Hölder Regularity for Certain Strongly Coupled Parabolic Systems

Le Dung

Center for Dynamical Systems and Nonlinear Studies, Atlanta, Georgia 30332-0109

Received October 13, 1997; revised May 26, 1998

We present a shorter proof to show Hölder continuity of bounded solutions to a general class of quasilinear parabolic equations. The proof will be extended to obtain regularity results for bounded solutions to certain strongly coupled (or cross-diffusion) quasilinear parabolic systems.

In this paper we study the Hölder continuity of bounded solutions to a class of certain strongly coupled quasilinear parabolic systems of the form

$$\frac{\partial u}{\partial t} \text{div}(A_i(x, t, u, Du)) + f_i(x, t, u, Du), \quad i = 1, \ldots, m$$

(0.1)

in a space-time cylinder \(\Omega_T = \Omega \times [0, T]\). Here \(\Omega\) is an open subset in \(\mathbb{R}^n\), \((x, t), x \in \Omega, t \in \mathbb{R}_+\), denotes a generic point in \(\Omega_T\), \(u = (u^1, \ldots, u^m)\) is a vector valued function defined in \(\Omega_T\) and \(\text{div}, D\) denotes the divergence and spatial derivative operators. \(A_i, f_i\) are accordingly vector valued functions.

To the author’s knowledge there are only few works on Hölder regularity of solutions to systems of the type (0.1). In contrast to the case of scalar equations or reaction–diffusion systems with the coupling occurs only in the reaction terms, counterexamples (see [26]) indicate that one cannot expect bounded solutions to general strongly coupled systems to be regular everywhere. Also, concerning the problem of global existence of solutions, a priori \(L^\infty\) bounds are not enough to conclude that the solutions exist on the infinite time interval. The works of Amann [1, 3, 2] show that, in important cases, it suffices to find a priori \(L^\infty\) bounds to guarantee global existence provided that we can also prove uniform Hölder continuity in space and time ([2, Theorem 4.1]).

Partial regularity results were obtained by Giaquinta and Struwe in [14] for a fairly general class of systems. Everywhere regularity results for bounded solutions were proven only in few situations assuming additional structure conditions on the system (0.1). Among these are diagonal systems (see...
[15, 21, 27)], triangular systems (see [2]) or strongly coupled systems of special form (see [30]).

The methods in [14, 27] based on a perturbation argument to compare solutions of (0.1) with those of related linear systems with constant coefficients. The operators $a^i$ should be linear with respect to $D u$. They showed that a bounded solution $u$ is regular on certain subset of $\Omega_T$ where the oscillation of $u$ is small enough. In [30], Wiegner gave an everywhere regularity result for a strongly coupled system having special structure. He employed an alternative analysis on level sets to show the smallness condition for the oscillation of solution so that the regularity result of [14] can be applied. Recently, in [19, 20], Kühner generalized the results on invariant regions in [24] to derive $L^\infty$ bounds for solutions to some strongly coupled systems, which also satisfy the structure conditions considered in [30], so that global existence results follow.

On the other hand, the approach in [2, 23, 25, 32, 31] based on semigroup theories associated to the systems and made use of many imbedding results in sofisticated interpolation and extrapolation theories of Banach spaces.

In this paper, we show that regularity results for strongly coupled quasilinear systems (0.1) can be obtained by an elementary technique which has been used in [11, 10, 12] for a scalar parabolic equation of the form

$$\frac{\partial u}{\partial t} = \text{div}(a(x, t, u, Du)) + b(x, t, u, Du), \quad (x, t) \in \Omega_T = \Omega \times [0, T]. \quad (0.2)$$

Some historical remarks should be made here. First, the regularity theory for nondegenerate scalar parabolic equation (0.2) was first proven by Moser [22] and then extended to quasilinear cases by Aronson and Serrin [4] and Trudinger [28]. Their methods bases on the Harnack principle which is itself very important theoretically but its derivation is truly complicated. This method seems not to be applicable to systems.

As a counterpart to the Moser method, the method of level sets or truncation technique of DeGiorgi had been generalized by Ladyzhenskaya et al. [21]. Roughly speaking, this method investigates two alternatives of the level sets of the solution and derives certain decay estimates for the oscillation of the solution in nested cylinders. The latter implies the Hölder continuity. The proof is somewhat complicated but the ideas can be extended to some systems (see [29, 30]).

