \oplus FD_1^p \oplus V'$. Then $\mathcal{G} = L[\alpha,\beta] + V'$, hence

$$\mathfrak{t}' \subset \mathfrak{c}_{\mathcal{G}}(\mathfrak{t}') = \mathfrak{c}_{L[\alpha,\beta]+V'}(\mathfrak{t}') = \Psi_{\alpha,\beta}(H') + V'.$$

Since $(\Psi_{\alpha,\beta}(H') + V')^3 \subset \Psi_{\alpha,\beta}(H')$, Jacobson's formula and induction on k enable us to deduce that $(\Psi_{\alpha,\beta}(H') + V')^{[p]^k} \subset (V')^{[p]^k} + \sum_{i=0}^k \Psi_{\alpha,\beta}(H')^{[p]^k}$ for all $k \geqslant 0$. From our earlier remarks we know that $V' \subset G_{(1)}$ consists of p-nilpotent elements of $\mathcal{G}$. Therefore, $(\Psi_{\alpha,\beta}(H') + V')^{[p]^e} \subset \sum_{i=0}^e \Psi_{\alpha,\beta}(H')^{[p]^i}$ for all sufficiently large e. Since H' is nilpotent, this forces $\mathfrak{t}' = (\mathfrak{t}')^{[p]^e} \subset (\Psi_{\alpha,\beta}(H'))^{[p]^e}$ for $e \gg 0$. But then δ' vanishes on $\mathfrak{t}'$. By contradiction, the claim follows.

Suppose $L[\alpha,\beta] \not\subset H(2;(2,1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)}x_2^{(p-1)})$ and pick $\mu \in F^\times$. Recall the elements $t_\mu \in \widetilde{S}$ and $h_\mu \in \mathfrak{c}_G(t_\mu)$ from Lemma 2.1. Our present assumption on $L[\alpha,\beta]$ implies that $\mathfrak{c}_{L[\alpha,\beta]}(t_\mu) \supsetneq C'_\mu$; see Lemma 2.2(i). As $L[\alpha,\beta] \subset G$ by our remarks earlier in the proof, $L[\alpha,\beta]$ contains an element from $(G \cap C_\mu) \setminus C'_\mu$; call it h. In view of Lemma 2.3(ii), we may assume that $h = h_\mu + sD_H(x_1^{(p^2)})$ for some $s \in F$.

Let h_0 denote the p-semisimple part of h in the p-envelope of $L[\alpha,\beta]$ in $\mathcal{G}$. It is immediate from Lemma 2.3(iv) that the elements h_0 and t_μ are linearly independent. This implies that $\mathfrak{t}_\mu := Fh_0 \oplus Ft_\mu$ is a torus of maximal dimension in $\mathcal{G}$. Recall that the restricted Lie algebra homomorphism $\Psi_{\alpha,\beta}$ takes $T + L(\alpha,\beta)_p$ into $\mathcal{G}$. Hence it follows from [St-F, Lemma 2.4.4(2)] that there exists a torus of maximal dimension T'' in L_p contained in $T + L(\alpha,\beta)_p$ and such that $\mathfrak{t}_\mu = \Psi_{\alpha,\beta}(T'')$ and $T \cap \ker\alpha \cap \ker\beta \subseteq T \cap T''$. We denote by H'' the centralizer of T'' in L. By construction, there exists $\tilde{h} \in H''$ with $\Psi_{\alpha,\beta}(\tilde{h}) = h$.

Set $T_0 := T \cap \ker\alpha \cap \ker\beta$. Because $L(\alpha,\beta) = \mathfrak{c}_L(T_0)$, it is straightforward to see that $L(\gamma'') = L(\alpha,\beta)(\gamma'')$ for every $\gamma'' \in \Gamma(L,T'')$ with $\gamma''(T_0) = 0$. Since $\Psi_{\alpha,\beta}(T'') = \mathfrak{t}_\mu$, there exists $\delta'' \in \Gamma(L,T'')$ such that $\delta''(T_0) = 0$, $\delta''(t_\mu) = 0$ and $\delta''(h_0) \neq 0$; see Lemma 2.5(4). Then $C'_\mu \subset \Psi_{\alpha,\beta}((L(\alpha,\beta))(\delta'')) \subset C_\mu$ and $\delta''(\tilde{h}) \neq 0$. Since $(L(\alpha,\beta))(\delta'') = L(\delta'')$ by the preceding remark, Lemma 2.2(i) shows that δ'' is a solvable root which does not vanish on H''. In view of [P-St 04, Proposition 3.8], this entails that every root space $L_{i\delta''} = (\operatorname{rad} L(\delta''))_{i\delta''}$, where $i \in \mathbb{F}_p$, consists of p-nilpotent elements of L_p. Since $\Psi_{\alpha,\beta}$ is a restricted Lie algebra homomorphism, this means that for every $\lambda \in F^\times$ all λ-eigenvectors of the linear operator $(\operatorname{ad} h)_{|C'_\mu}$ must act nilpotently on $\widetilde{S}$. As this contradicts Lemma 2.3(iv), we now derive that our present assumption is false. Thus, $L[\alpha,\beta] \subset H(2;(2,1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)}x_2^{(p-1)})$, completing the proof. $\square$

If $\widetilde{S}$ is a minimal ideal of $L[\alpha,\beta]$ and $TR(\widetilde{S}) = 1$, then [P-St 04, Theorem 4.4] implies the following:

Theorem 3.7. *Suppose $\widetilde{S}$ is a unique minimal ideal of $L(\alpha,\beta)$ and $TR(\widetilde{S}) = 1$. Then there exists $\delta \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$ such that $\Psi_{\alpha,\beta}(L_\gamma) \subset \widetilde{S}$ for all $\gamma \in \Gamma(L,T) \setminus \mathbb{F}_p\delta$. Moreover, one of the following holds*:

(1) $L[\alpha,\beta] = L[\eta]$ *for some* $\eta \in \Gamma(L,T) \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$;
(2) $\widetilde{S} \cong H(2;\underline{1})^{(2)}$, $L[\alpha,\beta] \subset \operatorname{Der} H(2;\underline{1})^{(2)}$ *and* $\dim \Psi_{\alpha,\beta}(T) = 2$;
(3) $S \otimes \mathcal{O}(m;\underline{1}) \subset L[\alpha,\beta] \subset (\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1}) \rtimes (\operatorname{Id} \otimes W(m;\underline{1}))$, *where S is one of* $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$, $\widetilde{S} \cong S \otimes \mathcal{O}(m;\underline{1})$, *and* $m > 0$.

In cases (1) *and* (2) *one can take $\delta = 0$, i.e. $\Psi_{\alpha,\beta}(L_\gamma) \subset \widetilde{S}$ for all $\gamma \in \Gamma(L,T)$.*

More information on the two-sections of L can be found in [P-St 04, Section 4].

4. Nonstandard tori of maximal dimension

From now on we assume that T is a nonstandard torus of maximal dimension in the semisimple p-envelope L_p of L. In light of [P 94, Theorem 1] this implies that $p = 5$. As explained in Section 2, the linear functions on T can be regarded as functions on the nilpotent restricted Lie algebra $\mathfrak{c}_{L_p}(T)$. Set $H := \mathfrak{c}_L(T)$ and define

$$\Omega = \Omega(L,T) := \{\delta \in \Gamma(L,T) \mid \delta(H^3) \neq 0\}.$$

As T is a torus of maximal dimension in L_p, it is immediate from [P 94, Theorem 1(ii)] that there exist $\mathbb{F}_p$-independent roots $\alpha,\beta \in \Gamma(L,T)$ for which $L[\alpha,\beta] \cong \mathcal{M}(1,1)$. By Lemmas 4.1 and 4.4 of [P 94], we then have $i\alpha + j\beta \in \Omega$ for all nonzero $(i,j) \in \mathbb{F}_p^2$. In particular, $\Omega \neq \emptyset$. In view of Schue's lemma [St 04, Proposition 1.3.6(1)], this yields

$$L_\gamma = \sum_{\delta \in \Omega} [L_\delta, L_{\gamma-\delta}] \quad (\forall \gamma \in \Gamma(L,T) \cup \{0\}). \tag{4.1}$$

Because of [P 94, Theorem 1(ii)] we can also assume that $TR(L) \geqslant 3$. Our main goal in this section is to give a preliminary description of the 2-sections of L relative to T. More precisely, we will go through all possible types of 2-sections (described in Section 3) and eliminate some of them by using our assumption on T.

Lemma 4.1. *For any nonsolvable $\alpha \in \Omega$ there exists $\beta \in \Gamma(L,T)$ such that $L[\alpha,\beta] \cong \mathcal{M}(1,1)$ and $\alpha([L_{i\alpha}, L_{-i\alpha}], [L_\beta, L_{-\beta}]) \neq 0$ for some $i \in \mathbb{F}_p^\times$.*

Proof. Since α is nonsolvable and $\alpha(H^3) \neq 0$, Theorem 3.3 implies that $L[\alpha] \cong H(2;\underline{1})^{(2)} \oplus F(1+x_1)^4\partial_1$. By [P-St 04, Theorem 3.5], there is $k \in \mathbb{F}_p^\times$ for which the set $\Omega_1 := \{\delta \in \Gamma(L,T) \mid \delta([L_{k\alpha}, L_{-k\alpha}]) \neq 0\}$ is nonempty. Since $\Psi_\alpha(H) \cap H(2;\underline{1})^{(2)}$ has codimension one in $\Psi_\alpha(H)$, Schue's lemma [St 04, Proposition 1.3.6(1)] implies that there exists $\beta \in \Omega_1$ with the property that

$$\Psi_\alpha(H) = \Psi_\alpha(H) \cap H(2;\underline{1})^{(2)} + \Psi_\alpha\big([L_\beta, L_{-\beta}]\big).$$

Hence there exist $h_1 \in L(\alpha)^{(\infty)} \cap H$ and $h_2 \in [L_\beta, L_{-\beta}]$ with $\alpha([h_2,[h_2,h_1]]) \neq 0$. Note that $\beta([h_2,[h_2,h_1]]) \in \beta([L_\beta, L_{-\beta}]^2) = 0$ by Lemma 3.1. In particular, α and β are linearly independent over $\mathbb{F}_p$. Since $\beta \in \Omega_1$, we then have

$$\beta\big(\big[h_2,[h_2,h_1]\big]\big) = 0; \qquad \alpha\big(\big[h_2,[h_2,h_1]\big]\big) \neq 0; \qquad \beta\big([L_{k\alpha}, L_{-k\alpha}]\big) \neq 0. \tag{4.2}$$

We now look more closely at the T-semisimple quotient $L[\alpha,\beta]$ of the 2-section $L(\alpha,\beta)$. Since α is nonsolvable, $L[\alpha,\beta] \neq (0)$. Let $\widetilde{\mathcal{S}}$ denote the p-envelope of the T-socle $\widetilde{S}$ of $L[\alpha,\beta]$ in $\operatorname{Der}\widetilde{S}$, and set $u := \Psi_{\alpha,\beta}([h_2,[h_2,h_1]])$. Given $x \in \widetilde{S}$ we write x_s for the p-semisimple part of x in $\widetilde{\mathcal{S}}$. Because the roots α,β are $\mathbb{F}_p$-independent, $h_1 \in L(\alpha)^{(\infty)} \cap H = \sum_{j\in\mathbb{F}_p^\times}[L_{j\alpha}, L_{-j\alpha}]$ and $h_2 \in [L_\beta, L_{-\beta}]$, it follows from Theorems 3.3, 3.5, 3.6 and 3.7 that $u \in \widetilde{S}$. Now relations (4.2) enable us to find $v \in \widetilde{S} \cap \Psi_{\alpha,\beta}([L_{k\alpha}, L_{k\alpha}])$ such that the span of u_s and v_s is 2-dimensional. This yields $\Psi_{\alpha,\beta}(T) \subset \widetilde{\mathcal{S}}$ showing that $TR(\widetilde{S}) = 2$. Since $\beta([L_{k\alpha}, L_{-k\alpha}]) \neq 0$, we also deduce that there are $\mathbb{F}_p$-independent $\delta_1,\delta_2 \in \Gamma(L,T)$ for which $[\Psi_{\alpha,\beta}(L_{\delta_1}), \Psi_{\alpha,\beta}(L_{\delta_2})] \neq 0$. In view of Theorem 3.5, this implies that $\widetilde{S}$ is a minimal ideal of $L[\alpha,\beta]$.

Theorem 3.6 now says that $\widetilde{S}$ is a simple Lie algebra and $\Psi_{\alpha,\beta}(L_\gamma) \subset \widetilde{S}$ for all $\gamma \in \Gamma(L,T) \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$. Since $\alpha(H,[L_{k\alpha}, L_{-k\alpha}]) \neq 0$, the torus $\Psi_{\alpha,\beta}(T) \subset \widetilde{\mathcal{S}} = \widetilde{S}_p$ is nonstandard. Applying [P 94, Theorem 1(ii)] we conclude that $L[\alpha,\beta] \cong \mathcal{M}(1,1)$, finishing the proof. □

Proposition 4.2. *If $\alpha \in \Omega$ and $\beta \in \Gamma(L,T)$, then one of the following occurs*:

(1) $L[\alpha,\beta] = (0)$.
(2) $L[\alpha,\beta] = L[\delta]$ *for some* $\delta \in \Gamma(L,T)$.
(3) $L[\delta_1]^{(1)} \oplus L[\delta_2]^{(1)} \subset L[\alpha,\beta] \subset L[\delta_1] \oplus L[\delta_2]$ *for some* $\delta_1,\delta_2 \in \Gamma(L,T)$.
(4) $S \otimes \mathcal{O}(m;\underline{1}) \subset L[\alpha,\beta] \subset (\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1}) \rtimes (\mathrm{Id} \otimes W(m;\underline{1}))$, *where S is one of* $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$, $\widetilde{S} \cong S \otimes \mathcal{O}(m;\underline{1})$, *and* $m > 0$.
(5) $H(2;(2,1))^{(2)} \subset L[\alpha,\beta] \subset H(2;(2,1))$ *and* $\widetilde{S} = H(2;(2,1))^{(2)} = L[\alpha,\beta]^{(1)}$. *Furthermore, each* $\eta \in \Gamma(L[\alpha,\beta], \Psi_{\alpha,\beta}(T))$ *is Hamiltonian*, $\eta(\Psi_{\alpha,\beta}(T) \cap \widetilde{S}) \neq 0$, *and* $\Gamma(L[\alpha,\beta], \Psi_{\alpha,\beta}(T)) = (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta) \setminus \{0\}$.
(6) $L[\alpha,\beta] \cong \mathcal{M}(1,1)$.

Proof. (a) Set $\overline{T} := \Psi_{\alpha,\beta}(T)$ and $\overline{H} := \Psi_{\alpha,\beta}(H)$. If $\Gamma(L[\alpha,\beta],\overline{T}) = \emptyset$, then $L(\alpha,\beta)$ is solvable, forcing $L[\alpha,\beta] = (0)$. If $\emptyset \neq \Gamma(L[\alpha,\beta],\overline{T}) \subset \mathbb{F}_p\delta$ for a single root δ, then for any $\delta' \in (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta) \setminus \mathbb{F}_p\delta$ we have that $L_{\delta'} \subset \operatorname{rad}_T L(\alpha,\beta)$. Then $L[\alpha,\beta] = L[\delta]$. So we may assume from now that $\Gamma(L[\alpha,\beta],\overline{T})$ contains two roots independent over $\mathbb{F}_p$. Then $L[\alpha,\beta]$ is described in Theorems 3.5, 3.6 and 3.7. Let $\widetilde{S}$ be the T-socle of $L[\alpha,\beta]$. If $\widetilde{S}$ is not a minimal ideal of $L[\alpha,\beta]$, then Theorem 3.5 says that we are in case (3) of this proposition. Thus, we may assume further that $\widetilde{S}$ is a minimal ideal of $L[\alpha,\beta]$.

(b) Suppose $TR(\widetilde{S}) = 2$. Then $L[\alpha,\beta]$ is described in Theorem 3.6. Since $\alpha(H^3) \neq 0$, there exists $\eta \in \Gamma(\widetilde{S},\overline{T})$ with $\eta(\overline{H}^3) \neq 0$. In cases (1)–(4) of Theorem 3.6 we have $\overline{H}^3 \subset (\overline{T} + \overline{H} \cap \widetilde{S})^3 = (\overline{H} \cap \widetilde{S})^3$, implying that $\overline{H}' = \mathfrak{c}_{\widetilde{S}}(\overline{T})$ acts nontriangulably on $\widetilde{S}$. But then [P 94, Theorem 1(ii)] shows that $\widetilde{S} \cong \mathcal{M}(1,1)$. This brings up case (6) of this proposition.

(c) Suppose $L[\alpha,\beta]$ is as in case (5) of Theorem 3.6. Then $\widetilde{S} \cong H(2;(2,1))^{(2)}$ and $L[\alpha,\beta] \subset H(2;(2,1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)}x_2^{(p-1)})$. Furthermore, $\overline{T} \subset \widetilde{S}_p$. If no root in $\Gamma(\widetilde{S},\overline{T})$ vanishes on $\overline{T} \cap \widetilde{S}$, then Lemma 2.5(2) shows that we are in case (5) of this proposition. So assume for a contradiction that there is $\delta \in \Gamma(\widetilde{S},\overline{T})$ with $\delta(\overline{T} \cap \widetilde{S}) = 0$. By Lemma 2.4, we have $\overline{T} \cap \widetilde{S} = Fu_2 \neq (0)$.

Since δ vanishes on $u_2 \in \overline{T} \cap \widetilde{S}$, we may assume without loss that u_2 is a toral element. As before, we put $G = H(2; (2,1))$ and $\mathcal{G} = \widetilde{S}_p \oplus V$, where $V \subset \operatorname{Der} \widetilde{S}$ is defined in Section 2. Since $\alpha \in \Omega$, the Lie algebra $\overline{H}^3$ acts nonnilpotently on S.

(c1) We first suppose that $\overline{T} \cap \widetilde{S} \not\subset S_{(0)}$. Then we can find $\Psi_{\alpha,\beta}$ such that $\overline{T} \cap \widetilde{S} = Ft_\mu$ where $\mu \in F$; see Lemma 2.1. Thus, no generality will be lost by assuming that $u_2 = t_\mu$. But then it follows from Lemma 2.2(i) that

$$\overline{H} \subset C_\mu \cap \big(H\big(2; (2,1)\big)^{(2)} \oplus FD_H\big(x_1^{(p^2)}\big) \oplus FD_H\big(x_1^{(p^2-1)} x_2^{(p-1)}\big)\big) = C'_\mu$$

and $[\overline{H}, \overline{H}] \subset [C'_\mu, C'_\mu] = (0)$. Since $\overline{H}$ acts nontriangulably on $\widetilde{S}$, this is impossible.

(c2) Now suppose that $\overline{T} \cap \widetilde{S} \subset S_{(0)}$. Then $\overline{T} \cap \overline{S}_{(0)}$ contains a nonzero p-semisimple element, say t; see Lemma 2.4. It follows from Lemma 2.4 and our earlier remarks that $\mathcal{G} = \overline{T} + G$. As $\operatorname{gr} t \in G_{(0)}/G_{(1)} \cong \mathfrak{sl}(2)$ acts invertibly on $G_{(-1)} = G/G_{(0)}$, this implies that $\overline{H} \subset \overline{T} + \mathfrak{c}_G(\overline{T}) = \overline{T} + \mathfrak{c}_{G_{(1)}}(\overline{T})$. But then $\overline{H}^{(1)} \subset G_{(1)}$ acts nilpotently on G, a contradiction.

As a result, no root in $\Gamma(\widetilde{S}, \overline{T})$ vanishes on $\overline{H} \cap \widetilde{S}$ and we are in case (5) of this proposition; see Lemma 2.5(2).

(d) If $L[\alpha,\beta]$ is as in case (1) of Theorem 3.7, then it is listed in the present proposition as type (2). If $L[\alpha,\beta]$ is as in case (2) of Theorem 3.7, then $\widetilde{S} = H(2; \underline{1})^{(2)}$, $L[\alpha,\beta] \subset \operatorname{Der} H(2; \underline{1})^{(2)}$, and $\overline{T}$ is a 2-dimensional torus in $\operatorname{Der} \widetilde{S}$. It is well known that any 2-dimensional torus in $\operatorname{Der} \widetilde{S}$ is self-centralizing; see [St 92, (III.1)] for instance. But then $\gamma(H^{(1)}) = 0$ for all $\gamma \in \mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta$. Thus, this case cannot occur in our situation. Finally, case (3) of Theorem 3.7 is listed as type (4) in the present proposition. $\square$

Corollary 4.3. *Let $\alpha \in \Omega$ and $\beta \in \Gamma(L,T)$. If $L[\alpha,\beta]$ is as in cases* (1)–(3), (5) *or* (6) *of Proposition* 4.2, *then $\sum_{i \in \mathbb{F}_p^\times} (\operatorname{rad} L(\gamma))_{i\gamma} \subset \operatorname{rad}_T L(\alpha,\beta)$ for all nonzero $\gamma \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$.*

Proof. If $L[\alpha,\beta]$ is of type (1) or (2), then all 1-sections of $L[\alpha,\beta]$ are semisimple and there is nothing to prove. If $L[\alpha,\beta]$ is of type (3), then there are $h_i \in \overline{H} \cap L[\delta_i]$ such that $\delta_i(h_i) \neq 0$, where $i = 1,2$ (recall that $\overline{H} = \Psi_{\alpha\beta}(H)$). It follows that $\operatorname{rad} L[\alpha,\beta](\delta_i) \subset \overline{H} + L[\delta_i]^{(1)}$. As each $L[\delta_i]^{(1)}$ is simple, we get $\operatorname{rad}(L[\alpha,\beta](\gamma)) \subset \overline{H}$ for all nonzero $\gamma \in \mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta$. If $L[\alpha,\beta]$ is of type (5) or (6), then all T-roots of $L[\alpha,\beta]$ are Hamiltonian and the corresponding root spaces are 5-dimensional (see Lemma 2.5 and [P 94, Lemmas 4.1 and 4.4]). Hence in these cases $\operatorname{rad}(L[\alpha,\beta](\gamma)) \subset \overline{H}$ for all $\gamma \in (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta) \setminus \{0\}$. $\square$

Lemma 4.4. *The following hold for every $\gamma \in \Gamma(L,T)$ with $\gamma(H) \neq 0$:*

(a) *All elements in $\bigcup_{i \in \mathbb{F}_p^\times} (H^3 \cap [(\operatorname{rad} L(\gamma))_{i\gamma}, L_{-i\gamma}])$ are p-nilpotent in L_p.*
(b) *If $\gamma \in \Omega$, then all elements in $\bigcup_{i \in \mathbb{F}_p^\times} ((\operatorname{rad} L(\gamma))_{i\gamma} \cup [(\operatorname{rad} L(\gamma))_{i\gamma}, L_{-i\gamma}])$ are p-nilpotent in L_p.*

Proof. We will treat both cases simultaneously. Set

$$\Omega' := \left\{ \alpha \in \Gamma(L,T) \;\middle|\; \alpha\left(\bigcup_{i \in \mathbb{F}_p^\times} \big(H^3 \cap \big[\big(\operatorname{rad} L(\gamma)\big)_{i\gamma}, L_{-i\gamma}\big]\big) \neq 0 \right) \right\},$$

$$\Omega'' := \left\{ \alpha \in \Gamma(L,T) \;\middle|\; \alpha\left(\bigcup_{i \in \mathbb{F}_p^\times} \big(\big(\operatorname{rad} L(\gamma)\big)_{i\gamma}^{[p]} \cup \big[\big(\operatorname{rad} L(\gamma)\big)_{i\gamma}, L_{-i\gamma}\big]\big) \right) \neq 0 \right\}.$$

Assume for a contradiction that either $\Omega' \neq \emptyset$ or $\gamma \in \Omega$ and $\Omega'' \neq \emptyset$. Note that $\Omega' \subset \Omega'' \cap \Omega$. Since $\gamma(H) \neq 0$, Schue's lemma [St 04, Proposition 1.3.6(1)] shows that there exists $\mu \in \Omega'$ or $\mu \in \Omega''$ for $\gamma \in \Omega$ such that

$$\gamma\big([L_\mu, L_{-\mu}]\big) \neq 0. \tag{4.3}$$

In both cases, the type of $L[\gamma,\mu]$ is determined by Proposition 4.2. If $L[\gamma,\mu]$ is as in cases (1), (2), (3), (5) or (6) of Proposition 4.2, then $\sum_{i\in\mathbb{F}_p^\times}(\operatorname{rad} L(\gamma))_{i\gamma} \subset \operatorname{rad}_T L(\gamma,\mu)$ by Corollary 4.3. Since $\mu \in \Omega''$ in both cases, this yields $L_{\pm\mu} \subset \operatorname{rad}_T L(\gamma,\mu)$. Easy induction on n based on (4.3) now gives

$$\sum_{i\in\mathbb{F}_p^\times}\big(\operatorname{rad} L(\gamma)\big)_{i\gamma} \subset \bigcap_{n\geqslant 1}\big(\operatorname{rad}_T L(\gamma,\mu)\big)^{(n)} = (0).$$

Since this contradicts our assumption that either Ω' or Ω'' is nonempty, $L[\gamma,\mu]$ must be of type (4). Then the minimal ideal of $L[\gamma,\mu]$ has the form $\widetilde{S} = S \otimes \mathcal{O}(m;\underline{1})$, where S is a restricted simple Lie algebra of absolute toral rank 1 and $m > 1$. According to [P-St 99, Theorem 3.2] we can choose $\Psi_{\gamma,\mu}$ such that $\overline{T} = \Psi_{\gamma,\mu}(T)$ has the form $F(h_0\otimes 1) \oplus F(d\otimes 1 + \mathrm{Id}_S \otimes t_0)$ for some $d \in \operatorname{Der} S$ and some nonzero toral elements $t_0 \in W(m;\underline{1})$ and $h_0 \in S$.

