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# The product in the Hochschild cohomology ring of preprojective algebras of Dynkin quivers

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**Abstract**

In this paper, we compute the cup product structure of the preprojective algebra Dynkin quivers of type $D$ and $E$ over a field of characteristic zero. This is a continuation of the work done in [P. Etingof, C. Eu, Hochschild and cyclic homology of preprojective algebras of ADE quivers, arXiv: math.AG/0609006] where the additive structure of the Hochschild cohomology (together with its grading) was computed. Together with the results in [K. Erdmann, N. Snashall, On Hochschild cohomology of preprojective algebras. I, J. Algebra 205 (2) (1998) 391–412, II, J. Algebra 205 (2) (1998) 413–434] (where the $A$-case was studied), this yields a complete description of the product in the Hochschild cohomology of $ADE$ quivers over a field of characteristic zero.



*Keywords:* Preprojective algebras; Dynkin quivers; Hochschild cohomology ring; Cup product

**Contents**



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## 1. Introduction

In this paper, we compute the product structure of the Hochschild cohomology of preprojective algebras of quivers of types $D$ and $E$ over a field of characteristic zero. This is a continuation of [EE2] where the cohomology spaces together with the grading induced by the natural grading (all arrows have degree 1) were computed.

Together with the results in [ES2] where it was done for type $A$ (over a field of any characteristic), this yields a complete description of the product in the Hochschild cohomology ring of preprojective algebras of $ADE$ quivers over a field of characteristic zero.

We note that this description is essentially uniform (i.e. does not refer to particular Dynkin-diagrams), while the proof uses case-by-case arguments.

For our computation, the same complex as in [ES2] is used, namely the one which we get by applying the *Hom*-functor to the Schofield resolution (which is periodic with period 6) of the algebra.

To compute the cup product, we use the same method as in [ES2]: via the isomorphism $HH^i(A) \equiv \underline{Hom}(\Omega^i A, A)$ (where for an $A$-bimodule $M$ we write $\Omega M$ for the kernel of its projective cover) and we identify elements in $HH^i(A)$ with equivalence classes of maps $\Omega^i(A) \to A$. For $[f] \in HH^i(A)$ and $[g] \in HH^j(A)$, the product is $[f][g] := [f \circ \Omega^i g]$ in $HH^{i+j}(A)$. We compute all products $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ for $0 \leqslant i \leqslant j \leqslant 5$. The remaining ones follow from the perodicity of the Schofield resolution and the graded commutativity of the multiplication. Some computations are similar to those in [ES2] for type $A$.

In the first part of the paper, we introduce a basis for each cohomology space explicitly (for each quiver). Then in the second part we compute the product in these bases. We use the results about the grading of the cohomology spaces from [EE2] to find the bases and the products.

The main result of the paper is Theorem 4.0.8, which explicitly gives the product structure in Hochschild cohomology of these preprojective algebras. The final section of the paper gives a description of the Hochschild cohomology by generators and relations.

Note that for connected non-Dynkin quivers, the Hochschild cohomology and its product structure were already calculated in [CBEG] where the situation is much easier because the homological dimension of the preprojective algebra is 2.

We leave out long computations in the journal-version and want to refer the reader to the web-version on arXiv: math.RT/0703568.

## 2. Preliminaries

### *2.1. Quivers and path algebras*

Let $Q$ be a quiver of $ADE$ type with vertex set $I$ and $|I| = r$. We write $a \in Q$ to say that $a$ is an arrow in $Q$.

We define $Q^*$ to be the quiver obtained from $Q$ by reversing all of its arrows. We call $\bar{Q} = Q \cup Q^*$ the *double* of $Q$.

Let $C$ be the adjacency matrix corresponding to the quiver $\bar{Q}$.

The concatenation of arrows generate the *nontrivial paths* inside the quiver $\bar{Q}$. We define $e_i$, $i \in I$ to be the *trivial path* which starts and ends at $i$. The *path algebra* $P_{\bar{Q}} = \mathbb{C}\bar{Q}$ of $\bar{Q}$ over

$\mathbb{C}$ is the $\mathbb{C}$-algebra with basis the paths in $\bar{Q}$ and the product $xy$ of two paths $x$ and $y$ is their concatenation if they are compatible and 0 if not. We define the *Lie bracket* $[x, y] = xy - yx$.

Let $R = \bigoplus_{i \in I} \mathbb{C}e_i$. Then $R$ is a commutative semisimple algebra, and $P_{\bar{Q}}$ is naturally an $R$-bimodule.

### 2.2. The preprojective algebra

Given a quiver $Q$, we define the *preprojective algebra* $\Pi_Q$ to be the quotient of the path algebra $P_{\bar{Q}}$ by the relation $\sum_{a \in Q}[a, a^*] = 0$.

Given a monomial $x = a_1 a_2 \cdots a_n \in P_{\bar{Q}}$, we write $x^*$ to be the monomial $a_n^* \cdots a_2^* a_1^*$, and we extend this definition linearly to all elements in $P_{\bar{Q}}$.

We introduce a grading, such that each trivial path has degree 0 and each arrow in $\bar{Q}$ has degree 1.

From now on, we assume that $Q$ is of ADE type, and we write $A = \Pi_Q$.

### 2.3. Graded spaces and Hilbert series

Let $M = \bigoplus_{d \geqslant 0} M(d)$ be a $\mathbb{Z}_+$-graded vector space, with finite dimensional homogeneous subspaces. We denote by $M[n]$ the same space with grading shifted by $n$. The graded dual space $M^*$ is defined by the formula $M^*(n) = M(-n)^*$.

**Definition 2.3.1** *(The Hilbert series of vector spaces).* Let $M = \bigoplus_{d \geqslant 0} M(d)$ be a $\mathbb{Z}_+$-graded vector space, with finite dimensional homogeneous subspaces. We define the *Hilbert series* $h_M(t)$ to be the series

$$h_M(t) = \sum_{d=0}^{\infty} \dim M(d) t^d.$$

**Definition 2.3.2** *(The Hilbert series of bimodules).* Let $M = \bigoplus_{d \geqslant 0} M(d)$ be a $\mathbb{Z}_+$-graded bimodule over the ring $R$, so we can write $M = \bigoplus_{i,j \in I} M_{i,j}$. We define the *Hilbert series* $H_M(t)$ to be a matrix valued series with the entries

$$H_M(t)_{i,j} = \sum_{d=0}^{\infty} \dim M(d)_{i,j} t^d.$$

### 2.4. Frobenius algebras and Nakayama automorphism

**Definition 2.4.1.** Let $\mathcal{A}$ be a finite dimensional unital $\mathbb{C}$-algebra, let $\mathcal{A}^* = Hom_{\mathbb{C}}(\mathcal{A}, \mathbb{C})$. We call it Frobenius if there is a linear function $f : \mathcal{A} \to \mathbb{C}$, such that the form $(x, y) := f(xy)$ is nondegenerate, or, equivalently, if there exists an isomorphism $\phi : \mathcal{A} \xrightarrow{\simeq} \mathcal{A}^*$ of left $\mathcal{A}$-modules: given $f$, we can define $\phi(a)(b) = f(ba)$, and given $\phi$, we define $f = \phi(1)$.

**Remark 2.4.2.** If $\tilde{f}$ is another linear function satisfying the same properties as $f$ from above, then $\tilde{f}(x) = f(xa)$ for some invertible $a \in \mathcal{A}$. Indeed, we define the form $\{a, b\} = \tilde{f}(ab)$. Then $\{-, 1\} \in \mathcal{A}^*$, so there is an $a \in \mathcal{A}$, such that $\phi(a) = \{-, 1\}$. Then $\tilde{f}(x) = \{x, 1\} = \phi(a)(x) = f(xa)$.

**Definition 2.4.3.** Given a Frobenius algebra $\mathcal{A}$ (with a function $f$ inducing a bilinear form $(-,-)$ from above), the automorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by the equation $(x, y) = (y, \eta(x))$ is called the *Nakayama automorphism* (corresponding to $f$).

**Remark 2.4.4.** We note that the freedom in choosing $f$ implies that $\eta$ is uniquely determined up to an inner automorphism. Indeed, let $\tilde{f}(x) = f(xa)$ and define the bilinear form $\{a, b\} = \tilde{f}(ab)$. Then

$$\begin{aligned}\{x, y\} &= \tilde{f}(xy) = f(xya) = (x, ya) = \bigl(ya, \eta(x)\bigr) = f\bigl(ya\eta(x)a^{-1}a\bigr)\\ &= \bigl(y, a\eta(x)a^{-1}\bigr).\end{aligned}$$

## 2.5. Root system parameters

Let $w_0$ be the longest element of the Weyl group $W$ of $Q$. Then we define $\nu$ to be the involution of $I$, such that $w_0(\alpha_i) = -\alpha_{\nu(i)}$ (where $\alpha_i$ is the simple root corresponding to $i \in I$). It turns out that $\eta(e_i) = e_{\nu(i)}$ ([S]; see [ES2]).

Let $m_i$, $i = 1, \ldots, r$, be the exponents of the root system attached to $Q$, enumerated in increasing order. Let $h = m_r + 1$ be the Coxeter number in $Q$, i.e. the order of a Coxeter element in $W$.

Let $P$ be the permutation matrix corresponding to the involution $\nu$. Let $r_+ = \dim\ker(P - 1)$ and $r_- = \dim\ker(P + 1)$. Thus, $r_-$ is half the number of vertices which are not fixed by $\nu$, and $r_+ = r - r_-$.

$A$ is finite dimensional, and the following Hilbert series is known from [MOV, Theorem 2.3]:

$$H_A(t) = \bigl(1 + Pt^h\bigr)\bigl(1 - Ct + t^2\bigr)^{-1}. \tag{2.5.1}$$

It turns out that the top degree of $A$ is $h-2$ (i.e. $A(d)$ vanishes for $d > h-2$), and for the top degree $A^{\mathrm{top}}$ part we get the following decomposition in 1-dimensional submodules:

$$A^{\mathrm{top}} = A(h-2) = \bigoplus_{i \in I} e_i A(h-2) e_{\nu(i)}. \tag{2.5.2}$$

It is known that $A$ is a Frobenius algebra (see e.g. [ES2,MOV]).

# 3. Hochschild cohomology

The Hochschild cohomology spaces of $A$ were computed in [EE2]. We recall the results:

**Definition 3.0.3.** We define the spaces

$$\begin{aligned}U &= \bigoplus_{d<h-2} HH^0(A)(d)[2],\\ L &= HH^0(A)(h-2)[-h+2],\\ K &= HH^2(A)[2],\\ Y &= HH^6(A)(-h-2)[h+2].\end{aligned}$$

**Theorem 3.0.4.**

(1) *U has the following Hilbert series*:

$$h_U(t) = \sum_{\substack{i=1 \\ m_i < \frac{h}{2}}}^{r} t^{2m_i}. \tag{3.0.5}$$

(2) *We have natural isomorphisms*

$$K \equiv \ker(P+1),$$
$$L \equiv \langle e_i \mid \nu(i) = i \rangle,$$

*and*

$$\dim Y = r_+ - r_- - \#\left\{i\colon m_i = \frac{h}{2}\right\}.$$

**Theorem 3.0.6.** *For the Hochschild cohomology spaces, we have the following natural isomorphisms*:

$$HH^0(A) = U[-2] \oplus L[h-2],$$
$$HH^1(A) = U[-2],$$
$$HH^2(A) = K[-2],$$
$$HH^3(A) = K^*[-2],$$
$$HH^4(A) = U^*[-2],$$
$$HH^5(A) = U^*[-2] \oplus Y^*[-h-2],$$
$$HH^6(A) = U[-2h-2] \oplus Y[-h-2],$$

*and* $HH^{6n+i}(A) = HH^i(A)[-2nh]$ $\forall i \geqslant 1$.

**Corollary 3.0.7.** *The center* $Z = HH^0(A)$ *of* $A$ *has Hilbert series*

$$h_Z(t) = \sum_{\substack{i=1 \\ m_i < \frac{h}{2}}}^{r} t^{2m_i - 2} + (r_+ - r_-)t^{h-2}.$$

## 4. Main results

From Theorem 3.0.6, we already know the additive structure of $HH^*(A)$. As the main result of this paper, we present the product structure in $HH^*(A)$. The rest of the paper is devoted to this computation. Since the product $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ is graded-commutative, we can assume $i \leqslant j$ here.

Let $(U[-2])_+$ be the positive degree part of $U[-2]$ (which lies in nonnegative degrees).

We have a decomposition $HH^0(A) = \mathbb{C} \oplus (U[-2])_+ \oplus L[-h-2]$ where we have the natural identification $(U[-2])(0) = \mathbb{C}$.

Let

- $z_0 = 1 \in \mathbb{C} \subset U[-2] \subset HH^0(A)$ (in lowest degree 0),
- $\theta_0$ the corresponding element in $HH^1(A)$ (in lowest degree 0),
- $\psi_0$ the dual element of $z_0$ in $U^*[-2] \subset HH^5(A)$ (in highest degree $-4$), i.e. $\psi_0(z_0) = 1$,
- $\zeta_0$ the corresponding element in $U^*[-2] \subset HH^4(A)$ (in highest degree $-4$), that is the dual element of $\theta_0$, $\zeta_0(\theta_0) = 1$, $\varphi_0 : HH^0(A) \to HH^6(A)$ the natural quotient map (which induces the natural isomorphism $U[-2] \to U[-2h-2]$) and
- $\phi$ the quotient map $L \to Y$ induced by $\varphi_0$ in Theorem 4.0.8.

**Theorem 4.0.8** *(The product structure in $HH^*(A)$ for quivers of types D and E).*

(1) *The multiplication by $\varphi_0(z_0)$ induces the natural isomorphisms $\varphi_i : HH^i(A) \to HH^{i+6}(A)$ $\forall i \geqslant 1$ and the natural quotient map $\varphi_0$. Therefore, it is enough to compute products $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ with $0 \leqslant i \leqslant j \leqslant 5$.*

(2) *The $HH^0(A)$-action on $HH^i(A)$:*

  (a) *($(U[-2])_+$-action)*
  *The action of $(U[-2])_+$ on $U[-2] \subset HH^1(A)$ corresponds to the multiplication*

$$\begin{aligned} \big(U[-2]\big)_+ \times U[-2] &\to U[-2], \\ (u, v) &\mapsto u \cdot v \end{aligned}$$

  *in $HH^0(A)$, projected on $U[-2] \subset HH^0(A)$.*
  *$(U[-2])_+$ acts on $U^*[-2] = HH^4(A)$ and $U^*[-2] \subset HH^5(A)$ the following way:*

$$\begin{aligned} \big(U[-2]\big)_+ \times U^*[-2] &\to U^*[-2], \\ (u, f) &\mapsto u \circ f, \end{aligned}$$

  *where $(u \circ f)(v) = f(uv)$.*
  *$(U[-2])_+$ acts by zero on $L[h-2] \subset HH^0(A)$, $HH^2(A)$, $HH^3(A)$ and $Y^*[-h-2] \subset HH^5(A)$.*

  (b) *($L[h-2]$-action)*
  *$L[h-2]$ acts by zero on $HH^i(A)$, $1 \leqslant i \leqslant 4$, and on $U^*[-2] \subset HH^5(A)$.*
  *The $L[h-2]$-action on $HH^5(A)$ restricts to*

$$\begin{aligned} L[h-2] \times Y^*[-h-2] &\to U^*[-2], \\ (a, y) &\mapsto y\big(\phi(a)\big)\psi_0. \end{aligned}$$

(3) *(Zero products)*
*All products $HH^i(A) \times HH^j(A) \to HH^{i+j}$, $1 \leqslant i \leqslant j \leqslant 5$, where $i + j \geqslant 6$ or $i, j$ are both odd are zero except the pairings*

$$HH^1(A) \times HH^5(A) \to HH^6(A)$$

*and*

$$HH^5(A) \times HH^5(A) \to HH^{10}(A).$$

(4) ($HH^1(A)$*-products*)
 (a) *The multiplication*

$$HH^1(A) \times HH^4(A) = U[-2] \times U^*[-2] \to HH^5(A)$$

*is the same one as the restriction of*

$$HH^0(A) \times HH^5(A) \to HH^5(A)$$

*on* $U[-2] \times U^*[-2]$.
 (b) *The multiplication of the subspace* $U[-2]_+ \subset HH^1(A)$ *with* $HH^i(A)$ *where* $i = 2, 5$ *is zero.*
 (c) *The multiplication by* $\theta_0$ *induces a symmetric isomorphism*

$$\alpha : HH^2(A) = K[-2] \to K^*[-2] = HH^3(A).$$

*On* $HH^5(A)$*, it induces a skew-symmetric isomorphism*

$$\beta : Y^*[-h-2] \to Y[-h-2] \subset HH^6(A),$$

*and acts by zero on* $U^*[-2] \subset HH^5(A)$. $\alpha$ *and* $\beta$ *will be given by explicit matrices* $M_\alpha$ *and* $M_\beta$ *later.*

(5) ($HH^2(A)$*-products*)

$$HH^2(A) \times HH^2(A) \to HH^4(A),$$
$$(a, b) \mapsto \langle a, b\rangle \zeta_0$$

*is given by* $\langle -,- \rangle = \alpha$ *where* $\alpha$ *is regarded as a symmetric bilinear form.*
$HH^2(A) \times HH^3(A) \to HH^5(A)$ *is the multiplication*

$$K[-2] \times K^*[-2] \to HH^5(A),$$
$$(a, y) \mapsto y(a)\psi_0.$$

(6) ($HH^5(A) \times HH^5(A) \to HH^{10}(A)$)
*The restriction of this product to*

$$Y^*[-h-2] \times Y^*[-h-2] \to HH^{10}(A),$$
$$(a, b) \mapsto \Omega(a, b)\varphi_4(\zeta_0)$$

*is given by* $\Omega(-,-) = -\beta$ *where* $\beta$ *is regarded as a skew-symmetric bilinear form.*
*The multiplication of the subspace* $U^*[-2] \subset HH^5(A)$ *with* $HH^5(A)$ *is zero.*

## 5. Some basic facts about preprojective algebras

### 5.1. Labeling of quivers

From now on, we use the following labellings for the different types of quivers, as shown in Figs. 1–4.