In Section 1, in order to explain the method used later for systems we study bounded solutions to the scalar equation (0.2) and their Hölder regularity. The result (Theorem 2) is not new but the proof is much shorter than the forementioned ones. The main ideas can be sketched as follows: We introduce the auxiliary logarithmic functions $w_1, w_2$ (see (1.5)) and
show that Hölder continuity of the solution $u$ follows from the boundedness of either $w_1$ or $w_2$. To estimate $w_i$, simple calculations (Lemma 1.6) show that they are subsolutions to some parabolic equation related to (0.2). We then make use of well-known supremum estimates to reduce the problem to that of finding bounds for local $L^2$ norms of $w_i$. We derive the latter by fairly elementary techniques of differential inequalities and conclude our proof.

Our proof makes use of the technique of logarithmic functions which was developed in our earlier works for elliptic equations (see [8, 9]) and scalar parabolic equations (see [11, 12]). The idea of using logarithmic function was originally employed by Moser and other authors ([29, 30]) in the aforementioned works to obtain auxiliary results but did not play a direct role to derive regularity results. In contrast, our proof makes use of these functions to get a simpler and straightforward proof compared to those of the methods mentioned above.

On the other hand, we allow the parameters in the structure conditions for (0.2) to be in a larger class of function spaces than those considered in literature (see the definition (1), (F) and (F')). This opens a possibility of studying parabolic equations with distribution data as in [8, 9]. However, our primary motivation for such a general setting comes from the local estimate for $L^2$ norm of the derivative of solution in Corollary 3 and the main goal of extending the technique to quasilinear systems.

In Section 2 we consider some strongly coupled quasilinear systems. The first one is a class of triangular systems whose prototype is the following problem

$$u_i = \text{div}(a(u) \, Du) + A(\phi(v) u) + f(u, v),$$

$$v_i = Av + g(u, v),$$

which includes the models studied in [6, 17, 23, 25] in the context of population dynamics. General triangular systems was also studied by Amann [2]. We briefly show (see Theorem 6) here that regularity results for this type of systems follow immediately from Theorem 4 and Corollary 3 of Section 1.

The second one is a strongly coupled system motivated by the work of Wiegner [30]

$$\frac{\partial u^i}{\partial t} = \text{div}(a'(x, t, u) \, Du^i + r'(x, t, u) + c'(x, t, u) \, DH) + f'(x, t, u, Du),$$

(0.3)

where $H = H(x, t, u)$ is some $C^2$ function. Here we will show that the proof of Theorem 2 can be adapted to this case by a simple change of variables
in the definition of auxiliary functions and hence give a slightly different proof (see Theorem 7) for Theorem 1 in [30].

However, the conditions in [30] did not cover the case when the ellipticity condition involving \( H \) in (0.3) is not fulfilled. This is the case, for example, when \( H \) is a linear function in \( u \), a situation frequently encountered in the context of population dynamics. We shall relax the conditions and show that our proof can be easily applied to this situation as we demonstrate in Theorem 8 and Example 2.2. Moreover, in this case, we also allow the self diffusion coefficients \( a' \) to be slightly different in (0.3) while they are required to be identical in [30].

Furthermore, although the results and proofs in this paper concern only the local (interior) regularity of solutions and no boundary condition will be specified, we want to point out that all of our results here can be extended to the whole domain to obtain estimates for Hölder norms of solutions. The boundary conditions can be of Dirichlet, Neumann or even nonlinear Robin types. The ideas and techniques are similar with only minor modifications. We refer to [12, Sect. 4, Chap. 4] for the results and proofs for the case of one scalar equation. Since our proof for systems in this paper is based on that for a scalar equation, one can see that the same idea can be used to get regularity up to the parabolic boundary, and thus, global estimates for the Hölder norms. We refrain from giving details here and refer to [12].

Finally, we should mention here that the method employed in this paper allows us to trace easily the dependency of the estimates for Hölder norms on their \( L^\infty \) norms. One can see that if the \( L^\infty \) norms of solutions are ultimately uniform with respect to initial data then so are their Hölder norms. This observation is important when we study the existence of global attractors of dynamical systems associated to (0.1).

1. HÖLDER REGULARITY FOR SCALAR PARABOLIC EQUATIONS

In this section we show that a bounded weak solution \( u \) to a non-degenerate quasilinear equation of the form

\[
\frac{\partial u}{\partial t} = \text{div}(a(x, t, u, Du)) + b(x, t, u, Du), \quad (x, t) \in \Omega_T, \tag{1.1}
\]

is Hölder continuous in the interior of \( \Omega_T \). Moreover, its Hölder norm is bounded by a constant depending uniformly on the (local) supremum norm of \( u \). We impose the following structure condition on (1.1):
\[ a(x, t, u, Du) \geq v_0 |Du|^2 - \psi_0(x, t), \]
\[ |a(x, t, u, Du)| \leq v_1 |Du| + \psi_1(x, t), \]
\[ |b(x, t, u, Du)| \leq v_2 |Du|^2 + \psi_2(x, t). \] (1.2)

**Remark 1.1.** For simplicity we consider only equations of divergence form (1.1). However, one can see that the same proof applies to equations of the form
\[ \frac{\partial u}{\partial t} = \sum_{i,j} D_i(a_{ij}(x, t, u, Du))D_ju + b(x, t, u, Du), \] (x, t) ∈ Ω_T,
which satisfy similar structure condition as (1.2).