Since $TR(L[\gamma,\mu]) = 2$, the roots γ and μ span the dual space of $\overline{T}$. Therefore, $\gamma(h_0\otimes 1) \neq 0$ or $\mu(h_0 \otimes 1) \neq 0$. It is straightforward to see that γ vanishes on all $(\operatorname{rad} L(\gamma))_{i\gamma}^{[p]}$ and $[(\operatorname{rad} L(\gamma))_{i\gamma}, L_{-i\gamma}]$ with $i \in \mathbb{F}_p^\times$. Because $\mu \in \Omega''$, this observation in conjunction with (4.3) shows that $\Psi_{\gamma,\mu}(L_{i\gamma+j\mu}) \subset S\otimes\mathcal{O}(m;\underline{1})$ for all nonzero $(i,j) \in (\mathbb{F}_p)^2$. There are in both cases

$$x \in \bigcup_{i\in\mathbb{F}_p^\times}\Big(\big(\operatorname{rad} L(\gamma)\big)_{i\gamma}^{[p]} \cup \big[\big(\operatorname{rad} L(\gamma)\big)_{i\gamma}, L_{-i\gamma}\big]\Big) \quad \text{and} \quad h \in [L_\mu, L_{-\mu}]$$

such that $\gamma(x^{[p]}) = 0$, $\mu(x^{[p]}) \neq 0$ and $\gamma(h) \neq 0$. But then $2 \leqslant TR(S\otimes\mathcal{O}(m;\underline{1})) = TR(S) = 1$, a contradiction. $\square$

Proposition 4.5. *Let $\alpha \in \Omega$ and $\beta \in \Gamma(L,T)$ be such that $L[\alpha,\beta]$ is as in case (4) of Proposition 4.2. Then $\widetilde{S} \cong S \otimes \mathcal{O}(1;\underline{1})$, where $S = H(2;\underline{1})^{(2)}$, and $\Psi_{\alpha,\beta}$ can be chosen such that $\overline{T} := \Psi_{\alpha,\beta}(T) = F(h_0\otimes 1) \oplus F(\mathrm{Id}_S \otimes (1+x_1)\partial_1)$ for some nonzero toral element $h_0 \in S$. Furthermore, $\Omega \neq \Gamma(L,T)$ and the following hold for $\gamma \in \Gamma(L[\alpha,\beta],\overline{T})$:*

$$\gamma \in \Omega \quad \Leftrightarrow \quad \gamma(h_0\otimes 1) \neq 0;$$
$$\gamma \notin \Omega \quad \Rightarrow \quad \alpha\big(L_\gamma^{[p]}\big) \neq 0 \quad \text{or} \quad \beta\big(L_\gamma^{[p]}\big) \neq 0.$$

Proof. By our assumption, $\widetilde{S} = S \otimes \mathcal{O}(m;\underline{1})$ where $m \geqslant 1$, S is one of $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$. Recall that $\Psi_{\alpha,\beta}$ takes $T + L(\alpha,\beta)_p$ into $\operatorname{Der}(S\otimes\mathcal{O}(m;\underline{1}))$. Let

$$\pi\colon\ \operatorname{Der}\big(S\otimes\mathcal{O}(m;\underline{1})\big) = (\operatorname{Der} S)\otimes\mathcal{O}(m;\underline{1}) \rtimes \big(\mathrm{Id}_S \otimes W(m;\underline{1})\big) \twoheadrightarrow W(m;\underline{1})$$

denote the canonical projection. According to [P-St 99, Theorem 3.2], we can choose $\Psi_{\alpha,\beta}$ such that

$$\overline{T} := \Psi_{\alpha,\beta}(T) = F(h_0\otimes 1) \oplus F(d\otimes 1 + \mathrm{Id}_S \otimes t_0),$$

where Fh_0 is a maximal torus of S, $d \in \operatorname{Der} S$ and t_0 is a toral element of $W(m;\underline{1})$. Moreover, if $t_0 \in W(m;\underline{1})_{(0)}$, then $t_0 = \sum_{i=1}^m s_i x_i \partial_i$, where $s_i \in \mathbb{F}_p$, and if $t_0 \notin W(m;\underline{1})_{(0)}$, then $d = 0$ and $t_0 = (1+x_1)\partial_1$.

Our argument is quite long and will be split into two parts, each part consisting of several intermediate statements. Given a subset X of $T + L(\alpha,\beta)_p$ we denote by $\overline{X}$ the set $\{\Psi_{\alpha,\beta}(x) \mid x \in X\}$. If $\{x_1,\ldots,x_m\}$ is a generating set of the maximal ideal $\mathcal{O}(m;\underline{1})_{(1)}$, then we sometimes invoke the notation $\mathcal{O}(m;\underline{1}) = F[x_1,\ldots,x_m]$.

Part A. We first consider the case where $t_0 \in W(m;\underline{1})_{(0)}$.

Claim 1. $\pi(\overline{H}) \subset W(m;\underline{1})_{(0)}$.

Indeed, suppose the contrary. Then Schue's lemma [St 04, Proposition 1.3.6(1)] shows that there exists $\kappa \in \Gamma(L,T)$ with $\kappa(H) \neq 0$ such that $\pi(\overline{[L_\kappa, L_{-\kappa}]}) \not\subset W(m;\underline{1})_{(0)}$. Then there is $E \in [L_\kappa, L_{-\kappa}]$ such that $\overline{E} = \overline{E}' + \mathrm{Id}_S \otimes \pi(\overline{E})$ with $\overline{E}' \in (\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1})$ and $\pi(\overline{E}) \equiv \sum_{i=1}^m a_i\partial_i \not\equiv 0 \pmod{W(m;\underline{1})_{(0)}}$ for some $a_i \in F$. No generality will be lost by assuming that $a_1 \neq 0$. Then

$$0 = \bigl[t_0, \pi(\overline{E})\bigr] \equiv \sum_{i=1}^m a_i s_i \partial_i \quad \bigl(\operatorname{mod} W(m;\underline{1})_{(0)}\bigr),$$

forcing $s_1 = 0$. But then $h_0 \otimes x_1^{p-1} \in \overline{H}$ and

$$(\operatorname{ad}\overline{E})^{p-1}\bigl(h_0 \otimes x_1^{p-1}\bigr) \in F^\times(h_0 \otimes 1) + S \otimes \mathcal{O}(m;\underline{1})_{(1)},$$

which implies that $[L_\kappa, L_{-\kappa}]^3 \not\subset \operatorname{nil} H_p$. As this contradicts Lemma 3.1, the claim follows.

Claim 2. *There exists $\nu \in \Gamma(L[\alpha,\beta],\overline{T})$ with $\pi(\overline{L}_\nu) \not\subset W(m;\underline{1})_{(0)}$ and $\nu(h_0 \otimes 1) = 0$.*

Indeed, $\widetilde{S}$ is derivation simple and $\pi(\overline{T} + \overline{H}) \subset W(m;\underline{1})_{(0)}$ by our general assumption in this part and Claim 1. Hence there is $\nu \in \Gamma(L[\alpha,\beta],\overline{T})$ with $\pi(\overline{L}_\nu) \not\subset W(m;\underline{1})_{(0)}$. Since $\pi([h_0 \otimes 1, \overline{L}_\nu]) = 0$, it must be that $\nu(h_0 \otimes 1) = 0$.

Claim 3. *If $\gamma \in \Gamma(L[\alpha,\beta],\overline{T})$, then $\gamma \in \Omega \Leftrightarrow \gamma(h_0 \otimes 1) \neq 0$.*

Let γ be any root in $\Gamma(L[\alpha,\beta],\overline{T})$ with $\gamma(h_0 \otimes 1) = 0$. As $h_0 \otimes 1 \in \overline{T}$ is a nonzero toral element, $\gamma \in \mathbb{F}_p^\times \nu$, where ν is the root from Claim 2. Hence there is $\overline{E} \in \overline{L}_{i\gamma}$ for some $i \in \mathbb{F}_p^\times$, such that $\pi(\overline{E}) \notin W(m;\underline{1})_{(0)}$. As before, we have that $\pi(\overline{E}) \equiv \sum_{i=1}^m a_i\partial_i \not\equiv 0 \pmod{W(m;\underline{1})_{(0)}}$, and it can be assumed that $a_1 \neq 0$. Then $h_0 \otimes x_1 \in \widetilde{S}_{-i\gamma}$. Note that $h_0 \otimes \mathcal{O}(m;\underline{1})$ is an abelian ideal of the centralizer of $h_0 \otimes 1$ in $\operatorname{Der}\widetilde{S}$. Consequently, $h_0 \otimes x_1 \in \operatorname{rad}(L[\alpha,\beta](\gamma))_{-i\gamma}$ and

$$a_1 h_0 \otimes 1 \equiv [\overline{E}, h_0 \otimes x_1] \quad \bigl(\operatorname{mod} S \otimes \mathcal{O}(m;\underline{1})_{(1)}\bigr).$$

It follows that $[L_{i\gamma}, (\operatorname{rad} L(\gamma))_{-i\gamma}]$ contains an element which is not p-nilpotent in L_p. Then $\gamma \notin \Omega$ by Lemma 4.4. Since $\alpha \in \Omega$, these considerations show that $\alpha(h_0 \otimes 1) \neq 0$. As a consequence,

$$i\alpha + j\gamma \in \Omega \quad \Leftrightarrow \quad (i\alpha + j\gamma)\bigl(H^3\bigr) \neq 0 \quad \Leftrightarrow \quad i \in \mathbb{F}_p^\times \quad \Leftrightarrow \quad (i\alpha + j\gamma)(h_0 \otimes 1) \neq 0,$$

hence the claim.

Claim 4. *The Lie algebra $\pi(\overline{H})^3$ consists of p-nilpotent elements of $W(m;\underline{1})$.*

Otherwise, there is $y \in \overline{H}^3$ with $y^{[p]^e} \in \overline{T} \setminus F(h_0 \otimes 1)$, so that $y^{[p]^e} = b_1(h_0 \otimes 1) + b_2(d \otimes 1 + \mathrm{Id}_S \otimes t_0)$ for some $b_1 \in F$ and $b_2 \in F^\times$. Let $\nu \in \Gamma(L[\alpha,\beta],\overline{T})$ be as in Claim 2. Then $\nu(h_0 \otimes 1) = 0$ and $\nu(d \otimes 1 + \mathrm{Id}_S \otimes t_0) \neq 0$, forcing $\nu(y^{[p]^e}) \neq 0$. It follows that $\nu \in \Omega$. This contradicts Claim 3, however.

Claim 5. $d \in Fh_0$.

Claim 1 in conjunction with our standing hypothesis in this part shows that there is a Lie algebra homomorphism

$$\Psi : (\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1}) + (\overline{H} + \overline{T}) \to \operatorname{Der} S$$

whose kernel is spanned by $(\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1})_{(1)}$ and those elements of $\overline{H} + \overline{T}$ which map $(\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1})$ into $(\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1})_{(1)}$. Suppose $d \notin Fh_0$. Then $\Psi(\overline{T}) = Fh_0 \oplus Fd$. Since d is a semisimple derivation of S, it follows that $S = H(2; \underline{1})^{(2)}$ and $\Psi(\overline{T})$ is a torus of maximal dimension in $\operatorname{Der} S$. Since every such torus is self-centralizing in $\operatorname{Der} S$, by [St 92, (III.1)], it must be that $\overline{H} \subset \overline{T} + \ker \Psi$. Note that

$$(\overline{H} + \overline{T}) \subset (\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1}) + F(\mathrm{Id}_S \otimes t_0) + \mathrm{Id}_S \otimes \pi(\overline{H})$$

and $F(\mathrm{Id}_S \otimes t_0) + \mathrm{Id}_S \otimes \pi(\overline{H}) \subset \ker \Psi$ by our assumption on t_0 and Claim 1. Hence

$$\begin{aligned} \overline{H} &\subset (\overline{T} + \ker \Psi) \cap \overline{H} \subset (\ker \Psi) \cap (\overline{H} + \overline{T}) + \overline{T} \\ &\subset (\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1})_{(1)} + F(\mathrm{Id}_S \otimes t_0) + \mathrm{Id}_S \otimes \pi(\overline{H}) + \overline{T}, \end{aligned}$$

forcing $\overline{H}^3 \subset (\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1})_{(1)} + \mathrm{Id}_S \otimes \pi(\overline{H})^3$. In view of Claim 4 the Lie algebra on the right acts nilpotently on $S \otimes \mathcal{O}(m; \underline{1})$. But then $\overline{H}^3$ acts nilpotently on $L[\alpha, \beta]$, a contradiction.

As a consequence, $\overline{H} \cap (S \otimes \mathcal{O}(m; \underline{1})) = \mathfrak{c}_S(h_0) \otimes \mathrm{Ann}_{\mathcal{O}(m;\underline{1})}(t_0)$ and we may take $d = 0$.

Claim 6. *Let ν be as in Claim 2. Then*

$$\overline{H} \cap \widetilde{S} \subset \Psi_{\alpha,\beta}\big(\big[\big(\operatorname{rad} L(\nu)\big)_{-\nu}, L_\nu\big]\big) + \overline{H} \cap \big(S \otimes \mathcal{O}(m; \underline{1})_{(1)}\big).$$

By definition, there is $\overline{E} \in \overline{L}_\nu$ such that

$$\pi(\overline{E}) \equiv \sum_{i=1}^{m} a_i \partial_i \not\equiv 0 \quad \big(\operatorname{mod} W(m; \underline{1})_{(0)}\big), \quad a_1 \neq 0.$$

We have shown in the course of the proof of Claim 3 that $\mathfrak{c}_S(h_0) \otimes x_1 \subset \overline{\operatorname{rad} L(\nu)}_{-\nu}$. Then $\mathfrak{c}_S(h_0) \otimes F \subset [\overline{E}, \widetilde{S}_{-\nu}] + \overline{H} \cap (\widetilde{S} \cap \mathcal{O}(m; \underline{1})_{(1)})$. As a consequence,

$$\begin{aligned} \overline{H} \cap \widetilde{S} &= \mathfrak{c}_S(h_0) \otimes \mathrm{Ann}_{\mathcal{O}(m;\underline{1})}(t_0) \subset \mathfrak{c}_S(h_0) \otimes F + \mathfrak{c}_S(h_0) \otimes \mathcal{O}(m; \underline{1})_{(1)} \\ &\subset \big[\overline{L}_\nu, \overline{\big(\operatorname{rad} L(\nu)\big)}_{-\nu}\big] + \overline{H} \cap \big(S \otimes \mathcal{O}(m; \underline{1})_{(1)}\big). \end{aligned}$$

Claim 7. *If ν is as in Claim 2, then $\nu(H) = 0$.*

As $S \otimes F$ is $\overline{T}$-stable and S is not nilpotent, there is $\mu \in \Gamma(\widetilde{S}, \overline{T})$ with $(S \otimes F)_\mu \neq (0)$. Then $\mu(\mathrm{Id}_S \otimes t_0) = 0$ and hence $\mu(h_0 \otimes 1) \neq 0$. It follows that

$$L[\alpha, \beta](\mu) \subset S \otimes \mathcal{O}(m; \underline{1}) + \overline{H} \subset (\operatorname{Der} S) \otimes \mathcal{O}(m; \underline{1}) + \mathrm{Id}_S \otimes W(m; \underline{1})_{(0)}.$$

Let $\Phi : L[\alpha,\beta](\mu) \to \operatorname{Der} S$ denote the natural $\overline{T}$-equivariant Lie algebra homomorphism with $\ker\Phi = L[\alpha,\beta](\mu) \cap ((\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1})_{(1)} + \mathrm{Id}_S \otimes W(m;\underline{1})_{(0)})$ and $S \subset \operatorname{im}\Phi$. Then [St 04, Theorems 1.2.8 and 1.3.11] show that

$$TR(\ker\Phi) \leqslant TR\big(L[\alpha,\beta](\mu)\big) - TR(S) \leqslant TR\big(L(\mu)\big) - TR(S) \leqslant 1 - TR(S) \leqslant 0,$$

implying that $\ker\Phi$ is a nilpotent ideal of $L[\alpha,\beta](\mu)$. As $\Phi(L[\alpha,\beta](\mu))$ contains S, it is semisimple, hence isomorphic to $L[\mu]$. Note that $\mu \in \Omega$ by Claim 3. As $L[\mu] \neq (0)$, Theorem 3.3 says that $p = 5$ and $\Phi(L[\alpha,\beta](\mu)) \cong H(2;\underline{1})^{(2)} \oplus F(1+x_1)^4\partial_2$. In particular, μ is Hamiltonian. Observe that

$$\big(\operatorname{ad}(1+x_1)^4\partial_2\big)^2 D_H\big((1+x_1)^3x_2^3\big) = D_H\big((1+x_1)x_2\big). \tag{4.4}$$

By (the proof of) Lemma 3.4, we may assume that $h_0 = D_H((1+x_1)x_2)$. Then (4.4) shows that there exists $\mathcal{D} \in \Phi(\overline{H})$ such that $[\mathcal{D},[\mathcal{D},\mathfrak{c}_S(h_0)]] \not\subset \operatorname{nil}\mathfrak{c}_S(h_0)$.

Note that $\operatorname{nil}\mathfrak{c}_S(h_0)$ has codimension 1 in $\mathfrak{c}_S(h_0)$. As $\ker\Phi$ acts nilpotently on $L[\alpha,\beta](\mu)$, there is $\widetilde{\mathcal{D}} \in \overline{H}$ with $\mu([\widetilde{\mathcal{D}},[\widetilde{\mathcal{D}},\widetilde{S} \cap \overline{H}]]) \neq 0$. Since $\overline{H} \cap (S \otimes \mathcal{O}(m;\underline{1})_{(1)})$ is an ideal of $\overline{H}$, Claim 6 entails that $\Psi_{\alpha,\beta}([(\operatorname{rad} L(\nu))_{-\nu}, L_\nu]) \cap \overline{H}^3$ does not consist of p-nilpotent elements of L_p. In view of Lemma 4.4(1), this yields that $\nu(H) = 0$.

Since $t_0 \in W(m;\underline{1})_{(0)}$, the 2-section $L[\alpha,\beta]$ is semisimple (not just T-semisimple), and $\widetilde{S}$ is the unique minimal ideal of $L[\alpha,\beta]$. On the other hand, applying Proposition 3.2 with $\mathfrak{t} = T \cap \ker\nu$ shows that the unique minimal ideal of $L[\alpha,\beta]$ is a simple Lie algebra (notice that $\mathfrak{c}_L(\mathfrak{t}) = L(\nu)$ is nilpotent by the Engel–Jacobson theorem). But then $m = 0$, a contradiction. This means that the case where $t_0 \in W(m;\underline{1})_{(0)}$ cannot occur.

Part B. Thus, we may assume that $t_0 \notin W(m;\underline{1})_{(0)}$. Because of [P-St 99, Theorem 3.2] it can be assumed further that $\overline{T} = F(h_0 \otimes 1) \oplus F(\mathrm{Id}_S \otimes (1+x_1)\partial_1)$. Then $\overline{H} \cap \widetilde{S} = \mathfrak{c}_S(h_0) \otimes F[x_2,\dots,x_m]$. Since α and β are $\mathbb{F}_p$-independent, there exists $\lambda \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$ such that $\lambda(h_0 \otimes 1) = 0$ and $\lambda(\mathrm{Id}_S \otimes (1+x_1)\partial_1) = 1$. Note that

$$Fh_0 \otimes (1+x_1)^i \subset \widetilde{S}_{i\lambda} \subset \overline{(\operatorname{rad} L(\lambda))}_{i\lambda} \quad \forall i \in \mathbb{F}_p^\times. \tag{4.5}$$

Hence $(\operatorname{rad} L(\lambda))_{i\lambda}$ contains nonnilpotent elements of L_p for all $i \in \mathbb{F}_p^\times$. Lemma 4.4(b) yields $\lambda \notin \Omega$. Since $S \otimes F$ is $\overline{T}$-stable and not nilpotent, there is $\kappa \in \Gamma(L[\alpha,\beta],\overline{T})$ with $(S \otimes F)_\kappa \neq (0)$. As $\kappa(\mathrm{Id}_S \otimes (1+x_1)\partial_1) = 0$, it must be that $\kappa(h_0 \otimes 1) \neq 0$.

Claim 1. *If $\gamma \in \Gamma(L[\alpha,\beta],\overline{T})$, then $\gamma \in \Omega \Leftrightarrow \gamma(h_0 \otimes 1) \neq 0$.*

As $\alpha \in \Omega$ and $\lambda \notin \Omega$, one has $i\alpha + j\lambda \in \Omega$ for all $i \in \mathbb{F}_p^\times$ and $j \in \mathbb{F}_p$. So

$$i\alpha + j\lambda \in \Omega \quad\Leftrightarrow\quad i \neq 0 \quad\Leftrightarrow\quad (i\alpha + j\lambda)(h_0 \otimes 1) \neq 0 \quad \forall i, j \in \mathbb{F}_p.$$

Since α and λ are $\mathbb{F}_p$-independent, their $\mathbb{F}_p$-span contains $\Gamma(L[\alpha,\beta],\overline{T})$.

It follows from Claim 1 and (4.5) that $\Gamma(L[\alpha,\beta],\overline{T}) \setminus \Omega = \mathbb{F}_p^\times\lambda$ and L_γ contains nonnilpotent elements of L_p for all $\gamma \in \Gamma(L[\alpha,\beta],\overline{T}) \setminus \Omega$. Thus, it remains to show that $m = 1$.

Claim 2. *The subspace $\sum_{j=2}^m S \otimes x_j\mathcal{O}(m;\underline{1})$ is $\overline{H}$-invariant.*

Note that $L[\alpha,\beta](\kappa) = \overline{H} + S \otimes F[x_2,\dots,x_m]$. In particular, κ is nonsolvable. Let $\psi : L[\alpha,\beta](\kappa) \twoheadrightarrow L[\kappa]$ denote the canonical homomorphism. By Theorem 3.3, the Lie algebra $L[\kappa]^{(1)}$ is simple. As the ideal $S \otimes F[x_2,\dots,x_m]$ is perfect, ψ maps it onto $L[\kappa]^{(1)}$. As a consequence, $S \otimes F[x_2,\dots,x_m]_{(1)} = \ker\psi \cap (S \otimes F[x_2,\dots,x_m])$, showing that $S \otimes F[x_2,\dots,x_m]_{(1)}$ is $\overline{H}$-invariant.

Claim 3. *$S \cong H(2;\underline{1})^{(2)}$ and $[D,[D,h]]$ acts nonnilpotently on $\widetilde{S}$ for some $D \in \overline{H}$ and $h \in \overline{H} \cap \widetilde{S}$.*

We have seen in the proof of Claim 2 that

$$L[\kappa] = \psi\big(L[\alpha,\beta](\kappa)\big) \cong L[\alpha,\beta](\kappa)/\operatorname{rad}\big(L[\alpha,\beta](\kappa)\big) \cong S + H/\big(H \cap \operatorname{rad} L(\kappa)\big).$$

Our choice of κ and Claim 1 imply that $\kappa \in \Omega$. So Theorem 3.3 implies that $L[\kappa] \cong H(2;\underline{1})^{(2)} \oplus F\overline{D}$ and there exists $\tilde{h} \in \sum_{i\in\mathbb{F}_p^\times}[L_{i\kappa}, L_{-i\kappa}]$ such that $[\overline{D},[\overline{D},\Psi_\kappa(\tilde{h})]]$ acts nonnilpotently on $L[\kappa]$. Pick $D \in \psi^{-1}(\overline{D}) \cap \overline{H}$ and set $h := \Psi_{\alpha,\beta}(\tilde{h})$. Standard toral rank considerations show that $\ker\psi$ acts nilpotently of $L[\alpha,\beta](\kappa)$ (see the proof of Claim 7 in Part A for a similar argument). In light of the preceding remark this implies that $\kappa([D,[D,h]]) \neq 0$.

Claim 4. $m = 1$.

We first note that $L[\alpha,\beta] \subset \overline{L(\lambda)} + (\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1})$. If all derivations from the set $\bigcup_{i\in\mathbb{F}_p^\times} \mathrm{Id}_S \otimes \pi(\overline{L}_{i\lambda})$ preserve the ideal $I := \sum_{j=2}^m S \otimes x_j \mathcal{O}(m;\underline{1})$ of $(\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1})$, then Claim 2 entails that I is a nilpotent $\overline{T}$-stable ideal of $L[\alpha,\beta]$. Since $L[\alpha,\beta]$ is T-semisimple, this would force $m = 1$.