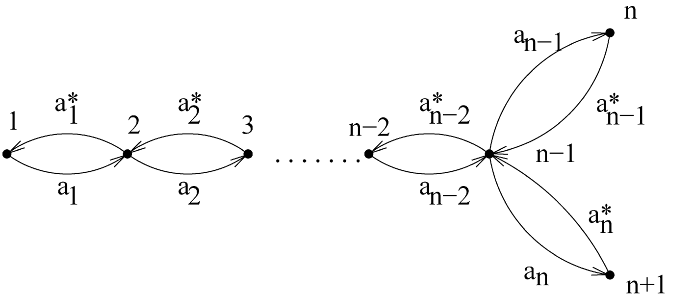


Fig. 1. $D_{n+1}$-quiver.

*5.1.1.* $Q = D_{n+1}$

$A$ is the path algebra modulo the relations

$$a_1^* a_1 = 0,$$
$$a_{i+1}^* a_{i+1} = a_i a_i^*, \quad 1 \leqslant i \leqslant n-3,$$
$$a_{n-1} a_{n-1}^* = a_n a_n^* = 0,$$
$$a_{n-1}^* a_{n-1} + a_n^* a_n = a_{n-2} a_{n-2}^*.$$

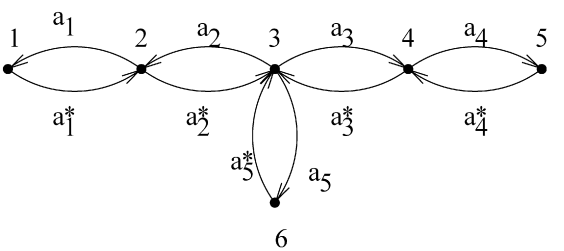


Fig. 2. $E_6$-quiver.

*5.1.2.* $Q = E_6$

$A$ is the path algebra modulo the relations

$$a_1 a_1^* = a_4 a_4^* = a_5 a_5^* = 0,$$
$$a_1^* a_1 = a_2 a_2^*,$$
$$a_4^* a_4 = a_3 a_3^*,$$
$$a_2^* a_2 + a_3^* a_3 + a_5^* a_5 = 0.$$

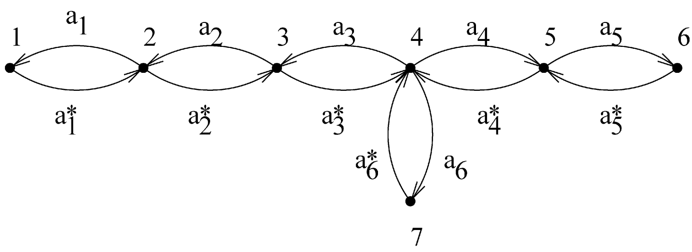


Fig. 3. $E_7$-quiver.

*5.1.3.* $Q = E_7$

$A$ is the path algebra modulo the relations

$$a_1a_1^* = a_5a_5^* = a_6a_6^* = 0,$$
$$a_1^*a_1 = a_2a_2^*,$$
$$a_2^*a_2 = a_3a_3^*,$$
$$a_5^*a_5 = a_4a_4^*,$$
$$a_3^*a_3 + a_4^*a_4 + a_6^*a_6 = 0.$$

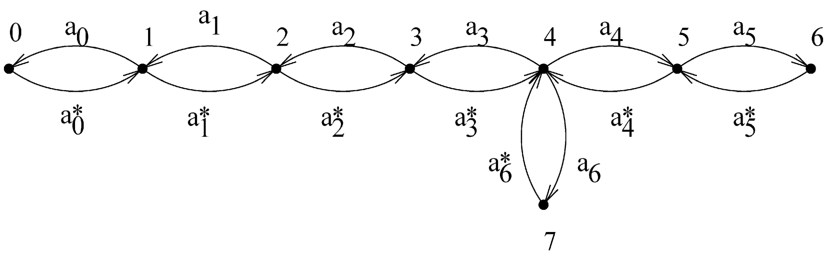


Fig. 4. $E_8$-quiver.

*5.1.4.* $Q = E_8$

$A$ is the path algebra modulo the relations

$$a_0a_0^* = a_5a_5^* = a_6a_6^* = 0,$$
$$a_0^*a_0 = a_1a_1^*,$$
$$a_1^*a_1 = a_2a_2^*,$$
$$a_2^*a_2 = a_3a_3^*,$$
$$a_5^*a_5 = a_4a_4^*,$$
$$a_3^*a_3 + a_4^*a_4 + a_6^*a_6 = 0.$$

*5.2. The Nakayama automorphism*

Recall that $A$ is a Frobenius algebra. The linear function $f : A \to \mathbb{C}$ is zero in the nontop degree part of $A$. It maps a top degree element $\omega_i \in e_i A^{\mathrm{top}} e_{\nu(i)}$ to 1. It is uniquely determined by the choice of one of these $\omega_i$ and a Nakayama automorphism.

For each quiver, we define a Nakayama automorphism $\eta$ and make a choice of one $\omega_i \in e_i A^{\mathrm{top}} e_{\nu(i)}$.

*5.2.1. $Q = D_{n+1}$, $n$ odd*

We define $\eta$ by

$$\eta(a_i) = -a_i, \tag{5.2.1}$$

$$\eta\big(a_i^*\big) = a_i^*, \tag{5.2.2}$$

and

$$\omega_1 = a_1^* \cdots a_{n-2}^* a_{n-1}^* a_{n-1} a_{n-2} \cdots a_1. \tag{5.2.3}$$

Let

$$\begin{aligned}
\overline{a_i} &= (-1)^i a_{i-1} \cdots a_1 a_1^* \cdots a_{n-1}^* a_{n-1} \cdots a_{i+1} \quad \forall 1 \leqslant i \leqslant n-2,\\
\overline{a_{n-1}} &= a_{n-2} \cdots a_1 a_1^* \cdots a_{n-1}^*,\\
\overline{a_n} &= -a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^* a_n^*,\\
\overline{a_i^*} &= a_{i+1}^* \cdots a_{n-1}^* a_{n-1} \cdots a_1 a_1^* \cdots a_{i-1}^* \quad \forall 1 \leqslant i \leqslant n-2,\\
\overline{a_{n-1}^*} &= a_{n-1} \cdots a_1 a_1^* \cdots a_{n-2}^*,\\
\overline{a_n^*} &= -a_n a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^*,
\end{aligned}$$

and $\omega_i = a_i^* \overline{a_i^*}$ $\forall 1 \leqslant i \leqslant n-1$ (where $\omega_1$ coincides with the expression in (5.2.3)), $\omega_n = a_{n-1}\overline{a_{n-1}}$, $\omega_{n+1} = a_n \overline{a_n}$. Then $\omega_{i+1} = a_i \overline{a_i}$ $\forall 1 \leqslant i \leqslant n-2$, and $\omega_i = \overline{a_i} \cdot (-a_i)$ $\forall 1 \leqslant i \leqslant n-1$, $\omega_n = \overline{a_n} \cdot (-a_n) = \overline{a_{n+1}} \cdot (-a_{n+1})$, $\omega_{i+1} = \overline{a_i^*} a_i^*$ $\forall 1 \leqslant i \leqslant n$.

These $\omega_i$ define the function $f$ (and the bilinear form) associated to the Frobenius algebra $A$. Since $\{\overline{a_1}, \ldots, \overline{a_n}, \overline{a_1^*}, \ldots, \overline{a_n^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1, \ldots, a_n, a_1^*, \ldots, a_n^*\}$ in $A(1)$ and $\{-a_1, \ldots, -a_n, a_1^*, \ldots, a_n^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1}, \ldots, \overline{a_n}, \overline{a_1^*}, \ldots, \overline{a_n^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by Eqs. (5.2.1) and (5.2.2).

*5.2.2. $Q = D_{n+1}$, $n$ even*

We define $\eta$ by

$$\begin{aligned}
\forall i \leqslant n-2\colon \quad & \eta(a_i) = -a_i,\\
\forall i \leqslant n-2\colon \quad & \eta\big(a_i^*\big) = a_i^*,\\
& \eta(a_{n-1}) = -a_n,
\end{aligned}$$

$$\eta(a_{n-1}^*) = a_n^*,$$
$$\eta(a_n) = -a_{n-1},$$
$$\eta(a_n^*) = a_{n-1}^*,$$

$$\omega_1 = a_1^* \cdots a_{n-2}^* a_{n-1}^* a_{n-1} a_{n-2} \cdots a_1. \tag{5.2.4}$$

Let

$$\begin{aligned}
\overline{a_i} &= (-1)^i a_{i-1} \cdots a_1 a_1^* \cdots a_{n-1}^* a_{n-1} \cdots a_{i+1} \quad \forall 1 \leqslant i \leqslant n-2,\\
\overline{a_{n-1}} &= a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^* a_n^*,\\
\overline{a_n} &= -a_{n-2} \cdots a_1 a_1^* \cdots a_{n-1}^*,\\
\overline{a_i^*} &= a_{i+1}^* \cdots a_{n-1}^* a_{n-1} \cdots a_1 a_1^* \cdots a_{i-1}^* \quad \forall 1 \leqslant i \leqslant n-2,\\
\overline{a_{n-1}^*} &= a_{n-1} \cdots a_1 a_1^* \cdots a_{n-2}^*,\\
\overline{a_n^*} &= -a_n a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^*
\end{aligned}$$

and $\omega_i = a_i^* \overline{a_i^*}$ $\forall 1 \leqslant i \leqslant n-1$ (where $\omega_1$ coincides with the expression in (5.2.4)), $\omega_n = a_{n-1}\overline{a_{n-1}}$, $\omega_{n+1} = a_n \overline{a_n}$. Then $\omega_{i+1} = a_i \overline{a_i}$ $\forall 1 \leqslant i \leqslant n-2$, $\omega_{n-1} = a_n^* \overline{a_n^*}$ and $\omega_{i+1} = \overline{a_i^*} a_i$ $\forall 1 \leqslant i \leqslant n-2$, $\omega_n = \overline{a_{n-1}^*} a_n^*$, $\omega_{n+1} = \overline{a_n^*} a_{n-1}^*$, $\omega_i = \overline{a_i} \cdot (-a_i)$ $\forall 1 \leqslant i \leqslant n-2$, $\omega_n = \overline{a_5} \cdot (-a_6)$, $\omega_{n+1} = \overline{a_n} \cdot (-a_{n-1})$.

Again, these $\omega_i$ define the function $f$ (and the bilinear form) associated to the Frobenius algebra $A$. Since $\{\overline{a_1}, \ldots, \overline{a_n}, \overline{a_1^*}, \ldots, \overline{a_n^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1, \ldots, a_n, a_1^*, \ldots, a_n^*\}$ in $A(1)$ and $\{-a_1, \ldots, -a_n, -a_{n-1}, a_1^*, \ldots a_n^*, a_{n-1}^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1}, \ldots, \overline{a_{n-1}}\overline{a_n}, \overline{a_1^*}, \ldots, \overline{a_{n-1}^*}, \overline{a_n^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by $\eta$ above.

*5.2.3.* $Q = E_6$

We define $\eta$ by

$$\eta(a_1) = -a_4,$$
$$\eta(a_1^*) = a_4^*,$$
$$\eta(a_2) = -a_3,$$
$$\eta(a_2^*) = a_3^*,$$
$$\eta(a_5) = -a_5,$$
$$\eta(a_5^*) = a_5^*,$$

and

$$\omega_3 = a_3^* a_3 \left(a_2^* a_2 a_3^* a_3\right)^2. \tag{5.2.5}$$

Let

$$\begin{aligned}
\overline{a_1} &= -a_2a_3^*a_4^*a_4a_3a_5^*a_5a_3^*a_4^*,\\
\overline{a_2} &= a_3^*a_4^*a_4a_3a_5^*a_5a_3^*a_4^*a_4,\\
\overline{a_3} &= a_5^*a_5a_3^*a_4^*a_4a_3a_2^*a_1^*a_1,\\
\overline{a_4} &= -a_3a_5^*a_5a_3^*a_4^*a_4a_3a_2^*a_1^*,\\
\overline{a_5} &= a_2^*a_1^*a_1a_2a_3^*a_4^*a_4a_3a_5^*,\\
\overline{a_1^*} &= -a_1a_2a_3^*a_4^*a_4a_3a_5^*a_5a_3^*,\\
\overline{a_2^*} &= -a_1^*a_1a_2a_3^*a_4^*a_4a_3a_5^*a_5,\\
\overline{a_3^*} &= -a_4^*a_4a_3a_5^*a_5a_3^*a_4^*a_4a_3,\\
\overline{a_4^*} &= -a_4a_3a_5^*a_5a_3^*a_4^*a_4a_3a_2^*,\\
\overline{a_5^*} &= a_5a_2^*a_1^*a_1a_2a_3^*a_4^*a_4a_3
\end{aligned}$$

and $\omega_1 = a_1\overline{a_1}$, $\omega_2 = a_2\overline{a_2}$, $\omega_3 = a_2^*\overline{a_2^*}$ (which coincides with the expression in (5.2.5)), $\omega_4 = a_3\overline{a_3}$, $\omega_5 = a_4\overline{a_4}$. $\omega_6 = a_5\overline{a_5}$. Then $\omega_2 = a_1^*\overline{a_1^*}$, $\omega_3 = a_3^*\overline{a_3^*} = a_5^*\overline{a_5^*}$, $\omega_4 = a_4^*\overline{a_4^*}$ and $\omega_1 = \overline{a_1^*}a_4^*$, $\omega_2 = \overline{a_2^*}a_3^* = \overline{a_1}\cdot(-a_4)$, $\omega_3 = \overline{a_2}\cdot(-a_3) = \overline{a_3}\cdot(-a_2) = \overline{a_5}\cdot(-a_5)$, $\omega_4 = \overline{a_3^*}a_2^* = \overline{a_4}\cdot(-a_1)$, $\omega_5 = \overline{a_4^*}a_1^*$, $\omega_6 = \overline{a_5^*}a_5^*$.

Again, these $\omega_i$ define the function $f$ (and the bilinear form) associated to the Frobenius algebra $A$. Since $\{\overline{a_1},\dots,\overline{a_5},\overline{a_1^*},\dots,\overline{a_5^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1,\dots,a_5,a_1^*,\dots,a_5^*\}$ in $A(1)$ and $\{-a_4,-a_3,-a_2,-a_1,-a_5,a_4^*,a_3^*,a_2^*,a_1^*,a_5^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1},\dots,\overline{a_5}\overline{a_1^*},\dots,\overline{a_5^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by $\eta$ above.

*5.2.4. $Q = E_7$*

We define $\eta$ by

$$\eta(a_i) = -a_i,$$
$$\eta\bigl(a_i^*\bigr) = a_i^*,$$

and

$$\omega_4 = \bigl(a_4^*a_4a_3^*a_3\bigr)^4. \tag{5.2.6}$$

Given the basis $\{a_1,\dots,a_6,a_1^*,\dots,a_6^*\}$ in $A(1)$, we claim that a dual basis $\{\overline{a_1},\dots,\overline{a_6},\overline{a_1^*},\dots,\overline{a_6^*}\}$ in $A(h-3)$ is given by

$$\begin{aligned}
\overline{a_1} &= -a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*,\\
\overline{a_2} &= a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1,\\
\overline{a_3} &= -a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2,
\end{aligned}$$

$$\overline{a_4} = -a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5,$$
$$\overline{a_5} = a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*,$$
$$\overline{a_6} = a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*,$$
$$\overline{a_1^*} = -a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*,$$
$$\overline{a_2^*} = -a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*,$$
$$\overline{a_3^*} = -a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4,$$
$$\overline{a_4^*} = -a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3,$$
$$\overline{a_5^*} = -a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*,$$
$$\overline{a_6^*} = a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3$$

and $\omega_i = a_i\overline{a_i}$ $\forall 1 \leqslant i \leqslant 3$, $\omega_{i+1} = a_i\overline{a_i}$ $\forall 4 \leqslant i \leqslant 6$, $\omega_4 = a_3^*\overline{a_3^*}$. Then $\omega_2 = a_1^*\overline{a_1^*}$, $\omega_3 = a_2^*\overline{a_2^*}$, $\omega_4 = a_4^*\overline{a_4^*} = a_6^*\overline{a_6^*}$ (which coincides with the expression (5.2.6)), $\omega_5 = a_5^*\overline{a_5^*}$ and $\omega_i = \overline{a_i^*}a_i^*$ $\forall 1 \leqslant i \leqslant 3$, $\omega_{i+1} = \overline{a_i^*}a_i^*$ $\forall 4 \leqslant i \leqslant 6$, $\omega_{i+1} = \overline{a_i} \cdot (-a_i)$ $\forall 1 \leqslant i \leqslant 3$, $\omega_i = \overline{a_i} \cdot (-a_i)$ $\forall 4 \leqslant i \leqslant 5$, $\omega_4 = \overline{a_6} \cdot (-a_6)$.

Again, these $\omega_i$ define the function $f$ (and the bilinear form) associated to the Frobenius algebra $A$. Since $\{\overline{a_1}, \dots, \overline{a_6}, \overline{a_1^*}, \dots, \overline{a_6^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1, \dots, a_6, a_1^*, \dots, a_6^*\}$ in $A(1)$ and $\{-a_1, \dots, -a_6, a_1^*, \dots, a_6^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1}, \dots, \overline{a_6}\overline{a_1^*}, \dots, \overline{a_6^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by $\eta$ above.