Let fix a point \((t_0, x_0)\) ∈ Ω_T and let \(R > 0\). We denote the cylinder \(Q(R, r) := Q(x_0, t_0, R, r) := B_R(x_0) \times [t_0 - r, t_0]\).
First, we define the function spaces for the parameters \(\psi_i\).

**Definition 1.** Let \(R, \varepsilon > 0\) and \(L(R, \varepsilon)\) be the collection of nonnegative measurable functions on \(Q(R, R^2)\) satisfy the following: If \(f \# L(R, \varepsilon)\) there exists a constant \(\|f\|_{L(R, \varepsilon)}\) such that for any measurable set \(A \subset Q(R, R^2)\) we have
\[ \int_A f(x, t) \, dx \, dt \leq \|f\|_{L(R, \varepsilon)} |A|^{\varepsilon}, \] (1.3)
where \(|A|\) denotes the Lebesgue measure of the set \(A\).

We then assume that
\[ (F) \] There is \( \nu > n/(n + 2) \) such that, for any \(R > 0\), the function \(\psi_0 + \psi_1^2 + \psi_2^2\) belong to the space \(L(R, \varepsilon)\). Moreover, we assume that \(v_0 > v_2 \sup_{\Omega_T} u\).

We set \(F(Q(R, R^2)) := \|\psi_0 + \psi_1^2 + \psi_2^2\|_{L(R, \varepsilon)}\). If \(K \subset \Omega_T\) is a compact set, we set \(F(K) = \sup_{\mathcal{A}} \mathcal{F}(Q(R, R^2))\) where \(\mathcal{A}\) is a finite collection of cylinders \(Q(R, R^2)\) covering \(K\).

**Remark 1.2.** if \(f \in L^q_w(\Omega_T)\) for some \(q\) then we see that \(f \in L^q_w(\Omega_T)\). Therefore, if \(q > 1 + n/2\) and \(\psi_0 \in L^q_w(\Omega_T)\) and \(\psi_1, \psi_2 \in L^q_w(\Omega_T)\) then the condition \((F)\) is verified.

The following regularity result is well-known in literature. We want to present here a shorter proof and also to streamline the technique used later for system.
Theorem 2. Let $u$ be a locally bounded weak solution of (1.1), and let (1.2) and (F) hold. Then $(x, t) \mapsto u(x, t)$ is locally Hölder continuous in the interior of $\Omega_T$. That is, for every compact subset $K$ of $\Omega_T$, there exists a constant $C = C(\|u\|_{\infty, K}, \mathcal{F}_K)$ and $\alpha = \alpha(\|u\|_{\infty, K}, \mathcal{F}_K)$ in $(0, 1)$ such that
\[
|u(x_1, t_1) - u(x_2, t_2)| \leq C(|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\alpha/2})
\]
for every pair of points $(x_1, t_1), (x_2, t_2) \in K$.

Remark 1.3. The assumption $v_0 > v_2 \sup_\Omega u$ in (F) is not essential for Hölder1 to be true. For scalar parabolic equations, it is well known ([21, 28]) that Hölder continuity for bounded weak solutions can be obtained without such assumption. It is also possible to remove this condition by modifying the logarithmic functions $w_i$ defined in (1.5) and by some extra analysis. However, since our primary interest is to study parabolic systems and to show that the technique in this section can be extended to such cases, we shall not pursue such a generality but try to keep the main ideas as simple as possible.

Remark 1.4. As we mentioned in the introduction, the above interior regularity result can be extended up to the parabolic boundary if Dirichlet, Neumann or even nonlinear Robin boundary conditions are specified there. Only minor modifications will be needed and we refer to [12] for details.

We have the following well-known (compared also to [30, Lemma 1]) local estimate on the supremum norm of $u$.