So assume for a contradiction that there exists $E \in L_{k\lambda}$ for some $k \in \mathbb{F}_p^\times$ such that $\mathrm{Id}_S \otimes \pi(\overline{E})$ does not preserve I. Since $\pi(\overline{E})$ is an eigenvector for $(1+x_1)\partial_1$ with eigenvalue $k \neq 0$, it has the form

$$\pi(\overline{E}) = f_1(x_2,\dots,x_m)(1+x_1)^{k+1}\partial_1 + \sum_{j=2}^m f_j(x_2,\dots,x_m)(1+x_1)^k\partial_j$$

for some $f_1,\dots,f_m \in F[x_2,\dots,x_m]$. As $\pi(\overline{E})$ does not stabilize I, it must be that $f_{j_0}(0) \neq 0$ for some $j_0 \geqslant 2$. After renumbering we may assume that $j_0 = 2$. Since $\mathfrak{c}_S(h_0) \otimes (1+x_1)^{p-k}x_2 \subset \widetilde{S}_{-k\lambda} \subset (\overline{\operatorname{rad} L(\lambda)})_{-k\lambda}$, we have that

$$\begin{aligned}\mathfrak{c}_S(h_0) \otimes F &\subset [\overline{E}, \widetilde{S}_{-k\lambda}] + \big(S \otimes F[x_1,\dots,x_m]_{(1)}\big) \cap \overline{H}\\ &= [\overline{E}, \widetilde{S}_{-k\lambda}] + \mathfrak{c}_S(h_0) \otimes F[x_1,\dots,x_m]_{(1)}.\end{aligned}$$

From this it follows that

$$\overline{H} \cap \widetilde{S} = \mathfrak{c}_S(h_0) \otimes F[x_1,\dots,x_m] \subset \big[\overline{L}_{k\lambda}, \big(\overline{\operatorname{rad} L(\lambda)}\big)_{-k\lambda}\big] + \mathfrak{c}_S(h_0) \otimes F[x_2,\dots,x_m]_{(1)}.$$

The subspace $I \cap \overline{H} = \mathfrak{c}_S(h_0) \otimes F[x_2,\dots,x_m]_{(1)}$ is $\overline{H}$-invariant by Claim 2 and acts nilpotently on $L[\alpha,\beta](\kappa)$. These observations in conjunction with Claim 3 imply that $(\operatorname{ad} D)^2([\overline{L}_{k\lambda}, (\overline{\operatorname{rad} L(\lambda)})_{-k\lambda}]) \subset \overline{H}^3 \cap [\overline{L}_{k\lambda}, (\overline{\operatorname{rad} L(\lambda)})_{-k\lambda}]$ does not consist of nilpotent derivations of $\widetilde{S}$. But then $\lambda(H) = 0$ by Lemma 4.4(a).

We now set $\mathfrak{t} := T \cap \ker\lambda$. Since $L(\lambda) = \mathfrak{c}_L(\mathfrak{t})$ is nilpotent by the Engel–Jacobson theorem, Proposition 3.2 says that $L(\alpha,\beta)/\operatorname{rad} L(\alpha,\beta)$ has a unique minimal ideal, S' say, which is a simple Lie algebra. Then S' must be the image of $\widetilde{S} = S \otimes \mathcal{O}(m;\underline{1})$ under the natural homomorphism $\phi : L[\alpha,\beta] \twoheadrightarrow L(\alpha,\beta)/\operatorname{rad} L(\alpha,\beta)$. As a consequence, $\ker\phi \cap \widetilde{S}$ coincides with the radical of $\widetilde{S}$. As the latter equals $S \otimes \mathcal{O}(m;\underline{1})_{(1)}$, we derive that $S \otimes \mathcal{O}(m;\underline{1})_{(1)} = \ker\phi \cap \widetilde{S}$ is an ideal of $L[\alpha,\beta]$. On the other hand, $\pi(\overline{E}) \notin W(m;\underline{1})_{(0)}$. This shows that our present assumption is false and $m = 1$.

The proof of the proposition is now complete. □

Corollary 4.6. *Let $\alpha \in \Omega$, $\beta \in \Gamma(L,T)$ and suppose $L[\alpha,\beta]$ is as in case (4) of Proposition 4.2. Then $\sum_{i\in\mathbb{F}_p^\times}(\operatorname{rad} L(\gamma))_{i\gamma} \subset \operatorname{rad}_T L[\alpha,\beta]$ for all $\gamma \in \Omega \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$.*

Proof. Pick $\gamma \in \Omega \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$ and view it as a $\overline{T}$-root of $L[\alpha,\beta]$. In the present case $L[\alpha,\beta](\gamma) = \overline{H} + \widetilde{S}(\gamma)$ and $\widetilde{S} = H(2;\underline{1})^{(2)} \otimes \mathcal{O}(1;\underline{1})$; see Proposition 4.5. Furthermore, in the notation of Proposition 4.5 we have that $\gamma = i\kappa + j\lambda$ for some $i \in \mathbb{F}_p^\times$ and $j \in \mathbb{F}_p$, where $\kappa, \lambda \in \overline{T}^*$ are such that $\kappa(h_0 \otimes 1) = r \in \mathbb{F}_p^\times$, $\kappa(\mathrm{Id}_S \otimes (1+x_1)\partial_1) = 0$, $\lambda(h_0 \otimes 1) = 0$ and $\lambda(\mathrm{Id}_S \otimes (1+x_1)\partial_1) = 1$. Let S_ℓ denote the ℓ-eigenspace of $\mathrm{ad}_S\, h_0$. Then

$$\widetilde{S}(\gamma) = \bigoplus_{k\in\mathbb{F}_p} S_{kir} \otimes (1+x_1)^{kj} \cong \bigoplus_{k\in\mathbb{F}_p} S_{kir} = H(2;\underline{1})^{(2)}$$

as Lie algebras. Hence $\mathrm{rad}(L[\alpha,\beta](\gamma)) = \mathrm{rad}(\overline{H} + \widetilde{S}(\gamma)) \subset \overline{H}$. The result follows. $\square$

We are now in a position to prove our first result on the global structure of L.

Theorem 4.7. *If $\alpha \in \Omega$, then α is Hamiltonian,* $\dim L_\alpha = 5$, *and* $\mathrm{rad}\, L(\alpha) \subset H$.

Proof. For $\gamma \in \Gamma(L,T)$ put $R_\gamma := (\mathrm{rad}\, L(\gamma))_\gamma$. Let $\mu \in \Omega$ be such that $\mathrm{rad}\, L(\mu) \not\subset H$. By Theorem 3.3, the radical of $L(\mu)$ is T-stable. Hence there is $a \in \mathbb{F}_p^\times$ such that $(\mathrm{rad}\, L(\mu))_{a\mu} \neq (0)$. Put $\nu := a\mu$ and note that $\nu \in \Omega$. For $k \in \mathbb{Z}_+$ define

$$I_0 := R_\nu, \qquad I_k := \sum_{\gamma_1,\dots,\gamma_k} \big[L_{\gamma_1}, \big[\cdots [L_{\gamma_k}, R_\nu]\cdots\big]\big], \qquad I := \sum_{k\geqslant 0} I_k.$$

Clearly, I is an ideal of L containing R_ν. We intend to show that $I \subsetneq L$. As a first step we are going to use induction on k to prove the following:

Claim. *If $\nu + \gamma_1 + \cdots + \gamma_k \in \Omega$, then $[L_{\gamma_1}, [\cdots [L_{\gamma_k}, R_\nu]\cdots]] \subset R_{\nu+\gamma_1+\cdots+\gamma_k}$.*

The claim is obviously true for $k = 0$, and it also holds for $k = 1$ thanks to Corollaries 4.3 and 4.6. Suppose it is true for all $k < n$ and let $\gamma_1, \dots, \gamma_n \in \Gamma(L,T)$ be such that $\nu + \gamma_1 + \cdots + \gamma_n \in \Omega$. If $\nu + \gamma_i \in \Omega$ or $\nu + \gamma_i \notin \Gamma(L,T)$ for some $i \leqslant n$, then applying Corollaries 4.3 and 4.6 gives

$$\begin{aligned}\big[L_{\gamma_1}, \big[\cdots [L_{\gamma_n}, R_\nu]\cdots\big]\big] &\subset \big[L_{\gamma_1}, \big[\cdots \big[\widehat{L_{\gamma_i}} \cdots \big[L_{\gamma_n}, [L_{\gamma_i}, R_\nu]\big]\cdots\big]\cdots\big]\big] + I_{n-1} \\ &\subset \big[L_{\gamma_1}, \big[\cdots \big[\widehat{L_{\gamma_i}} \cdots [L_{\gamma_n}, R_{\nu+\gamma_i}]\cdots\big]\cdots\big]\big] + I_{n-1}.\end{aligned}$$

In this case the claim holds by our induction hypothesis. So assume from now that $\nu + \gamma_i \in \Gamma(L,T) \setminus \Omega$ for all $i \leqslant n$. We may also assume that $\tilde{\nu} := \nu + \gamma_1 + \cdots + \gamma_n$ is not solvable, for otherwise we are done. According to Lemma 4.1 there is $\kappa \in \Gamma(L,T)$ such that $L[\tilde{\nu},\kappa] \cong \mathcal{M}(1,1)$. Moreover, it follows from [P 94, Lemmas 4.1 and 4.4] that the radical of every 1-section $L[\tilde{\nu},\kappa](\delta)$ is contained in $\Psi_{\tilde{\nu},\kappa}(T)$ and

$$(\mathbb{F}_p\tilde{\nu} + \mathbb{F}_p\kappa) \setminus \{0\} \subset \Omega. \tag{4.6}$$

Take an arbitrary $\kappa' \in (\mathbb{F}_p\tilde{\nu} + \mathbb{F}_p\kappa) \setminus \mathbb{F}_p\tilde{\nu}$. It follows from (4.6) that $\tilde{\nu} + \mathbb{F}_p\kappa' \subset \Gamma(L,T)$. Note that the rule

$$\gamma \asymp \gamma' \quad \Leftrightarrow \quad (\gamma - \gamma')_{|H^3} = 0$$

defines an equivalence relation on the set of all F-valued functions on H. Since $\gamma_i \asymp -\nu$ for all $i \leqslant n$, we have that $\tilde{\nu} \asymp (1-n)\nu$. If $\nu + \kappa' \asymp 0$, then $\tilde{\nu} + (1-n)\kappa' \notin \Omega$. As $\tilde{\nu} + (1-n)\kappa' \neq 0$ by our choice of κ', this is not true; see (4.6). Thus, $\nu + \kappa' \not\asymp 0$, showing that $\nu + \kappa' \in \Omega$ whenever $\nu + \kappa' \in \Gamma(L,T)$. But then $[R_\nu, L_{\kappa'}] \subset R_{\nu+\kappa'}$ by Corollaries 4.3 and 4.6. As $\nu + \gamma_i \asymp 0$ and $\kappa' \in \Omega$ by (4.6), we also have that $\nu + (\gamma_i + \kappa') \in \Omega$ whenever $\nu + (\gamma_i + \kappa') \in \Gamma(L, T)$ for all $i \leqslant n$. So arguing as above one now obtains that $[[L_{\gamma_i}, L_{\kappa'}], R_\nu] \subset R_{\nu+\gamma_i+\kappa'}$. This implies that

$$\bigl[\bigl[L_{\gamma_1}, \bigl[\cdots [L_{\gamma_n}, R_\nu]\cdots\bigr]\bigr], L_{\kappa'}\bigr] \subset R_{\tilde{\nu}+\kappa'} \subset \operatorname{rad} L(\tilde{\nu}, \kappa').$$

As $\mathcal{M}(1,1)$ is a simple Lie algebra, Schue's lemma [St 04, Proposition 1.3.6(1)] yields

$$\bigl[\Psi_{\tilde{\nu},\kappa'}\bigl(\bigl[L_{\gamma_1}, \bigl[\cdots [L_{\gamma_n}, R_\nu]\cdots\bigr]\bigr]\bigr), \mathcal{M}(1,1)\bigr] = (0),$$

forcing $[L_{\gamma_1}, [\cdots [L_{\gamma_n}, R_\nu]\cdots]] \subset (\operatorname{rad} L(\tilde{\nu}, \kappa'))_{\tilde{\nu}} \subset (\operatorname{rad} L(\tilde{\nu}))_{\tilde{\nu}} = R_{\tilde{\nu}}$. This completes the induction step.

As a consequence, $I_\gamma \subset R_\gamma$ for all $\gamma \in \Omega$. On the other hand, it follows from [P 94, Lemma 3.8] that Ω contains at least one Hamiltonian root, λ say. Then $I_\lambda \neq L_\lambda$, implying $I \neq L$. Then $I = (0)$, proving that $\operatorname{rad} L(\mu) \subset H$ for all $\mu \in \Omega$. As a consequence, all roots in Ω are nonsolvable.

Now let $\alpha \in \Omega$. Because α is nonsolvable, it follows from Theorem 3.3 that α is Hamiltonian. Since $\operatorname{rad} L(\alpha) \subset H$, this gives $\dim L_\alpha = 5$. $\square$

5. Further reductions

In this section we are going to prove that no root in $\Gamma(L, T)$ vanishes on H^3. Theorem 4.7 will play a crucial role in our arguments.

Lemma 5.1. *If $\gamma \in \Gamma(L, T)$ does not vanish on H, then $\gamma \in \Omega$.*

Proof. Suppose there is $\beta \in \Gamma(L, T) \setminus \Omega$ such that $\beta(H) \neq 0$. By (4.1), there is $\alpha \in \Omega$ such that $\beta([L_\alpha, L_{-\alpha}]) \neq 0$. Then $[L_\beta, [L_\alpha, L_{-\alpha}]] = L_\beta$, implying that $\alpha + \beta \in \Gamma(L, T)$ or $-\alpha + \beta \in \Gamma(L, T)$. Since $\beta \notin \Omega$ by our assumption, we have that $\alpha + \beta \in \Omega$ or $-\alpha + \beta \in \Omega$. Theorem 4.7 then shows that $\{\alpha, \alpha + \beta\}$ or $\{\alpha, -\alpha + \beta\}$ consists of nonsolvable roots. Then $L[\alpha, \beta]$ cannot be of type (1) or (2) of Proposition 4.2.

Suppose $L[\alpha, \beta]$ is as in case (3) of Proposition 4.2 and set $\delta_1 := \alpha, \delta_2 := \alpha + \beta$ if $\alpha + \beta \in \Gamma(L, T)$ and $\delta_1 := \alpha, \delta_2 := \alpha - \beta$ if $-\alpha + \beta \in \Gamma(L, T)$. In either case, we can find elements $h_1, h_2 \in H^3$ such that $\delta_i(h_j) = \delta_{ij}$ for $i, j \in \{1, 2\}$. As a consequence, $\alpha(h_2) = 0$ and $\beta(h_2) \neq 0$. But then $\beta \in \Omega$, a contradiction.

Suppose $L[\alpha, \beta]$ is as in case (4) of Proposition 4.2. Then Proposition 4.5 applies. As $\alpha \in \Omega$, Proposition 4.5 says that $\alpha(h_0 \otimes 1) \neq 0$. This forces $\Psi_{\alpha,\beta}(L_{\pm\alpha}) \subset \widetilde{S}$. Since $\beta([L_\alpha, L_{-\alpha}]) \neq 0$, we now deduce that β does not vanish on $\Psi_{\alpha,\beta}(H) \cap \widetilde{S}$. This forces $\beta(h_0 \otimes 1) \neq 0$. Applying Proposition 4.5 once again we obtain $\beta \in \Omega$, a contradiction.

Suppose $L[\alpha, \beta]$ is of type (5) of Proposition 4.2. Then $\widetilde{S} = H(2; (2,1))^{(2)}$ and $L[\alpha, \beta] \subset H(2; (2,1))$. In this case $\Psi_{\alpha,\beta}(H)^3 \subset \widetilde{S}$, and it follows from Lemma 2.5 and Demuškin's description of maximal tori in $H(2; \underline{1})^{(2)}$ that $\Psi_{\alpha,\beta}(H) \cap \widetilde{S}$ is abelian and $\operatorname{nil}(\Psi_{\alpha,\beta}(H) \cap \widetilde{S})$ has codimension 1 in $\Psi_{\alpha,\beta}(H) \cap \widetilde{S}$; see [St 04, Theorem 7.5.8] for instance. As $\alpha \in \Omega$, this means that $\Psi_{\alpha,\beta}(H) \cap \widetilde{S} = \Psi_{\alpha,\beta}(H)^3 + \operatorname{nil}(\Psi_{\alpha,\beta}(H) \cap \widetilde{S})$. As a consequence, $\gamma \in \Gamma(L[\alpha, \beta], \Psi_{\alpha,\beta}(T))$ is in Ω if and only if $\gamma(\Psi_{\alpha,\beta}(H) \cap \widetilde{S}) \neq 0$. As $\alpha \in \Omega$, Theorem 4.7 implies that α does not vanish on $[\Psi_{\alpha,\beta}(L_\alpha), \Psi_{\alpha,\beta}(L_{-\alpha})]$. As $\Psi_{\alpha,\beta}(L_{\pm\alpha}) \subset \widetilde{S}$, this shows that

$$\Psi_{\alpha,\beta}(H) \cap \widetilde{S} = \bigl[\Psi_{\alpha,\beta}(L_\alpha), \Psi_{\alpha,\beta}(L_{-\alpha})\bigr] + \operatorname{nil}\bigl(\Psi_{\alpha,\beta}(H) \cap \widetilde{S}\bigr).$$

But then $\beta(\Psi_{\alpha,\beta}(H) \cap \widetilde{S}) \neq 0$ by our choice of β, implying that $\beta \in \Omega$. Since this contradicts our choice of β, we derive that $L[\alpha, \beta]$ cannot be of type (5).

If $L[\alpha, \beta]$ is as in case (6) of Proposition 4.2, then $(\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\} \subset \Omega$ by [P 94, Lemmas 4.1 and 4.4]. So this case cannot occur either, and our proof is complete. $\square$

Proposition 5.2. *If $\mu \in \Gamma(L, T)$ vanishes on H, then L_μ consists of p-nilpotent elements of L_p.*

Proof. Suppose for a contradiction that there is $\mu \in \Gamma(L,T)$ with $\mu(H)=0$ such that $\alpha(L_\mu^{[p]}) \neq 0$ for some $\alpha \in \Gamma(L,T)$. It follows from (4.1) that every root is the sum of two roots in Ω. Therefore, we may assume that $\alpha \in \Omega$. Since α is nonsolvable by Theorem 4.7, there exists $\beta \in \Omega$ such that $L[\alpha,\beta] \cong \mathcal{M}(1,1)$ and $\alpha([L_{i\alpha}, L_{-i\alpha}], [L_\beta, L_{-\beta}]) \neq 0$ for some $i \in \mathbb{F}_p^\times$; see Lemma 4.1. Lemma 5.1 shows that $\beta \in \Omega$.

We now consider the T-semisimple 3-section $L[\alpha,\beta,\mu]$. Set $\overline{T} := \Psi_{\alpha,\beta,\mu}(T)$, $\overline{H} := \Psi_{\alpha,\beta,\mu}(H)$ and $\widetilde{S} := \widetilde{S}(\alpha,\beta,\mu)$. Given a Lie subalgebra M of $L[\alpha,\beta,\mu]$ we denote by $M_{[p]}$ the p-envelope of M in $\operatorname{Der}\widetilde{S}$. Note that the restricted Lie algebra $\overline{T} + L[\alpha,\beta,\mu]_{[p]} \subset \operatorname{Der}\widetilde{S}$ is centerless. As T is a torus of maximal dimension in $T + L(\alpha,\beta,\mu)_p$, it follows from [St 04, Theorem 1.2.8(4a)] that $\overline{T}$ is a torus of maximal dimension in $\overline{T} + L[\alpha,\beta,\mu]_{[p]}$. Let J be a minimal T-invariant ideal of $L[\alpha,\beta,\mu]$. Then $TR(J) \leqslant TR(L[\alpha,\beta,\mu]) \leqslant 3$; see [St 04, Theorems 1.2.7(1) and 1.3.11(3)].

(a) Suppose $TR(J)=3$. Then it follows from [St 04, Theorem 1.2.9(3)] that the restricted Lie algebra $(\overline{T} + L[\alpha,\beta,\mu]_{[p]})/J_{[p]}$ is p-nilpotent. From this it is immediate that $\overline{T} \subset J_{[p]}$, $J = \widetilde{S}$ and $L[\alpha,\beta,\mu] = \overline{H} + \widetilde{S}$. By Block's theorem, $\widetilde{S} = S \otimes \mathcal{O}(m;\underline{1})$, where S is a simple Lie algebra and $m \in \mathbb{Z}_+$. Let π denote the canonical projection

$$\operatorname{Der}\big(S \otimes \mathcal{O}(m;\underline{1})\big) = (\operatorname{Der} S) \otimes \mathcal{O}(m;\underline{1}) \rtimes \mathrm{Id}_S \otimes W(m;\underline{1}) \twoheadrightarrow W(m;\underline{1}).$$

In the present situation [P-St 99, Theorem 2.6] implies that the torus $\overline{T}$ is conjugate under $\operatorname{Aut}(S \otimes \mathcal{O}(m;\underline{1}))$ to $T_0 \otimes F$ for some torus T_0 in S_p. Hence we can choose $\Psi_{\alpha,\beta,\mu}$ such that $\overline{T} = T_0 \otimes F$. Then $L[\alpha,\beta,\mu](\alpha) = \overline{H} + S(\alpha) \otimes \mathcal{O}(m;\underline{1})$. Since α is nonsolvable, there is a surjective homomorphism $\psi : L[\alpha,\beta,\mu](\alpha) \twoheadrightarrow L[\alpha] \neq (0)$. By Theorem 3.3, $(\operatorname{im}\psi)^{(1)}$ is a simple Lie algebra and the unique minimal ideal of $\operatorname{im}\psi$. Since T_0 is a torus of maximal dimension in S_p, Theorem 3.3 also applies to the 1-section $S[\alpha]$. So it must be that $(\operatorname{im}\psi)^{(1)} \cong S[\alpha]^{(1)}$. As a consequence,

$$\widetilde{S}(\alpha)^{(1)} \cap \ker\psi = \big(\operatorname{rad} S(\alpha) \cap S(\alpha)^{(1)}\big) \otimes F + S(\alpha)^{(1)} \otimes \mathcal{O}(m;\underline{1})_{(1)}$$

is $\overline{H}$-invariant. As $S(\alpha)$ is not solvable, it follows that $\pi(\overline{H}) \subset W(m;\underline{1})_{(0)}$. But then $S \otimes \mathcal{O}(m;\underline{1})_{(1)}$ is an ideal of $L[\alpha,\beta,\mu]$. As $L[\alpha,\beta,\mu]$ is T-semisimple and $T = T_0 \otimes F$, we now obtain that $m=0$ and $L[\alpha,\beta,\mu] = \overline{H} + \widetilde{S}$.

As a consequence, $\Psi_{\alpha,\beta,\mu}(L_\gamma) \subset \widetilde{S}$ for all $\gamma \in \Gamma(L[\alpha,\beta,\mu],\overline{T})$. This implies that $L[\alpha,\beta] \cong \mathcal{M}(1,1)$ is a homomorphic image of the 2-section $\widetilde{S}(\alpha,\beta)$, showing that $\overline{H} \cap \widetilde{S}$ is a nontriangulable subalgebra of $\widetilde{S}$. We now set $\mathfrak{t} := \Psi_{\alpha,\beta,\mu}(T \cap \ker\mu)$ and $\mathfrak{h} := \widetilde{S}(\mu)$. Then $\widetilde{S}$ is simple, $\mathfrak{t}$ is a torus of dimension at most 2 in $\widetilde{S}_p$, and $\overline{H} \cap \widetilde{S} \subset \mathfrak{h}$. This inclusion in conjunction with our assumption on μ and the Engel–Jacobson theorem shows that $\mathfrak{h}$ is a nontriangulable nilpotent subalgebra of $\widetilde{S}$. But then [P 94, Theorem 1(ii)] yields $\widetilde{S} \cong \mathcal{M}(1,1)$. As $TR(\mathcal{M}(1,1))=2$ by [P 94, Lemma 4.3], we reach a contradiction thereby establishing that $TR(J) \leqslant 2$.

(b) We now put $T' := \overline{T} \cap J_{[p]}$ and observe that

$$\dim T' \geqslant TR\big(J_{[p]}, \overline{T} + L[\alpha,\beta,\mu]_{[p]}\big) = TR(J_{[p]}) \neq 0;$$

see [St 04, Theorems 1.2.9 and 1.2.8(2)] (one should also keep in mind that $\overline{T} + L[\alpha,\beta,\mu]_{[p]}$ is centerless).

Suppose $\mu(T') \neq 0$. Then $\Psi_{\alpha,\beta,\mu}(L_{i\mu}) \subset J$ for all $i \in \mathbb{F}_p^\times$ and hence $\Psi_{\alpha,\beta,\mu}(L_\alpha) \subset J$ by our choice of α. Since $L[\alpha,\beta] \cong \mathcal{M}(1,1)$ is simple, it follows that $\Psi_{\alpha,\beta,\mu}(L_{i\alpha+j\beta}) \subset J$ for all nonzero $(i,j) \in \mathbb{F}_p^2$. As a consequence, the p-envelope of $\overline{H} \cap J$ in $J_{[p]}$ contains a torus of dimension at least 2. This torus must be smaller than T', because μ vanishes on H. But then $TR(J) > 2$ which is not true.