*5.2.5.* $Q = E_8$

We define $\eta$ by

$$\eta(a_i) = -a_i,$$
$$\eta(a_i^*) = a_i^*,$$

and

$$\omega_4 = \left(a_4^*a_4a_3^*a_3\right)^7. \tag{5.2.7}$$

Then

$$\overline{a_0} = a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*,$$
$$\overline{a_1} = -a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0,$$
$$\overline{a_2} = a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1,$$
$$\overline{a_3} = -a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2,$$
$$\overline{a_4} = -a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*,$$
$$\overline{a_5} = a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*,$$

$$\overline{a_6} = a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*,$$
$$\overline{a_0^*} = a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*,$$
$$\overline{a_1^*} = a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*,$$
$$\overline{a_2^*} = a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*,$$
$$\overline{a_3^*} = a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4,$$
$$\overline{a_4^*} = -a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6,$$
$$\overline{a_5^*} = a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*,$$
$$\overline{a_6^*} = a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4$$

and $\omega_i = a_i\overline{a_i}$ $\forall 0 \leqslant i \leqslant 3$, $\omega_{i+1} = a_i\overline{a_i}$ $\forall 4 \leqslant i \leqslant 6$, $\omega_4 = a_3^*\overline{a_3^*}$. Then $\omega_2 = a_1^*\overline{a_1^*}$, $\omega_3 = a_2^*\overline{a_2^*}$, $\omega_4 = a_4^*\overline{a_4^*} = a_6^*\overline{a_6^*}$ (which coincides with the expression (5.2.7)), $\omega_5 = a_5^*\overline{a_5^*}$ and $\omega_i = \overline{a_i^*}a_i^*$ $\forall 0 \leqslant i \leqslant 3$, $\omega_{i+1} = \overline{a_i^*}a_i^*$ $\forall 4 \leqslant i \leqslant 6$, $\omega_{i+1} = \overline{a_i} \cdot (-a_i)$ $\forall 0 \leqslant i \leqslant 3$, $\omega_i = \overline{a_i} \cdot (-a_i)$ $\forall 4 \leqslant i \leqslant 5$, $\omega_4 = \overline{a_6} \cdot (-a_6)$.

Again, these $\omega_i$ define the function $f$ (and the bilinear form) associated to the Frobenius algebra $A$. Since $\{\overline{a_0}, \dots, \overline{a_6}, \overline{a_0^*}, \dots, \overline{a_6^*}\}$ in $A(h-3)$ is a dual basis of $\{a_0, \dots, a_6, a_0^*, \dots, a_6^*\}$ in $A(1)$ and $\{-a_0, \dots, -a_6, a_0^*, \dots, a_6^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_0}, \dots, \overline{a_6}, \overline{a_0^*}, \dots, \overline{a_6^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by $\eta$ above.

### 5.3. *Preprojective algebras by numbers*

We summarize useful numbers associated to preprojective algebras, by quiver:

| $Q$ | | Exponents $m_i$ | $h$ | $\deg A^{\mathrm{top}}$ | Degrees $HH^0(A)$ |
|---|---|---|---|---|---|
| $D_{n+1}$ | $n$ odd | $1, 3, \dots, 2n-1, n$ | $2n$ | $2n-2$ | $0, 4, \dots, 2n-6, 2n-2$ |
| | $n$ even | | | | $0, 4, \dots, 2n-4, 2n-2$ |
| $E_6$ | | $1, 4, 5, 7, 8, 11$ | 12 | 10 | $0, 6, 8, 10$ |
| $E_7$ | | $1, 5, 7, 9, 11, 13, 17$ | 18 | 16 | $0, 8, 12, 16$ |
| $E_8$ | | $1, 7, 11, 13, 17, 19, 23, 29$ | 30 | 28 | $0, 12, 20, 24, 28$ |

We see that for quivers of types $D$ and $E$, the degrees of the space $U$ (which are $2m_i$, $m_i < \frac{h}{2}$) are even and range from 0 to $h-2$.

We get the following degree ranges for the Hochschild cohomology:

$$\begin{aligned}
&HH^0(A) = U[-2] \oplus L[h-2], && 0 \leqslant \deg HH^0(A) \leqslant h-2,\\
&HH^1(A) = U[-2], && 0 \leqslant \deg HH^1(A) \leqslant h-4,\\
&HH^2(A) = K[-2], && \deg HH^2(A) = -2,\\
&HH^3(A) = K^*[-2], && \deg HH^3(A) = -2,\\
&HH^4(A) = U^*[-2], && -h \leqslant \deg HH^4(A) \leqslant -4,\\
&HH^5(A) = U^*[-2] \oplus Y^*[-h-2], && -h-2 \leqslant \deg HH^5(A) \leqslant -4,\\
&HH^6(A) = U[-2h-2] \oplus Y[-h-2], && -2h \leqslant \deg HH^6(A) \leqslant -h-2.
\end{aligned}$$

### 5.4. The Schofield resolution

We recall the Schofield resolution of $A$ from [S].

Define the $A$-bimodule $\mathcal{N}$ obtained from $A$ by twisting the right action by $\eta$, i.e., $\mathcal{N} = A$ as a vector space, and $\forall a, b \in A$, $y \in \mathcal{N}$: $a \cdot y \cdot b = ay\eta(b)$. Introduce the notation $\epsilon_a = 1$ if $a \in Q$, $\epsilon_a = -1$ if $a \in Q^*$. We write $B$ for a set of all homogeneous basis elements of $A$, let $(x_i)_{x_i \in B}$ be a homogeneous basis of $A$ and $(x_i^*)_{x_i \in B}$ the dual basis under the form attached to the Frobenius algebra $A$. Let $V$ be the bimodule spanned by the arrows of $\bar{Q}$.

We start with the following exact sequence:

$$0 \to \mathcal{N}[h] \xrightarrow{i} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0,$$

where $P_2 = A \otimes_R A[2]$, $P_1 = A \otimes_R V \otimes_R A$, $P_0 = A \otimes_R A$,

$$\begin{aligned} d_0(u \otimes v) &= uv, \\ d_1(u \otimes v \otimes w) &= uv \otimes w - u \otimes vw, \\ d_2(u \otimes v) &= \sum_{a \in \bar{Q}} \epsilon_a ua \otimes a^* \otimes v + \sum_{a \in \bar{Q}} \epsilon_a u \otimes a \otimes a^* v, \\ i(a) &= a \sum_{x_i \in B} x_i \otimes x_i^*, \end{aligned}$$

where $B$ is a homogeneous basis of $A$.

Since $\eta^2 = 1$, we can make a canonical identification $A = \mathcal{N} \otimes_A \mathcal{N}$ (via $x \mapsto x \otimes 1$), so by tensoring the above exact sequence with $\mathcal{N}$, connecting with the original exact sequence and repeating this process, we get the Schofield resolution

$$\cdots \to P_6 \xrightarrow{d_6} P_5 \xrightarrow{d_5} \to P_4 \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0,$$

with

$$P_{i+3} = (P_i \otimes_R \mathcal{N})[h].$$

We will work with the Hochschild cohomology complex obtained from this resolution, which is given explicitly in [EE2, Subsection 4.5]. This allows us to identify $HH^i(A)$ with quotients of subsets of $A^R$, $\mathcal{N}^R$, $(V \otimes A)^R$ and $(V \otimes \mathcal{N})^R$, where denote $U^R$ as the space of $R$-invariants in $U$. For an element $v$ in $A^R$, $\mathcal{N}^R$, $(V \otimes A)^R$ or $(V \otimes \mathcal{N})^R$, we denote its cohomology class by $[v]$.

### 5.5. Basis of the preprojective algebra for $Q = D_{n+1}$

We need to work with the Hilbert series and with an explicit basis of $A$. We do this for each type of quiver separately.

We write $B$ for a set of all homogeneous basis elements of $A$, $B_{i,-}$ for a homogeneous basis of $e_i A$, $B_{-,j}$ for a homogeneous basis of $Ae_j$, $B_{i,j}$ for a basis of $e_i Ae_j$ and $B_{i,j}(d)$ for a basis of $e_i Ae_j(d)$.

A basis of $A$ is given by the following elements:
For $\underline{k, j \leqslant n-1}$:

$$B_{k,n} = \left\{ \left(a_{k-1}a_{k-1}^*\right)^l a_k^* \cdots a_{n-2}^* a_{n-1}^* \;\middle|\; 0 \leqslant l \leqslant k-1 \right\},$$
$$B_{k,n+1} = \left\{ \left(a_{k-1}a_{k-1}^*\right)^l a_k^* \cdots a_{n-2}^* a_n^* \;\middle|\; 0 \leqslant l \leqslant k-1 \right\},$$
$$B_{n,n} = \left\{ \left(a_{n-1}a_n^* a_n a_{n-1}^*\right)^l \;\middle|\; 0 \leqslant l \leqslant \left\{ \begin{array}{ll} \frac{n-1}{2} & n\ odd, \\ \frac{n-2}{2} & n\ even \end{array} \right\} \right\},$$
$$B_{n+1,n+1} = \left\{ \left(a_n a_{n-1}^* a_{n-1} a_n^*\right)^l \;\middle|\; 0 \leqslant l \leqslant \left\{ \begin{array}{ll} \frac{n-1}{2} & n\ odd, \\ \frac{n-2}{2} & n\ even \end{array} \right\} \right\},$$
$$B_{n+1,n} = \left\{ a_n a_{n-1}^* \left(a_{n-1}a_n^* a_n a_{n-1}^*\right)^l \;\middle|\; 0 \leqslant l \leqslant \left\{ \begin{array}{ll} \frac{n-3}{2} & n\ odd, \\ \frac{n-2}{2} & n\ even \end{array} \right\} \right\},$$
$$B_{n,n+1} = \left\{ a_{n-1} a_n^* \left(a_n a_{n-1}^* a_{n-1} a_n^*\right)^l \;\middle|\; 0 \leqslant l \leqslant \left\{ \begin{array}{ll} \frac{n-3}{2} & n\ odd, \\ \frac{n-2}{2} & n\ even \end{array} \right\} \right\},$$
$$B_{n,j} = \left\{ a_{n-1}a_{n-2} \cdots a_j \left(a_{j-1}a_{j-1}^*\right)^l \;\middle|\; 0 \leqslant l \leqslant j-1 \right\},$$
$$B_{n+1,j} = \left\{ a_n a_{n-2} \cdots a_j \left(a_{j-1}a_{j-1}^*\right)^l \;\middle|\; 0 \leqslant l \leqslant j-1 \right\}.$$

For $\underline{k \leqslant j \leqslant n-1}$,

$$\begin{aligned} B_{k,j} = {} & \left\{ \left(a_{k-1}a_{k-1}^*\right)^l a_k^* \cdots a_{j-1}^* \;\middle|\; 0 \leqslant l \leqslant \min\{k-1, n-j-1\} \right\} \\ & \cup \left\{ \left(a_{k-1}a_{k-1}^*\right)^l a_k^* \cdots a_{n-1}^* a_{n-1} a_{n-2} \cdots a_j \;\middle|\; 0 \leqslant l \leqslant k-1 \right\} \\ & \cup \left\{ \left(a_{k-1}a_{k-1}^*\right)^l a_k^* \cdots a_n^* a_n a_{n-2} \cdots a_j \;\middle|\; 0 \leqslant l \leqslant k-1+j-n \right\}. \end{aligned}$$

For $\underline{j < k \leqslant n-1}$,

$$\begin{aligned} B_{k,j} = {} & \left\{ a_{k-1} \cdots a_j \left(a_j^* a_j\right)^l \;\middle|\; 0 \leqslant l \leqslant \min\{n-k-1, j-1\} \right\} \\ & \cup \left\{ a_k^* \cdots a_{n-2}^* a_{n-1}^* a_{n-1} a_{n-2} \cdots a_j \left(a_j^* a_j\right)^l \;\middle|\; 0 \leqslant l \leqslant j-1 \right\} \\ & \cup \left\{ a_k^* \cdots a_{n-2}^* a_n^* a_n a_{n-2} \cdots a_j \left(a_j^* a_j\right)^l \;\middle|\; 0 \leqslant l \leqslant j-1+k-n \right\}. \end{aligned}$$

### 5.6. *Hilbert series of the preprojective algebra for $Q = E_6$*

We give the columns of the Hilbert series $H_A(t)$ which can be calculated from (2.5.1):

$$\left(H_A(t)_{i,1}\right)_{1 \leqslant i \leqslant 6} = \begin{pmatrix} 1 + t^6 \\ t + t^5 + t^7 \\ t^2 + t^4 + t^6 + t^8 \\ t^3 + t^5 + t^9 \\ t^4 + t^{10} \\ t^3 + t^7 \end{pmatrix},$$

$$\big(H_A(t)_{i,2}\big)_{1\leqslant i\leqslant 6}=\begin{pmatrix} t+t^5+t^7\\ 1+t^2+t^4+2t^6+t^8\\ t+2t^3+2t^5+2t^7+t^9\\ t^2+2t^4+t^6+t^8+t^{10}\\ t^3+t^5+t^9\\ t^2+t^4+t^6+t^8 \end{pmatrix},$$

$$\big(H_A(t)_{i,3}\big)_{1\leqslant i\leqslant 6}=\begin{pmatrix} t^2+t^4+t^6+t^8\\ t+2t^3+2t^5+2t^7+t^9\\ 1+2t^2+3t^4+3t^6+2t^8+t^{10}\\ t+2t^3+2t^5+2t^7+t^9\\ t^2+t^4+t^6+t^8\\ t+t^3+2t^5+t^7+t^9 \end{pmatrix},$$

$$\big(H_A(t)_{i,4}\big)_{1\leqslant i\leqslant 6}=\begin{pmatrix} t^3+t^5+t^9\\ t^2+2t^4+t^6+t^8+t^{10}\\ t+2t^3+2t^5+2t^7+t^9\\ 1+t^2+t^4+2t^6+t^8\\ t+t^5+t^7\\ t^2+t^4+t^6+t^8 \end{pmatrix},$$

$$\big(H_A(t)_{i,5}\big)_{1\leqslant i\leqslant 6}=\begin{pmatrix} t^4+t^{10}\\ t^3+t^5+t^9\\ t^2+t^4+t^6+t^8\\ t+t^5+t^7\\ 1+t^6\\ t^3+t^7 \end{pmatrix},$$

$$\big(H_A(t)_{i,6}\big)_{1\leqslant i\leqslant 6}=\begin{pmatrix} t^3+t^7\\ t^2+t^4+t^6+t^8\\ t+t^3+2t^5+t^7+t^9\\ t^2+t^4+t^6+t^8\\ t^3+t^7\\ 1+t^4+t^6+t^{10} \end{pmatrix}.$$

## 6. $HH^0(A) = Z$

From the Hilbert series (Corollary 3.0.7) we see that we have one (unique up to a constant factor) central element of degree $2m_i - 2$ for each exponent $m_i < \frac{h}{2}$. We will denote a $\deg i\,(< h-2)$ central element by $z_i$.

From (2.5.2) and from the Hilbert series we can also see that the top degree ($= \deg h - 2$) center is spanned by one element $\omega_i$ in each $e_i A e_i$, such that $\nu(i) = i$.

The $\omega_i \in L[h-2]$ are already given in Section 2.4, and we will find the $z_i \in U[-2]$ for each Dynkin quiver separately.

### 6.1. $Q = D_{n+1}$

We define the nonzero elements

$$
\begin{aligned}
b_{i,0} &= e_i, \\
b_{i,j} &= a_i^* \cdots a_{i+j-1}^* a_{i+j-1} \cdots a_i \quad \bigl(\text{where } 1 \leqslant j \leqslant \min\{i-1, n-1-i\}\bigr), \\
c_{i,j} &= a_i^* \cdots a_{n-2}^* \bigl(a_{n-2} a_{n-2}^*\bigr)^j a_{n-2} \cdots a_i \quad (\text{where } 1 \leqslant i \leqslant n-2,\ 1 \leqslant j \leqslant i-1), \\
c_{n-1,j} &= \bigl(a_{n-2} a_{n-2}^*\bigr)^j, \quad 1 \leqslant j \leqslant n-2, \\
c_i' &= a_i^* \cdots a_{n-2}^* a_{n-1}^* a_{n-1} \bigl(a_{n-2} a_{n-2}^*\bigr)^{i-1} a_{n-2} \cdots a_i, \quad 1 \leqslant i \leqslant n-1, \\
d_0 &= e_n, \\
d_j &= \bigl(a_{n-1} a_n^* a_n a_{n-1}^*\bigr)^j \quad \text{for } 1 \leqslant j < \frac{n}{2}, \\
d_0' &= e_{n+1}, \\
d_j' &= \bigl(a_n a_{n-1}^* a_{n-1} a_n^*\bigr)^j \quad \text{for } 1 \leqslant j < \frac{n}{2}
\end{aligned}
$$

and extend this notation for any other $j$, where $b_{i,j}$, $c_{i,j}$, $d_j$ and $d_j'$ are zero.

The exponents $m_i$ are $1, 3, \ldots, 2n-1, n$ and $h = 2n$. From Corollary 3.0.7 we get the Hilbert series of $Z$, depending on the parity of $n$, since $r_+ = n+1$ for $n$ odd and $r_+ = n-1$ for $n$ even:

$$
\begin{aligned}
n \text{ odd:} \quad & h_Z(t) = 1 + t^4 + t^8 + \cdots + t^{2n-6} + (n+1)t^{2n-2}, \\
n \text{ even:} \quad & h_Z(t) = 1 + t^4 + t^8 + \cdots + t^{2n-4} + (n-1)t^{2n-2}.
\end{aligned}
$$

The central elements of degree $4j < 2n-2$ are

$$
z_{4j} = \sum_{i=2j+1}^{n-1-2j} b_{i,2j} + \sum_{i=0}^{2j-1} c_{n-1-i,2j-i} + d_j + d_j'.
$$

The top degree central elements are $\omega_i = c_i'$ $(1 \leqslant i \leqslant n-1)$, and additionally $\omega_n = d_{\frac{n-1}{2}}$, $-\omega_{n+1} = d'_{\frac{n-1}{2}}$ if $n$ is odd.

For $j + k < \frac{n-1}{2}$ we get the following product:

$$
z_{4j} z_{4k} = z_{4(j+k)}.
$$

If $n$ is odd and $j + k = \frac{n-1}{2}$, the multiplication becomes

$$
z_{4j} z_{4k} = d_{\frac{n-1}{2}} + d'_{\frac{n-1}{2}} = \omega_n - \omega_{n+1}.
$$

### 6.2. $Q = E_6$

The Coxeter number is $h = 12$, and the exponents $m_i < \frac{h}{2} = 6$ are 1, 4, 5, $r_+ = 2$. For the center, we get the following Hilbert series (from Corollary 3.0.7):

$$
h_Z(t) = 1 + t^6 + t^8 + 2t^{10}.
$$

From the degrees, we see that the product of any two positive degree central elements is always 0. The central elements are $z_0 = 1$, $z_6$, $z_8$, $\omega_3$ and $\omega_6$.