Lemma 1.5. Assume (F), for any $\sigma > 0$, there exists a positive continuous function $C$ such that
\[
\sup_{Q(R, \sigma R^2)} u \leq C \left( \frac{1}{\sigma R^{2\sigma+2}} \int_{Q(2R, 2\sigma R^2)} (u^+)^2 \, dx \, dt, \mathcal{F}_{Q(R, \sigma R^2)} \right),
\]
where $u^+ = \max\{u, 0\}$.

Proof. We follow the standard truncation technique of [7, 21]. Consider the sequences
\[
\begin{align*}
R_n &= 2R - \frac{1}{2^n} R, \quad r_n = 2\sigma R^2 - \frac{\sigma}{2^n} R^2, \\
R_n &= \frac{R_n + R_{n+1}}{2}, \quad r_n = \frac{r_n + r_{n+1}}{2},
\end{align*}
\]
and the corresponding cylinders $Q_n := Q(R_n, r_n), \tilde{Q}_n := Q(\tilde{R}_n, \tilde{r}_n)$. Obviously, $Q_{n+1} \subset \tilde{Q}_n \subset Q_n$. Let $k$ be a constant, which is to be determined, we consider the increasing sequence

$$k_n = k - \frac{1}{2^n} k.$$ 

Introduce the cutoff functions $\eta_n$ that satisfies: $\eta_n$ vanishes on the parabolic boundary of $Q_n$, $\eta_n = 1$ in $\tilde{Q}_n$, $|D_x \eta_n| \leq 2^{n+2}/R$ and $0 \leq D_t \eta_n \leq 2^{n+2}/R^2$. Multiplying the equation of $u$ by $(u - k_{n+1})_+ \eta_n^2$ and integrating over $Q_0$, we obtain in a standard way that

$$\text{sup}_{t \in [t_0, t_0]} \int_{B(R)} (u - k_{n+1})_+ \eta_n^2 dx + \int_{Q_n} |D(u - k_{n+1})_+ \eta_n|^2 dx dt \leq C 2^{2n} \int_{Q_n} (u - k_{n+1})_+ \eta_n^2 dx dt + \int_{A_{n+1}} \Phi dx dt,$$

where $\Phi = \psi_0 + \psi_1^2 + \psi_2^2$ and $A_{n+1} = \{(x, t) \in Q_n : u(x, t) > k\}$. By (F), we can estimate the last term in the above inequality by $\mathcal{F}_{\theta(R)}, A_{n+1}^*$. Because $v > n/(n+2)$, we now can follow the lines of [7, p. 131] to conclude the proof.

Let $R > 0$. For $i = 1, 2, \ldots$ we denote $Q_{iR} = B_{iR} \times [t_0, t_0]$ and

$$M_i = \sup_{Q_{iR}} u, \quad m_i = \inf_{Q_{iR}} u, \quad \text{and} \quad \omega_i = M_i - m_i.$$ 

In $Q_{4R}$, we consider the following logarithmic functions

$$w_1(x, t) = \log \left( \frac{\omega_4 + \rho_x}{N_1(u)} \right), \quad w_2(x, t) = \log \left( \frac{\omega_4 + \rho_x}{N_2(u)} \right), \quad (1.5)$$

where $N_1(u) = \theta(M_4 - u) + \rho_x$, $N_2(u) = \theta(u - m_4) + \rho_x$, for some constants $\theta \geq 2$ and $\rho > 0$ to be determined later. Note also that $w_i > -\log \theta$. Suppose that we can find some finite constant $C$ which is independent of $R$ such that

$$w_1(x, t) \leq C \quad \text{or} \quad w_2(x, t) \leq C, \quad \forall (x, t) \in Q_R. \quad (1.6)$$

Then it is easy to see that either of the above inequality implies

$$\omega_4 \leq C(\omega_4 - \omega_1 + \rho_x), \quad (1.7)$$
Indeed, suppose that the first estimate of (1.6) is true then we can find a universal constant $C$ such that
\[
\omega_4 + R^s \leq C\theta M_4 - C\theta u + CR^s, \quad \forall (x, t) \in Q_R. \tag{1.7}
\]

Taking the infimum over $Q_R$, we deduce
\[
\omega_4 \leq C\theta M_4 - C\theta M_1 + (C - 1)R^s = C\theta \omega_4 - C\theta (M_1 - m_4) + (C - 1)R^s.
\]

Because $m_4 \leq m_1$ we can replace the quantity $m_4$ in the right hand side by $m_1$ and obtain (1.7). If the second part of (1.6) holds, we can argue similarly using the fact that $M_1 \leq M_4$ to have (1.7) again.