Thus, $\mu(T')=0$. Then $\alpha(T') \neq 0$ or $\beta(T') \neq 0$. Relying on the simplicity of $L[\alpha,\beta] \cong \mathcal{M}(1,1)$ and arguing as before, we derive that $J(\alpha,\beta)/\operatorname{rad} J(\alpha,\beta) \cong \mathcal{M}(1,1)$. As $\mu(T')=0$, it follows that $\dim T' = TR(J) = 2$. By Block's theorem, $J = J' \otimes \mathcal{O}(k;\underline{1})$ for some simple Lie algebra J' and some $k \in \mathbb{Z}_+$. The above shows that $TR(J')=2$. The natural homomorphism $J \twoheadrightarrow J/J' \otimes \mathcal{O}(k;\underline{1})_{(1)} \cong J'$ maps $J(\alpha,\beta)$ onto a subalgebra $\mathfrak{g}$ of J' such that $\mathfrak{g}/\operatorname{rad}\mathfrak{g} \cong \mathcal{M}(1,1)$. As $TR(J')=2$, this implies that J_p contains a nonstandard 2-dimensional torus. Applying [P 94, Theorem 1(ii)] now yields $J' \cong \mathcal{M}(1,1)$. Since this holds for every minimal $\overline{T}$-invariant ideal of $L[\alpha,\beta,\mu]$ and $TR(L[\alpha,\beta,\mu]) \leqslant 3$, we may conclude at this point that the T-socle $\widetilde{S} = \widetilde{S}(\alpha,\beta,\mu) = S \otimes \mathcal{O}(m;\underline{1})$ is the unique minimal ideal of $L[\alpha,\beta,\mu]$.

Recall that all derivations of $S = \mathcal{M}(1,1)$ are inner; see [St 04, Theorem 7.1.4] for instance. In this situation [P-St 99, Theorem 3.2] says that $\Psi_{\alpha,\beta,\mu}$ can be chosen such that $\overline{T} = (T_0 \otimes 1) + F(\mathrm{Id}_S \otimes t_0)$, where T_0 is a 2-dimensional torus in $S_p = S$ and $t_0 \in W(m;\underline{1})$. Furthermore, $L[\alpha,\beta,\mu] = \mathcal{M}(1,1) \otimes \mathcal{O}(m;\underline{1}) \rtimes \mathrm{Id}_S \otimes \mathfrak{d}$ for some Lie subalgebra $\mathfrak{d}$ of $W(m;\underline{1})$. Note that $T' = \overline{T} \cap \widetilde{S} = T_0 \otimes 1$. Using the simplicity of $L[\alpha,\beta]$ and arguing as before, we observe that $\Psi_{\alpha,\beta,\mu}(L_{i\alpha+j\beta}) \subset \widetilde{S}$ for all nonzero $(i,j) \in \mathbb{F}_p^\times$. By the choice of β, we then have $\alpha([\widetilde{S}_{i\alpha}, \widetilde{S}_{-i\alpha}], [\widetilde{S}_\beta, \widetilde{S}_{-\beta}]) \neq 0$ for some $i \in \mathbb{F}_p^\times$. This means that T_0 is a nonstandard torus in $S = \mathcal{M}(1,1)$.

If $t_0 \notin W(m;\underline{1})_{(0)}$, then we may assume further that $t_0 = (1+x_1)\partial_1$; see [P-St 99, Theorem 3.2]. Choose $h, h' \in \mathfrak{c}_S(T_0)$ such that $[h,h']$ acts nonnilpotently on S. Recall that $\mu(T_0 \otimes F) = 0$. Then $\mu(\mathrm{Id}_S \otimes t_0) \neq 0$ and hence there exists $r \in \mathbb{F}_p^\times$ such that $h \otimes (1+x_1) \in \widetilde{S}_{r\mu}$ and $h' \otimes (1+x_1)^{p-1} \in \widetilde{S}_{-r\mu}$. Clearly, the element

$$\left[h \otimes (1+x_1), h' \otimes (1+x_1)^{p-1}\right] \in [\widetilde{S}_{r\mu}, \widetilde{S}_{-r\mu}]$$

acts nonnilpotently on $\widetilde{S}$.

Suppose $t_0 \in W(m;\underline{1})_{(0)}$. Since $\widetilde{S}$ is $(\mathrm{Id}_S \otimes (Ft_0 + \mathfrak{d}))$-simple, there is $r \in \mathbb{F}_p$ such that $\mathfrak{d}_{r\mu} \not\subset W(m;\underline{1})_{(0)}$ (here $\mathfrak{d}_0 = \pi(\overline{H})$ is the centraliser of t_0 in $\mathfrak{d}$). On the other hand, looking at the 1-section $L[\alpha,\beta,\mu](\alpha) = \overline{H} + S(\alpha) \otimes \mathcal{O}(m;\underline{1})$ and applying Theorem 3.3 to $L[\alpha] \neq (0)$ one observes that $\pi(\overline{H}) \subset W(m;\underline{1})_{(0)}$ (see part (a) for a similar argument). So it must be that $t_0 \neq 0$ and $r \in \mathbb{F}_p^\times$.

Let $E \in L_{r\mu}$ be such that $\pi(\Psi_{\alpha,\beta,\mu}(E)) \equiv \sum_{j=1}^m a_i\partial_i \pmod{W(m;\underline{1})_{(0)}}$, where not all a_j are zero. We may assume after renumbering and rescaling that $a_1 = 1$. In the present situation [P-St 99, Theorem 3.2] says that $\Psi_{\alpha,\beta,\gamma}$ can be chosen such that $t_0 = \sum_{j=1}^m s_i x_j \partial_j$ for some $s_j \in \mathbb{F}_p$. As $[t_0, \pi(\Psi_{\alpha,\beta,\mu}(E))]$ is a *nonzero* multiple of $\pi(\Psi_{\alpha,\beta,\mu}(E))$, it must be that $s_1 \neq 0$. Therefore, $\mathfrak{c}_S(T_0) \otimes x_1 \subset \widetilde{S}_{-r\mu}$, implying that $[\Psi_{\alpha,\beta,\mu}(L_{r\mu}), \widetilde{S}_{-r\mu}]$ contains nonnilpotent elements of $\widetilde{S}$.

(c) We have thus shown that there is $r \in \mathbb{F}_p^\times$ such that $[L_{r\mu}, L_{-r\mu}]$ contains nonnilpotent elements of L_p. Therefore, the set

$$\Omega_1 := \left\{\gamma \in \Gamma(L,T) \;\middle|\; \gamma\left([L_{r\mu}, L_{-r\mu}]\right) \neq 0\right\}$$

is nonempty. By Lemma 5.1, we have the inclusion $\Omega_1 \subset \Omega$. Also, $\mu \notin \Omega_1$, because $\mu(H) = 0$. Since $\mu \neq 0$, there is $\gamma \in \Gamma(L,T)$ such that $\mu(L_\gamma^{[p]}) \neq 0$.

Suppose $\gamma \in \Omega$. Since $\mu(L_\gamma^{[p]}) \neq 0$, all elements from $\mu + \mathbb{F}_p\gamma$ are in $\Gamma(L,T)$. Since $\mu(H) = 0$, we then have $\mu + \mathbb{F}_p^\times\gamma \subset \Omega$. Since all roots in Ω are nonsolvable by Theorem 4.7, the T-semisimple 2-section $L[\gamma,\mu]$ cannot be as in cases (1), (2) or (3) of Proposition 4.2. If $L[\gamma,\mu]$ is of type (4), then Proposition 4.5 implies that $\Psi_{\gamma,\mu}(L_\gamma) \subset \widetilde{S}$. As $\mu(L_\gamma^{[p]}) \neq 0$, this forces $\Psi_{\gamma,\mu}(L_{i\mu}) \subset \widetilde{S}$ for all $i \in \mathbb{F}_p^\times$. Since μ vanishes on H, it follows from the description of $\Psi_{\gamma,\mu}(T)$ given in Proposition 4.5 that

$$\sum_{i \in \mathbb{F}_p^\times} \Psi_{\gamma,\mu}(L_{i\mu}) \subset \mathfrak{c}_{H(2;\underline{1})^{(2)}}(h_0) \otimes \mathcal{O}(1;\underline{1}).$$

As the subalgebra on the right is abelian and $\Psi_{\gamma,\mu}(L_{i\gamma}) \neq (0)$ for all $i \in \mathbb{F}_p^\times$, this contradicts our choice of μ. So $L[\gamma,\mu]$ is not of that type. If $L[\gamma,\mu]$ is as in case (5) or case (6) of Proposition 4.2, then Corollary 4.3 shows that no root in $\Gamma(L[\gamma,\mu], \Psi_{\gamma,\mu}(T)) = (\mathbb{F}_p\gamma \oplus \mathbb{F}_p\mu) \setminus \{0\}$ vanishes on $\Psi_{\gamma,\mu}(H)$. As $\mu(H) = 0$, this is false.

Thus, $\gamma \notin \Omega$. Schue's lemma [St 04, Proposition 1.3.6(1)] yields $L_\gamma = \sum_{\delta\in\Omega_1}[L_\delta, L_{\gamma-\delta}]$. If $x_1 \ldots, x_d \in L_\gamma$, then

$$\left(\sum_{j=1}^{d} x_j\right)^{[p]} \equiv \sum_{j=1}^{d} x_j^{[p]} \pmod{H},$$

by Jacobson's formula. Note that the set $H \cup (\bigcup_{\delta\in\Omega_1, k\geqslant 1}[L_\delta, L_{-\delta}]^{[p]^k})$ is weakly closed. Since μ vanishes on H, the Engel–Jacobson theorem implies that there is $\kappa \in \Omega_1$ such that $\mu([L_\kappa, L_{\gamma-\kappa}]^{[p]}) \neq 0$. Note that κ and $\gamma-\kappa$ are both in Ω, hence $\Psi_{\gamma,\kappa,\mu}(L_\kappa) \neq (0)$ and $\Psi_{\gamma,\kappa,\mu}(L_{\gamma-\kappa}) \neq (0)$ by Theorem 4.7. Let $\widetilde{S} = \widetilde{S}(\gamma, \kappa, \mu)$ and let J be any minimal ideal of $L[\gamma, \kappa, \mu]$. Put $T_1 := \Psi_{\gamma,\kappa,\mu}(T) \cap J_{[p]}$, where $J_{[p]}$ is the p-envelope of J in $\operatorname{Der}\widetilde{S}$. Since $J_{[p]}$ is centerless, it follows from [St 04, Theorem 1.2.8(a)] that T_1 is a torus of maximal dimension in $J_{[p]}$.

Suppose $\mu(T_1) = 0$. Then either $\kappa(T_1) \neq 0$ or $(\gamma - \kappa)(T_1) \neq 0$, for $T_1 \neq (0)$. In any event, $\Psi_{\gamma,\kappa,\mu}([L_\kappa, L_{\gamma-\kappa}]) \subset J$ and therefore $\mu(J_\gamma^{[p]}) \neq 0$. But then $\mu(T_1) \neq 0$, a contradiction. Thus, $\mu(T_1) \neq 0$, forcing $\sum_{i\in\mathbb{F}_p^\times} \Psi_{\gamma,\kappa,\mu}(L_{i\mu}) \subset J$. As $\kappa \in \Omega_1$, this yields $\sum_{i\in\mathbb{F}_p^\times} \Psi_{\gamma,\kappa,\mu}(L_{i\kappa}) \subset J$. As a result, the nilpotent subalgebra $J(\mu)$ acts nontriangulably on J. As $\kappa([L_{r\mu}, L_{-r\mu}]) \neq 0$ and $\Psi_{\gamma,\kappa,\mu}([L_\kappa, L_{\gamma-\kappa}]^{[p]}) \subset J_{[p]}$, we have that $TR(J) = \dim T_1 \geqslant 2$ (one should keep in mind that μ vanishes on H but not on $[L_\kappa, L_{\gamma-\kappa}]^{[p]}$).

Since $\kappa \in \Omega$, we can now argue as in part (a) of this proof to deduce that $TR(J) \leqslant 2$. As a result, $TR(J) = 2$ for any minimal ideal J of $L[\gamma, \kappa, \mu]$. As $TR(L[\gamma, \kappa, \mu]) \leqslant 3$, this shows that $\widetilde{S} = S \otimes \mathcal{O}(m; \underline{1})$ is the unique minimal ideal of $L[\gamma, \kappa, \mu]$ and $TR(\widetilde{S}) = TR(S) = 2$. According to [P-St 99, Theorem 2.6], we can choose $\Psi_{\gamma,\kappa,\mu}$ such that

$$\Psi_{\gamma,\kappa,\mu}(T) = \left(T_0' \otimes 1\right) + F(d \otimes 1 + \mathrm{Id}_S \otimes t_0), \quad T_0' \subset S_p,\ d \in \operatorname{Der} S,\ t_0 \in W(m; \underline{1}).$$

Moreover, if d is an inner derivation of S, then we can assume further that $d = 0$. Since $T_1 = T_0' \otimes 1$, we get $\dim T_0' = 2$. Set $\mathfrak{t} := T_0' + Fd$, a torus in $\operatorname{Der} S$. The subalgebra $S \otimes F$ of $\widetilde{S}$ is invariant under the action of $\Psi_{\gamma,\kappa,\mu}(T)$. Given $\delta \in \Gamma((S \otimes F), \Psi_{\gamma,,\kappa,\mu}(T))$ we denote by $\bar{\delta}$ the unique $\mathfrak{t}$-root in $\Gamma(S, \mathfrak{t})$ for which $S_{\bar{\delta}} \otimes F = (S \otimes F)_\delta$.

(d) Suppose $t_0 \in W(m; \underline{1})_{(0)}$. Because $\widetilde{S}$ and $S \otimes \mathcal{O}(m; \underline{1})_{(1)}$ are both T-invariant, T acts on $S \cong \widetilde{S}/(S \otimes \mathcal{O}(m; \underline{1})_{(1)})$ as the torus $\mathfrak{t} \subset \operatorname{Der} S$. Since $\widetilde{S}_\kappa \neq (0)$ and $\kappa \in \Omega_1$, we also have that $\Psi_{\gamma,\kappa,\mu}(L_{\pm r\mu}) \neq (0)$. We mentioned above that $\Psi_{\gamma,\kappa,\mu}(L_{\pm r\mu}) \subset \widetilde{S}$. Define $\mathfrak{t}_0 := \mathfrak{t} \cap \ker \bar{\mu}$. Then $\dim \mathfrak{t}_0 \leqslant 2$ and $\mathfrak{c}_S(\mathfrak{t}_0) = S(\bar{\mu})$. Because $S_p \otimes \mathcal{O}(m; \underline{1})_{(1)}$ is p-nilpotent and $\widetilde{S}(\mu)$ acts nontriangulably on $\widetilde{S}$ by our discussion in part (c), the subalgebra $S(\bar{\mu})$ is nilpotent and acts nontriangulably on S. Applying [P 94, Theorem 2(ii)] now yields $S \cong \mathcal{M}(1, 1)$. But then all derivations of S are inner; see [St 04, Theorem 7.1.4] for example. Then $d = 0$ and $\mathfrak{t}$ is a torus of maximal dimension in S_p. It follows that $S(\bar{\mu}) = \mathfrak{c}_S(\mathfrak{t}_0)$ is a Cartan subalgebra of toral rank 1 in S. Since such Cartan subalgebras are triangulable by [P 94, Theorem 2], our assumption on t_0 is false.

Thus, $t_0 \notin W(m; \underline{1})_{(0)}$. Recall that μ and κ are both nonzero on $T_1 = T_0' \otimes 1$. Since μ vanishes on H and the nonsolvable root κ does not vanish on $\Psi_{\gamma,\kappa,\mu}([L_{i\kappa}, L_{-i\kappa}]) \subset \Psi_{\gamma,\kappa,\mu}(H) \cap \widetilde{S}$ for some $i \in \mathbb{F}_p^\times$, the roots μ and κ are linearly independent on T_1. Hence

$$\Psi_{\gamma,\kappa,\mu}(T) = T_1 \oplus \left(\Psi_{\gamma,\kappa,\mu}(T) \cap \ker \mu \cap \ker \kappa\right),$$

implying that $\pi(\Psi_{\gamma,\kappa,\mu}(T) \cap \ker \mu \cap \ker \kappa) \not\subset W(m; \underline{1})_{(0)}$. In that case [P-St 99, Theorem 2.6] says that $\Psi_{\gamma,\kappa,\mu}$ can be selected such that $d = 0$, $t_0 = (1 + x_1)\partial_1$, and $\Psi_{\gamma,\kappa,\mu}(T) \cap \ker \mu \cap \ker \kappa = F(\mathrm{Id}_S \otimes t_0)$.

Then $\widetilde{S}(\kappa, \mu) = S \otimes F[x_2, \ldots, x_m]$ and the evaluation map $\mathrm{ev}: \widetilde{S}(\kappa, \mu) \twoheadrightarrow S$, taking $s \otimes f \in S \otimes F[x_2, \ldots, x_m]$ to $f(0)s \in S$, is T-equivariant. As before, $S(\bar{\mu})$ acts nontriangulably on S. Since in the present case $\mathfrak{t}$ is a torus of maximal dimension in S_p, its 1-section $S(\bar{\mu})$ has toral rank 1 in S. Since such a Cartan subalgebra must act triangulably on S by [P 94, Theorem 2], we reach a contradiction, thereby proving the proposition. □

Corollary 5.3. *The following are true:*

(i) $\Gamma(L,T)=\Omega$.
(ii) *If* $\alpha,\beta\in\Gamma(L,T)$, *then* $L[\alpha,\beta]$ *is not as in case* (4) *of Proposition* 4.2.

Proof. (1) Suppose $\Gamma(L,T)\neq\Omega$ and let $\lambda\in\Gamma(L,T)\setminus\Omega$. Take any $\alpha\in\Omega$ and consider the T-semisimple 2-section $L[\alpha,\lambda]$. By Theorem 4.7, $L(\alpha)$ is not solvable, hence $L[\alpha,\lambda]$ is not as in case (1) of Proposition 4.2. Because of Lemma 5.1 we have $\lambda(H)=0$, hence $L(\lambda)$ is solvable. If $L[\alpha,\lambda]$ is as in cases (2), (3), (5) or (6) of Proposition 4.2, then $L_\lambda\subset\operatorname{rad}_T L(\alpha,\lambda)$ by Corollary 4.3, hence

$$[L_\alpha,L_\lambda]\subset\big(\operatorname{rad}_T L(\alpha,\lambda)\big)\cap L_{\alpha+\lambda}\subset\big(\operatorname{rad}_T L(\alpha+\lambda)\big)_{\alpha+\lambda}=(0)$$

by Theorem 4.7 (because $\alpha+\lambda\in\Omega$). If $L[\alpha,\lambda]$ is as in case (4) of Proposition 4.2, then it follows from Proposition 4.5 that L_λ contains nonnilpotent elements of L_p. Since this contradicts Proposition 5.2, we see that $L[\alpha,\lambda]$ is not of that type. As a consequence, $[L_\alpha,L_\lambda]=0$ for all $\alpha\in\Omega$. But then (4.1) yields that L_λ is contained in the center of L. This contradiction proves the first statement.

(2) If $L[\alpha,\beta]$ is as in case (4) of Proposition 4.2, then Proposition 4.5 implies that one of the roots in $\Gamma(L,T)\cap(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)$ is not contained in Ω. Since this is impossible by part (1), our proof is complete. $\square$

Corollary 5.4. *For every* $\alpha\in\Gamma(L,T)$ *the radical of* $L(\alpha)$ *lies in the center of* H.

Proof. Recall that $\operatorname{rad}L(\alpha)\subset H$ by Theorem 4.7 and Corollary 5.3. Set

$$\Omega_2:=\big\{\gamma\in\Gamma(L,T)\;\big|\;\gamma\big(\big[H,\operatorname{rad}L(\alpha)\big]\big)\neq 0\big\}.$$

Suppose $\Omega_2\neq\emptyset$ and let $\beta\in\Omega_2$. Since $\alpha,\beta\in\Omega$ by Corollary 5.3, Proposition 4.2 applies to $L[\alpha,\beta]$. Since α vanishes on $[H,\operatorname{rad}L(\alpha)]$, the roots α and β are $\mathbb{F}_p$-independent. As α and β are both nonsolvable by Theorem 4.7, $L[\alpha,\beta]$ cannot be as in case (1) or case (2) of Proposition 4.2. It cannot be governed by case (5) or case (6) either, because in case (5) the radical of $L[\alpha,\beta](\alpha)$ is trivial by Proposition 2.5(2) and in case (6) the radical of $L[\alpha,\beta](\alpha)$ is contained in $\Psi_{\alpha,\beta}(T)$; see [P 94, Lemmas 4.1 and 4.4].

Thus, $L[\alpha,\beta]$ is as in case (3) of Proposition 4.2. But then $L[\alpha,\beta]=L[\alpha,\beta](\alpha)+L[\alpha,\beta](\beta)$ and $[L_\alpha,L_\beta]\subset\operatorname{rad}_T L(\alpha,\beta)$. Since $(\alpha+\beta)([H,\operatorname{rad}L(\alpha)])\neq 0$, and $L(\alpha+\beta)$ is solvable, it must be that $\alpha+\beta\notin\Gamma(L,T)$. We now derive that $[L_\alpha,L_\beta]=(0)$ for all $\beta\in\Omega_2$. In view of Schue's lemma [St 04, Proposition 1.3.6(1)], this means that L_α lies in the center of L.

This contradiction shows that $\Omega_2=\emptyset$. Hence the ideal $H_\alpha:=[H,\operatorname{rad}L(\alpha)]$ of H consists of p-nilpotent elements of L_p. Now let β be any root in $\Gamma(L,T)$. Since $H_\alpha\subset H^{(1)}$, it follows from Theorem 3.3 and (the proof of) Lemma 3.4 that $\Psi_\beta(H_\alpha)=(0)$. Then $[H_\alpha,L(\beta)]\subset\operatorname{rad}L(\beta)$, forcing $[H_\alpha,L_\beta]=(0)$; see Theorem 4.7. As a result, $[H_\alpha,L]=(0)$, and hence $H_\alpha=(0)$ by the simplicity of L. This proves the corollary. $\square$

We are finally in a position to describe the 2-sections of L with respect to T. Let $\mathfrak{z}(H)$ denote the center of $H=\mathfrak{c}_L(T)$.

Theorem 5.5. *The following are true*:

(i) $H^4=(0)$ *and* $H^{[p]}\subset T$.
(ii) $\dim H^2=3$ *and* $\dim H^3=2$.
(iii) $H^3\subset T$ *and* $\dim H/\mathfrak{z}(H)=3$.
(iv) $\mathfrak{z}(H)=H\cap T$.

Proof. (a) Let $\alpha \in \Gamma(L,T)$. Then $\alpha \in \Omega$ by Corollary 5.3(i). It is immediate from Theorem 3.3 that $H^4 \subset \operatorname{rad} L(\alpha)$. Then $[H^4, L_\alpha] \subset (\operatorname{rad} L(\alpha))_\alpha = (0)$ by Theorem 4.7. Since this holds for every root α and L is simple, we derive $H^4 = (0)$.

Let $\mathcal{N}(H_p)$ denote the set of all p-nilpotent elements of H_p. Since $\dim L_\gamma = 5$ for all $\gamma \in \Gamma(L,T)$ any p-nilpotent element $x \in \mathcal{N}(H_p)$ has the property that $(\operatorname{ad} x)^5(\sum_{\gamma\in\Gamma(L,T)} L_\gamma) = 0$. Then $x^{[p]} = 0$ by the simplicity of L. The Jordan–Chevalley decomposition in H_p now yields $(H_p)^{[p]} \subset T$, forcing $H^{[p]} \subset T$. As a result, statement (i) follows, and we also deduce that $\mathcal{N}(H_p) = \{x \in H_p \mid x^{[p]} = 0\}$ and $H_p \subset H + T$.

Since $H^4 = (0)$ and $[T,H] = 0$, Jacobson's formula implies that $(x+y)^{[5]} = x^{[5]} + y^{[5]}$ for all $x, y \in H_p$. Therefore, $\mathcal{N}(H_p)$ is a subspace of H. By the Jordan–Chevalley decomposition in H_p, we also get $H_p \subset \mathcal{N}(H_p) \oplus T$.

(b) Since $\Gamma(L,T) = \Omega$, it follows from Theorem 3.3 and (the proof of) Lemma 3.4 that $H^2 + \operatorname{rad} L(\alpha)$ has codimension 2 in H for every $\alpha \in \Gamma(L,T)$. Since $\operatorname{rad} L(\alpha) \subset \mathfrak{z}(H)$ by Corollary 5.4, there exist $x, y \in H$ such that $H = Fx + Fy + H^2 + \mathfrak{z}(H)$. As a consequence, $H^2 = F[x,y] + H^3$ and $H^3 = F[x,[x,y]] + F[y,[y,x]] + H^4$. As $H^4 = (0)$, this gives $\dim H^3 \leqslant 2$ and $\dim H^2 = 1 + \dim H^3$.

Let $\alpha, \beta \in \Gamma(L,T)$ be such that $L[\alpha,\beta] \cong \mathcal{M}(1,1)$ (such a pair of roots exists by [P 94, Theorem 1(ii)]). It is immediate from [P 94, Lemmas 4.1 and 4.4] that $\dim \Psi_{\alpha,\beta}(H^3) = 2$. Hence $\dim H^3 \geqslant 2$. In conjunction with the above remarks, this gives $\dim H^3 = 2$ and $\dim H^2 = 3$. Statement (ii) follows.