We give the central elements $z_6$ and $z_8$ explicitly (it can be easily checked that they are central):

**Proposition 6.2.1.**

(1) *The central element of* deg 6 *is*

$$z_6 = a_1a_2a_3^*a_3a_2^*a_1^* - a_2\left(a_3^*a_3\right)^2a_2^* - a_5^*a_5a_3^*a_3a_5^*a_5 + a_3\left(a_2^*a_2\right)^2a_3^* - a_4a_3a_2^*a_2a_3^*a_4^*,$$

(2) *the central element of* deg 8 *is*

$$z_8 = -a_2a_5^*a_5a_3^*a_3a_5^*a_5a_2^* - a_5^*a_5\left(a_3^*a_3\right)^2a_5^*a_5 - a_3a_5^*a_5a_2^*a_2a_5^*a_5a_3^*.$$

*6.3. $Q = E_7$*

The Coxeter number is $h = 18$, the exponents $m_i < \frac{h}{2} = 9$ are $1, 5, 7$, $r_+ = 7$, and the Hilbert series of the center is (see Corollary 3.0.7):

$$h_Z(t) = 1 + t^8 + t^{12} + 7t^{16}.$$

The center is spanned by $z_0 = 1$, $z_8$, $z_{12}$, $\omega_1, \dots, \omega_7$. The only interesting product to compute is $z_8^2$ which lies in the top degree.

We give $z_8$ and $z_{12}$ explicitly:

**Proposition 6.3.1.**

(1) *The central element of degree* 8 *is*

$$z_8 = -a_1a_2a_3a_6^*a_6a_3^*a_2^*a_1^* - a_2a_3\left(a_4^*a_4\right)^2a_3^*a_2^* - a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^* \\ - a_4^*a_4\left(a_3^*a_3\right)^2a_4^*a_4 - a_4a_4^*a_4a_6^*a_6a_4^*a_4a_4^* + a_6a_4^*a_4a_6^*a_6a_4^*a_4a_6^*.$$

(2) *The central element of degree* 12 *is*

$$z_{12} = -a_3\left(a_4^*a_4a_6^*a_6\right)^2a_4^*a_4a_3^* - a_4^*a_4a_6^*a_6\left(a_4^*a_4\right)^2a_6^*a_6a_4^*a_4 \\ + a_4\left(a_6^*a_6a_4^*a_4\right)^2a_6^*a_6a_4^* + a_6\left(a_4^*a_4a_6^*a_6\right)^2a_4^*a_4a_6^*.$$

**Proposition 6.3.2.** *We get*

$$z_8^2 = \omega_1 + \omega_3 - \omega_7.$$

*6.4. $Q = E_8$*

The Coxeter number $h = 30$, and the exponents $m_i < \frac{h}{2} = 15$ are $1, 7, 11, 13$, $r_+ = 8$. For the center, we get the following Hilbert series (from Corollary 3.0.7):

$$h_Z(t) = 1 + t^{12} + t^{20} + t^{24} + 8t^{28}.$$

The center is spanned by $z_0 = 1, z_{12}, z_{20}, z_{24}, \omega_1, \ldots, \omega_8$. The only interesting product is $z_{12}^2$.

**Proposition 6.4.1.**

(1) *The central element of degree* 12 *is*

$$\begin{aligned} z_{12} = {} & a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_2^*a_1^* + a_2a_3a_4^*a_4\left(a_3^*a_3\right)^2a_4^*a_4a_3^*a_2^* \\ & + a_3\left(a_4^*a_4a_6^*a_6\right)^2a_4^*a_4a_3^* + \left(a_3^*a_3a_4^*a_4a_3^*a_3\right)^2 - a_4\left(a_6^*a_6a_4^*a_4\right)^2a_6^*a_6a_4^* \\ & + a_5a_4a_6^*a_6\left(a_4^*a_4\right)^2a_6^*a_6a_4^*a_5^* - a_6\left(a_4^*a_4a_6^*a_6\right)^2a_4^*a_4a_6^*. \end{aligned}$$

(2) *The central element of degree* 20 *is*

$$\begin{aligned} z_{20} = {} & -a_1a_2a_3\left(a_4^*a_4\right)^2\left(a_3^*a_3\right)^3\left(a_4^*a_4\right)^2a_3^*a_2^*a_1^* - a_2a_3\left(a_6^*a_6a_4^*a_4\right)^2\left(a_4^*a_4a_6^*a_6\right)^2a_3^*a_2^* \\ & + a_3\left(a_6^*a_6a_4^*a_4\right)^4a_6^*a_6a_3^* - \left(a_4^*a_4a_6^*a_6\right)^5 + \left(a_6^*a_6\left(a_4^*a_4\right)^2\right)^3a_6^*a_6 \\ & - \left(a_6^*a_6a_4^*a_4\right)^5 - a_4\left(a_4^*a_4a_6^*a_6a_4^*a_4\right)^3a_4^* - a_6\left(a_4^*a_4a_6^*a_6\right)^4a_4^*a_4a_6^*. \end{aligned}$$

(3) *The central element of degree* 24 *is*

$$z_{24} = z_{12}^2.$$

## 7. $HH^1(A)$

Recall Theorem 3.0.6 where we know that $HH^1(A)$ is isomorphic to the nontopdegree part of $HH^0(A)$. In fact, $HH^1(A)$ is generated by the central elements in the following way:

**Proposition 7.0.2.** *$HH^1(A)$ is spanned by maps*

$$\begin{aligned} &\theta_k : (A \otimes V \otimes A) \to A, \\ &\theta_k(1 \otimes a_i \otimes 1) = 0, \\ &\theta_k\left(1 \otimes a_i^* \otimes 1\right) = a_i^* z_k. \end{aligned}$$

**Proof.** These maps clearly lie in $\ker d_2^*$: Recall

$$\begin{aligned} &A \otimes A \xrightarrow{d_2} A \otimes V \otimes A, \\ &x \otimes y \longmapsto \sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* \otimes y + \sum_{a \in \bar{Q}} \epsilon_a x \otimes a \otimes a^* y, \end{aligned}$$

then

$$d_2^* \circ \theta_k(1 \otimes 1) = \theta_k\left(\sum_{a \in \bar{Q}} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a 1 \otimes a \otimes a^*\right)$$
$$= \sum_{i \in I} a_i a_i^* z_k - \sum_{i \in I} a_i^* a_i z_k = \sum_{i \in I} [a_i, a_i^*] z_k = 0.$$

We will later see in section 10 that $HH^4(A)$ is generated by $\zeta_k$ where $\zeta_k(\theta_k) = 1$ under the duality $HH^4(A) = (HH^1(A))^*$ established in [EE2], so $\theta_k$ is nonzero in $HH^1(A)$. □

## 8. $HH^2(A)$

We know from Theorem 3.0.6 that $HH^2(A) = K[-2]$ lies in degree $-2$, i.e. in the lowest degree of $A^R[-2]$ (using the identifications in [EE2, Section 4.5]), that is in $R[-2]$. Since the image of $d_2^*$ lies in degree $> -2$, $HH^2(A) = \ker d_3^*$.

**Proposition 8.0.3.** *$HH^2(A)$ is given by the kernel of the matrix $H_A(1)$, where we identify $\mathbb{C}^I = R = \bigoplus_{i \in I} Re_i$.*

**Proof.** Recall

$$d_3^*(y) = \sum_{x_i \in B} x_i y x_i^* = \sum_{j,k \in I} \sum_{x_i \in B_{j,k}} x_i y x_i^*.$$

For each $x_i \in e_k A e_j$, we see that $x_i e_l x_i^* = \delta_{jl} \omega_k$.

It follows that for $y = \sum_{i \in I} \lambda_i e_i$ the map is given by

$$d_3^*(y) = \sum_{i \in I} \mu_i \omega_i,$$

where the vectors $\lambda = (\lambda_i)_{i \in I} \in \mathbb{C}^I$ and $\mu = (\mu_i)_{i \in I} \in \mathbb{C}^I$ satisfy the equation

$$H_A(1)\lambda = \mu. \tag{8.0.4}$$

So the kernel of $d_3^*$ is given by the kernel of $H_A(1)$. □

Now, we find the elements in $HH^2(A)$ for the quivers separately.

*8.1.* $Q = D_{n+1}$, *n even*

$$H_A(1) = \begin{pmatrix} 2 & 2 & 2 & 2 & \dots & \dots & 2 & 1 & 1 \\ 2 & 4 & 4 & 4 & \dots & \dots & 4 & 2 & 2 \\ 2 & 4 & 6 & 6 & \dots & \dots & 6 & 3 & 3 \\ 2 & 4 & 6 & 8 & \dots & \dots & 8 & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & 8 & \dots & \dots & 2(n-1) & n-1 & n-1 \\ 1 & 2 & 3 & 4 & \dots & \dots & n-1 & \frac{n}{2} & \frac{n}{2} \\ 1 & 2 & 3 & 4 & \dots & \dots & n-1 & \frac{n}{2} & \frac{n}{2} \end{pmatrix} \tag{8.1.1}$$

with kernel $\langle e_n - e_{n+1} \rangle$. So a basis of $HH^2(A)$ is given by

$$\{ f_n = [e_n - e_{n+1}] \}.$$

*8.2.* $Q = E_6$

$$H_A(1) = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 4 & 8 & 12 & 8 & 4 & 6 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 2 & 3 & 4 & 3 & 2 & 2 \\ 2 & 4 & 6 & 4 & 2 & 4 \end{pmatrix} \tag{8.2.1}$$

with kernel $\langle e_1 - e_5, e_2 - e_4 \rangle$. So a basis of $HH^2(A)$ is given by

$$\{ f_1 = [e_1 - e_5], f_2 = [e_2 - e_4] \}.$$

## 9. $HH^3(A)$

We know that $HH^3(A)$ lives in degree $-2$. Using the notations and identifications in [EE2, Subsection 4.5], we see that the kernel of $d_4^*$ has to be the top degree part of $\mathcal{N}^R[-h]$ (since $\operatorname{Im} d_3^*$ lives in degree $-2$), so

$$HH^3(A) = \mathcal{N}^R[-h](-2) / \operatorname{Im} d_3^*.$$

**Proposition 9.0.2.** *$HH^3(A)$ is given by the cokernel of the matrix $H_A(1)$, where we identify $\mathbb{C}^I = A^{\text{top}} = \bigoplus_{i \in I} e_i A^{\text{top}} e_{\nu(i)}$.*

**Proof.** This follows immediately from the discussion in the previous section because $d_3^*$ is given by $H_A(1)$. □

Note that $HH^3(A) = (HH^2(A))^*$ under the duality in [EE2]. We choose a basis $h_i$ of $HH^3(A)$, so that $h_i(f_j) = \delta_{i,j}$.

*9.1. $Q = D_{n+1}$, n even*

From $H_A(1)$ in (8.1.1) we see that:

$$d_3^*(2e_1 - e_2) = 2\omega_1,$$
$$d_3^*(-e_{i-1} + 2e_i - e_{i+1}) = 2\omega_i \quad \forall 2 \leqslant i \leqslant n-2,$$
$$d_3^*\big((-n-1)e_{n-2} + 2(n-1)e_{n-1} - 2(n-1)e_n\big) = (n-1)\omega_{n-1},$$
$$d_3^*(2e_n - e_{n-1}) = \omega_n + \omega_{n+1},$$

so

$$HH^3(A) = \big(\mathcal{N}^R\big)^{\mathrm{top}}[-h]/(\omega_1 = \omega_2 = \cdots = \omega_{n-1} = 0,\ \omega_n + \omega_{n+1} = 0)$$

with basis

$$\big\{h_n = [\omega_n]\big\}.$$

*9.2. $Q = E_6$*

From $H_A(1)$ in (8.2.1) we see that:

$$d_3^*(2e_1 - e_2) = \omega_1 + \omega_5,$$
$$d_3^*(-e_1 + 2e_2 - e_3) = \omega_2 + \omega_4,$$
$$d_3^*(-2e_2 + 2e_3 - e_6) = 2\omega_3,$$
$$d_3^*(-e_3 + 2e_6) = 2\omega_6,$$

so

$$HH^3(A) = \big(\mathcal{N}^R\big)^{\mathrm{top}}[-h]/(\omega_3 = \omega_6 = \omega_1 + \omega_5 = \omega_2 + \omega_3 = 0)$$

with basis

$$\big\{h_1 = [\omega_1],\ h_2 = [\omega_2]\big\}.$$

## 10. $HH^4(A)$

We have $HH^4(A) = U^*[-2]$, so its top degree is $-4$, and its generators sit in degrees $-4 - \deg z_k$ for each central element, one in each degree.

**Proposition 10.0.1.** *Let $\zeta_0 \in \ker d_5^*$ be a top degree element in $(V \otimes \mathcal{N})^R[-h-2]$, such that $m(\zeta_0)$ is nonzero, where $m$ is the multiplication map. Then $HH^4(A)$ is generated by elements $\zeta_k \in \ker d_5^*$ which satisfy $\zeta_k z_k = \zeta_0$.*

**Proof.** If $x \in \mathcal{N}^R[-h]$ lies in degree $-4$, then $m(d_4^*(x)) = 0$, so $\zeta_0$ is nonzero in $HH^4(A)$.

For every nontopdegree central element $z_k$ we can find a $\zeta_k$ satisfying the properties above, which is done for each quiver separately below.

For any central element $z \in A$, we have that $d_4^*(zy) = d_4^*(y)z$. If $\zeta_k = d_4^*(y)$, then by construction $\zeta_0 = \zeta_k z_k = d_4^*(z_k y)$ which is a contradiction.

So these $\zeta_k$ are all nonzero in $HH^4(A)$, and also generate this cohomology space. □

A basis of $HH^4(A)$ is given by these $\zeta_k$, and we choose them so that $\zeta_k(\theta_k) = 1$ under the duality $HH^4(A) = (HH^1(A))^*$ in [EE2].

*10.1. $Q = D_{n+1}$, n odd*

We define

$$\begin{aligned}
\zeta_0 &= \big[a_{n-1}^* \otimes a_{n-1}a_n^*a_n\big(a_{n-1}^*a_{n-1}a_n^*a_n\big)^{\frac{n-3}{2}} \\
&\quad + a_{n-1} \otimes a_n^*a_na_{n-1}^*\big(a_{n-1}a_n^*a_na_{n-1}^*\big)^{\frac{n-3}{2}}\big], \\
\zeta_{4k} &= \frac{1}{2}\big[a_{n-1}^* \otimes a_{n-1}a_n^*a_n\big(a_{n-1}^*a_{n-1}a_n^*a_n\big)^{\frac{n-3}{2}-k} \\
&\quad + a_{n-1} \otimes a_n^*a_na_{n-1}^*\big(a_{n-1}a_n^*a_na_{n-1}^*\big)^{\frac{n-3}{2}-k} \\
&\quad - a_n^* \otimes a_na_{n-1}^*a_{n-1}\big(a_n^*a_na_{n-1}^*a_{n-1}\big)^{\frac{n-3}{2}-k} \\
&\quad - a_n \otimes a_{n-1}^*a_{n-1}a_n^*\big(a_na_{n-1}^*a_{n-1}a_n^*\big)^{\frac{n-3}{2}-k}\big].
\end{aligned}$$

*10.2. $Q = D_{n+1}$, n even*

We define

$$\begin{aligned}
\zeta_0 &= \big[a_{n-1}^* \otimes a_{n-1}\big(a_n^*a_na_{n-1}^*a_{n-1}\big)^{\frac{n-2}{2}-k} \\
&\quad + a_{n-1} \otimes a_n^*\big(a_na_{n-1}^*a_{n-1}a_n^*\big)^{\frac{n-2}{2}-k}\big], \\
\zeta_{4k} &= \frac{1}{2}\big[a_{n-1}^* \otimes a_{n-1}\big(a_n^*a_na_{n-1}^*a_{n-1}\big)^{\frac{n-2}{2}-k} \\
&\quad + a_{n-1} \otimes a_n^*\big(a_na_{n-1}^*a_{n-1}a_n^*\big)^{\frac{n-2}{2}-k} \\
&\quad - a_n^* \otimes a_n\big(a_{n-1}^*a_{n-1}a_n^*a_n\big)^{\frac{n-2}{2}-k} \\
&\quad - a_n \otimes a_{n-1}^*\big(a_{n-1}a_n^*a_na_{n-1}^*\big)^{\frac{n-2}{2}-k}\big].
\end{aligned}$$

*10.3. $Q = E_6$*

We define

$$\zeta_0 = \big[a_3^* \otimes a_3\big(a_2^*a_2a_3^*a_3\big)^2 + a_3 \otimes a_2^*\big(a_2a_3^*a_3a_2^*\big)^2\big],$$

$$\zeta_6 = \frac{1}{4}\big[-a_3^* \otimes a_3a_2^*a_2 - a_3 \otimes a_2^*a_2a_2^* + a_2^* \otimes a_2a_2^*a_2 + a_2 \otimes a_2^*a_2a_3^*$$
$$- a_2^* \otimes a_2a_3^*a_3 - a_2 \otimes a_3^*a_3a_3^* + a_3^* \otimes a_3a_3^*a_3 + a_3 \otimes a_3^*a_3a_2^*\big],$$
$$\zeta_8 = \frac{1}{2}\big[a_3^* \otimes a_3 + a_3 \otimes a_2^* - a_2^* \otimes a_2 - a_2 \otimes a_3^*\big].$$

*10.4.* $Q = E_7$

We define

$$\zeta_0 = \big[a_4^* \otimes a_4a_3^*a_3\big(a_4^*a_4a_3^*a_3\big)^3 + a_4 \otimes a_3^*a_3a_4^*\big(a_4a_3^*a_3a_4^*\big)^3\big],$$
$$\zeta_8 = \frac{1}{2}\big[a_4^* \otimes a_4a_3^*a_3a_4^*a_4a_3^*a_3 + a_4 \otimes a_3^*a_3a_4^*a_4a_3^*a_3a_4^*$$
$$- a_3^* \otimes a_3a_4^*a_4a_3^*a_3a_4^*a_4 - a_3 \otimes a_4^*a_4a_3^*a_3a_4^*a_4a_3^*\big],$$
$$\zeta_{12} = \frac{1}{2}\big[a_4^* \otimes a_4a_3^*a_3 + a_4 \otimes a_3^*a_3a_4^* - a_3^* \otimes a_3a_4^*a_4 - a_3 \otimes a_4^*a_4a_3^*\big].$$

*10.5.* $Q = E_8$

We define

$$\zeta_0 = \big[a_4^* \otimes a_4a_3^*a_3\big(a_4^*a_4a_3^*a_3\big)^6 + a_4 \otimes a_3^*a_3a_4^*\big(a_4a_3^*a_3a_4^*\big)^6\big],$$
$$\zeta_{12} = \frac{1}{2}\big[a_4^* \otimes a_4a_3^*a_3\big(a_4^*a_4a_3^*a_3\big)^3 + a_4 \otimes a_3^*a_3a_4^*\big(a_4a_3^*a_3a_4^*\big)^3$$
$$- a_3^* \otimes a_3a_4^*a_4\big(a_3^*a_3a_4^*a_4\big)^3 - a_3 \otimes a_4^*a_4a_3^*\big(a_3a_4^*a_4a_3^*\big)^3\big],$$
$$\zeta_{20} = \frac{1}{2}\big[a_4^* \otimes a_4a_3^*a_3a_4^*a_4a_3^*a_3 + a_4 \otimes a_3^*a_3a_4^*a_4a_3^*a_3a_4^*$$
$$- a_3^* \otimes a_3a_4^*a_4a_3^*a_3a_4^*a_4 - a_3 \otimes a_4^*a_4a_3^*a_3a_4^*a_4a_3^*\big],$$
$$\zeta_{24} = \frac{1}{2}\big[a_4^* \otimes a_4a_3^*a_3 + a_4 \otimes a_3^*a_3a_4^* - a_3^* \otimes a_3a_4^*a_4 - a_3 \otimes a_4^*a_4a_3^*\big].$$

## 11. $HH^5(A)$

We have $HH^5(A) = U^*[-2] \oplus Y^*[-h-2]$. We discuss these two subspaces separately.