(c) Since $H^4 = (0)$, we have that $H^3 \subset \mathfrak{z}(H)$. If the nilpotent Lie algebra $H/\mathfrak{z}(H)$ has codimension < 3 in H, then it is abelian. In this case $H^2 \subset \mathfrak{z}(H)$, forcing $H^3 = (0)$. This contradiction shows that $\mathfrak{z}(H)$ has codimension $\geqslant 3$ in H. Since $H^3 \neq (0)$ has codimension 1 in H^2, the equality $H^2 \cap \mathfrak{z}(H) = H^3$ holds. Therefore,

$$
\begin{aligned}
3 \leqslant \dim H/\mathfrak{z}(H) &= \dim H/\big(H^2 + \mathfrak{z}(H)\big) + \dim H^2/H^3 \\
&\leqslant \dim H/\big(H^2 + \operatorname{rad} L(\alpha)\big) + \dim H^2/H^3 = 3.
\end{aligned}
$$

This implies that $\mathfrak{z}(H)$ has codimension 3 in H.

Let $h \in \mathfrak{z}(H)$ and write $h = h_s + h_n$ with $h_s \in T$ and $h_n \in \mathcal{N}(H_p)$. In view of our earlier remarks, $h_n \in \mathfrak{z}(H) \cap (T + H)$. Because $\Gamma(L,T) = \Omega$, Theorem 3.3 shows that for every $\gamma \in \Gamma(L,T)$ the element $\Psi_\gamma(h_n) \in \Psi_\gamma(T) + \Psi_\gamma(H) = \Psi_\gamma(H)$ of $L[\gamma] \cong H(2;\underline{1})^{(2)} \oplus F(1+x_1)^4\partial_2$ is p-nilpotent in $L[\gamma]$ and commutes with $\Psi_\alpha(H)$. Arguing as in the proof of Lemma 3.4 it is now straightforward to see that $\Psi_\gamma(h_n) = 0$. Then $[h_n, L(\gamma)] \subset \operatorname{rad} L(\gamma)$. In view of Theorem 4.7, this entails that $[h_n, L_\gamma] = 0$ for all $\gamma \in \Gamma(L,T)$. As a consequence, $h_n = 0$, forcing $\mathfrak{z}(H) = H \cap T$. Combined with our remarks in part (b) this gives (iii), completing the proof. $\square$

Corollary 5.6. *Let $\alpha, \beta \in \Gamma(L,T)$. Then case* (3) *of Proposition* 4.2 *does not occur for $L[\alpha,\beta]$.*

Proof. Indeed, otherwise the T-socle of $L[\alpha,\beta]$ has the form $S_1 \oplus S_2 = S_1(\delta_1) \oplus S_2(\delta_2)$. Then $\Psi_{\alpha,\beta}(H) \cap S_i(\delta_i) \cong \Psi_{\delta_i}(H)$ for $i = 1,2$. As $\delta_1, \delta_2 \in \Omega$ by Corollary 5.3(i), it follows from Theorem 3.3 that $S_i(\delta_i) \cong H(2;\underline{1})^{(2)} \oplus F(1+x_2)^4\partial_2$ and $\Psi_{\delta_i}(H)$ is a nonabelian Cartan subalgebra of $S_i(\delta_i)$. Then Lemma 3.4 implies that $\dim \Psi_{\delta_i}(H^2) = 2$. As a consequence, $\Psi_{\alpha,\beta}(H^2) \cap S_i(\delta_i)$ is 2-dimensional for $i = 1,2$. But then $\dim H^2 \geqslant 4$ contrary to Theorem 5.5(ii). The result follows. $\square$

Corollary 5.7. *The following are true*:

(1) *$\Gamma(L,T) \cup \{0\}$ is an $\mathbb{F}_p$-subspace of T^*.*
(2) *The p-envelope of H^3 in L_p coincides with T.*
(3) *$H_p = H + T$.*

Proof. (1) Since every $\gamma \in \Gamma(L,T)$ is Hamiltonian by Theorem 4.7, we have $\mathbb{F}_p^\times \gamma \subset \Gamma(L,T)$. Let $\alpha, \beta \in \Gamma(L,T)$ be $\mathbb{F}_p$-independent. Then $\Gamma(L[\alpha,\beta], \Psi_{\alpha,\beta}(T))$ contains two nonsolvable roots. In view of Corollary 5.6, this implies that $L[\alpha,\beta]$ is determined by case (5) or case (6) of Proposition 4.2. In both cases, $\Gamma(L[\alpha,\beta],\Psi_{\alpha,\beta}(T))\cup\{0\}=\mathbb{F}_p\alpha+\mathbb{F}_p\beta$; see Lemma 2.5(4) and [P 94, Lemmas 4.1 and 4.4]. As a consequence, $\alpha+\beta\in\Gamma(L,T)$. Statement (1) follows.

(2) By Theorem 5.5(3), $H^3\subset T$. Denote by T_0 the p-envelope of H^3 in T and suppose that $T_0\neq T$. Then T_0 is a proper subtorus of T. By part (1), there exists $\gamma\in\Gamma(L,T)$ such that $\gamma(T_0)=0$. Then $\gamma(H^3)=0$ contrary to Corollary 5.3(i). Therefore, $(H^3)_p=T$.

(3) It is immediate from Theorem 5.5(i) that $H_p\subset H+T$. Since $T=(H^3)_p\subset H_p$ by part (2), we now derive that $H_p=H+T$. □

We now summarize the results of this section:

Theorem 5.8. *Let L, T and H be as above. Then the following hold*:

(1) $\Gamma(L,T)\cup\{0\}$ *is an $\mathbb{F}_p$-subspace of T^* and no root in $\Gamma(L,T)$ vanishes on H^3.*
(2) $H^3\subset T$, $\mathfrak{z}(H)=H\cap T$, $H_p=H+T$, $\dim H/(H\cap T)=3$, $\dim H^2=3$, *and* $\dim H^3=2$. *The p-envelope of H^3 in L_p coincides with T.*
(3) $\operatorname{rad}L(\alpha)=H\cap T\cap\ker\alpha$, $\dim L_\alpha=5$, *and* $L[\alpha]\cong H(2;\underline{1})^{(2)}\oplus F(1+x_1)^4\partial_2$ *for every* $\alpha\in\Gamma(L,T)$.
(4) *If $\alpha,\beta\in\Gamma(L,T)$ are $\mathbb{F}_p$-independent, then either $L[\alpha,\beta]\cong\mathcal{M}(1,1)$ or*

$$H\bigl(2;(2,1)\bigr)^{(2)}\subset L[\alpha,\beta]\subset H\bigl(2;(2,1)\bigr).$$

Furthermore, $L[\alpha,\beta]\cong L(\alpha,\beta)/H\cap T\cap\ker\alpha\cap\ker\beta$.

Proof. Parts (1) and (2) are just reformulations of our earlier results. In order to get (3) and (4) it suffices to observe that $\operatorname{rad}L(\alpha)\subset\mathfrak{z}(H)=H\cap T$; see Corollary 5.4 and Theorem 5.5(iv). □

6. Some properties of the restricted Melikian algebra

In order to proceed further with our investigation, we now need more information on central extensions and irreducible representations of the Melikian algebra $\mathcal{M}(1,1)$.

Proposition 6.1. *Every Melikian algebra $\mathcal{M}(\underline{n})$, where $\underline{n}=(n_1,n_2)$, possesses a nondegenerate invariant symmetric bilinear form.*

Proof. Adopt the notation of [St 04, Section 4.3] and consider the natural grading

$$\mathcal{M}(\underline{n})=\mathcal{M}_{-3}\oplus\mathcal{M}_{-2}\oplus\mathcal{M}_{-1}\oplus\mathcal{M}_0\oplus\mathcal{M}_1\oplus\cdots\oplus\mathcal{M}_s,\quad s=3\bigl(5^{n_1}+5^{n_2}\bigr)-7$$

of the Melikian algebra $\mathcal{M}=\mathcal{M}(\underline{n})$. Recall that $\mathcal{M}_0=\bigoplus_{i,j=1}^2 x_i\partial_j\cong\mathfrak{gl}(2)$, $\mathcal{M}_{-3}=F\partial_1\oplus F\partial_2$ and $\mathcal{M}_s=Fx^{(\tau(\underline{n}))}\tilde{\partial}_1\oplus Fx^{(\tau(\underline{n}))}\tilde{\partial}_2$, where $\tau(\underline{n})=(5^{n_1}-1,5^{n_2}-1)$. Both $\mathcal{M}_{-3}$ and $\mathcal{M}_s$ are 2-dimensional irreducible $\mathcal{M}_0$-modules. Using the multiplication table [St 04, (4.3.1)], it is easy to observe that

$$\bigl[x_1\partial_1,x^{(\tau(\underline{n}))}\tilde{\partial}_1\bigr]=(-2+2)x^{(\tau(\underline{n}))}\tilde{\partial}_1=0,\qquad \bigl[x_2\partial_1,x^{(\tau(\underline{n}))}\tilde{\partial}_1\bigr]=0,$$
$$\bigl[x_2\partial_2,x^{(\tau(\underline{n}))}\tilde{\partial}_1\bigr]=(-1+2)x^{(\tau(\underline{n}))}\tilde{\partial}_1.$$

This shows that $x^{(\tau(\underline{n}))}\tilde{\partial}_1$ is a primitive vector of weight $(0,1)$ for the Borel subalgebra $\mathfrak{b}:=Fx_1\partial_1\oplus Fx_2\partial_2\oplus Fx_2\partial_1$ of $\mathcal{M}_0$. Now let f be the linear function on $\mathcal{M}_{-3}$ such that $f(\partial_1)=0$ and $f(\partial_2)=1$. Then $(x_1\partial_1)(f)=-f\circ(x_1\partial_1)=0$, $(x_2\partial_2)(f)=-f\circ(x_2\partial_2)=f$ and $(x_2\partial_1)(f)=-f\circ(x_2\partial_1)=0$, showing that $f\in(\mathcal{M}_{-3})^*$ is a primitive vector of weight $(0,1)$ for the Borel subalgebra $\mathfrak{b}$. From this it is immediate that $(\mathcal{M}_{-3})^*\cong\mathcal{M}_s$ as $\mathcal{M}_0$-modules. As $\mathcal{M}$ is an irreducible graded $\mathcal{M}_p$-module, [P 85, Lemma 4] shows that there exists a module isomorphism $\theta:\mathcal{M}\xrightarrow{\sim}\mathcal{M}^*$ sending $\mathcal{M}_i$ onto $(\mathcal{M}_{s-3-i})^*$

for all $i \in \{-3, \dots, s\}$ (as usual, we identify $(\mathcal{M}_i)^*$ with the subspace of $\mathcal{M}^*$ consisting of all linear functions vanishing on all $\mathcal{M}_k$ with $k \neq i$).

Define a bilinear form $b : \mathcal{M} \times \mathcal{M} \to F$ by setting $b(x, y) := (\theta(x))(y)$ for all $x, y \in \mathcal{M}$. Since θ is an isomorphism of $\mathcal{M}$-modules, the form b is nondegenerate and $\mathcal{M}$-invariant. Next we define a bilinear skew-symmetric form b' on $\mathcal{M}$ by setting $b'(x, y) := b(x, y) - b(y, x)$ for all $x, y \in \mathcal{M}$. As $\mathcal{M}$ is a simple Lie algebra, the invariant form b' is either nondegenerate or zero. As $\dim \mathcal{M} = 5^{n_1+n_2+1}$ is odd, it must be that $b' = 0$. Therefore, the form b is symmetric. $\square$

From now on we denote by $\mathcal{M}$ the restricted Melikian algebra $\mathcal{M}(1, 1)$.

Proposition 6.2. *If $\widetilde{\mathcal{M}}$ is a Lie algebra with center $\mathfrak{z} = \mathfrak{z}(\widetilde{\mathcal{M}})$ such that $\widetilde{\mathcal{M}}/\mathfrak{z} \cong \mathcal{M}$, then $\widetilde{\mathcal{M}}^{(1)} \cong \mathcal{M}$ and $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}^{(1)} \oplus \mathfrak{z}$.*

Proof. We need to show that the second cohomology group $\mathrm{H}^2(\mathcal{M}, F)$ vanishes. Let b be the nondegenerate bilinear form from the proof of Proposition 6.1. By a standard argument explained in detail in [P 94, p. 681], for every 2-cocycle $\varphi : \mathcal{M} \times \mathcal{M} \to F$ there exists a derivation $d \in \operatorname{Der} \mathcal{M}$ such that $b(d(x), y) = -b(x, d(y))$ and $\varphi(x, y) = b(d(x), y)$ for all $x, y \in \mathcal{M}$. Moreover, φ is a 2-coboundary if and only if the derivation d is inner. Since $\operatorname{Der} \mathcal{M} = \operatorname{ad} \mathcal{M}$ by [St 04, Theorem 7.1.4], for instance, we now obtain $\mathrm{H}^2(\mathcal{M}, F) = 0$, as desired. $\square$

If V is an irreducible module over a finite-dimensional restricted Lie algebra $\mathcal{L}$ over F, then there exists a linear function $\chi = \chi_V \in \mathcal{L}^*$ such that for every $x \in \mathcal{L}$ the central element $x^p - x^{[p]}$ of $U(\mathcal{L})$ acts on V as the scalar operator $\chi(x)^p \operatorname{Id}_V$. The linear function χ is called the *p-character* of V. Given $f \in \mathcal{L}^*$ we denote by $\mathfrak{z}_{\mathcal{L}}(f)$ the stabilizer of f in $\mathcal{L}$. Recall that $\mathfrak{z}_{\mathcal{L}}(f) = \{x \in \mathcal{L} \mid f([x, \mathcal{L}]) = 0\}$ is a restricted subalgebra of even codimension in $\mathcal{L}$.

For our constructions in the final sections of this work we need some information on the p-characters of irreducible representations of dimension $\leqslant 125$ of the restricted Melikian algebra $\mathcal{M} = \bigoplus_{i=-3}^{s} \mathcal{M}_i$.

Proposition 6.3. *If V is an irreducible $\mathcal{M}$-module of dimension $\leqslant 125$, then the p-character of V vanishes on the subspace $\bigoplus_{i \geqslant -2} \mathcal{M}_i$. If V has a nonzero p-character, then* $\dim V = 125$.

Proof. Write $\mathcal{M}^* = \bigoplus_{i=-3}^{s} (\mathcal{M}_i)^*$, where $(\mathcal{M}_i)^* = \{f \in \mathcal{M}^* \mid \bigoplus_{j \neq i} \mathcal{M}_j \subset \ker f\}$ and $s = 3(5+5) - 7 = 23$. Let χ be the p-character of the $\mathcal{M}$-module V. If $\chi = 0$, then there is nothing to prove; so suppose $\chi \neq 0$. Then $\chi = \sum_{i=-3}^{d} \chi_i$, where $\chi_i \in (\mathcal{M}_i)^*$ and $\chi_d \neq 0$.

(a) We first suppose that $d > 0$ and let $2q = \operatorname{codim}_{\mathcal{M}} \mathfrak{z}_{\mathcal{M}}(\chi_d)$. Then [P-Sk 99, Proposition 5.5] yields that $5^q \mid \dim V$. Since $\dim V \leqslant 5^3$, it follows that $\mathfrak{z}_{\mathcal{M}}(\chi_d)$ has codimension $\leqslant 6$ in $\mathcal{M}$. Let b be the $\mathcal{M}$-invariant nondegenerate bilinear form from the proof of Proposition 6.1. Then $\chi_d = b(z, \cdot) = \theta(z)$ for some nonzero $z \in \mathcal{M}_{s-3-d}$ and $\mathfrak{z}_{\mathcal{M}}(\chi_d) = \mathfrak{c}_{\mathcal{M}}(z)$. It follows that the set

$$\mathcal{X} := \left\{x \in \mathcal{M}_{s-3-d} \;\middle|\; \operatorname{codim}_{\mathcal{M}} \mathfrak{c}_{\mathcal{M}}(x) \leqslant 6\right\}$$

is nonzero. It is straightforward to see that $\mathcal{X}$ is a Zariski closed, conical subset of $\mathcal{M}_{s-3-d}$ invariant under the subgroup $\operatorname{Aut}_0 \mathcal{M}$ of all automorphisms of $\mathcal{M}$ preserving the natural grading of $\mathcal{M}$. Let $\mathbb{P}(\mathcal{X})$ be the closed subset of the projective space $\mathbb{P}(\mathcal{M}_{s-3-d})$ corresponding to $\mathcal{X}$ and let $\mathbf{T}$ denote the 2-dimensional torus of the algebraic group $\operatorname{Aut}_0 \mathcal{M}$ whose group of rational characters is described in [Sk 01, p. 72]. Note that the Lie algebra of $\mathbf{T}$ equals $F(\operatorname{ad} x_1 \partial_1) \oplus F(\operatorname{ad} x_2 \partial_2)$.

The connected abelian group $\mathbf{T}$ acts regularly on $\mathcal{X}$, hence fixes a point in $\mathbb{P}(\mathcal{X})$ by Borel's theorem. This means that there exists a nonzero $x_0 \in \mathcal{M}_{s-3-d}$ such that $\mathfrak{c}_{\mathcal{M}}(x_0)$ has codimension $\leqslant 6$ in $\mathcal{M}$ and $\mathbf{T} \cdot x_0 \subset F x_0$. Let $\mathfrak{n}_0$ denote the normalizer of $F x_0$ in $\mathcal{M}$ and set $\mathfrak{t} := F(x_1 \partial_1) \oplus F(x_2 \partial_2)$, a 2-dimensional torus in $\mathcal{M}$. By our choice of x_0 (and $\mathbf{T}$) we have that $[\mathfrak{t}, x_0] \subset F x_0$.

Suppose $[\mathfrak{t}, x_0] \neq 0$. Then $\mathfrak{n}_0 \supsetneq \mathfrak{c}_{\mathcal{M}}(x_0)$. As a consequence, $\mathfrak{n}_0$ is a proper subalgebra of codimension $\leqslant 5$ in $\mathcal{M}$. By a result of Kuznetsov [Kuz 91, Theorem 4.7], every proper subalgebra of $\mathcal{M}$ has codimension $\geqslant 5$ and every subalgebra of codimension 5 contains $\bigoplus_{i\geqslant 1}\mathcal{M}_i$ (see also [St 04, Theorem 4.3.3] and [Sk 01, Section 1]). Since the subalgebra $\bigoplus_{i\geqslant 1}\mathcal{M}_i$ of $\mathfrak{n}_0$ acts nilpotently on $\mathcal{M}$, it must annihilate Fx_0. On the other hand, it is immediate from the simplicity of the graded Lie algebra $\mathcal{M}$ that the graded subspace $\operatorname{Ann}_{\mathcal{M}}(\bigoplus_{i>0}\mathcal{M}_i)$ coincides with $\mathcal{M}_s$. So $x_0 \in \mathcal{M}_s$ forcing $d = -3$, a contradiction.

Now suppose $[\mathfrak{t}, x_0] = 0$. Using [St 04, (4.3.1)] one checks immediately that $\mathfrak{c}_{\mathcal{M}}(\mathfrak{t}) = \mathfrak{t} \oplus Fx_1^3x_2^3 \oplus Fx_1^4x_2^3\tilde{\partial}_1 \oplus Fx_1^3x_2^4\tilde{\partial}_2$. In view of [St 04, p. 200], we have that $\mathfrak{t} \subset \mathcal{M}_0$, $x_1^2x_2^2 \in \mathcal{M}_{10}$ and $Fx_1^4x_2^3\tilde{\partial}_1 \oplus Fx_1^3x_2^4\tilde{\partial}_2 \subset \mathcal{M}_{20}$. As $d > 0$ by our present assumption, we have $s - 3 - d = 23 - 3 - d < 20$. Rescaling x_0 if need be we thus may assume that either $x_0 = x_1^2x_2^2$ or $x_0 = x_1\partial_1 + \alpha x_2\partial_2$ for some $\alpha \in F$ (by symmetry). Applying [St 04, (4.3.1)] it is easy to observe that $\mathfrak{c}_{\mathcal{M}_i}(x_1^2x_2^2) = (0)$ for $i < 0$ and $\mathfrak{c}_{\mathcal{M}_0}(x_1^2x_2^2) = \mathfrak{t}$. This shows that the case $x_0 = x_1^2x_2^2$ is impossible (as $\mathfrak{c}_{\mathcal{M}}(x_0)$ has codimension $\leqslant 6$ in $\mathcal{M}$). If $x_0 = x_1\partial_1 + \alpha x_2\partial_2$, then $[x_0, \mathcal{M}]$ contains all $x_1^i\partial_1$ with $i \in \{0, 2, 3, 4\}$ and all $x_1^jx_2\partial_2$ with $j \in \{1, 2, 3, 4\}$. It follows that $\operatorname{codim}_{\mathcal{M}} \mathfrak{c}_{\mathcal{M}}(x_0) \geqslant 8$ in this case, showing that the case where $d > 0$ cannot occur.

(b) Thus $d \leqslant 0$. Recall from [Sk 01, p. 72] that the group of rational characters of $\mathbf{T}$ has $\mathbb{Z}$-basis $\{\varepsilon_1, \varepsilon_2\}$ and the $\mathbf{T}$-weight vectors $\partial_1, \partial_2 \in \mathcal{M}_{-3}$, $1 \in \mathcal{M}_{-2}$, $\tilde{\partial}_1, \tilde{\partial}_2 \in \mathcal{M}_{-1}$ and $x_1\partial_2, x_2\partial_1 \in \mathcal{M}_0$ have weights $-2\varepsilon_1 - \varepsilon_2, -\varepsilon_1 - 2\varepsilon_2, -\varepsilon_1 - \varepsilon_2, -\varepsilon_1, -\varepsilon_2$ and $\varepsilon_1 - \varepsilon_2, -\varepsilon_1 + \varepsilon_2$, respectively.

Assume that $\chi_0(x_1\partial_2) \neq 0$ and consider the cocharacter $\varepsilon_1^*: F^\times \to \operatorname{Aut}\mathcal{M}$ such that $(\varepsilon_1^*(t))(x) = t^n x$ for all $t \in F^\times$ and all weight vectors $x \in \mathcal{M}_{n\varepsilon_1 + m\varepsilon_2}$, where $m, n \in \mathbb{Z}$. Let $\mathcal{M} = \bigoplus_{i\in\mathbb{Z}}\mathcal{M}(i)$ be the $\mathbb{Z}$-grading of $\mathcal{M}$ induced by ε_1^*. Since $d \leqslant 0$ and $\chi_0(x_1\partial_2) \neq 0$ by our assumption, we have that $\chi = \chi(-2) + \chi(-1) + \chi(0) + \chi(1)$, where $\chi(i) \in \mathcal{M}(i)^*$ and $\chi(1) \neq 0$. Applying [P-Sk 99, Proposition 5.5] to the graded Lie algebra $\bigoplus_{i\in\mathbb{Z}}\mathcal{M}(i)$ we deduce that $\mathfrak{z}_{\mathcal{M}}(\chi(1))$ has codimension $\leqslant 6$ in $\mathcal{M}$. Since in the present case $x_1\partial_1 \in \mathfrak{n}_{\mathcal{M}}(F\chi(1)) \setminus \mathfrak{z}_{\mathcal{M}}(\chi(1))$, the normalizer $\mathfrak{n}_{\mathcal{M}}(F\chi(1))$ has codimension $\leqslant 5$ in $\mathcal{M}$. Using Kuznetsov's description of subalgebras of codimension 5 in $\mathcal{M}$ and arguing as in part (a) we now obtain that $\chi(1) = b(y, \cdot)$ for some $y \in \mathcal{M}_s$. Since in the present case $s - 3 - d \neq s$, we reach a contradiction, thereby showing that $\chi_0(x_1\partial_2) = 0$. Arguing in a similar fashion one obtains that χ_0 vanishes on $x_2\partial_1$.

(c) Thus we may assume from now that $d \leqslant 0$ and χ_0 vanishes on $F(x_1\partial_2) \oplus F(x_2\partial_1)$. In this situation [P-Sk 99, Proposition 5.5] is no longer useful, so we have to argue differently. Denote by $\mathfrak{g}$ the Lie subalgebra of $\mathcal{M}$ generated by the graded components $\mathcal{M}_{\pm 1}$. Using [St 04, (4.3.1)] it is easy to check that $\mathcal{M}_1 = Fx_1 \oplus Fx_2$, $\mathcal{M}_1^2 = F(x_1\tilde{\partial}_1 + x_2\tilde{\partial}_2)$, $\mathcal{M}_1^3 = F(x_1^2\partial_1 + x_1x_2\partial_2) \oplus F(x_1x_2\partial_1 + x_2^2\partial_2)$ and $\mathcal{M}_1^4 = (0)$. Then it is immediate from [St 04, Theorem 5.4.1] that $\mathfrak{g}$ is a 14-dimensional simple Lie algebra of type G_2. We identify χ with its restriction to $\mathfrak{g}$, denote by $\mathbf{G}$ the simple algebraic group $\operatorname{Aut}\mathfrak{g}$, and regard $\mathbf{L} := \operatorname{Aut}_0\mathcal{M}$ as a Levi subgroup of $\mathbf{G}$. Clearly, $\mathbf{T}$ is a maximal torus of $\mathbf{G}$ contained in $\mathbf{L}$. Also, $\operatorname{Lie}(\mathbf{G}) = \operatorname{ad}\mathfrak{g}$ and 5 is a good prime for the root system $\Phi = \Phi(\mathbf{G}, \mathbf{T})$. Since the Killing form κ of the Lie algebra $\mathfrak{g}$ is nondegenerate, we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the $\mathbf{G}$-equivariant map sending $x \in \mathfrak{g}$ to the linear function $\kappa(x, \cdot) \in \mathfrak{g}^*$.

Let $\mathbf{P}$ be the parabolic subgroup of $\mathbf{G}$ with $\operatorname{Lie}(\mathbf{P}) = \operatorname{ad}(\bigoplus_{i\geqslant 0}\mathfrak{g}_i)$, where $\mathfrak{g}_i = \mathfrak{g} \cap \mathcal{M}_i$, and let Φ^+ be a positive system in Φ containing the $\mathbf{T}$-weights of $\bigoplus_{i>0}\mathfrak{g}_i$. Let $\{\alpha_1, \alpha_2\}$ be the basis of simple roots of Φ contained in Φ^+. Adopting Bourbaki's numbering we will assume that $\mathfrak{g}_0$ is spanned by $\mathfrak{t}$ and root vectors $e_{\pm\alpha_2}$ and $\mathfrak{g}_1$ is spanned by root vectors e_{α_1} and $e_{\alpha_1+\alpha_2}$. We stress that α_1 is a *short* root of Φ.

Since $g(\chi_0) = \chi_0$ for all $g \in \mathbf{T}$ and $\chi_{-1} + \chi_{-2} + \chi_{-3}$ is a linear combination of $\mathbf{T}$-weight vectors corresponding to positive roots, the Zariski closure of $\mathbf{T} \cdot \chi$ contains χ_0. It follows that $\dim \mathbf{G} \cdot \chi \geqslant \dim \mathbf{G} \cdot \chi_0$. Since χ_0 vanishes on all root vectors $e_\alpha \in \mathfrak{g}$ with $\alpha \in \Phi$ and 5 is a good prime for Φ, the stabilizer $Z_{\mathbf{G}}(\chi_0)$ of χ_0 in $\mathbf{G}$ is a Levi subgroup of $\mathbf{G}$; see [P 95, (3.1)] and references therein. Since the $\mathfrak{g}$-module V has p-character χ, the Kac–Weisfeiler conjecture proved in [P 95] shows that $5^{(\dim \mathbf{G}\cdot\chi)/2} \mid \dim V$.

Suppose $\chi_0 \neq 0$. Then $Z_{\mathbf{G}}(\chi_0)$ is a proper Levi subgroup of $\mathbf{G}$. Since any Levi subgroup of $\mathbf{G}$ is conjugate to a standard Levi subgroup, this implies that $\dim Z_{\mathbf{G}}(\chi_0) \leqslant 4$. As a consequence,

$$\dim \mathbf{G} \cdot \chi \geqslant \dim \mathbf{G} \cdot \chi_0 = \dim \mathbf{G} - \dim Z_{\mathbf{G}}(\chi_0) \geqslant 10.$$

But then $5^5 \mid \dim V$, a contradiction. Thus, $\chi = \kappa(y_1 + y_2 + y_3, \cdot)$ for some $y_i \in \mathfrak{g}_i$.

Suppose $y_1 \neq 0$. Since y is a nilpotent element of $\mathfrak{g}$, all nonzero scalar multiples of y are $\mathbf{G}$-conjugate. From this it is immediate that the Zariski closure of $\mathbf{G} \cdot y$ contains y_1, implying $\dim \mathbf{G} \cdot y \geqslant \dim \mathbf{G} \cdot y_1$. As all nonzero elements of $\mathfrak{g}_1$ are conjugate under the action of $\mathbf{L}$, we may assume that $y_1 = e_{\alpha_1}$. As $\dim \mathfrak{c}_{\mathfrak{g}}(e_{\alpha_1}) = 6$, it follows that

$$\dim \mathbf{G} \cdot \chi = \dim \mathbf{G} \cdot y \geqslant \dim \mathbf{G} \cdot y_1 = \dim \mathbf{G} - \dim Z_{\mathbf{G}}(y_1) \geqslant \dim \mathbf{G} - \dim \mathfrak{c}_{\mathfrak{g}}(y_1) = 8.$$

Applying [P 95, Theorem I] now gives $5^4 \mid \dim V$. Since this is false, it must be that $y_1 = 0$. If $y_2 \neq 0$, then y_2 is a nonzero multiple of $e_{2\alpha_1+\alpha_2}$ (for $\mathfrak{g}_2 = [\mathcal{M}_1, \mathcal{M}_1] = Fe_{2\alpha_1+\alpha_2}$). As $y = y_2 + y_3$, it is easy to see that the orbit $\mathbf{P} \cdot y$ contains $e_{2\alpha_1+\alpha_2}$. As $2\alpha_1 + \alpha_2$ is a short root of Φ, we can argue as before to obtain $5^4 \mid \dim V$, a contradiction.

As a result, $y = y_3$. Then $\chi = \chi_{-3}$ vanishes on $\bigoplus_{i \geqslant -2} \mathcal{M}_i$ as stated. If $\chi \neq 0$, then we can assume that $y = e_{3\alpha_1+2\alpha_2}$ (for all nonzero elements in $\mathfrak{g}_3 = [e_{2\alpha_1+\alpha_2}, \mathcal{M}_1]$ are conjugate under the action of $\mathbf{L}$). Since $\dim \mathfrak{c}_{\mathfrak{g}}(e_{3\alpha_1+2\alpha_2}) = 8$, it follows from [P 95, Theorem I] that $5^3 \mid \dim V$. Then $\dim V = 125$, completing the proof. $\square$

7. Melikian pairs

Set $\Gamma := \Gamma(L, T)$. According to Theorem 5.8(4), if $\alpha, \beta \in \Gamma$ are $\mathbb{F}_p$-independent, then either $L[\alpha, \beta] \cong \mathcal{M}$ or $H(2;(2,1))^{(2)} \subset L[\alpha, \beta] \subset H(2;(2,1))$. If $L[\alpha, \beta] \cong \mathcal{M}$ we say that $(\alpha, \beta) \in \Gamma^2$ is a *Melikian pair*. Recall from Theorem 5.8(2) that H^3 is a 2-dimensional subspace of T.

Lemma 7.1. *A pair $(\alpha, \beta) \in \Gamma^2$ is Melikian if and only if $H^3 \cap \ker \alpha \neq H^3 \cap \ker \beta$, i.e. if and only if $\alpha_{|H^3}$ and $\beta_{|H^3}$ are linearly independent over F.*

Proof. Suppose $H(2;(2,1))^{(2)} \subset L[\alpha, \beta] \subset H(2;(2,1))$. Recall from Section 2 that $H(2;(2,1)) = H(2;(2,1))^{(2)} \oplus V$ and $V^3 = (0)$. Then $L[\alpha, \beta]^3 \subset H(2;(2,1))^{(2)}$, forcing $\Psi_{\alpha,\beta}(H)^3 \subset H(2;(2,1))^{(2)}$. But then $\Psi_{\alpha,\beta}(H^3) \subset \Psi_{\alpha,\beta}(T) \cap H(2;(2,1))^{(2)}$ has dimension $\leqslant 1$ by Lemma 2.4. In view of Theorem 5.8(4) and the inclusion $H^3 \subset T$, this means that $H^3 \cap \ker \alpha \cap \ker \beta$ has codimension $\geqslant 1$ in H^3. It follows that α and β are linearly dependent as linear functions on H^3.

Now suppose that $L[\alpha, \beta] \cong \mathcal{M}$. In view of Theorem 5.8(1), both α and β are in Ω. Therefore, $\Psi_{\alpha,\beta}(T)$ is a nonstandard 2-dimensional torus in $L[\alpha, \beta] \cong \operatorname{Der} L[\alpha, \beta]$. Applying [P 94, Lemmas 4.1 and 4.4] now gives $\dim \Psi_{\alpha,\beta}(H)^3 = 2$, which in conjunction with Theorem 5.8(5) yields that $H^3 \cap \ker \alpha \cap \ker \beta$ has codimension $\leqslant 2$ in H^3. So α and β must be linearly independent on H^3. $\square$

Corollary 7.2. *For any $\alpha \in \Gamma$ there exists $\beta \in \Gamma$ such that (α, β) is a Melikian pair.*

Proof. It follows from Theorem 5.8 that $H^3 \cap \ker \alpha = Ft$ for some nonzero $t \in H^3$. Since $H^3 \subset T$ and L is centerless, there is a $\beta \in \Gamma$ with $\beta(t) \neq 0$. Then (α, β) is a Melikian pair by Lemma 7.1. $\square$

Lemma 7.3. *If (α, β) is a Melikian pair, then*

$$L_p(\alpha, \beta) = L(\alpha, \beta)^{(1)} \oplus T \cap \ker \alpha \cap \ker \beta, \qquad L_p(\alpha, \beta)^{(1)} = L(\alpha, \beta)^{(1)} \cong \mathcal{M}.$$

Proof. (a) Since $\operatorname{rad}_T L(\alpha, \beta) = H \cap T \cap \ker \alpha \cap \ker \beta$ by Theorem 5.8(5), we have that $\operatorname{rad}_T L(\alpha, \beta) = \mathfrak{z}(L(\alpha, \beta))$. Hence

$$(0) \to H \cap T \cap \ker \alpha \cap \ker \beta \to L(\alpha, \beta) \to \mathcal{M} \to (0)$$

is a central extension $\mathcal{M}$. By Proposition 6.2, this extension splits; that is, $L(\alpha, \beta) = L(\alpha, \beta)^{(1)} \oplus H \cap T \cap \ker \alpha \cap \ker \beta$ and $L(\alpha, \beta)^{(1)} \cong \mathcal{M}$.

(b) Note that $L_p(\alpha,\beta) = \widetilde{H} + L(\alpha,\beta)$, where $\widetilde{H} = \mathfrak{c}_{L_p}(T)$, and $[\widetilde{H}, L(\alpha,\beta)^{(1)}] \subset L(\alpha,\beta)^{(1)}$. Hence $\widetilde{H}$ acts on $L(\alpha,\beta)^{(1)}$ as derivations. As all derivations of $L(\alpha,\beta)^{(1)} \cong \mathcal{M}$ are inner by [St 04, Theorem 7.1.4], it must be that $\widetilde{H} = H' \oplus \widetilde{H}_0$, where $\widetilde{H}_0 = \mathfrak{c}_{\widetilde{H}}(L(\alpha,\beta)^{(1)})$ and $H' = L(\alpha,\beta)^{(1)} \cap H$. From part (a) of this proof it follows that $H \subset T + H'$. Consequently, $[H, \widetilde{H}_0] = 0$.

Put $\Gamma' := \{\gamma \mid \gamma(H') \neq 0\}$ and let μ be any root in Γ'. Recall that $\dim L_\mu = 5$; see Theorem 5.8(3). As H' is a nontriangulable Cartan subalgebra of $L(\alpha,\beta)^{(1)} \cong \mathcal{M}$ by [P 94, Lemmas 4.1 and 4.4], the H'-module L_μ is irreducible. But then $\widetilde{H}_0$ acts on L_μ as scalar operators. On the other hand, it follows from Schue's lemma [St 04, Proposition 1.3.6(1)] that L is generated by the root spaces L_γ with $\gamma \in \Gamma'$. It follows that $\widetilde{H}_0$ acts semisimply on L, implying $\widetilde{H}_0 \subset T$. From this it is immediate that $\widetilde{H}_0 = T \cap \ker\alpha \cap \ker\beta$. As a result,

$$L_p(\alpha,\beta) = L(\alpha,\beta)^{(1)} + \widetilde{H}_0 = L(\alpha,\beta)^{(1)} \oplus T \cap \ker\alpha \cap \ker\beta,$$

finishing the proof. $\square$

Let (α,β) be a Melikian pair. Note that $T_0 := T \cap \ker\alpha \cap \ker\beta$ is a restricted ideal of $L_p(\alpha,\beta)$ and $T = H^3 \oplus T_0$. So the Lie algebra $L_p(\alpha,\beta)/T_0$ inherits a pth power map from $L_p(\alpha,\beta)$. Since $L_p(\alpha,\beta)/T_0 \cong \mathcal{M}$ by Lemma 7.3 and both Lie algebras are centerless and restricted, every isomorphism between $L_p(\alpha,\beta)/T_0$ and $\mathcal{M}$ is an isomorphism of restricted Lie algebras. Any such isomorphism maps the torus T/T_0 of the restricted Lie algebra $L_p(\alpha,\beta)/T_0$ onto a 2-dimensional nonstandard torus of $\mathcal{M}$. According to [P 94, Lemmas 4.1 and 4.4], any such torus is conjugate under $\operatorname{Aut}\mathcal{M}$ to the torus $\mathfrak{t} := F(1+x_1)\partial_1 \oplus F(1+x_2)\partial_2$.

Recall from Section 6 the natural grading of the Lie algebra $\mathcal{M}$. For $i \geqslant -3$, we set $\mathcal{M}_{(i)} := \bigoplus_{j\geqslant i} \mathcal{M}_i$. The decreasing filtration $(\mathcal{M}_{(i)})_{i\geqslant -3}$ of the Lie algebra $\mathcal{M}$ can be regarded as a standard (Weisfeiler) filtration of $\mathcal{M}$ associated with its maximal subalgebra $\mathcal{M}_{(0)}$. It is referred to as the *natural* filtration of $\mathcal{M}$, because $\mathcal{M}_{(0)}$ is the only subalgebra of codimension 5 and depth 3 in $\mathcal{M}$. All components $\mathcal{M}_{(i)}$ of this filtration are invariant under the automorphism group of $\mathcal{M}$; see [St 04, Theorem 4.3.3(2) and Remark 4.3.4] for more detail. Note that $\mathcal{M} = \mathfrak{t} \oplus \mathcal{M}_{(-2)}$.

Regard $\widetilde{\mathcal{M}} := \mathcal{M} \oplus T_0$ as a direct sum of Lie algebras and define a pth power map $u \mapsto u^p$ on $\widetilde{\mathcal{M}}$ by setting $u^p = u^{[p]}$ for all $u \in \mathcal{M}$ and $u^p = 0$ for all $u \in T_0$ (here $u \mapsto u^{[p]}$ is *the* pth power map on $\mathcal{M}$). The above discussion in conjunction with Lemma 7.3 shows that there exists a Lie algebra isomorphism

$$\Phi : L_p(\alpha,\beta) \xrightarrow{\sim} \widetilde{\mathcal{M}} = \mathcal{M}_{(-2)} \oplus \Phi(T) \tag{7.1}$$

such that

$$\Phi\big(L(\alpha,\beta)^{(1)}\big) = \mathcal{M}, \qquad \Phi\big(H^3\big) = \mathfrak{t}, \qquad \Phi_{|T_0} = \mathrm{Id}_{T_0}. \tag{7.2}$$

Note that Φ maps $L_p(\alpha,\beta)^{(1)}$ onto $\widetilde{\mathcal{M}}^{(1)} = \mathcal{M}$. We stress that H^3 is not a restricted subalgebra of $L_p(\alpha,\beta)$, whilst $\Phi(H^3)$ is a maximal torus of $\widetilde{\mathcal{M}}$. There exists a p-linear mapping $\Lambda : \widetilde{\mathcal{M}} \to \mathfrak{z}(\widetilde{\mathcal{M}}) = T_0$ such that

$$\Lambda(u) = \Phi^{-1}(u)^{[p]} - \Phi^{-1}\big(u^p\big) \quad (\forall u \in \widetilde{\mathcal{M}}),$$

where $\Phi^{-1}(u) \longmapsto \Phi^{-1}(u)^{[p]}$ is the pth power map in L_p.

Lemma 7.4. *The p-linear mapping Λ vanishes on the subspace $\mathcal{M}_{(-2)}$ of $\widetilde{\mathcal{M}}$.*

Proof. Suppose $\Lambda(u) \neq 0$ for some $u \in \mathcal{M}_{(-2)}$. Then there is $\gamma \in \Gamma$ which does not vanish on $\Lambda(u) \in T_0 \setminus \{0\}$. Since $\Lambda(u) \subset T \cap \ker\alpha \cap \ker\beta$, the root γ is $\mathbb{F}_p$-independent of α and β. Let $M(\gamma;\alpha,\beta) := \bigoplus_{i,j\in\mathbb{F}_p} L_{\gamma+i\alpha+j\beta}$. By Theorem 5.8, $M(\gamma;\alpha,\beta)$ is a 125-dimensional submodule of the

$(T + L(\alpha,\beta)_p)$-module L. The map $\operatorname{ad}\circ\Phi^{-1}$ gives $M(\gamma;\alpha,\beta)$ an $\mathcal{M}$-module structure. Note that T_0 acts on $M(\gamma;\alpha,\beta)$ as scalar operators. This means that the $\mathcal{M}$-module $M(\gamma;\alpha,\beta)$ has a p-character; we call it χ. It is straightforward to see that $\Lambda(x) = \chi(x)^p$ for all $x \in \mathcal{M}$. But then χ does not vanish on $\mathcal{M}_{(-2)}$. Since $\dim M(\gamma;\alpha,\beta) = 125$, this contradicts Proposition 6.3. The result follows. $\square$

We now set $(L_p(\alpha,\beta)^{(1)})_{(i)} := \Phi^{-1}(\mathcal{M}_{(i)})$ for all $i \geqslant -3$. Then the following hold:

- $(L_p(\alpha,\beta)^{(1)})_{(-3)} = L_p(\alpha,\beta)^{(1)}$;
- $(L_p(\alpha,\beta)^{(1)})_{(0)}$ is a subalgebra of codimension 5 in $L_p(\alpha,\beta)^{(1)}$;
- $u^{[p]} \in L_p(\alpha,\beta)^{(1)}$ for all $u \in (L_p(\alpha,\beta)^{(1)})_{(-2)}$;
- $(L_p(\alpha,\beta)^{(1)})_{(0)}$ is a restricted subalgebra of $L_p(\alpha,\beta)$.

Since the natural filtration of $\mathcal{M}$ is invariant under all automorphisms of $\mathcal{M}$ (see [St 04, Remark 4.3.4(3)]), the above definition of the subspaces $(L_p(\alpha,\beta)^{(1)})_{(i)}$ is independent of the choice of Φ satisfying (7.1) and (7.2).

8. Describing $L_p(\alpha)$

Fix $\alpha \in \Gamma$ and pick $\beta \in \Gamma$ be such that (α,β) is a Melikian pair; see Corollary 7.2. As before, we put $T_0 := T \cap \ker\alpha \cap \ker\beta$ and let Φ be a map satisfying (7.1) and (7.2). It gives rise to the *restricted* Lie algebra isomorphism

$$\overline{\Phi} : L_p(\alpha,\beta)/T_0 \xrightarrow{\sim} \mathcal{M} = \mathcal{M}_{(-2)} \oplus \overline{\Phi}\big(H^3\big), \quad \overline{\Phi}\big(H^3\big) = \mathfrak{t}.$$

By Theorem 5.8(1), no root in Γ vanishes on H^3. As $\dim H^3 = 2$, there exists a nonzero $h_\alpha \in H^3$ such that $Fh_\alpha = H^3 \cap \ker\alpha$. As $\overline{\Phi}(Fh_\alpha)$ is a 1-dimensional subtorus of the nonstandard torus $\mathfrak{t}$, it follows from [Sk 01, Theorem 2.1] that there is an automorphism of $\mathcal{M}$ which maps $\mathfrak{t}$ onto itself and $F\overline{\Phi}(h_\alpha)$ onto $F(1+x_1)\partial_1$. Hence we may assume without loss of generality that

$$\Phi\big(L_p(\alpha)\big) = \mathfrak{c}_{\mathcal{M}}\big((1+x_1)\partial_1\big) \oplus T_0, \qquad \Phi(T) = \mathfrak{t} \oplus T_0, \qquad \overline{\Phi}(h_\alpha) = (1+x_1)\partial_1. \tag{8.1}$$

For $f \in \mathcal{O}(2;(1,1))_{(0)}$ set $f^{(k)} := f^k/k!$ for $0 \leqslant k \leqslant 4$ and $f^{(k)} := 0$ for $k < 0$ and $k \geqslant 5$. Direct computations show that $\mathfrak{c}_{\mathcal{M}}((1+x_1)\partial_1)$ has basis

$$\big\{x_2^{(r)}\partial_2, x_2^{(r)}(1+x_1)\partial_1, x_2^{(r)}(1+x_1)^2, x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(r)}(1+x_1)^4\tilde{\partial}_1 \bigm| 0 \leqslant r \leqslant 4\big\}.$$

Using the multiplication table in [St 04, (4.3.1)] it is easy to observe that

$$\big[x_2^{(r)}\partial_2, x_2^{(s)}\partial_2\big] = \left[\binom{r+s-1}{r} - \binom{r+s-1}{s}\right] x_2^{(r+s-1)}\partial_2;$$

$$\big[x_2^{(r)}(1+x_1)\partial_1, x_2^{(s)}\partial_2\big] = -\binom{r+s-1}{s} x_2^{(r+s-1)}(1+x_1)\partial_1;$$

$$\big[x_2^{(r)}(1+x_1)\partial_1, x_2^{(s)}(1+x_1)\partial_1\big] = 0;$$

$$\big[x_2^{(r)}(1+x_1)^2, x_2^{(s)}\partial_2\big] = -\left[\binom{r+s-1}{s} - 2\binom{r+s-1}{s-1}\right] x_2^{(r+s-1)}(1+x_1)^2;$$

$$\big[x_2^{(r)}(1+x_1)^2, x_2^{(s)}(1+x_1)\partial_1\big] = -\left[2\binom{r+s}{s} - 2\binom{r+s}{s}\right] x_2^{(r+s)}(1+x_1)^2 = 0;$$

$$\big[x_2^{(r)}(1+x_1)^2, x_2^{(s)}(1+x_1)^2\big] = 2\left[-\binom{r+s-1}{r} + \binom{r+s-1}{s}\right] x_2^{(r+s-1)}(1+x_1)^4\tilde{\partial}_1;$$

$$\big[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}\partial_2\big] = -\binom{r+s}{r}x_2^{(r+s-1)}(1+x_1)^3\tilde{\partial}_2;$$

$$\big[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}(1+x_1)\partial_1\big] = \binom{r+s-1}{r}x_2^{(r+s-1)}(1+x_1)^4\tilde{\partial}_1;$$

$$\big[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}(1+x_1)^2\big] = -\binom{r+s}{r}x_2^{(r+s)}\partial_2;$$

$$\big[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}(1+x_1)^3\tilde{\partial}_2\big] = 0;$$

$$\big[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}\partial_2\big] = -\left[\binom{r+s-1}{s}+2\binom{r+s-1}{s-1}\right]x_2^{(r+s-1)}(1+x_1)^4\tilde{\partial}_1;$$

$$\big[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)\partial_1\big] = -\left[3\binom{r+s}{r}+2\binom{r+s}{s}\right]x_2^{(r+s)}\tilde{\partial}_1 = 0;$$

$$\big[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)^2\big] = -\binom{r+s}{r}x_2^{(r+s)}(1+x_1)\partial_1;$$

$$\big[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)^3\tilde{\partial}_2\big] = \binom{r+s}{r}x_2^{(r+s)}(1+x_2)^2;$$

$$\big[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)^4\tilde{\partial}_1\big] = 0.$$

In order to obtain a more invariant description of $L_p(\alpha)$ we now consider a vector space $R = R' \oplus C$ over F with $\dim C = \dim T - 2$ such that R' has basis $\{x_1^{(i)}x_2^{(j)} \mid 0 \leqslant i, j \leqslant 4, 1 \leqslant i+j \leqslant 7\} \cup \{x_2^{(5)}\} \cup \{z\}$. We give R a Lie algebra structure by setting

$$\big[x_1^{(i)}x_2^{(j)}, x_1^{(k)}x_2^{(l)}\big] := \left[\binom{i+k-1}{i-1}\binom{j+l-1}{j} - \binom{i+k-1}{i}\binom{j+l-1}{j-1}\right]x_1^{(i+k-1)}x_2^{(j+l-1)}$$

for all i, j, k, l with $3 \leqslant i+j+k+l \leqslant 7$ such that $(j,l) \neq (0,0)$ whenever $i+k=5$, and by requiring that $[Fz + C, R] = 0$ and

$$\big[x_1^{(i)}x_2^{(j)}, x_1^{(k)}x_2^{(l)}\big] := \begin{cases} 0 & \text{if } i+j+k+l \leqslant 2, \\ (-1)^i z & \text{if } j=l=0 \text{ and } i+k=5. \end{cases}$$

The Lie algebra R is a (nonsplit) central extension of $H(2;\underline{1})^{(2)} \oplus FD_H(x_2^{(5)})$. Computations show that

$$\big[x_1x_2^{(r)}, x_1x_2^{(s)}\big] = \left[\binom{r+s-1}{r} - \binom{r+s-1}{s}\right]x_1x_2^{(r+s-1)};$$

$$\big[-x_1^{(4)}x_2^{(r-1)}, x_1x_2^{(s)}\big] = \begin{cases} -\binom{r+s-1}{s}(-x_1^{(4)}x_2^{(r+s-2)}) & \text{if } r+s \geqslant 2, \\ -z & \text{if } r=1,\ s=0; \end{cases}$$

$$\big[-x_1^{(4)}x_2^{(r-1)}, -x_1^{(4)}x_2^{(s-1)}\big] = 0;$$

$$\big[x_1^{(2)}x_2^{(r)}, x_1x_2^{(s)}\big] = -\left[\binom{r+s-1}{s} - 2\binom{r+s-1}{s-1}\right]x_1^{(2)}x_2^{(r+s-1)};$$

$$\big[x_1^{(2)}x_2^{(r)}, -x_1^{(4)}x_2^{(s-1)}\big] = 0;$$

$$\big[x_1^{(2)}x_2^{(r)}, x_1^{(2)}x_2^{(s)}\big] = 2\left[-\binom{r+s-1}{r} + \binom{r+s-1}{s}\right]x_1^{(3)}x_2^{(r+s-1)};$$

$$\big[x_2^{(r+1)}, x_1x_2^{(s)}\big] = -\binom{r+s}{r}x_2^{(r+s)};$$

$$\begin{aligned}
\big[x_2^{(r+1)}, -x_1^{(4)}x_2^{(s-1)}\big] &= \binom{r+s-1}{r} x_1^{(3)}x_2^{(r+s-1)};\\
\big[x_2^{(r+1)}, x_1^{(2)}x_2^{(s)}\big] &= -\binom{r+s}{r} x_1 x_2^{(r+s)};\\
\big[x_2^{(r+1)}, x_2^{(s+1)}\big] &= 0;\\
\big[x_1^{(3)}x_2^{(r)}, x_1 x_2^{(s)}\big] &= -\left[\binom{r+s-1}{s} + 2\binom{r+s-1}{s-1}\right] x_1^{(3)}x_2^{(r+s-1)};\\
\big[x_1^{(3)}x_2^{(r)}, -x_1^{(4)}x_2^{(s-1)}\big] &= 0;\\
\big[x_1^{(3)}x_2^{(r)}, x_1^{(2)}x_2^{(s)}\big] &= \begin{cases} -\binom{r+s}{r}\big(-x_1^{(4)}x_2^{(r+s-1)}\big) & \text{if } r+s \geqslant 1,\\ -z & \text{if } r=s=0;\end{cases}\\
\big[x_1^{(3)}x_2^{(r)}, x_2^{(s+1)}\big] &= \binom{r+s}{r} x_1^{(2)}x_2^{(r+s)};\\
\big[x_1^{(3)}x_2^{(r)}, x_1^{(3)}x_2^{(s)}\big] &= 0.
\end{aligned}$$

By comparing the displayed multiplications tables it is straightforward to see that the following statement holds:

Proposition 8.1. *Any linear map $\Theta' : \mathfrak{c}_{\widetilde{\mathcal{M}}}((1+x_1)\partial_1) \to R$ which takes T_0 isomorphically onto C and satisfies the conditions*

$$\begin{aligned}
\Theta'\big(x_2^{(r)}(1+x_1)\partial_1\big) &= \begin{cases} -x_1^{(4)}x_2^{(r-1)} & \text{if } 1 \leqslant r \leqslant 4,\\ z & \text{if } r=0,\end{cases}\\
\Theta'\big(x_2^{(r)}\partial_2\big) &= x_1 x_2^{(r)}, \quad 0 \leqslant r \leqslant 4,\\
\Theta'\big(x_2^{(r)}(1+x_1)^2\big) &= x_1^{(2)}x_2^{(r)}, \quad 0 \leqslant r \leqslant 4,\\
\Theta'\big(x_2^{(r)}(1+x_1)^3\tilde{\partial}_2\big) &= x_2^{(r+1)}, \quad 0 \leqslant r \leqslant 4,\\
\Theta'\big(x_2^{(r)}(1+x_1)^4\tilde{\partial}_1\big) &= x_1^{(3)}x_2^{(r)}, \quad 0 \leqslant r \leqslant 4,
\end{aligned}$$

is an isomorphism of Lie algebras.