*11.1.* $U^*[-2]$

In $U^*[-2]$, like in $HH^4(A)$, we have generators coming from the center in some dual sense. We have $d_6^*(U^*[-2]) = 0$.

**Proposition 11.1.1.** *Let $\psi_0$ be a top degree element $[\omega_i]$ in some $e_i\mathcal{N}^R e_i[-h-2]$. Then $HH^5(A)$ is generated by $\psi_k \in \mathcal{N}^R$ which satisfy $\psi_k z_k = \psi_0$.*

**Proof.** If $\sum_{a\in\bar{Q}} a \otimes x_a \in V \otimes \mathcal{N}^R$ lies in degree $-4$, then the image of $d_5^*(x) = \sum_a ax_a - x_a\eta(a)$, under the linear map $f$ (which is associated to $A$ as a Frobenius algebra) is zero where $f(\omega_i) = 1$. So $\psi_0$ is nonzero in $HH^5(A)$.

For every nontopdegree central element $z_k$ we can find a $\zeta_k$ satisfying the properties above, which is done for each quiver separately in Subsection 11.3.

For any central element $z \in A$, we have that $d_5^*(zy) = d_5^*(y)z$. If $\psi_k = d_5^*(y)$, then by construction $\psi_0 = \psi_k z_k = d_4^*(z_k y)$ which is a contradiction.

So these $\psi_k$ are nonzero in $HH^5(A)$ and generate this cohomology space. □

The relation $ax_a = x_a\eta(a)$ then gives us that all $\omega_i$'s are equivalent in $HH^5(A)$.

*11.2. $Y^*[-h-2]$*

We have to introduce some new notations.

**Definition 11.2.1.** We define $F$ to be the set of vertices in $I$ which are fixed by $\nu$, i.e.

$$F = \{i \in I \mid \nu(i) = i\}.$$

**Definition 11.2.2.** Let $\eta_{ij}$ be the restriction of $\eta$ on $e_i A e_j$ $(i, j \in F)$. Let $n_{ij}^+ = \dim\ker(\eta_{ij} - 1)$ and $n_{ij}^- = \dim\ker(\eta_{ij} + 1)$.

We define the *signed truncated dimension matrix* $(H_A^\eta)_{i,j\in F}$ in the following way:

$$(H_A^\eta)_{ij} = n_{ij}^+ - n_{ij}^-.$$

Now we can make the following statement:

**Proposition 11.2.3.** *$Y^*[-h-2]$ is given by the kernel of the matrix $H_A^\eta$, where we identify $\mathbb{C}^F = \bigoplus_{i\in F} Re_i$.*

**Proof.** $Y^*[-h-2]$ is the kernel of the restriction $d_6^*|_{\mathcal{N}^R[-h-2](-h-2)=R_F[-h-2]} \to A^R[-2h]$, where $R_F$ is the linear span of $e_i$'s, such that $i$ is fixed by $\nu$,

$$d_6^*(y) = \sum_{x_j\in B} x_j y \eta(x_j^*) = \sum_{x_j\in B} \eta(x_j) y x_j^*,$$

then

$$d_6^* : R_F[-h-2] \to (A^{\mathrm{top}})^R[-2h]$$

can also be written as a matrix multiplication

$$H_A^\eta : \mathbb{C}^F \to \mathbb{C}^F$$

under the identifications $R_F = \mathbb{C}^F = \bigoplus_{i\in F} e_i A^{\mathrm{top}} e_i$. □

We compute the matrices $H_A^\eta$ and their kernels for each quiver separately.

Recall that $\dim Y = r_+ - r_- - \#\{m_i \mid m_i = \frac{h}{2}\} = \dim R_F - \#\{m_i \mid m_i = \frac{h}{2}\}$. We will find $Y^*$ explicitly for each quiver.

*11.2.1.* $Q = E_6, E_8$

$\frac{h}{2}$ is not an exponent, so $Y^* = R_F$.

*11.2.2.* $Q = D_{n+1}$, *n odd*

All basis elements of $e_k A e_j$ given in Section 5.5 are eigenvectors of $\eta_{kj}$.

For any of these basis elements $x$, $\eta(x) = (-1)^{n_x} x$ where $n_x$ is the number of no-star letters in the monomial expression of $x$. So $H_A^\eta$ can be computed directly, and we get

$$H_A^\eta = \begin{pmatrix} 2 & 0 & \cdots & 2 & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & \cdots & 2 & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 1 & 0 & \frac{n+1}{2} & -\frac{n-1}{2} \\ 1 & 0 & \cdots & 1 & 0 & -\frac{n+1}{2} & \frac{n+1}{2} \end{pmatrix},$$

and the kernel is given by

$$\left\langle e_{2k-1} - e_1,\ e_{2k}, (e_n + e_{n+1}) - e_1 \;\middle|\; k \leqslant \frac{n-1}{2} \right\rangle.$$

*11.2.3.* $Q = D_{n+1}$, *n even*

Since $F = \{1, \ldots, n-1\}$, we work only with $e_k A e_j$ for $j, k \leqslant n-1$, and we have to work with a modified basis, so that they are all eigenvectors of $\eta$:

For $\underline{k \leqslant j \leqslant n-1}$,

$$\begin{aligned} B_{k,j} = {} & \big\{ \big(a_{k-1} a_{k-1}^*\big)^l a_k^* \cdots a_{j-1}^* \;\big|\; 0 \leqslant l \leqslant \min\{k-1, n-j-1\} \big\} \\ & \cup \big\{ \big(a_{k-1} a_{k-1}^*\big)^l a_k^* \cdots \big(a_{n-1}^* a_{n-1} - a_n^* a_n\big) a_{n-2} a_j \;\big|\; 0 \leqslant l \leqslant k-1 \big\} \\ & \cup \big\{ \big(a_{k-1} a_{k-1}^*\big)^l a_k^* \cdots \big(a_{n-1}^* a_{n-1} + a_n^* a_n\big) a_{n-2} a_j \;\big|\; 0 \leqslant l \leqslant k-1+j-n \big\}. \end{aligned}$$

For $\underline{j < k \leqslant n-1}$,

$$\begin{aligned} B_{k,j} = {} & \big\{ a_{k-1} \cdots a_j \big(a_j^* a_j\big)^l \;\big|\; 0 \leqslant l \leqslant \min\{n-k-1, j-1\} \big\} \\ & \cup \big\{ a_k^* \cdots a_{n-2}^* \big(a_{n-1}^* a_{n-1} - a_n^* a_n\big) a_{n-2} \cdots a_j \big(a_j^* a_j\big)^l \;\big|\; 0 \leqslant l \leqslant j-1 \big\} \\ & \cup \big\{ a_k^* \cdots a_{n-2}^* \big(a_{n-1}^* a_{n-1} + a_n^* a_n\big) a_{n-2} \cdots a_j \big(a_j^* a_j\big)^l \;\big|\; 0 \leqslant l \leqslant j-1+k-n \big\}. \end{aligned}$$

From that, we can calculate the matrix:

$$H_A^\eta = \begin{pmatrix} 2 & 0 & \cdots & 2 & 0 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 0 & \cdots & 2 & 0 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & \cdots & 2 & 0 & 2 \end{pmatrix},$$

and we get immediately its kernel

$$\left\langle e_{2k+1} - e_1, e_{2k} \;\middle|\; 1 \leqslant k \leqslant \frac{n}{2} \right\rangle.$$

*11.2.4.* $Q = E_7$

We do not use an explicit basis of $A$ here. All we have to know is the number of no-star letters in the monomial basis elements which can be directly obtained from the Hilbert series $H_A(t)$ in the following way: given a monomial $x$ of length $l$ in $e_k A e_j$, $n_{kj}$ the number of arrows in $Q$ on the shortest path from $j$ to $k$ of length $d(k, j)$, $x$ contains $n_{k,j} + \frac{l-d(k,j)}{2}$ arrows in $Q$.

So we obtain the formula

$$\left(H_A^\eta\right)_{k,j} = (-1)^n_{k,j} \frac{H_A(t)_{k,j}}{t^{d(k,j)}}\bigg|_{t=\sqrt{(-1)}},$$

where we can get $H_A(\sqrt{-1})$ from (2.5.1) and compute

$$H_A^\eta = \begin{pmatrix} 3 & 0 & 3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -3 & 0 & 0 & 0 & 3 \end{pmatrix},$$

and its kernel is

$$\langle e_1 + e_7, e_2, e_3 + e_7, e_4, e_5, e_6 \rangle.$$

*11.3. Result*

Now we give explicit bases for each quiver where $\psi_i \in U^*[-2]$ satisfy the properties given in Section 11.1 and $\varepsilon_i \in Y^*[-h-2]$ are taken from Section 11.2.

Note the duality $HH^6(A) = (HH^5(A))^*$ which was established in [EE2], $\phi_0(z_0) \in U[-2h-2]$, $\varphi_0(\omega_i) \in Y[-h-2]$. We choose $\psi_0$ such that $\psi_0(\varphi_0(z_0)) = 1$ (from that follows $\psi_k(\varphi_0(z_k)) = z_k \psi_k(\varphi_0(z_0)) = \psi_0(\varphi_0(z_0)) = 1$) and $\varepsilon_i$ such that $\varepsilon_i(\phi_0(\omega_j)) = \delta_{ij}$.

*11.3.1. $Q = D_{n+1}$, n odd*

We define

$$\psi_{4k} = \left[\left(a_{n-1}^* a_{n-1} a_n^* a_n\right)^{\frac{n-1}{2}-k}\right],$$
$$\varepsilon_{2k-1} = [e_{2k-1} - e_1], \qquad \varepsilon_{2k} = [e_{2k}], \qquad \varepsilon_n = \left[(e_n + e_{n+1}) - e_1\right], \quad k \leqslant \frac{n-1}{2}.$$

*11.3.2. $Q = D_{n+1}$, n even*

We define

$$\psi_{4k} = \left[a_{n-1}^* a_{n-1}\left(a_n^* a_n a_{n-1}^* a_{n-1}\right)^{\frac{n-2}{2}-k}\right],$$
$$\varepsilon_{2k+1} = [e_{2k+1} - e_1], \qquad \varepsilon_{2k} = [e_{2k}], \quad 1 \leqslant k \leqslant \frac{n}{2} - 1.$$

*11.3.3. $Q = E_6$*

We define

$$\psi_0 = \left[a_3^* a_3\left(a_2^* a_2 a_3^* a_3\right)^2\right],$$
$$\psi_6 = \left[-a_3^* a_3 a_2^* a_2\right],$$
$$\psi_8 = \left[a_3^* a_3 - a_2^* a_2\right],$$
$$\varepsilon_3 = [e_3], \quad \varepsilon_6 = [e_6].$$

*11.3.4. $Q = E_7$*

We define

$$\psi_0 = \left[\left(a_4^* a_4 a_3^* a_3\right)^4\right],$$
$$\psi_8 = \left[\left(a_4^* a_4 a_3^* a_3\right)^2\right],$$
$$\psi_{12} = \left[a_4^* a_4 a_3^* a_3\right],$$
$$\varepsilon_1 = [e_1 + e_7], \qquad \varepsilon_2 = [e_2], \qquad \varepsilon_3 = [e_3 + e_7],$$
$$\varepsilon_4 = [e_4], \qquad \varepsilon_5 = [e_5], \qquad \varepsilon_6 = [e_6].$$

*11.3.5. $Q = E_8$*

We define

$$\psi_0 = \left[\left(a_4^* a_4 a_3^* a_3\right)^7\right],$$
$$\psi_{12} = \left[\left(a_4^* a_4 a_3^* a_3\right)^4\right],$$
$$\psi_{20} = \left[\left(a_4^* a_4 a_3^* a_3\right)^2\right],$$
$$\psi_{24} = \left[a_4^* a_4 a_3^* a_3\right],$$
$$\varepsilon_1 = [e_1], \qquad \varepsilon_2 = [e_2], \qquad \varepsilon_3 = [e_3], \qquad \varepsilon_4 = [e_4],$$
$$\varepsilon_5 = [e_5], \qquad \varepsilon_6 = [e_6], \qquad \varepsilon_7 = [e_7], \qquad \varepsilon_8 = [e_8].$$

## 12. $HH^6(A)$

$HH^6(A) = U[-2h-2] \oplus Y[-h-2] = HH^0(A)/\operatorname{Im}(d_6^*)$, and $\operatorname{Im}(d_6^*)$ is spanned by the columns of the matrices $H_A^\eta$ which were computed in the previous section.

This gives us the following result:

**Proposition 12.0.1.** *$HH^6(A)$ is a quotient of $HH^0(A)$. In particular,*

$$HH^6(A) = \begin{cases} HH^0(A), & Q = E_6, E_8, \\ HH^0(A)/(\sum_{\substack{i=1\\ odd}}^{n-2} \omega_i = 0, \omega_n = \omega_{n+1}), & Q = D_{n+1},\ n \text{ odd}, \\ HH^0(A)/(\sum_{\substack{i=1\\ odd}}^{n-1} \omega_i = 0), & Q = D_{n+1},\ n \text{ even}, \\ HH^0(A)/(\omega_1 + \omega_3 - \omega_7 = 0), & Q = E_7. \end{cases}$$

## 13. Products involving $HH^0(A) = Z$

Recall the decomposition $HH_0(A) = \mathbb{C} \oplus (U[-2])_+ \oplus L[h-2]$. It is clear that the $\mathbb{C}$-part acts on $HH^i(A)$ as the usual multiplication with $\mathbb{C}$, with $z_0$ as identity. From the periodicity of the Schofield resolution with period 6, it follows that the multiplication with $\varphi(z_0) \in HH^6(A)$ gives the natural isomorphism $HH^i(A) \to HH^{i+6}(A)$ for $i \geqslant 1$.

We summarize all products not involving the $\mathbb{C}$-part.

### *13.1. $HH^0(A) \times HH^0(A) \twoheadrightarrow HH^0(A)$*

This is already done in the $HH^0(A)$-section of this paper. We state the results:

*13.1.1. $Q = D_{n+1}$, $n$ odd*

The products are

$$z_{4j} z_{4k} = \begin{cases} z_{4(j+k)}, & j+k < \frac{n-1}{2}, \\ \omega_n - \omega_{n+1}, & j+k = \frac{n-1}{2}, \\ 0, & j+k > \frac{n-1}{2}. \end{cases}$$

*13.1.2. $Q = D_{n+1}$, $n$ even*

The products are

$$z_{4j} z_{4k} = \begin{cases} z_{4(j+k)}, & j+k < \frac{n-1}{2}, \\ 0, & j+k \geqslant \frac{n-1}{2}. \end{cases}$$

*13.1.3. $E_6$*

All products are zero.

*13.1.4. $E_7$*

The only nonzero product is $z_8^2 = \omega_1 + \omega_3 - \omega_7$.

*13.1.5.* $E_8$

The only nonzero product is $z_{12}^2 = z_{24}$.

*13.2.* $HH^0(A) \times HH^1(A) \twoheadrightarrow HH^1(A)$

From the definition of the maps $\theta_k$ (which are generated by the central elements $z_k$), it follows that the $Z$-action is natural, i.e. the multiplication rule is the same as with the $z_k$ counterpart: $z_k\theta_0 = \theta_k$.

We state the other nonzero products:

*13.2.1.* $Q = D_{n+1}$

We have $z_{4j}\theta_{4k} = \theta_{4(j+k)}$ if $j + k < \frac{n-1}{2}$.

*13.2.2.* $E_8$

We have $z_{12}\theta_{12} = \theta_{24}$.

*13.3.* $HH^0(A) \times HH^i(A) \twoheadrightarrow HH^i(A)$, *$i = 2$ or 3*

$HH^2(A) = K[-2]$ and $HH^3(A) = K^*[-2]$ live in only one degree, so $(U[-2])_+ \subset HH^0(A)$ acts by zero.

*13.4.* $HH^0(A) \times HH^4(A) \twoheadrightarrow HH^4(A)$

We defined $\zeta_k$, such that $z_k\zeta_k = \zeta_0$ holds. By degree arguments, only these other products are nonzero:

*13.4.1.* $Q = D_{n+1}$

For $l < k$, $z_{4l}\zeta_{4k} = \zeta_{4(k-l)}$ (since $z_{4(k-l)}(z_{4l}\zeta_{4k}) = (z_{4(k-l)}z_{4l})\zeta_{4k} = \zeta_0$, and $\zeta_{4(k-l)}$ is (up to a multiple) the only one element of degree $-4 - 4(k-l)$ in $HH^4(A)$).

*13.4.2.* $Q = E_8$

We have $z_{12}\zeta_{24} = \zeta_{12}$ (since $z_{12}(z_{12}\zeta_{24}) = (z_{12}z_{12})\zeta_{24} = \zeta_0$, and $\zeta_{12}$ is (up to a multiple) the only element of degree $-16$ in $HH^4(A)$).