We now fix Θ' described in Proposition 8.1 and set $\Theta := \Theta' \circ \Phi_{|L_p(\alpha)}$, where $\Phi : L_p(\alpha,\beta) \xrightarrow{\sim} \widetilde{\mathcal{M}}$ is a Lie algebra isomorphism satisfying (7.1), (7.2) and (8.1). Clearly, $\Theta : L_p(\alpha) \xrightarrow{\sim} R$ is a Lie algebra isomorphism. We give R a pth power map by setting

$$r^p := \Theta\big(\Theta^{-1}(r)^{[p]}\big) \quad (\forall r \in R). \tag{8.2}$$

This turns Θ into an isomorphism of restricted Lie algebras. Because the p-linear map $\Lambda : \widetilde{\mathcal{M}} \to T_0$ vanishes on the subspace $\mathcal{M}_{(-2)}$ of $\widetilde{\mathcal{M}}$ by Lemma 7.4 and Θ is defined via Φ, the explicit description of Θ' in Proposition 8.1 shows that the map (8.2) has the following properties:

$$\begin{aligned}
\big(x_2^{(r+1)}\big)^p &= 0 \quad \text{if } 0 \leqslant r \leqslant 4;\\
\big(x_1 x_2^{(r)}\big)^p &= 0 \quad \text{if } r \neq 0, 1;\\
\big(x_1^{(2)}x_2^{(r)}\big)^p &= 0 \quad \text{if } 0 \leqslant r \leqslant 4;
\end{aligned}$$

$$\big(x_1^{(3)}x_2^{(r)}\big)^p = 0 \quad \text{if } 0 \leqslant r \leqslant 4;$$
$$\big(x_1^{(4)}x_2^{(r-1)}\big)^p = 0 \quad \text{if } 1 \leqslant r \leqslant 4;$$
$$(x_1x_2)^p = x_1x_2, \quad \text{i.e. } x_1x_2 \text{ is toral} \tag{8.3}$$

(we refer to [Sk 01] for more detail on the p-structure in the restricted Melikian algebra). Note that $(x_1)^p$ and z^p lie in $\Theta(T) = Fz \oplus C$. Moreover, $Fz = \Theta(H^3 \cap \ker\alpha)$ coincides the image of $F(1+x_1)\partial_1$ under Φ^{-1} and $\Theta'((1+x_2)\partial_2) = x_1 + x_1x_2$.

We stress that all constructions of Sections 7 and 8 depend on the choice of a Melikian pair.

9. The subalgebra $Q(\alpha)$

The results obtained so far apply to all nonstandard tori of maximal dimension in L_p. However, such tori need not be conjugate under the automorphism group of L. In order to identify L with one of the Melikian algebras, we will require a sufficiently generic nonstandard torus of maximal dimension in L_p.

Proposition 9.1. *There exists a nonstandard torus T' of maximal dimension in L_p for which $(\mathfrak{c}_L(T'))^3$ contains no nonzero toral elements of L_p.*

Proof. Let T and Γ be as Section 8 and let $(\alpha,\beta) \in \Gamma^2$ be a Melikian pair. Choose an isomorphism $\Phi: L_p(\alpha,\beta) \xrightarrow{\sim} \widetilde{\mathcal{M}}$ satisfying (7.1) and (7.2). Then $H^3 = \Phi^{-1}(\mathfrak{t})$. Set $q_i := \Phi^{-1}(x_i\partial_i)$, $n_i = \Phi^{-1}(\partial_i)$ and $h_i := n_i^{[p]}$, where $i = 1,2$. As the elements $x_i\partial_i$ are toral in $\mathcal{M}$, Lemma 7.4 says that both q_1 and q_2 are toral elements of L_p. Note that $T = F(q_1+n_1) \oplus F(q_2+n_2) \oplus T_0$, where $T_0 = T \cap \ker\alpha \cap \ker\beta$.

As Φ is a Lie algebra isomorphism, it is straightforward to see that $[q_i,n_i] = -n_i$ and $h_i \in T_0$ for $i = 1,2$. So it follows from Jacobson's formula that $(q_i+n_i)^{[p]^k} = q_i + n_i + \sum_{j=0}^{k-1} h_i^{[p]^j}$ for all $k \geqslant 1$. Since $(H^3)_p = T$ by Theorem 5.8(3) and $H^3 = F(q_1+n_1) \oplus F(q_2+n_2)$, it follows that the p-closure of $Fh_1 + Fh_2$ coincides with T_0.

Recall that $\dim T_0 \geqslant 1$. Let $\{t_1,\dots,t_s\}$ be a basis of T_0 consisting of toral elements of L_p. For $x = \sum_{j=1}^s \alpha_j t_j \in T_0$ define $\mathrm{Supp}(x) := \{j \mid \alpha_j \neq 0\}$. Write $h_1 = \sum_{j=1}^s \lambda_i t_i$ and $h_2 = \sum_{j=1}^s \mu_j t_j$ with $\lambda_j, \mu_j \in F$. Since the $[p]$th powers of h_1 and h_2 span T_0, it must be that

$$\mathrm{Supp}(h_1) \cup \mathrm{Supp}(h_2) = \{1,\dots,s\}.$$

In particular, $h_1 \neq 0$ or $h_2 \neq 0$. Recall from Section 6 the maximal torus $\mathbf{T}$ of the group $\mathrm{Aut}_0\,\mathcal{M}$ of all automorphisms of $\mathcal{M}$ preserving the natural grading of $\mathcal{M}$. For every $\sigma \in \mathrm{Aut}_0\,\mathcal{M}$ the subalgebra $\Phi^{-1}(\sigma(\mathfrak{t}) + T_0)$ is a nonstandard torus of maximal dimension in L_p and the elements $(\Phi^{-1}\circ\sigma)(x_1\partial_1)$ and $(\Phi^{-1}\circ\sigma)(x_2\partial_2)$ are toral in L_p by Lemma 7.4. Since the group $\mathrm{Aut}_0\,\mathcal{M}$ acts transitively on the set of bases of $\mathcal{M}_{-3}$, there is $\tau \in \mathrm{Aut}_0\,\mathcal{M}$ such that the elements $((\Phi^{-1}\circ\tau)(\partial_1))^{[p]}$ and $((\Phi^{-1}\circ\tau)(\partial_2))^{[p]}$ are both nonzero. Replacing $\mathfrak{t}$ by $\tau(\mathfrak{t})$ and renumbering the t_i's if necessary, we thus may assume that λ_1 and μ_1 are both nonzero.

Since F is infinite, there exist $a,b \in F^\times$ such that the elements $a^p\lambda_1$ and $b^p\mu_1$ of F are linearly independent over $\mathbb{F}_p$. Applying a suitable automorphism from the subgroup $\mathbf{T}$ of $\mathrm{Aut}_0\,\mathcal{M}$ one observes that $\mathfrak{t}' := F(a+x_1)\partial_1 \oplus F(b+x_2)\partial_2$, is a 2-dimensional nonstandard torus in $\mathcal{M}$ and $\mathfrak{t}' = (\mathfrak{c}_{\mathcal{M}}(\mathfrak{t}'))^3$ (alternatively, one can apply [P 94, Lemmas 4.1 and 4.4]). This entails that

$$T' := \Phi^{-1}(\mathfrak{t}' \oplus T_0) = F(q_1+an_1) \oplus F(q_2+bn_2) \oplus T_0$$

is a nonstandard torus of maximal dimension in L_p with $F(q_1+an_1) \oplus F(q_2+bn_2) = (\mathfrak{c}_L(T'))^3$. Suppose

$$\big(x(q_1+an_1) + y(q_2+bn_2)\big)^{[p]} = x(q_1+an_1) + y(q_2+bn_2) \tag{9.1}$$

for some $x, y \in F$. Applying Φ to both sides of (9.1) gives

$$\big(x(a+x_1)\partial_1 + y(b+x_2\partial_2)\big)^{[p]} = x(a+x_1)\partial_1 + y(b+x_2)\partial_2.$$

As both $(a+x_1)\partial_1$ and $(b+x_2)\partial_2$ are toral elements of $\mathcal{M}$, we get $x, y \in \mathbb{F}_p$. Hence

$$\begin{aligned} x(q_1+an_1) + y(q_2+bn_2) &= \big(x(q_1+an_1) + y(q_2+bn_2)\big)^{[p]} \\ &= x\big(q_1+an_1+a^p h_1\big) + y\big(q_2+bn_2+b^p h_2\big), \end{aligned}$$

implying $xa^p h_1 + yb^p h_2 = 0$. As a consequence, $xa^p\lambda_j + yb^p\mu_j = 0$ for all $j \leqslant s$. But then $a^p\lambda_1$ and $b^p\mu_1$ are linearly dependent over $\mathbb{F}_p$, a contradiction. We conclude that $(\mathfrak{c}_L(T'))^3$ contains no nonzero toral elements of L_p. $\square$

Retain the notation introduced in Sections 7 and 8. In view of Proposition 9.1, we may assume that for every $\alpha \in \Gamma$ no nonzero element of $H^3 \cap \ker\alpha$ is toral in L_p.

The map $\Theta : L_p(\alpha) \xrightarrow{\sim} R$ defined in Section 8 induces a natural Lie algebra isomorphism

$$\overline{\Theta} : L_p(\alpha)/\mathfrak{z}\big(L_p(\alpha)\big) \xrightarrow{\sim} R/\mathfrak{z}(R) \cong H(2;\underline{1})^{(2)} \oplus FD_H\big(x_2^{(5)}\big).$$

Let $(R/\mathfrak{z}(R))_{(i)}$ denote the ith component of the standard filtration of the Cartan type Lie algebra $R/\mathfrak{z}(R)$, where $i \geqslant -1$, and denote by $L_p(\alpha)_{(i)}$ the inverse image of $(R/\mathfrak{z}(R))_{(i)}$ under $\overline{\Theta}$. We thus obtain a filtration $\{L_p(\alpha)_{(i)} \mid i \geqslant -1\}$ of the Lie algebra $L_p(\alpha)$ with $\bigcap_{i\geqslant -1} L_p(\alpha)_{(i)} = T \cap \ker\alpha$ and $\dim(L_p(\alpha)/L_p(\alpha)_{(0)}) = 2$. This filtration is, in fact, independent of the choice of $\overline{\Theta}$, because $(R/\mathfrak{z}(R))_{(0)}$ is the unique subalgebra of codimension 2 in the Cartan type Lie algebra $R/\mathfrak{z}(R)$. Since $\overline{\Theta}$ is a restricted Lie algebra isomorphism, all $L_p(\alpha)_{(i)}$ are restricted subalgebras of $L_p(\alpha)$. We denote by $\mathrm{nil}_{[p]}(L_p(\alpha)_{(i)})$ the maximal ideal of $L_p(\alpha)_{(i)}$ consisting of p-nilpotent elements of L_p.

Definition 9.1. *Define*

$$\begin{aligned} W &:= \big\{u \in L_p(\alpha)^{(1)} \cap L_p(\alpha)_{(0)} \;\big|\; u^{[p]} \in L_p(\alpha)^{(1)}\big\}; \\ P &:= \big\{u \in W \;\big|\; [u, W] \subset W\big\}; \\ Q(\alpha) &:= P + \mathrm{nil}_{[p]}\big(L_p(\alpha)_{(3)}\big). \end{aligned}$$

Because of the uniqueness of the filtration $\{L_p(\alpha)_{(i)} \mid i \geqslant -1\}$ this definition is independent of the choices made earlier. The main result of this section is the following:

Proposition 9.2. *If (α, β) is a Melikian pair in Γ^2, then*

$$Q(\alpha) = L_p(\alpha) \cap \big(L_p(\alpha,\beta)^{(1)}\big)_{(0)}.$$

Proof. (a) Choose any Lie algebra isomorphism $\Phi : L_p(\alpha,\beta) \xrightarrow{\sim} \widetilde{\mathcal{M}} = \mathcal{M} \oplus T_0$ satisfying (7.1), (7.2) and (8.1). Then $\Phi(L_p(\alpha) \cap (L_p(\alpha,\beta)^{(1)})_{(0)})$ is spanned by

$$\big\{x_2^{(r)}\partial_2, x_2^{(r)}(1+x_1)\partial_1, x_2^{(r)}(1+x_1)^2, x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(r)}(1+x_1)^4\tilde{\partial}_1 \;\big|\; 1 \leqslant r \leqslant 4\big\}.$$

Let $\Theta = \Phi \circ \Theta' : L_p(\alpha) \xrightarrow{\sim} R$ be the isomorphism associated with Φ. The explicit formulae for Θ' yield that $\Theta(L_p(\alpha) \cap (L_p(\alpha,\beta)^{(1)})_{(0)})$ is spanned by the set

$$\big\{x_1x_2^{(r)}, x_1^{(2)}x_2^{(r)}, x_1^{(3)}x_2^{(r)} \;\big|\; 1 \leqslant r \leqslant 4\big\} \cup \big\{x_1^{(4)}x_2^{(r)} \;\big|\; 0 \leqslant r \leqslant 3\big\} \cup \big\{x_2^{(r)} \;\big|\; 2 \leqslant r \leqslant 5\big\};$$

see Proposition 8.1.

(b) Next we are going to determine $\Theta(W)$, $\Theta(P)$ and $\Theta(Q(\alpha))$ by using Definition 9.1. First we observe that

$$\Theta\bigl(L_p(\alpha)^{(1)} \cap L_p(\alpha)_{(0)}\bigr) = Fz \oplus \biggl(\bigoplus_{0\leqslant i,\, j\leqslant 4,\, 2\leqslant i+j\leqslant 7} Fx_1^{(i)}x_2^{(j)}\biggr);$$

see Proposition 8.1. It is immediate from Eqs. (8.3) that

$$\bigl(x_1^{(i)}x_2^{(j)}\bigr)^p \in \Theta\bigl(L_p(\alpha)^{(1)} \cap L_p(\alpha)_{(0)}\bigr) \quad \text{whenever } i+j \geqslant 2.$$

Recall that Θ is an isomorphism of restricted Lie algebras. In conjunction with Jacobson's formula, this shows that $\Theta(W)$ is a subspace of R. As a consequence, we have the inclusion

$$\bigoplus_{0\leqslant i,\, j\leqslant 4,\, 2\leqslant i+j\leqslant 7} Fx_1^{(i)}x_2^{(j)} \subset \Theta(W).$$

On the other hand, if $z \in \Theta(W)$, then the definition of Θ' and our assumption on Φ yield $H^3 \cap \ker\alpha \subset W$. Then $h_\alpha \in W$. As $Fh_\alpha = H^3 \cap \ker\alpha = F\Theta^{-1}(z)$, our assumption on h_α in (8.1) yields $h_\alpha = \Phi^{-1}((1+x_1)\partial_1)$. It follows that $h_\alpha^{[p]} - h_\alpha \in L_p(\alpha)^{(1)} \cap T_0$. As $h_\alpha^{[p]} \neq h_\alpha$ by our choice of T, this entails $L_p(\alpha,\beta)^{(1)} \cap T_0 \neq (0)$ contradicting Lemma 7.3. We conclude that

$$\Theta(W) = \bigoplus_{0\leqslant i,\, j\leqslant 4,\, 2\leqslant i+j\leqslant 7} Fx_1^{(i)}x_2^{(j)}.$$

Let $u = \sum_{i,j} s_{i,j}x_1^{(i)}x_2^{(j)} \in \Theta(P)$. Since $x_1^{(2)}, x_1^{(3)} \in \Theta(W)$ and $[x_1^{(2)}, x_1^{(3)}] = z$, it follows readily from the definition of P that $s_{2,0} = s_{3,0} = 0$. The multiplication table for R given Section 8 now shows that $\Theta(P)$ is spanned by

$$\bigl\{x_1^{(4)}, x_2^{(2)}, x_2^{(3)}\bigr\} \cup \bigl\{x_1^{(i)}x_2, x_1^{(i)}x_2^{(2)}, x_1^{(i)}x_2^{(3)} \bigm| 1 \leqslant i \leqslant 4\bigr\} \cup \bigl\{x_1^{(i)}x_2^{(4)} \bigm| 0 \leqslant i \leqslant 3\bigr\}.$$

(c) Finally, the nilpotent subalgebra $\Theta(L_p(\alpha)_{(3)})$ is spanned by

$$\bigl\{x_1^{(i)}x_2^{(4)} \bigm| 0 \leqslant i, j \leqslant 4;\ 5 \leqslant i+j \leqslant 7\bigr\} \cup \bigl\{x_2^{(5)}, z\bigr\} \cup C.$$

By (8.3), the Lie product of any two elements in this set is p-nilpotent in R. Since Θ is an isomorphism of restricted Lie algebras, it follows that $\Theta(\mathrm{nil}_{[p]}(L_p(\alpha)_{(3)}))$ is spanned by $\{x_1^{(i)}x_2^{(4)} \mid 0 \leqslant i, j \leqslant 4; 5 \leqslant i+j \leqslant 7\} \cup \{x_2^{(5)}\}$. Comparing the spanning set of $\Theta(L_p(\alpha) \cap (L_p(\alpha,\beta)^{(1)})_{(0)})$ from part (a) of this proof with that of $\Theta(Q(\alpha)) = \Theta(P) + \Theta(\mathrm{nil}_{[p]}(L_p(\alpha)_{(3)}))$ we now obtain that

$$\Theta\bigl(L_p(\alpha) \cap \bigl(L_p(\alpha,\beta)^{(1)}\bigr)_{(0)}\bigr) = \Theta\bigl(Q(\alpha)\bigr).$$

Since Θ is an isomorphism, the proposition follows. □

Remark 9.3. Proposition 9.2 implies that $Q(\alpha)$ is a subalgebra of $L(\alpha)$.

At the end of Section 8 we mentioned that $\Theta'((1+x_2)\partial_2) = x_1(1+x_2)$. In what follows we require some computations in the subalgebra $\Theta(H) \subset \mathfrak{c}_R(x_1(1+x_2))$. It follows from the multiplication table for R that $\mathfrak{c}_R(x_1(1+x_2))$ contains $x_1^{(2)}(1+x_2)^2$ and $x_1^{(3)}(1+x_2)^3$. Set $w := x_2 - x_2^{(2)} + 2x_2^{(3)} - x_2^{(4)} - x_2^{(5)}$ and observe that

$$\begin{aligned}\big[x_1(1+x_2), w\big] = [x_1, w] + [x_1x_2, w] = \big(-x_2 + 2x_2^{(2)} - x_2^{(3)} - x_2^{(4)}\big)\\ + \left(x_2 - \binom{2}{1}x_2^{(2)} + 2\binom{3}{1}x_2^{(3)} - \binom{4}{1}x_2^{(4)}\right) = 0.\end{aligned} \tag{9.2}$$

Applying Proposition 8.1 it is now easy to see that

$$\bigoplus_{i=1}^{3} Fx_1^{(i)}(1+x_2)^i \oplus Fw \oplus Fz \subset \Theta\big(L_p(\alpha,\beta)^{(1)} \cap H_p\big) \subset \Theta(H).$$

Direct computations show that

$$\begin{aligned}\big[x_1^{(2)}(1+x_2)^2, w\big] &= x_1(1+x_2)^2\big(1 - x_2 + 2x_2^{(2)} - x_2^{(3)} - x_2^{(4)}\big)\\ &= x_1(1+x_2)^6 = x_1(1+x_2); &(9.3)\\ \big[x_1^{(3)}(1+x_2)^3, w\big] &= x_1^{(2)}(1+x_2)^3\big(1 - x_2 + 2x_2^{(2)} - x_2^{(3)} - x_2^{(4)}\big)\\ &= x_1^{(2)}(1+x_2)^7 = x_1^{(2)}(1+x_2)^2. &(9.4)\end{aligned}$$

Proposition 9.4. *Let α be an arbitrary root of Γ. Then for any $r \in \mathbb{F}_p^\times$ there exists a linear map $l_{r\alpha}: L_{r\alpha} \to H$ such that $x - l_{r\alpha}(x) \in Q(\alpha)$ for all $x \in L_{r\alpha}$. Furthermore, $H \cap Q(\alpha) = (0)$ and $L(\alpha) = H + Q(\alpha)$.*

Proof. In order to perform computations in $L_p(\alpha)$ we are going to invoke the isomorphism $\Theta = \Theta' \circ \Phi$; see Proposition 8.1. Recall that

$$\Theta(T) = Fx_1(1+x_2) \oplus Fz \oplus C.$$

Replacing α by an $\mathbb{F}_p^\times$-multiple of α, if necessary, we may assume that $\alpha(x_1(1+x_2)) = 1$. Using the multiplication table for R it is then straightforward to see that

$$\Theta(L_{r\alpha}) = \bigoplus\nolimits_{i=1}^{3} Fx_1^{(i)}(1+x_2)^{r+i} \oplus F\big(x_1^{(4)}(1+x_2)^{r-1} - r^{-1}z\big) \oplus F\big((1+x_2)^r - 1\big)$$

for all $r \in \mathbb{F}_p^\times$ and that $\Theta(H)$ is sandwiched between $\bigoplus_{i=1}^{3} Fx_1^{(i)}(1+x_2)^i \oplus Fw \oplus Fz$ and $\Theta(H_p) = \bigoplus_{i=1}^{3} Fx_1^{(i)}(1+x_2)^i \oplus Fw \oplus Fz \oplus C$. We now define a linear map $l_{r\alpha}: L_{r\alpha} \to H$ by the formula $l_{r\alpha} = \Theta^{-1} \circ m_r \circ \Theta$, where m_r is the linear map from $\Theta(L_{r\alpha})$ into $\Theta(H)$ given by

$$\begin{aligned}m_r\big(x_1^{(4)}(1+x_2)^{r-1} - r^{-1}z\big) &= -r^{-1}z;\\ m_r\big(x_1^{(i)}(1+x_2)^{r+i}\big) &= x_1^{(i)}(1+x_2)^i, \quad 1 \leqslant i \leqslant 3;\\ m_r\big((1+x_2)^r - 1\big) &= rw.\end{aligned}$$

Using the spanning set of $\Theta(Q(\alpha))$ from the proof of Proposition 9.2 one observes that $w - x_2 \in \Theta(Q(\alpha))$ and $x_1^{(i)}(1+x_2)^i - x_1^{(i)} \in \Theta(Q(\alpha))$ for $1 \leqslant i \leqslant 3$. By the same token, one finds that the subspace $\bigoplus_{i=1}^{3} Fx_1^{(i)} \oplus Fx_2 \oplus Fz \oplus C$ of R complements $\Theta(Q(\alpha))$. Since $x_1^{(4)}(1+x_2)^{r-1} \in \Theta(Q(\alpha))$ for all $r \in \mathbb{F}_p^\times$, this implies that $y - m_r(y) \in \Theta(Q(\alpha))$ for all $y \in \Theta(L_{r\alpha})$ and $R = \Theta(H_p) \oplus \Theta(Q(\alpha))$.

As a result, $x - l_{r\alpha}(x) \in Q(\alpha)$ for all $r \in \mathbb{F}_p^\times$ and all $x \in L_{r\alpha}$. Consequently, $L_p(\alpha) = H_p \oplus Q(\alpha)$. Since $Q(\alpha) \subset L(\alpha)$, this yields $L(\alpha) = H \oplus Q(\alpha)$ and the proposition follows. □

Proposition 9.5. *Let $\mathcal{N}(H)$ denote the set of all p-nilpotent elements of L_p contained in H. Then the following hold*:

(1) *$\mathcal{N}(H)$ is a 3-dimensional subspace of H.*
(2) *There exists a unique 2-dimensional subspace $H_{(-1)}$ in $\mathcal{N}(H)$ satisfying the condition $[H_{(-1)}, H_{(-1)}] \subset \mathcal{N}(H)$. Moreover, $[H_{(-1)}, [H_{(-1)}, H_{(-1)}]] = H^3$.*
(3) *For every $\alpha \in \Gamma$ the subspace $H_{(-1)} + Q(\alpha)$ is stable under the adjoint action of $Q(\alpha)$.*

Proof. Jacobson's formula together with (8.3) and the multiplication table for R shows that the subspace $N := Fx_1^{(2)}(1+x_2)^2 \oplus Fx_1^{(3)}(1+x_2)^3 \oplus Fw$ consists of p-nilpotent elements of R. On the other hand, it is clear from our remarks in the proof of Proposition 9.4 that $\Theta(H_p) = \Theta(T) \oplus N$. Since $\Theta(T)$ is a torus, this entails that N coincides with the set of all p-nilpotent elements of the restricted Lie algebra $\Theta(H_p)$. Since $\Theta : L_p(\alpha) \xrightarrow{\sim} R$ is an isomorphism of restricted Lie algebras, we deduce that $\mathcal{N}(H) = \Theta^{-1}(N)$ is a 3-dimensional subspace of H.

The elements $D_H(x_1^{(2)}(1+x_2)^2)$ and $D_H(x_1^{(3)}(1+x_2)^3)$ of the Hamiltonian algebra $H(2;\underline{1})^{(2)}$ commute. Therefore, in our central extension R we have the equality

$$\big[x_1^{(2)}(1+x_2)^2, x_1^{(3)}(1+x_2)^3\big] = \big[x_1^{(2)}, x_2^{(3)}\big] = z. \tag{9.5}$$

Now take any linearly independent elements $u_1 = a_1x_1^{(2)}(1+x_2)^2 + b_1x_1^{(3)}(1+x_2)^3 + c_1w$ and $u_2 = a_2x_1^{(2)}(1+x_2)^2 + b_2x_1^{(3)}(1+x_2)^3 + c_2w$ in N such that $[u_1, u_2] \in N$. Then (9.5) together with (9.3) and (9.4) yields

$$N \ni [u_1, u_2] = (a_1b_2 - a_2b_1)z + (a_1c_2 - a_2c_1)x_1(1+x_2) + (b_1c_2 - b_2c_1)x_1^{(2)}(1+x_2)^2,$$

forcing $a_1b_2 = a_2b_1$ and $a_1c_2 = a_2c_1$. If $a_1 \neq 0$, then $u_2 = \frac{a_2}{a_1}u_2$ which is false. Therefore, $a_1 = 0$. Arguing similarly, one obtains $a_2 = 0$. This shows that $H_{(-1)} := \Theta^{-1}(Fx_1^{(3)}(1+x_2)^3 \oplus Fw)$ is the only 2-dimensional subspace of $\mathcal{N}(H)$ with the property that $[H_{(-1)}, H_{(-1)}] \subset \mathcal{N}(H)$. Combining (9.4), (9.3) and (9.5) one derives that $[H_{(-1)}, [H_{(-1)}, H_{(-1)}]] = H^3$.

Using the spanning set for $\Theta(Q(\alpha))$ displayed in part (a) the proof of Proposition 9.2 and the multiplication table for R, it is routine to check that

$$\big[\Theta\big(Q(\alpha)\big), Fx_1^{(3)}(1+x_2)^3 \oplus Fw\big] \subset \Theta\big(Q(\alpha)\big) + Fx_1^{(3)}(1+x_2)^3 \oplus Fw.$$

This implies that $H_{(-1)} + Q(\alpha)$ is invariant under the adjoint action of $Q(\alpha)$. □

10. Conclusion

For any $\gamma \in \Gamma$ we fix a map $l_\gamma : L_\gamma \to H$ satisfying the conditions of Proposition 9.4. Given $x \in L_\gamma$ we set $\tilde{x} := x - l_\gamma(x)$, an element of $Q(\alpha)$. Define

$$L_{(0)} := \sum_{\gamma \in \Gamma} Q(\gamma),$$

a subspace of L. We are going to show that $L_{(0)}$ is actually a subalgebra of L. Since it follows from Remark 9.3 that $[Q(\gamma), Q(\gamma)] \subset L_{(0)}$ for all $\gamma \in \Gamma$, we just need to check that $[Q(\alpha), Q(\beta)] \subset L_{(0)}$ for all $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma$.

Lemma 10.1. *Let (α, β) be an arbitrary Melikian pair in Γ^2 and let $x \in L_\alpha$, $y \in L_\beta$. Then $[\tilde{x}, \tilde{y}] \in L_{(0)}$ and*

$$[\tilde{x}, \tilde{y}] \equiv \widetilde{[x, y]} \quad \big(\mathrm{mod}\ Q(\alpha) + Q(\beta)\big).$$

Proof. Set $\Delta := \{\alpha\} \cup (\beta + \mathbb{F}_p\alpha)$. Proposition 9.4 says that $L(\delta) = H \oplus Q(\delta)$ for any $\delta \in \Delta$. In conjunction with Proposition 9.2, this gives

$$\big(L_p(\alpha,\beta)^{(1)}\big)(\delta) = \big(H \cap L_p(\alpha,\beta)^{(1)}\big) \oplus Q(\delta) \quad (\forall \delta \in \Delta). \tag{10.1}$$

Recall that $\Phi : L_p(\alpha,\beta)^{(1)} \xrightarrow{\sim} \mathcal{M}$ is a Lie algebra isomorphism taking $H \cap L_p(\alpha,\beta)^{(1)}$ onto $\mathfrak{c}_{\mathcal{M}}(\mathfrak{t})$ and $(L_p(\alpha,\beta)^{(1)})_{(0)}$ onto $\mathcal{M}_{(0)}$. Therefore,

$$\dim H \cap L_p(\alpha,\beta)^{(1)} = 5, \qquad \dim\big(L_p(\alpha,\beta)^{(1)}\big)_{(0)} = 120. \tag{10.2}$$

Combining (10.2) and (10.1) we now deduce that for every $\delta \in \Delta$ the subalgebra $Q(\delta) = L_p(\delta) \cap (L_p(\alpha,\beta)^{(1)})_{(0)}$ has codimension 5 in the 1-section $(L_p(\alpha,\beta)^{(1)})(\delta)$. Since $L_p(\alpha,\beta)^{(1)} \cong \mathcal{M}$, it follows from [P 94, Lemmas 4.1 and 4.4], for instance, that $\dim(L_p(\alpha,\beta)^{(1)})(\delta) = 25$. Therefore, $\dim Q(\delta) = 20$ for all $\delta \in \Delta$.

For any $\mu \in \Delta$ one has

$$Q(\mu) \cap \Big(\sum_{\delta \in \Delta\setminus\{\mu\}} Q(\delta)\Big) \subset Q(\mu) \cap \Big(\sum_{\delta \in \Delta\setminus\{\mu\}} L(\delta)\Big) \subset Q(\mu) \cap H = (0).$$

This shows that the sum $Q(\alpha) + \sum_{j=0}^{4} Q(\beta + j\alpha)$ is direct. But then

$$\dim\Big(Q(\alpha) \oplus \bigoplus_{j=0}^{4} Q(\beta + j\alpha)\Big) = 6 \cdot 20 = 120 = \dim\big(L_p(\alpha,\beta)^{(1)}\big)_{(0)},$$

implying that $(L_p(\alpha,\beta)^{(1)})_{(0)} = Q(\alpha) + \sum_{j \in \mathbb{F}_p} Q(\beta + j\alpha)$. As a consequence,

$$\begin{aligned}\big[Q(\alpha), Q(\beta)\big] &\subset \big[\big(L_p(\alpha,\beta)^{(1)}\big)_{(0)}, \big(L_p(\alpha,\beta)^{(1)}\big)_{(0)}\big] \subset \big(L_p(\alpha,\beta)^{(1)}\big)_{(0)} \\ &= Q(\alpha) + \bigoplus_{j=0}^{4} Q(\beta + j\alpha) \subset L_{(0)}.\end{aligned} \tag{10.3}$$

This shows that $[\tilde{x}, \tilde{y}] \in L_{(0)}$. Computing modulo $Q(\alpha) + Q(\beta)$ we get

$$\begin{aligned}[\tilde{x}, \tilde{y}] &= \big([x,y] - l_{\alpha+\beta}\big([x,y]\big) - \big[x, l_\beta(y)\big] + l_\alpha\big(\big[x, l_\beta(y)\big]\big) - \big[l_\alpha(x), y\big] + l_\beta\big(\big[l_\alpha(x), y\big]\big) \\ &\quad + \big[l_\alpha(x), l_\beta(y)\big]\big) + \big(l_{\alpha+\beta}\big([x,y]\big) - l_\alpha\big(\big[x, l_\beta(y)\big]\big) - l_\beta\big(\big[l_\alpha(x), y\big]\big)\big) \\ &= \widetilde{[x,y]} - \widetilde{\big[x, l_\beta(y)\big]} - \widetilde{\big[l_\alpha(x), y\big]} + \tilde{h} \\ &\equiv \widetilde{[x,y]} + \tilde{h},\end{aligned}$$

where $\tilde{h} = l_{\alpha+\beta}([x,y]) - l_\alpha([x, l_\beta(y)]) - l_\beta([l_\alpha(x), y]) + [l_\alpha(x), l_\beta(y)]$. As $\widetilde{[x,y]} \in L_{(0)}$, it must be that $\tilde{h} \in H \cap L_{(0)} = H \cap (\sum_{\gamma \in \Gamma} Q(\gamma))$. Expressing $\tilde{h} = \sum_{\gamma \in \Gamma}(v_\gamma - l_\gamma(v))$ with $v_\gamma \in L_\gamma$ we see that $v_\gamma = 0$ for all γ, whence $l_\gamma(v_\gamma) = 0$ and $\tilde{h} = 0$. The result follows. □

Theorem 10.2. *$L_{(0)}$ is a proper subalgebra of L.*

Proof. By our earlier remark in this section, we need to show that $[Q(\alpha), Q(\beta)] \subset L_{(0)}$ for all pairs $(\alpha, \beta) \in \Gamma^2$ such that α and β are $\mathbb{F}_p$-independent. If (α, β) is a Melikian pair, this follows from Lemma 10.1.

Take any $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma$ for which (α, β) is not a Melikian pair. Then $H^3 \cap \ker\alpha = H^3 \cap \ker\beta$; see Lemma 7.1. Recall that $H^3 \cap \ker\alpha = Fh_\alpha$ for some nonzero $h_\alpha \in H^3$. Put $\Gamma(\alpha) := \{\gamma \in \Gamma \mid \gamma(h_\alpha) \neq 0\}$. Since $H^3 \subset T$, the set $\Gamma(\alpha)$ is nonempty. Then it follows from Schue's lemma [St 04, Proposition 1.3.6(1)] that

$$L_\beta = \sum_{\gamma \in \Gamma(\alpha)} [L_\gamma, L_{\beta-\gamma}]. \tag{10.4}$$

Let γ be an arbitrary root in $\Gamma(\alpha)$. Since $\alpha(h_\alpha) = \beta(h_\alpha) = 0$, it is immediate from Lemma 7.1 that (α, γ) and $(\alpha, \beta - \gamma)$ are Melikian pairs in Γ^2.

Suppose $(\alpha + \gamma, \beta - \gamma)$ is not a Melikian pair. Then $(\beta - \gamma)(h_{\alpha+\gamma}) = 0$ by Lemma 7.1. As $(\beta - \gamma)(h_\alpha) = -\gamma(h_\alpha) \neq 0$ and $\dim H^3 = 2$ by Theorem 5.8(2), this yields $H^3 = Fh_\alpha \oplus Fh_{\alpha+\gamma}$. Also, $(\alpha + \beta)(h_\alpha) = 0$ and $(\alpha + \beta)(h_{\alpha+\gamma}) = ((\alpha + \gamma) + (\beta - \gamma))(h_{\alpha+\gamma}) = 0$ by our assumption on $(\alpha + \gamma, \beta - \gamma)$. This shows that $\alpha + \beta$ vanishes on H^3 and hence on $(H^3)_p = T$; see Theorem 5.8(2). But then $\alpha + \beta = 0$, a contradiction. Thus, $(\alpha + \gamma, \beta - \gamma)$ is a Melikian pair.

If $(\gamma, \alpha + \beta - \gamma)$ is not a Melikian pair, then $\gamma(h_{\alpha+\beta-\gamma}) = 0$. As $\gamma \in \Gamma(\alpha)$, we then have $H^3 = Fh_\alpha \oplus Fh_{\alpha+\beta-\gamma}$. But then $\alpha + \beta = \gamma + (\alpha + \beta - \gamma)$ vanishes on $(H^3)_p$, a contradiction. So $(\gamma, \alpha + \beta - \gamma)$ is a Melikian pair, too.

We now take arbitrary $u \in L_\alpha$ and $v \in L_\beta$. By (10.4), there exist $\gamma_1, \dots, \gamma_N \in \Gamma(\alpha)$ such that $v = \sum_{i=1}^N [x_i, y_i]$ for some $x_i \in L_{\gamma_i}$ and $y_i \in L_{\beta-\gamma_i}$, where $1 \leqslant i \leqslant N$. Applying Lemma 10.1 and the preceding remarks we obtain

$$\begin{aligned}
[\tilde{u}, \tilde{v}] &\in \sum_{i=1}^N \big[\tilde{u}, [\tilde{x}_i, \tilde{y}_i]\big] + \sum_{i=1}^N \big[Q(\alpha), Q(\gamma_i) + Q(\beta - \gamma_i)\big] \\
&\subset \sum_{i=1}^N \big(\big[[\tilde{u}, \tilde{x}_i], \tilde{y}_i\big] + \big[\tilde{x}_i, [\tilde{u}, \tilde{y}_i]\big]\big) + L_{(0)} \\
&\subset \sum_{i=1}^N \big(\big[Q(\alpha + \gamma_i), Q(\beta - \gamma_i)\big] + \big[Q(\gamma_i), Q(\alpha + \beta - \gamma_i)\big]\big) + L_{(0)} \subset L_{(0)}.
\end{aligned}$$

Consequently, $[Q(\alpha), Q(\beta)] \subset L_{(0)}$ in all cases. The argument at the end of the proof of Lemma 10.1 shows that $L_{(0)} \cap H = (0)$. Hence $L_{(0)}$ is a proper subalgebra of L. □

Recall the subspace $H_{(-1)}$ from Proposition 9.5(2). According to Proposition 9.5(3), $[Q(\gamma), H_{(-1)}] \subset H_{(-1)} + Q(\gamma) \subset H_{(-1)} + L_{(0)}$ for all $\gamma \in \Gamma$. In view of Theorem 10.4, this means that

$$\big[L_{(0)}, H_{(-1)} + L_{(0)}\big] = \left[\sum_{\gamma \in \Gamma} Q(\gamma), H_{(-1)} + \sum_{\delta \in \Gamma} Q(\delta)\right] \subset H_{(-1)} + L_{(0)}.$$

Thus, $L_{(-1)} := H_{(-1)} + L_{(0)}$ is stable under the adjoint action of the subalgebra $L_{(0)}$.

We have finally come to the end of this tale. Let L' denote the subalgebra of L generated by $L_{(-1)}$. Proposition 9.5(2) shows that $H^3 \subset L'$. Then the p-envelope of L' in L_p contains $(H^3)_p = T$; see Theorem 5.8(2). As a consequence, L' is T-stable. Let γ be any root in Γ. Then $[T, x - l_\gamma(x)] \subset L'$ for all $x \in L_\gamma$, implying $L_\gamma \subset L'$. As this holds for all $\gamma \in \Gamma$ and L is simple, we deduce that $L' = L$.

It follows from Theorem 10.4 that $L_{(-1)} \supsetneq L_{(0)}$. We now consider the standard filtration of L associated with the pair $(L_{(-1)}, L_{(0)})$ (it is defined recursively by setting $L_{(i)} := \{x \in L_{(i-1)} \mid [x, L_{(i-1)}] \subset L_{(i-1)}\}$ and $L_{(-i)} := [L_{(-1)}, L_{(-i+1)}] + L_{(-i+1)}$ for all $i > 0$). Since L is simple and finite-dimensional, this filtration is exhaustive and separating. Let $G = \bigoplus_{i\in\mathbb{Z}} G_i$ denote the associated graded Lie algebra, where $G_i = \mathrm{gr}_i(L) = L_{(i)}/L_{(i+1)}$.

Since $L_{(-1)} = H_{(-1)} + L_{(0)}$, we have that $L_{(-i)} = L_{(0)} + \sum_{j=1}^{i}(H_{(-1)})^j$ for all $i > 0$. Since $(H_{(-1)})^3 \subset H^3 \subset \mathfrak{z}(H)$ by Theorem 5.8(2), this shows that $L_{(-4)} = L_{(-3)}$, i.e. $G_{-4} = (0)$. As $\dim H_{(-1)} = 2$, we obtain by the same token that $\dim G_{-2} \leqslant 1$ and $\dim G_{-3} \leqslant 2$.

Let (α, β) be any Melikian pair in Γ^2. By our remarks in the proof of Lemma 10.1, $(L_p(\alpha,\beta)^{(1)}) \cap L_{(0)} = (L_p(\alpha,\beta)^{(1)})_{(0)}$, while from the explicit description of $\Theta(H_{(-1)})$ in the proof of Propositions 9.5 and 8.1 we see that

$$H_{(-1)} + \bigl(L_p(\alpha,\beta)^{(1)}\bigr)_{(0)} = \bigl(L_p(\alpha,\beta)^{(1)}\bigr)_{(-1)}. \tag{10.5}$$

In particular, $H_{(-1)} \subset L_p(\alpha,\beta)^{(1)}$. It follows that the filtration of $L_p(\alpha,\beta)^{(1)} \cong \mathcal{M}$ induced by that of L has the property that

$$L_{(i)} = \bigl(L_p(\alpha,\beta)^{(1)} \cap L_{(i)}\bigr) + L_{(i-1)}, \quad i = -1, -2, -3.$$

In view of (10.5), this entails that $\dim G_{-1} = \dim G_{-3} = 2$ and $\dim G_2 = 1$.

As $\dim G_{-1} = 2$, and G_0 acts faithfully on G_{-1}, we have an embedding $G_0 \subset \mathfrak{gl}(2)$. As $(L_p(\alpha,\beta)^{(1)})_{(0)}$ acts on $(L_p(\alpha,\beta)^{(1)})_{(-1)}/(L_p(\alpha,\beta)^{(1)})_{(0)}$ as $\mathfrak{gl}(2)$, it follows from (10.5) that $(L_{(0)} \cap L_p(\alpha,\beta)^{(1)})/(L_{(1)} \cap L_p(\alpha,\beta)^{(1)}) \cong \mathfrak{gl}(2)$. As a consequence, $G_0 \cong \mathfrak{gl}(2)$. Finally, (10.5) yields that $L_p(\alpha,\beta)^{(1)} \cap L_{(4)} \neq (0)$, giving $G_4 \neq (0)$.

Applying [St 04, Theorem 5.4.1] we now obtain that the graded Lie algebra G is isomorphic to a Melikian algebra $\mathcal{M}(m,n)$ regarded with its natural grading. By a result of Kuznetsov [Kuz 91], any filtered deformation of the naturally graded Lie algebra $\mathcal{M}(m,n)$ is isomorphic to $\mathcal{M}(m,n)$; see [St 04, Theorem 6.7.3]. Thus, $L \cong \mathcal{M}(m,n)$, completing the proof of Theorem 1.2.

Corollary 10.3. *Let L be a finite-dimensional simple Lie algebra of Cartan type over an algebraically closed field of characteristic $p > 3$ and let T be a torus of maximal dimension in $L_p \subset \operatorname{Der} L$. Then the centralizer of T in L_p acts triangulably on L.*

Proof. This is an immediate consequence of [P-St 04, Theorem A] and Theorem 1.2. □

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References

[B-W 82] R.E. Block, R.L. Wilson, The simple Lie p-algebras of rank two, Ann. of Math. 115 (1982) 93–186.
[B-W 88] R.E. Block, R.L. Wilson, Classification of the restricted simple Lie algebras, J. Algebra 114 (1988) 115–259.
[Kuz 91] M.I. Kuznetsov, The Melikyan algebras as Lie algebras of type G_2, Comm. Algebra 19 (1991) 1281–1312.
[P 85] A.A. Premet, Algebraic groups associated with Lie p-algebras of Cartan type, Mat. Sb. 122 (1983) 82–96 (in Russian); English transl.: Math. USSR Sb. 50 (1985) 85–97.
[P 89] A.A. Premet, Regular Cartan subalgebras and nilpotent elements in restricted Lie algebras, Mat. Sb. 180 (1989) 542–557 (in Russian); English transl.: Math. USSR Sb. 66 (1990) 555–570.
[P 94] A. Premet, A generalization of Wilson's theorem on Cartan subalgebras of simple Lie algebras, J. Algebra 167 (1994) 641–703.
[P 95] A. Premet, Irreducible representations of Lie algebras of reductive groups and the Kac–Weisfeiler conjecture, Invent. Math. 121 (1995) 79–117.
[P-Sk 99] A. Premet, S. Skryabin, Representations of restricted Lie algebras and families of associative $\mathcal{L}$-algebras, J. Reine Angew. Math. 507 (1999) 189–218.
[P-St 97] A. Premet, H. Strade, Simple Lie algebras of small characteristic: I. Sandwich elements, J. Algebra 189 (1997) 419–480.
[P-St 99] A. Premet, H. Strade, Simple Lie algebras of small characteristic II. Exceptional roots, J. Algebra 216 (1999) 190–301.

[P-St 01] A. Premet, H. Strade, Simple Lie algebras of small characteristic: III. The toral rank 2 case, J. Algebra 242 (2001) 236–337.

[P-St 04] A. Premet, H. Strade, Simple Lie algebras of small characteristic IV. Solvable and classical roots, J. Algebra 278 (2004) 766–833.

[P-St 05] A. Premet, H. Strade, Simple Lie algebras of small characteristic V. The non-Melikian case, J. Algebra 314 (2007) 664–692.

[P-St 06] A. Premet, H. Strade, Classification of finite dimensional simple Lie algebras in prime characteristics, in: Contemp. Math., vol. 413, 2006, pp. 185–214.

[Sk 01] S. Skryabin, Tori in the Melikian algebra, J. Algebra 243 (2001) 69–95.

[St 89] H. Strade, The classification of the simple modular Lie algebras: I. Determination of the two-sections, Ann. of Math. 130 (1989) 643–677.

[St 91] H. Strade, The classification of the simple modular Lie algebras: III. Solution to the classical case, Ann. of Math. 133 (1991) 577–604.

[St 92] H. Strade, The classification of the simple modular Lie algebras: II. The toral structure, J. Algebra 151 (1992) 425–475.

[St 93] H. Strade, The classification of the simple modular Lie algebras: IV. Determining the associated graded algebra, Ann. of Math. 138 (1993) 1–59.

[St 94] H. Strade, The classification of the simple modular Lie algebras: V. Algebras with Hamiltonian two-sections, Abh. Math. Sem. Univ. Hamburg 64 (1994) 167–202.

[St 98] H. Strade, The classification of the simple modular Lie algebras: VI. Solving the final case, Trans. Amer. Math. Soc. 350 (1998) 2552–2628.

[St 04] H. Strade, Simple Lie Algebras over Fields of Positive Characteristic, vol. I: Structure Theory, de Gruyter Exp. Math., vol. 38, de Gruyter, Berlin, 2004.

[St-F] H. Strade, R. Farnsteiner, Modular Lie Algebras and Their Representations, Marcel Dekker Monogr. Textbooks, vol. 116, Marcel Dekker, New York, 1988.