*13.5.* $HH^0(A) \times HH^5(A) \twoheadrightarrow HH^5(A)$

By definition, $z_k\psi_k = \psi_0$ holds. Since $\psi_i \in U^*[-2]$ corresponds to $\zeta_i \in U^*[-2]$ in $HH^4(A)$ with the rule $z_k\psi_k = \psi_0$ corresponding to $z_k\zeta_k = \zeta_0$ above, the multiplication rules of $\psi_k$ with elements in $HH^0(A)$ can be derived from above.

Products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and
$\varepsilon_j = \sum_{k \in F} \lambda_k e_k \in Y^*[-h-2]$ are easy to calculate: $\omega_i\varepsilon_j = \lambda_i[\omega_i] = \lambda_i\psi_0$.

**Proposition 13.5.1.** *The multiplication* $((U[-2])_+) \times Y^*[-h-2] \to HH^5(A)$ *is zero.*

We will show this for any quiver separately.

*13.5.1. $Q = D_{n+1}$, n odd*

For $l < k$, $z_{4l}\psi_{4k} = \psi_{4(k-l)}$.

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_{2k-1}\varepsilon_{2k-1} = \omega_{2k}\varepsilon_{2k} = \omega_n\varepsilon_n = \omega_{n+1}\varepsilon_n = \omega_1\varepsilon_{2k-1} = \omega_1\varepsilon_n = \psi_0,$$

$$\omega_1\varepsilon_{2k-1} = \omega_1\varepsilon_n = -\psi_0.$$

We show $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$: by degree argument, $z_{4k}\varepsilon_i = \lambda\psi_{2n-2-4k}$. Then $z_{2n-2-4k}(z_{4k}\varepsilon_i) = \lambda z_{2n-2-4k}\psi_{2n-2-4k} = \lambda\psi_0$, and by associativity this equals $(z_{2n-2-4k}z_{4k})\varepsilon_i = (\omega_n - \omega_{n+1})\varepsilon_i = 0$, so $\lambda = 0$.

*13.5.2. $Q = D_{n+1}$, n even*

For $l < k$, $z_{4l}\psi_{4k} = \psi_{4(k-l)}$.

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_{2k+1}\varepsilon_{2k+1} = \omega_{2k}\varepsilon_{2k} = \psi_0,$$

$$\omega_1\varepsilon_{2k+1} = -\psi_0.$$

We show $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$: by degree argument, $z_{4k}\varepsilon_i = \lambda\psi_{2n-2-4k}$. Then $z_{2n-2-4k}(z_{4k}\varepsilon_i) = \lambda z_{2n-2-4k}\psi_{2n-2-4k} = \lambda\psi_0$, and this equals $(z_{2n-2-4k}z_{4k})\varepsilon_i = 0$, so $\lambda = 0$.

*13.5.3. $Q = E_6$*

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_3\varepsilon_3 = \omega_6\varepsilon_6 = \psi_0.$$

By degree argument, $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$.

*13.5.4. $Q = E_7$*

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_1\varepsilon_1 = \omega_2\varepsilon_2 = \omega_3\varepsilon_3 = \omega_4\varepsilon_4 = \omega_5\varepsilon_5 = \omega_6\varepsilon_6 = \omega_7\varepsilon_1 = \omega_7\varepsilon_3 = \psi_0.$$

We show $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$: by degree argument, only products involving $z_8$ may eventually be nontrivial,

$$z_8\varepsilon_i = \lambda\psi_8, \quad \lambda \in \mathbb{C}.$$

Then

$$z_8(z_8\varepsilon_i) = \lambda z_8\psi_8 = \lambda\psi_0,$$

and by associativity this equals

$$z_8^2\varepsilon_i = (\omega_1 + \omega_3 - \omega_7)\varepsilon_i = 0,$$

so $\lambda = 0$.

## 14. Products involving $HH^1(A)$

### 14.1. $HH^1(A) \times HH^1(A) \xrightarrow{0} HH^2(A)$

This follows by degree argument since $\deg HH^1(A) > 0$, $\deg HH^2(A) = -2$.

### 14.2. $HH^1(A) \times HH^2(A) \twoheadrightarrow HH^3(A)$

$HH^2(A)$ and $HH^3(A)$ are trivial for $Q = D_{n+1}$ where $n$ is odd and for $Q = E_7, E_8$.
We know that $HH^1(A)$ is generated by maps $\theta_k$ and $HH^2(A)$ by $f_i$ $(i \neq \nu(i))$, and we lift

$$f_i : A \otimes A[2] \to A, \quad 1 \otimes 1 \mapsto e_i - e_{\nu(i)}$$

to

$$\hat{f}_i : A \otimes A[2] \to A \otimes A, \quad 1 \otimes 1 \mapsto e_i \otimes e_i - e_{\nu(i)} \otimes e_{\nu(i)}.$$

Then

$$\hat{f}_i d_3(1 \otimes 1) = \hat{f}_i \left( \sum_{x_j \in B} x_j \otimes x_j^* \right) = \sum_{x_j \in B} x_j e_i \otimes e_i x_j^* - x_j e_{\nu(i)} \otimes e_{\nu(i)} x_j^*.$$

To compute the lift $\Omega f_i$, we need to find out the preimage of $\sum x_j e_i \otimes e_i x_j^* - x_j e_{\nu(i)} \otimes e_{\nu(i)} x_j^*$ under $d_1$.

**Definition 14.2.1.** Let $b_1, \ldots, b_k$ be arrows, $p$ the monomial $\pm b_1 \cdots b_k$ and define

$$v_p := \pm(1 \otimes b_1 \otimes b_2 \cdots b_k + b_1 \otimes b_2 \otimes b_3 \cdots b_k + \cdots + b_1 \cdots b_{k-1} \otimes b_k \otimes 1),$$

and for $i < j$,

$$v_p^{(i,j)} := \pm \sum_{l=i}^{j} b_1 \cdots b_{l-1} \otimes b_l \otimes b_{l+1} \cdots b_k.$$

We will use the following lemma in our computations.

**Lemma 14.2.2.** *In the above setting,*

$$d_1(v_p) = \pm(b_1 \cdots b_k \otimes 1 - 1 \otimes b_1 \cdots b_k).$$

From that, we see immediately that when assuming all $x_j$ are monomials (which we can do), then

$$\hat{f}_i\left(\sum_{x_j\in B} x_j\otimes x_j^*\right)=d_1\left(\sum_{x_j\in B} v^{(1,\deg(x_j))}_{x_je_ix_j^*}-v^{(1,\deg(x_j))}_{x_je_{\nu(i)}x_j^*}\right)+1\otimes\underbrace{\sum_{x_j\in B}\left(x_je_ix_j^*-x_je_{\nu(i)}x_j^*\right)}_{=0},$$

so we have

$$\begin{aligned}\Omega f_i:\Omega^3(A)&\to\Omega(A),\\ 1\otimes 1&\mapsto\sum_{x_j\in B} v^{(1,\deg(x_j))}_{x_je_ix_j^*}-v^{(1,\deg(x_j))}_{x_je_{\nu(i)}x_j^*}.\end{aligned}$$

Then

$$\theta_k\left(\sum_{x_j\in B} v^{(1,\deg(x_j))}_{x_je_ix_j^*}-v^{(1,\deg(x_j))}_{x_je_{\nu(i)}x_j^*}\right)=z_k\left(\sum_{x_j\in B_{-,i}} s(x_j)x_jx_j^*-\sum_{x_j\in B_{-,\nu(i)}} s(x_j)x_jx_j^*\right),$$

where $s(x_j)$ is the number of arrows in $Q^*$ in the monomial expression of $x_j$.

So we get

$$(\theta_k\circ\Omega f_i)(1\otimes 1)=z_k\left(\sum_{x_j\in B_{-,i}} s(x_j)x_jx_j^*-\sum_{x_j\in B_{-,\nu(i)}} s(x_j)x_jx_j^*\right).$$

Under our identification in [EE2, Subsection 4.5],

$$\theta_k f_i=\left[z_k\left(\sum_{l\in I}\sum_{x_j\in B_{l,i}} s(x_j)\omega_l-\sum_{l\in I}\sum_{x_j\in B_{l,\nu(i)}} s(x_j)\omega_l\right)\right]\in HH^3(A).$$

All products are zero if $z_k$ lies in a positive degree, so we only have to calculate the products where $k=0$.

We make the following

**Proposition 14.2.3.** *The multiplication with $\theta_0$ induces a symmetric isomorphism*

$$\alpha: HH^2(A)=K[-2]\xrightarrow{\cong}K^*[-2]=HH^3(A).$$

Now we have to work with explicit basis elements $x_j\in Ae_i$, $i\neq\nu(i)$, so we treat the Dynkin quivers separately and find the matrix $M_\alpha$ which represents this map.

*14.2.1. $Q=D_{n+1}$, n even*

We can work with the basis given in Section 5.5 and compute

$$\theta_0 f_n=\frac{n}{2}\left([\omega_{n+1}]-[\omega_n]\right)=-nh_n \tag{14.2.4}$$

because of the relation $[\omega_n]+[\omega_{n+1}]=0$ in $HH^3(A)$. $\alpha$ is given by the matrix

$$M_\alpha=(-n).$$

*14.2.2. $E_6$*

We will write out the basis elements of $Ae_1$, $Ae_5$:

$$\begin{aligned}
B_{1,1} &= \langle e_1, a_1a_2a_5^*a_5a_2^*a_1^*\rangle,\\
B_{2,1} &= \langle a_1^*, a_2a_5^*a_5a_2^*a_1^*, a_2a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,\\
B_{3,1} &= \langle a_2^*a_1^*, a_3^*a_3a_2^*a_1^*, a_3^*a_3a_3^*a_3a_2^*a_1^*, a_5^*a_5a_3^*a_3a_3^*a_3a_2^*a_1^*\rangle,\\
B_{4,1} &= \langle a_3a_2^*a_1^*, a_3a_5^*a_5a_2^*a_1^*, a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,\\
B_{5,1} &= \langle a_4a_3a_2^*a_1^*, a_4a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,\\
B_{6,1} &= \langle a_5a_2^*a_1^*, a_5a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,
\end{aligned}$$

and

$$e_iAe_5 = \langle \eta(x) \mid x \in e_{\nu(i)}Ae_1\rangle,$$

where $\eta(a) = -\epsilon_a\bar{a}$ and for any arrow $a : i \to j$, $\bar{a}$ is the arrow $j \to i$, so $\eta$ preserves the number of star letters of a monomial $x$. From this, we obtain

$$\theta_0 f_1 = -4[\omega_1] - 2[\omega_2] + 2[\omega_4] + 4[\omega_5] = -8h_1 - 4h_2$$

because of the relations $[\omega_1] + [\omega_4] = [\omega_2] + [\omega_3] = 0$ in $HH^3(A)$.

We do the same thing for $Ae_2$ and $Ae_4$:

$$\begin{aligned}
B_{1,2} &= \langle a_1, a_1a_2a_5^*a_5a_2^*, a_1a_2a_5^*a_5a_3^*a_3a_2^*\rangle,\\
B_{2,2} &= \langle e_2, a_2a_2^*, a_2a_5^*a_5a_2^*, a_2a_3^*a_3a_5^*a_5a_2^*, a_2a_5^*a_5a_3^*a_3a_2^*, a_2a_5^*a_5a_3^*a_3a_5^*a_5a_2^*\rangle,\\
B_{3,2} &= \langle a_2^*, a_5^*a_5a_2^*, a_3^*a_3a_2^*, a_5^*a_5a_3^*a_3a_2^*, a_3^*a_3a_5^*a_5a_2^*,\\
&\qquad a_3^*a_3a_5^*a_5a_3^*a_3a_2^*, a_5^*a_5a_3^*a_3a_5^*a_5a_2^*, a_5^*a_5a_3^*a_3a_5^*a_5a_3^*a_3a_2^*\rangle,\\
B_{4,2} &= \langle a_3a_2^*, a_3a_5^*a_5a_2^*, a_3a_3^*a_3a_2^*, a_3a_3^*a_3a_5^*a_5a_2^*,\\
&\qquad a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*, a_3a_3^*a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*\rangle,\\
B_{5,2} &= \langle a_4a_3a_2^*, a_4a_3a_5^*a_5a_2^*, a_4a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*\rangle,\\
B_{6,2} &= \langle a_5a_2^*, a_5a_3^*a_3a_2^*, a_5a_3^*a_3a_5^*a_5a_2^*, a_5a_3^*a_3a_5^*a_5a_3^*a_3a_2^*\rangle,
\end{aligned}$$

and we get the basis elements for $e_iAe_3$ from $\eta(x_j)$ where $x_j \in e_{\nu(i)}Ae_4$. Since $\eta$ preserves the number of star-letters of a monomial, we can immediately calculate

$$\theta_0 f_2 = -2[\omega_1] - 4[\omega_2] + 4[\omega_4] + 2[\omega_5] = -4h_1 - 8h_2$$

because of the relations $[\omega_1] + [\omega_4] = [\omega_2] + [\omega_3] = 0$ in $HH^3(A)$.

So $\alpha$ is given by the symmetric, nondegenerate matrix

$$M_\alpha = \begin{pmatrix} -8 & -4 \\ -4 & -8 \end{pmatrix}. \tag{14.2.5}$$

*14.3.* $HH^1(A) \times HH^3(A) \xrightarrow{0} HH^4(A)$

This follows by degree argument: $\deg HH^1(A) \geqslant 0$, $\deg HH^3(A) = -2$, but $\deg HH^4(A) \leqslant -4$.

*14.4.* $HH^1(A) \times HH^4(A) \to HH^5(A)$

**Proposition 14.4.1.** *Given $\theta_k \in HH^1(A)$ and $\zeta_l \in HH^4(A)$, we get the following cup product*:

$$\theta_k \zeta_l = \psi_l z_k. \tag{14.4.2}$$

**Proof.** It is enough to show $\theta_0\zeta_0 = \psi_0$: $z_l(\theta_0\zeta_l) = \theta_0\zeta_0\psi_0$ implies that $(\theta_0\zeta_l) = \psi_l$, and the equation above follows from $\theta_k = z_k\theta_0$.

Let in general $x = \sum_{a\in\bar{Q}} a \otimes x_a \in HH^4(A)$. Then $x$ represents the map

$$\begin{aligned} x := A \otimes V \otimes \mathcal{N}[h] &\to A, \\ 1 \otimes a_i \otimes 1 &\mapsto -x_{a_i^*}, \\ 1 \otimes a_i^* \otimes 1 &\mapsto x_{a_i}, \end{aligned}$$

and it lifts to

$$\begin{aligned} \hat{x} : A \otimes V \otimes \mathcal{N}[h] &\to A \otimes A, \\ 1 \otimes a \otimes 1 &\mapsto -1 \otimes x_{a^*}, \\ 1 \otimes a^* \otimes 1 &\mapsto 1 \otimes x_a. \end{aligned}$$

Then

$$\begin{aligned} (\hat{x} \circ d_5)(1 \otimes 1) &= \hat{x}\Bigg(\sum_{a\in\bar{Q}} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a\in\bar{Q}} \epsilon_a 1 \otimes a \otimes a^*\Bigg) \\ &= \sum_{a\in Q} a \otimes x_a - \sum_{a\in Q} 1 \otimes x_a \eta(a) + \sum_{a\in Q} a^* \otimes x_{a^*} - \sum_{a\in Q} 1 \otimes x_{a^*}\eta\big(a^*\big) \\ &= \sum_{a\in Q} a \otimes x_a - \sum_{a\in Q} 1 \otimes a x_a \sum_{a\in Q} a^* \otimes x_{a^*} - \sum_{a\in Q} 1 \otimes a^* x_{a^*} \\ &= d_1\Bigg(\sum_{a\in Q} 1 \otimes a \otimes x_a + 1 \otimes a^* \otimes x_{a^*}\Bigg), \end{aligned}$$

so we have

$$\begin{aligned} \Omega x : \Omega^5(A) &\to \Omega(A), \\ 1 \otimes 1 &\mapsto \sum_{a\in Q} 1 \otimes a \otimes x_a + 1 \otimes a^* \otimes x_{a^*}, \end{aligned}$$

and this gives us

$$(\theta_0 \circ x)(1 \otimes 1) = \sum_{a \in Q} a^* x_{a^*},$$

so the cup product is

$$\theta_0 \cdot x = \sum_{a \in Q} a^* x_{a^*}. \tag{14.4.3}$$

It can be easily checked by using explicit elements that the RHS is $\psi_0$ for $x = \zeta_0$, but we the reason here why this is true: for $x = \sum_{a \in \bar{Q}} a \otimes x_a = \zeta_0$, the RHS becomes

$$\sum_{a \in Q} a^* x_{a^*} = \sum_{a \in Q} \bigl(a^*, x_{a^*}\bigr)[\omega_{t(a)}],$$

where $(-,-) : A \times A \to \mathbb{C}$ is the bilinear form attached to $A$ as a Frobenius algebra (see 2.4).

But under the bilinear form on $V \otimes A$, given in [EE2, Subsection 4.3] which induces the duality $HH^4(A) = (HH^1(A))^*$,

$$(a \otimes x_a, b \otimes x_b) = \delta_{a,b^*} \epsilon_a (x_a, x_b),$$

$$\sum_{a \in Q} \bigl(a^*, x_{a^*}\bigr) = (\theta_0, \zeta_0) = 1.$$

So for $x = \zeta_0$, Eq. (14.4.3) becomes

$$\theta_0 \zeta_0 = (\theta_0, \zeta_0)\psi_0 = \psi_0, \tag{14.4.4}$$

because $[\omega_i] = \psi_0$ in $HH^5(A)$ for all $i \in I$. □

*14.5. $HH^1(A) \times HH^5(A) \to HH^6(A)$*

We know that

$$0 \leqslant \deg\bigl(HH^1(A)\bigr) \leqslant h - 4,$$
$$-h - 2 \leqslant \deg\bigl(HH^5(A)\bigr) \leqslant -2,$$
$$-2h \leqslant \deg\bigl(HH^6(A)\bigr) \leqslant -h - 2,$$

so the product is trivial unless we pair the lowest degree parts of $HH^1(A)$ (generated by $\theta_0$) and $HH^5(A)$ (which is $Y^*[-h-2]$). The product will then live in degree $-h-2$ which is the top degree part of $HH^6(A)$, the space $Y[-h-2]$.

Given an element $\psi \in HH^5(A)(-h-2)$ which has the form

$$\psi : A \otimes \mathcal{N}[h+2] \to A,$$
$$1 \otimes 1 \mapsto \sum_{i \in F} \lambda_i e_i \in R,$$

this lifts to

$$\hat{\psi} : A \otimes \mathcal{N}[h+2] \to A \otimes A,$$
$$1 \otimes 1 \mapsto \sum_{i \in F} \lambda_i e_i \otimes e_i.$$

Then

$$\begin{aligned}
\hat{\psi}\big(d_6(1 \otimes 1)\big) &= \hat{\psi}\Big(\sum_{x_j \in B} x_j \otimes x_j^*\Big) = \hat{\psi}\Big(\sum_{x_j \in B} \eta(x_j) \otimes \eta\big(x_j^*\big)\Big) \\
&= \sum_{x_j \in B} \sum_{i \in F} \lambda_i \eta(x_j) e_i \otimes e_i x_j^* \\
&= d_1\Big(\sum_{i \in F} \sum_{x_j \in B} \lambda_i v_{\eta(x_j) e_i x_j^*}^{(1,\deg(x_j))}\Big) + 1 \otimes \underbrace{\sum_{x_j \in B} \sum_{i \in F} \lambda_i \eta(x_j) e_i x_j^*}_{=0},
\end{aligned}$$

so $\psi$ lifts to

$$\Omega\psi : \Omega^6(A) \to \Omega(A),$$
$$1 \otimes 1 \mapsto \sum_{i \in F} \sum_{x_j \in B} \lambda_i v_{\eta(x_j) e_i x_j^*}^{(1,\deg(x_j))}.$$

We get

$$(\theta_0 \circ \Omega\psi)(1 \otimes 1) = \sum_{i \in F} \sum_{x_j \in B_{-,i}} \lambda_i s(x_j) \eta(x_j) x_j^*,$$

where $s(x_j)$ is the number of arrows in $Q^*$ in the monomial expression of $x_j$ (or in general if $x_j$ is a homogeneous polynomial where each monomial term has the same number of arrows in $Q^*$, then $s(x_j)$ is the number of $Q^*$-arrows in each monomial term).

Under our identifications in [EE2, Subsection 4.5],

$$\theta_0 \psi = \sum_{i \in F} \sum_{x_j \in B_{-,i}} \lambda_i s(x_j) \eta(x_j) x_j^* = \sum_{i,k \in F} \sum_{x_j \in B_{ki}} \lambda_i s(x_j) \eta(x_j) x_j^*.$$

To simplify this computation, we will choose a basis, such that all $x_j \in e_k A e_l$ for some $k, l \in I$ and that additionally $x_j$ is an eigenvector of $\eta$ for $k, l \in F$ (since $\eta$ is an involution on $e_k A e_l$ for $k, l \in F$). Let $B_{k,l}^+$ be a basis of $(e_k A e_l)_+ = \ker(\eta|_{e_k A e_l} - 1)$ and $B_{k,l}^-$ a basis of $(e_k A e_l)_- = \ker(\eta|_{e_k A e_l} + 1)$.

Let us define

$$\kappa_{k,l} = \sum_{x_j \in B_{k,l}^+} s(x_j) - \sum_{x_j \in B_{k,l}^-} s(x_j). \tag{14.5.1}$$

Then the above equation becomes

$$\theta_0\psi = \sum_{l\in F}\lambda_l \sum_{k\in F}\kappa_{k,l}\varphi_0(\omega_k). \tag{14.5.2}$$

**Proposition 14.5.3.** *The multiplication by $\theta_0$ induces a skew-symmetric isomorphism*

$$\beta : Y^*[-h-2] \xrightarrow{\cong} Y[-h-2].$$

We will treat the Dynkin quivers separately and find the matrix $M_\beta$ which represents $\beta$ for each of these quivers.

*14.5.1. $Q = D_{n+1}$, $n$ odd*

We use the same basis as given in Section 5.5. Recall that these basis elements have the property $\eta(x) = (-1)^{n_x}x$ where $n_x$ is the number of $Q$-arrows in the monomial expression of $x$.

We can compute that for $k, l \leqslant n-1$,

$$\kappa_{k,l} = \begin{cases} n-k+l-1, & k \text{ odd}, \quad l \text{ odd},\\ l-n, & k \text{ odd}, \quad l \text{ even},\\ -k, & k \text{ even}, \quad l \text{ odd},\\ 0 & k \text{ even}, \quad l \text{ even},\end{cases}$$

$$\kappa_{k,n} = \kappa_{k,n+1} = \begin{cases} n-\frac{k+1}{2}, & k \text{ odd},\\ -\frac{k}{2}, & k \text{ even},\end{cases}$$

$$\kappa_{n,l} = \kappa_{n+1,l} = \begin{cases} n-\frac{l-1}{2}, & l \text{ odd},\\ \frac{l}{2}, & k \text{ even},\end{cases}$$

$$\kappa_{n,n} = \kappa_{n+1,n+1} = \frac{n^2-1}{4},$$

$$\kappa_{n+1,n} = \kappa_{n,n+1} = -\left(\frac{n-1}{2}\right)^2.$$

$Y^*[-h-2]$ has basis $\varepsilon_{2k+1} = [e_{2k+1} - e_1]$ $(0 \leqslant k \leqslant \frac{n-3}{2})$, $\varepsilon_{2k} = [e_{2k}]$ $(k \leqslant \frac{n-1}{2})$, $\varepsilon_n = [e_n + e_{n+1} - e_1]$, and we can calculate the products

$$\begin{aligned}\theta_0\varepsilon_{2k+1} &= \sum_{i\in F}(\kappa_{i,2k+1} - \kappa_{i,1})\varphi_0(\omega_i)\\ &= 2k\sum_{\substack{i=1\\ \text{odd}}}^{n-2}\varphi_0(\omega_i) - n\sum_{\substack{i=2\\ \text{even}}}^{2k}\varphi_0(\omega_i) + k\varphi_0(\omega_n + \omega_{n+1}),\\ \theta_0\varepsilon_{2k} &= \sum_{i\in F}(\kappa_{i,2k+1})\varphi_0(\omega_i)\\ &= (2k-n)\sum_{\substack{i=1\\ \text{odd}}}^{2k-1}\varphi_0(\omega_i) + 2k\sum_{\substack{i=2k+1\\ \text{odd}}}^{n-2}\varphi_0(\omega_i) + k\varphi_0(\omega_n + \omega_{n+1}),\end{aligned}$$

$$\begin{aligned}\theta_0\varepsilon_n &= \sum_{i\in F}(\kappa_{i,n}+\kappa_{n+1,1}-\kappa_{i,1})\varphi_0(\omega_i)\\ &= (n-1)\sum_{\substack{i=1\\ \text{odd}}}^{n-2}\varphi_0(\omega_i) - n\sum_{\substack{i=2\\ \text{even}}}^{n-1}\varphi_0(\omega_i) + \frac{n-1}{2}\varphi_0(\omega_n+\omega_{n+1}).\end{aligned}$$

We use the defining relations in $Y[-h-2]$,

$$\begin{aligned}\varphi_0(\omega_1) &= -\varphi_0\Bigg(\sum_{\substack{i=3\\ \text{odd}}}^{n-2}\varphi_0(\omega_i)-\varphi_0(\omega_n)\Bigg),\\ \varphi_0(\omega_{n+1}) &= \varphi_0(\omega_n)\end{aligned}$$

to write the RHS of the above cup product calculations in terms of the basis $(\omega_i)_{2\leqslant i\leqslant n}$:

$$\theta_0\varepsilon_{2k+1} = -n\sum_{\substack{i=2\\ \text{even}}}^{2k}\varphi_0(\omega_i),$$

$$\theta_0\varepsilon_{2k} = n\sum_{\substack{i=2k+1\\ \text{odd}}}^{n-2}\varphi_0(\omega_i) + n\varphi_0(\omega_n),$$

$$\theta_0\varepsilon_n = -n\sum_{\substack{i=2\\ \text{even}}}^{n-1}\varphi_0(\omega_i).$$

$\beta$ is given by the skew-symmetric, nondegenerate matrix

$$M_\beta = \begin{pmatrix} 0 & -n & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n\\ n & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n\\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & \dots & -n & 0 & -n & 0 & -n\\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -n & 0 & -n\\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & -n\\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & n & 0 \end{pmatrix}$$

with respect to the chosen basis $\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n$ of $Y^*[-h-2]$ and the dual basis $\varphi_0(\omega_2), \varphi_0(\omega_3), \dots, \varphi_0(\omega_n)$ of $Y[-h-2]$.

*14.5.2. $Q = D_{n+1}$, n even*

We use the same basis as in Section 11.2.3 for our computations.

For $k, l \leqslant n-1$,

$$\kappa_{k,l} = \begin{cases} n-k+l-1, & k \text{ odd}, \quad l \text{ odd}, \\ l-n, & k \text{ odd}, \quad l \text{ even}, \\ -k, & k \text{ even}, \quad l \text{ odd}, \\ 0, & k \text{ even}, \quad l \text{ even}. \end{cases}$$

$Y^*[-h-2]$ has basis $\varepsilon_{2k} = [e_{2k}]$, $\varepsilon_{2k+1} = [e_{2k+1} - e_1]$ $(1 \leqslant k \leqslant \frac{n-2}{2})$, and we calculate the products

$$\begin{aligned} \theta_0 \varepsilon_{2k+1} &= \sum_{i \in F} (\kappa_{i,2k+1} - \kappa_{i,1}) \varphi_0(\omega_i) \\ &= 2k \sum_{\substack{i=1 \\ \text{odd}}}^{n-1} \varphi_0(\omega_i) - n \sum_{\substack{i=2 \\ \text{even}}}^{2k} \varphi_0(\omega_i), \\ \theta_0 \varepsilon_{2k} &= \sum_{i \in F} (\kappa_{i,2k}) \varphi_0(\omega_i) \\ &= (2k-n) \sum_{\substack{i=1 \\ \text{odd}}}^{2k-1} [\omega_i] + 2k \sum_{i=2k+1}^{n-2} \varphi_0(\omega_i), \end{aligned}$$

and we use the defining relation of $Y[-h-2]$,

$$\varphi_0(\omega_1) = -\sum_{\substack{i=3 \\ \text{odd}}}^{n-2} \varphi(\omega_i)$$

to write the results of the cup product calculations in terms of the basis $\varphi_0(\omega_2), \varphi_0(\omega_3), \ldots, \varphi_0(\omega_{n-1})$. We get

$$\begin{aligned} \theta_0 \varepsilon_{2k+1} &= -n \sum_{\substack{i=2 \\ \text{even}}}^{2k} \varphi_0(\omega_i), \\ \theta_0 \varepsilon_{2k} &= n \sum_{\substack{i=2k+1 \\ \text{odd}}}^{n-1} \varphi_0(\omega_i). \end{aligned}$$

$\beta$ is given by the matrix

$$M_\beta = \begin{pmatrix} 0 & -n & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\ n & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & -n & 0 & -n & 0 & -n \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -n & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & n & 0 \end{pmatrix}$$

with respect to the basis $\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n-1}$ and its dual basis $\varphi_0(\omega_2), \varphi_0(\omega_3), \dots, \varphi_0(\omega_{n-1})$.

*14.5.3.* $Q = E_6$

We work with the bases

$$\begin{aligned} B^+_{3,3} &= \{e_3, a_3^*a_3 - a_2^*a_2, (a_3^*a_3 - a_2^*a_2)^2, a_5^*a_5a_3^*a_3a_5^*a_5, \\ &\quad a_5^*a_5a_3^*a_3a_5^*a_5a_3^*a_3, a_3^*a_3a_5^*a_5a_3^*a_3a_5^*a_5a_3^*a_3\}, \\ B^-_{3,3} &= \{a_5^*a_5, a_3^*a_3a_5^*a_5, a_5^*a_5a_3^*a_3, a_3^*a_3a_5^*a_5a_3^*a_3, \\ &\quad a_5^*a_5a_3a_3^*(a_3^*a_3 - a_2^*a_2)^2, a_3^*a_3a_5^*a_5a_3a_3^*(a_3^*a_3 - a_2^*a_2)^2\}, \\ B^+_{6,3} &= \{a_5a_3^*a_3a_5^*a_5, a_5a_3^*a_3a_5^*a_5a_3^*a_3, a_5a_3^*a_3a_5^*a_5a_3a_3^*(a_3^*a_3 - a_2^*a_2)\}, \\ B^-_{6,3} &= \{a_5, a_5a_3^*a_3, a_5a_3^*a_3(a_3^*a_3 - a_2^*a_2)\}, \\ B^+_{3,6} &= \{a_5^*, a_3^*a_3a_5^*, (a_3^*a_3 - a_2^*a_2)a_3^*a_3a_5^*\}, \\ B^-_{3,6} &= \{a_5^*a_5a_3^*a_3a_5^*, a_3^*a_3a_5^*a_5a_3^*a_3a_5^*, a_3^*a_3a_5^*a_5(a_3^*a_3)^2a_5^*\}, \\ B^+_{6,6} &= \{e_6, a_5a_3^*a_3a_5^*a_5(a_3^*a_3)^2a_5^*\}, \\ B^-_{6,6} &= \{a_5a_3^*a_3a_5^*, a_5(a_3^*a_3)^2a_5^*\}. \end{aligned}$$

We immediately get the matrix

$$M_\beta = \begin{pmatrix} \kappa_{3,3} & \kappa_{3,6} \\ \kappa_{6,3} & \kappa_{6,6} \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix}$$

which represents the $\beta$ with respect to the basis $\varepsilon_3, \varepsilon_6$ and dual basis $\varphi_0(\omega_3), \varphi_0(\omega_6)$.

*14.5.4. $E_7$*

For $E_7$ and $E_8$ we do not have to work with an explicit basis to calculate $\kappa_{k,l}$ since for any basis element $x$, $\eta(x) = \pm x$. It is enough to know the following: given any monomial $x \in e_k A e_j$ of length $l$, $n_{k,j}$ the number of arrows $x \in Q$ and $d(k,j)$ the distance between the vertices $k, j$, we know that $x$ contains $n_{k,j} + \frac{l-d(k,j)}{2}$ arrows in $Q$ and $d(k,j) - n_{k,j} + \frac{l-d(k,j)}{2}$ arrows in $\bar{Q}$.

We can derive the following formula:

$$\kappa_{k,j} = (-1)^{n_{k,j}} \left( \left(d(k,j) - n_{k,j}\right) \frac{H_A(t)}{t^{d(k,j)}}\bigg|_{t=\sqrt{-1}} + \frac{1}{2} t \frac{d}{dt} \frac{H_A(t)}{t^{d(k,j)}}\bigg|_{t=\sqrt{-1}} \right). \tag{14.5.4}$$

The resulting matrix is

$$(\kappa_{k,j})_{k,j} = \begin{pmatrix} 12 & 6 & 9 & 3 & 0 & 3 & -9 \\ -6 & 0 & 3 & 0 & 0 & 0 & -3 \\ 15 & -3 & 12 & 3 & 0 & 3 & -12 \\ -3 & 0 & -3 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & -9 & 0 \\ -3 & 0 & -3 & 0 & 9 & 0 & -6 \\ -15 & 3 & -12 & 6 & 0 & 6 & 12 \end{pmatrix}.$$

A basis of $Y^*[-h-2]$ is given by

$$\varepsilon_1 = [e_1 + e_7], \qquad \varepsilon_2 = [e_2], \qquad \varepsilon_3 = [e_3 + e_7], \qquad \varepsilon_4 = [e_4], \qquad \varepsilon_5 = [e_5], \qquad \varepsilon_6 = [e_6].$$

$(\theta_0 \varepsilon_i)_{1 \leqslant 1 \leqslant 6}$ is given by

$$\begin{pmatrix} 3 & 6 & 0 & 3 & 0 & 3 \\ -9 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & 0 & 3 & 0 & 3 \\ -9 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \\ -9 & 0 & -9 & 0 & 9 & 0 \\ -3 & 3 & 0 & 6 & 0 & 6 \end{pmatrix} \begin{pmatrix} \varphi_0(\omega_1) \\ \varphi_0(\omega_2) \\ \varphi_0(\omega_3) \\ \varphi_0(\omega_4) \\ \varphi_0(\omega_5) \\ \varphi_0(\omega_6) \\ \varphi_0(\omega_7) \end{pmatrix}.$$

Now use the defining relation of $Y[-h-2]$,

$$\varphi_0(\omega_7) = \varphi_0(\omega_1) + \varphi_0(\omega_3)$$

to obtain the matrix

$$M_\beta = \begin{pmatrix} 0 & 9 & 0 & 9 & 0 & 9 \\ -9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 9 \\ -9 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \\ -9 & 0 & -9 & 0 & 9 & 0 \end{pmatrix}$$

which represents $\beta$ with respect to the basis $\varepsilon_1, \ldots, \varepsilon_6$ and its dual basis $\varphi_0(\omega_1), \ldots, \varphi_0(\omega_6)$.

*14.5.5.* $E_8$

We can use (14.5.4) and get the matrix

$$M_\beta = (\kappa_{k,j})_{k,j} = \begin{pmatrix} 0 & 15 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & -15 \\ 15 & 0 & 15 & 0 & 15 & 0 & 15 & 0 \end{pmatrix}$$

which represents $\beta$ with respect to the basis $\varepsilon_1, \ldots, \varepsilon_8$ and its dual basis $\varphi_0(\omega_1), \ldots, \varphi_0(\omega_8)$.

**Remark 14.5.5.** With respect to our chosen bases $(\varepsilon_i)_{i\in I'}$ and $\phi_0(\omega_i)_{i\in I'}$, such that the vertex set $I' \subset I$, together with the arrows in $I$ form a connected subquiver $\bar{Q}'$, $M_\beta$ can be written in this general form:

$$M_\beta = \frac{h}{2} \cdot (C')^\epsilon, \tag{14.5.6}$$

where we call $(C')^\epsilon$ the *signed adjacency matrix* of the subquiver $\bar{Q}'$, that is

$$(C')_{ij} = \begin{cases} 0 & \text{if } i, j \text{ are not adjacent,} \\ +1 & \text{if arrow } i \leftarrow j \text{ lies in } Q^*, \\ -1 & \text{if arrow } i \leftarrow j \text{ lies in } Q. \end{cases} \tag{14.5.7}$$

In the $D_{n+1}$-case, we have

$$M_\beta = n \cdot \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}^{-1},$$

in the $E_6$-case, we have

$$M_\beta = 6 \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1},$$

in the $E_7$-case, we have

$$M_\beta = 9 \cdot \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}^{-1},$$

and in the $E_8$-case, we have

$$M_\beta = 15 \cdot \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}^{-1}.$$

## 15. Products involving $HH^2(A)$

We start with $HH^2(A) \times HH^3(A) \to HH^5(A)$ first and then deduce $HH^2(A) \times HH^2(A) \to HH^4(A)$ from associativity.

### *15.1.* $HH^2(A) \times HH^3(A) \to HH^5(A)$

We will prove the following general proposition:

**Proposition 15.1.1.** *For the basis elements $f_i \in HH^2(A)$, $h_j \in HH^3(A)$, the cup product is*

$$f_i h_j = \delta_{ij} \psi_0. \tag{15.1.2}$$

**Proof.** Recall the maps

$$h_j : A \otimes \mathcal{N} \to A, \quad 1 \otimes 1 \mapsto \omega_j$$

and lift it to

$$\hat{h}_j : A \otimes \mathcal{N} \to A \otimes A, \quad 1 \otimes 1 \mapsto 1 \otimes \omega_j.$$

Then

$$\hat{h}_j\big(d_4(1 \otimes a \otimes 1)\big) = \hat{h}_j(a \otimes 1 - 1 \otimes a) = a \otimes \omega_j = d_1(1 \otimes a \otimes \omega_j),$$

so

$$\Omega h_j : \Omega^4(A) \to \Omega(A),$$
$$1 \otimes a \otimes 1 \mapsto 1 \otimes a \otimes \omega_j.$$

Then we have

$$\Omega h_j\big(d_5(1 \otimes 1)\big) = \Omega h_j\left(\sum_{a \in \bar{Q} Q} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a 1 \otimes a \otimes a^*\right)$$
$$= \sum_{a \in \bar{Q}} \epsilon_a \otimes a^* \otimes \omega_j = d_2(1 \otimes \omega_j),$$

so

$$\Omega^2 h_j : \Omega^5(A) \to \Omega^2(A),$$
$$1 \otimes 1 \mapsto 1 \otimes \omega_j.$$

This gives us

$$f_i\big(\Omega^2 h_j\big)(1 \otimes 1) = f_i(1 \otimes \omega_j) = \delta_{ij}\omega_j,$$

i.e. the cup product

$$f_i h_j = \delta_{ij}[\omega_j] = \delta_{ij}\psi_0. \qquad \square$$

*15.2.* $HH^2(A) \times HH^2(A) \to HH^4(A)$

Since $\deg HH^2(A) = -2$, their product has degree $-4$ (i.e. lies in *span*$(\zeta_0)$), so it can be written as

$$HH^2(A) \times HH^2(A) \to HH^4(A),$$
$$(a, b) \mapsto \langle a, b\rangle \zeta_0,$$

where $\langle -,- \rangle : HH^2(A) \times HH^2(A) \to \mathbb{C}$ is a bilinear form. We prove the following proposition:

**Proposition 15.2.1.** *The cup product* $HH^2(A) \times HH^2(A) \to HH^4(A)$ *is given by* $\langle -,- \rangle = \alpha$*, where* $\alpha$ *(from Proposition* 14.2.3*) is regarded as a symmetric bilinear form.*

**Proof.** We use (14.4.2) to get

$$\theta_0(f_i f_j) = \theta_0\big(\langle f_i, f_j\rangle \zeta_0\big) = \langle f_i, f_j\rangle \psi_0. \tag{15.2.2}$$

On the other hand, by Propositions 14.2.3 and 15.1.1,

$$(\theta_0 f_i) f_j = \alpha(f_i) f_j = \sum (M_\alpha)_{li} h_l f_j = (M_\alpha)_{ji}\psi_0 = (M_\alpha)_{ij}\psi_0. \tag{15.2.3}$$

By associativity of the cup product, we can equate (15.2.2) and (15.2.3) to get

$$\langle f_i, f_j\rangle = (M_\alpha)_{ij}. \qquad \square \tag{15.2.4}$$

*15.3.* $HH^2(A) \times HH^4(A) \xrightarrow{0} HH^6(A)$

This computation uses the Batalin–Vilkovisky structure on Hochschild cohomology: We have $\deg HH^2(A) = -2$, $\deg HH^4(A) \geqslant -h$ and $\deg HH^6(A) \leqslant -h-2$. So we know by degree argument that

$$f_k\zeta_l = \begin{cases} 0, & l > h-4, \\ \sum_s \lambda_s \varphi(\omega_s), & l = h-4. \end{cases} \tag{15.3.1}$$

We use [Eu3, (6.0.12)] and the isomorphism $HH^i(A) = HH_{6m+2-i}(A)$ to get for the Gerstenhaber bracket on $HH^*(A)$:

$$\begin{aligned}[f_k, \zeta_l] &= \Delta(f_k\zeta_l) - \underbrace{\Delta(f_k)}_{=0}\zeta_l - f_k\underbrace{\Delta(\zeta_l)}_{=0} \\ &= \sum_s \lambda_s\left(\frac{1}{2}+m\right)h\beta^{-1}\bigl(\varphi(\omega_s)\bigr).\end{aligned}$$

The Gerstenhaber bracket has to be independent of the choice of $m \geqslant 0$. This implies that the RHS has to be zero, so all $\lambda_s = 0$. This shows that

$$f_k\zeta_{h-4} = 0, \tag{15.3.2}$$

so we have that the cup product of $HH^2(A)$ with $HH^4(A)$ is zero.

*15.4.* $HH^2(A) \times HH^5(A) \xrightarrow{0} HH^7(A)$

Let $a \in HH^2(A)$ and $b \in HH^5(A)$ be homogeneous elements, then $ab = \lambda\theta_k \in HH^7(A) = U[-2h-2]$, $\lambda \in \mathbb{C}$. Then

$$\lambda\psi_0 = \lambda\psi_k z_k = \lambda\theta_k\zeta_k = \lambda b(a\zeta_k) = 0,$$

the last equality coming from the product $a\zeta_k \in HH^2(A) \cup HH^4(A) = 0$.

## 16. Products involving $HH^3(A)$

*16.1.* $HH^3(A) \times HH^3(A) \xrightarrow{0} HH^6(A)$

This follows by degree argument: $\deg HH^3(A) = -2$, $\deg HH^6(A) \leqslant -h-2 < -4$.

*16.2.* $HH^3(A) \times HH^4(A) \xrightarrow{0} HH^7(A)$

This follows by degree argument: $\deg HH^3(A) = -2$, $\deg HH^4(A) \geqslant -h$, $\deg HH^7(A) \leqslant -h-4 < -h-2$.

*16.3.* $HH^3(A) \times HH^5(A) \xrightarrow{0} HH^8(A)$

This follows by degree argument: $\deg HH^3(A) = -2$, $\deg HH^5(A) \geqslant -h-2$, $\deg HH^8(A) = -2h-2 < -h-4$.

## 17. Products involving $HH^4(A)$

*17.1.* $HH^4(A) \times HH^4(A) \xrightarrow{0} HH^8(A)$

This follows by degree argument: $\deg HH^4(A) \geqslant -h$, $\deg HH^8(A) = -2h-2 < -2h$.

*17.2.* $HH^4(A) \times HH^5(A) \xrightarrow{0} HH^9(A)$

This is clear for $Q = D_{n+1}$, $n$ odd, $Q = E_7,\ E_8$ where $HH^9(A) = K[-2h-2] = 0$.

Let $Q = D_{n+1}$, $n$ even or $Q = E_6$. Let $a \in HH^4(A)$, $b \in HH^5(A)$. The product $HH^2(A) \times HH^3(A) \to HH^5(A)$, $(x, y) \mapsto \langle x, y\rangle\zeta_0$ induces a nondegenerate bilinear form $\langle -,-\rangle$. If $ab \in HH^9(A) = HH^3(A)[-2h]$ is nonzero, then we can find a $c \in HH^2(A)$, such that $c(ab) = \zeta_0$. But this equals $(ca)b = 0$ since $HH^2(A) \times HH^4(A) \xrightarrow{0} HH^6(A)$ which gives us a contradiction.

## 18. $HH^5(A) \times HH^5(A) \to HH^{10}(A)$

**Proposition 18.0.1.** *The multiplication of the subspace $U[-2]^*$ with $HH^5(A)$ is zero.*

*The pairing on $Y^*[-h-2]$ is*

$$\begin{aligned} Y^*[-h-2] \times Y^*[-h-2] &\to HH^{10}(A), \\ (a, b) &\mapsto \Omega(a, b)\varphi_4(\zeta_0), \end{aligned} \tag{18.0.2}$$

*where the skew-symmetric bilinear form $\Omega(-,-)$ is given by the matrix $-M_\beta$ from Subsection* 14.5.

**Proof.** We have $\deg HH^5(A) \geqslant -h-2$ and $\deg HH^{10}(A) \leqslant -2h-4$, so we can get a nonzero multiplication only by pairing bottom degree parts of $HH^5(A)$ which is $Y^*[-h-2]$. The product lies in the top degree part of $HH^{10}(A) = HH^4(A)[-2h]$ which is spanned by $\varphi_4(\zeta_0)$. This gives us the pairing of the form (18.0.2).

We want to find the matrix $(\Omega(\varepsilon_i, \varepsilon_j))_{i,j}$ where $\varepsilon_i$ are a basis of $Y^*[-h-2]$, given in the section about $HH^5(A)$. Recall that the multiplication $HH^1(A) \times HH^5(A) \to HH^6(A)$ was given by a skew-symmetric matrix $((M_\beta)_{i,j})_{i,j\in F}$, so that $\theta_0\varepsilon_i = \sum_{k\in F}(M_\beta)_{k,i}\varphi_0(\omega_k)$.

We multiply $\varepsilon_i\varepsilon_j = \Omega(\varepsilon_i, \varepsilon_j)\varphi_4(\zeta_0)$ with $\theta_0$ (see 14.4.2):

$$\theta_0(\varepsilon_i\varepsilon_j) = \Omega(\varepsilon_i, \varepsilon_j)\varphi_5(\psi_0). \tag{18.0.3}$$

Using associativity, this equals

$$(\theta_0\varepsilon_i)\varepsilon_j = \sum_{k\in F}(M_\beta)_{k,i}\varphi_0(\omega_k)\varepsilon_j = (M_\beta)_{j,i}\psi_0 = -(M_\beta)_{i,j}\varphi_5(\psi_0). \tag{18.0.4}$$

We see from Eqs. (18.0.3) and (18.0.4) that

$$\Omega(\varepsilon_i, \varepsilon_j) = -(M_\beta)_{i,j}.$$

This completes the cup product computation of $HH^*(A)$. □

## 19. Presentation of $HH^*(A)$

For each quiver, we give a presentation of $HH^*(A)$ as an algebra over $\mathbb{C}$ by generators and relations. We write $X$ for the element $\phi_0(z_0) \in HH^6(A)$.

*19.1. $Q = D_{n+1}$, $n$ odd*

$HH^*(A)$ is generated by

$$1, z_4, \omega_1, \ldots, \omega_n, \theta_0, \zeta_{2n-6}, \varepsilon_2, \ldots, \varepsilon_n, X$$

with relations ($\forall i, j = 2, \ldots, n$, $\forall k, l = 1, \ldots, n$)

$$(z_4)^{\frac{n+1}{2}} = \theta_0^2 = \zeta_{2n-6}^2 = z_4\varepsilon_i = 0,$$

$$z_4\omega_k = \theta_0\omega_k = \zeta_{2n-6}\omega_k = \omega_l\omega_k = X \sum_{\substack{m=1 \\ m \text{ odd}}}^{n-2} \omega_m = X z_4^{\frac{n-1}{2}} = 0,$$

$$\omega_i\varepsilon_j = \delta_{ij} z_4^{\frac{n-3}{2}} \theta_0 \zeta_{2n-6},$$

$$\varepsilon_i\varepsilon_j = -\Omega(\varepsilon_i, \varepsilon_j) X z_4^{\frac{n-3}{2}} \zeta_{2n-6},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form given by the matrix

$$\begin{pmatrix}
0 & -n & 0 & -n & \ldots & \ldots & -n & 0 & -n & 0 & -n \\
n & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -n & \ldots & \ldots & -n & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & -n & 0 & -n & 0 & -n \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n & 0 & n & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \ldots & \ldots & 0 & n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & -n \\
n & 0 & n & 0 & \ldots & \ldots & 0 & n & 0 & n & 0
\end{pmatrix}.$$

*19.2. $Q = D_{n+1}$, n even*

$HH^*(A)$ is generated by

$$1, z_4, \omega_1, \ldots, \omega_{n-1}, \theta_0, f_n, \zeta_{2n-4}, \varepsilon_2, \ldots, \varepsilon_{n-1}, X$$

with relations ($\forall i, j = 2, \ldots, n-1$, $\forall k, l = 1, \ldots, n-1$)

$$(z_4)^{\frac{n}{2}} = \theta_0^2 = z_4 f_n = \zeta_{2n-4}^2 = \zeta_{2n-4} f_n = 0,$$
$$z_4 \varepsilon_i = f_n \varepsilon_i = 0,$$
$$z_4\omega_k = \theta_0\omega_k = f_n\omega_k = \zeta_{2n-4}\omega_k = \omega_l\omega_k = X \sum_{\substack{m=1 \\ m \text{ odd}}}^{n-1} \omega_m = 0,$$
$$f_n^2 = -n z_4^{\frac{n-2}{2}} \zeta_{2n-4}$$
$$\omega_i \varepsilon_j = \delta_{ij} z_4^{\frac{n-2}{2}} \theta_0 \zeta_{2n-4},$$
$$\varepsilon_i \varepsilon_j = -\Omega(\varepsilon_i, \varepsilon_j) X z_4^{\frac{n-2}{2}} \zeta_{2n-4},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form given by the matrix

$$\begin{pmatrix}
0 & -n & 0 & -n & \ldots & \ldots & -n & 0 & -n & 0 & -n \\
n & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -n & \ldots & \ldots & -n & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & -n & 0 & -n & 0 & -n \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n & 0 & n & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \ldots & \ldots & 0 & n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 & -n \\
n & 0 & n & 0 & \ldots & \ldots & 0 & n & 0 & n & 0
\end{pmatrix}.$$

*19.3. $Q = E_6$*

$HH^*(A)$ is generated by

$$1, z_6, z_8, \omega_3, \omega_6, \theta_0, f_1, f_2, \zeta_6, \zeta_8, \varepsilon_3, \varepsilon_6, X$$

with relations (for $u, v \in \{6, 8\}$, $k, l \in \{3, 6\}$, $i, j \in \{1, 2\}$)

$$z_u z_v = \theta_0^2 = z_u f_i = \zeta_u \zeta_v = \zeta_u f_i = z_u \varepsilon_k = f_i \varepsilon_k = 0,$$

$$z_u \omega_k = \theta_0 \omega_k = f_i \omega_k = \zeta_u \omega_k = \omega_l \omega_k = 0,$$

$$z_8\zeta_8 = z_6\zeta_6, \qquad \omega_k\varepsilon_l = \delta_{kl}\theta_0 z_8\zeta_8,$$

$$f_i f_j = \langle f_i, f_j\rangle z_8\zeta_8,$$

where $\langle -,-\rangle$ is the symmetric bilinear form, given by the matrix

$$\begin{pmatrix} -8 & -4 \\ -4 & -8 \end{pmatrix},$$

$$\varepsilon_k\varepsilon_l = -\Omega(\varepsilon_k, \varepsilon_l) X z_8\zeta_8,$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form, given by the matrix

$$\begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix}.$$

*19.4.* $Q = E_7$

$HH^*(A)$ is generated by

$$1, z_8, z_{12}, \omega_1, \ldots, \omega_6, \theta_0, \zeta_8, \zeta_{12}, \varepsilon_1, \ldots, \varepsilon_6, X$$

with relations (for $u, v \in \{8, 12\}$, $k, l \in \{1, \ldots, 6\}$)

$$z_u z_{12} = \theta_0^2 = z_u^3 = z_u\varepsilon_k = 0,$$

$$z_u\omega_k = \theta_0\omega_k = \omega_l\omega_k = X z_8^2 = 0,$$

$$z_8\zeta_8 = z_{12}\zeta_{12}, \qquad \omega_k\varepsilon_l = \delta_{kl}\theta_0 z_{12}\zeta_{12},$$

$$\varepsilon_k\varepsilon_l = -\Omega(\varepsilon_k, \varepsilon_l) X z_{12}\zeta_{12},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form, given by the matrix

$$\begin{pmatrix} 0 & 9 & 0 & 9 & 0 & 9 \\ -9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 9 \\ -9 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \\ -9 & 0 & -9 & 0 & 9 & 0 \end{pmatrix}.$$

*19.5.* $Q = E_8$

$HH^*(A)$ is generated by

$$1, z_{12}, z_{20}, \omega_1, \ldots, \omega_8, \theta_0, \zeta_{20}, \zeta_{24}, \varepsilon_1, \ldots, \varepsilon_6, X$$

with relations (for $u, v \in \{12, 20\}$, $k, l \in \{1, \dots, 8\}$)

$$z_u z_{20} = \theta_0^2 = z_u^3 = z_u \varepsilon_k = z_{12}^3 = 0,$$

$$z_u \omega_k = \theta_0 \omega_k = \omega_l \omega_k = 0,$$

$$z_{12}^2 \zeta_{24} = z_{20} \zeta_{20}, \qquad \omega_k \varepsilon_l = \delta_{kl} \theta_0 z_{20} \zeta_{20},$$

$$\varepsilon_k \varepsilon_l = -\Omega(\varepsilon_k, \varepsilon_l) X z_{20} \zeta_{20},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form, given by the matrix

$$\begin{pmatrix} 0 & 15 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & -15 \\ 15 & 0 & 15 & 0 & 15 & 0 & 15 & 0 \end{pmatrix}.$$

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