Obviously, (1.7) is equivalent to
\[
\omega_1 \leq \omega_4 + CR^s
\]
with $\varepsilon = (C\theta - 1)/(C\theta < 1)$ and $C$ are positive constants independent of $u, R$. This and an elementary lemma in [16, Lemma 8.23] (see also [13, Lemma 2.1, p. 86]) give immediately the uniform estimate for the Hölder norm of $u$. Actually, following the proof of [13, Lemma 2.1, p. 86], we choose $\tau = 1/4$ and $\gamma, \alpha$ such that $\tau^\gamma = \varepsilon, \alpha < \gamma$ and we can conclude that
\[
\text{osc}_{\partial \Omega^e} u \leq C_1 \left( \frac{\rho}{R} \right)^\gamma \text{osc}_{\partial \Omega^r} u + C_2 \rho^\alpha \tag{1.8}
\]
for some universal constants $C_1, C_2$.

Hence, to prove Theorem 2 we need only to show (1.6).

First, we have

**Lemma 1.6.** For $R$ sufficiently small and for any nonnegative test function $\eta$, the functions $w_1, w_2$ satisfy an inequality of the form
\[
\int_Q \frac{\partial w}{\partial t} \eta + \int_Q a(x, t, Dw) D\eta \, dx \leq \int_Q b(x, t, Dw) \eta \, dx. \tag{1.9}
\]

The functions $a, b$ satisfy the following structure conditions
\[
a(x, t, Dw) \leq v_1 |Dw| + \bar{\psi}_1
\]
\[
a(x, t, Dw) \geq v_0 |Dw|^2 - \check{\psi}_0 \tag{1.10}
\]
\[
b(x, t, Dw) \leq \check{\psi}_2
\]
with $\bar{\psi}_0 = \theta^2 \psi_0/N^2(u)$, $\check{\psi}_1 = \theta \psi_1/N(u)$ and $\check{\psi}_2 = \theta(\psi_0/N^2(u) + \psi_2/N(u))$. Moreover, we can choose $\alpha$ such that the functions $\check{\psi}_1$ satisfy (F)
Proof. If \( w = w_1 \), we denote \( N(u) = N_1(u) \) and observe that
\[
D_xw = \frac{\partial D_x u}{N(u)}, \quad \frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \frac{\partial u}{\partial t}.
\]
We multiply the equation of \( u \) by \( \phi = \eta/N(u) \) and integrate over \( \Omega \) to get
\[
\frac{1}{\theta} \int_\Omega \frac{\partial w}{\partial t} \eta \, dx + \int_\Omega \left( \frac{a(...)}{N(u)} \frac{\partial \eta}{\partial t} + \frac{a(...)}{N^2(u)} \eta \frac{D u}{N(u)} \right) dx = \int_\Omega \frac{b(...)}{N(u)} \eta \, dx.
\]
(1.11)
Hence, \( w \) satisfies the inequality (1.9) with
\[
\begin{align*}
\tilde{a}(x, t, Dw) &= \frac{\partial a(x, t, u(x, t), Du)}{N(u(x, t))}, \\
\tilde{b}(x, t, Dw) &= \theta \left( \frac{b(x, t, u(x, t), Du)}{N(u(x, t))} - \frac{\partial a(...)}{N^2(u)} \right).
\end{align*}
\]
From (1.2), we see that \( \tilde{a} \) satisfies
\[
|\tilde{a}(x, t, Dw)| \leq v_1 |Dw| + \frac{\theta \psi_1}{N(u)}
\]
\[
\tilde{a}(x, t, Dw) Dw = \frac{\partial^2 a(x, t, u, Du)}{N^2(u)} \geq v_0 |Dw|^2 - \frac{\theta^2 \psi_0}{N^2(u)}.
\]
Moreover, since
\[
\frac{\partial a(...)}{N^2(u)} \geq v_0 |Dw|^2 - \frac{\theta^2 \psi_0}{N^2(u)}, \quad (1.12)
\]
and since \( N(u) \leq \theta \sup_{\partial \Omega} |u| + R^* \), we also have
\[
\frac{b(x, t, u(x, t), Du)}{N(u(x, t))} \leq v_1 (\theta M_4 + R^*) |Du|^2 + \frac{\psi_2}{N(u)}. \quad (1.13)
\]
So, because \( v_0 \geq v_2 M_4 \), if \( R \) is small enough such that \( v_0 \theta > v_2 (\theta M_4 + R^*) \) then \( \tilde{b} \) satisfies (1.10). Similar arguments show that \( w = w_2 \) also satisfies an inequality of the same form as (1.9).

The last statement is straightforward. For example, if \( \psi_0 \in L(R, v) \) then there is a constant \( C \) such that for any measurable set \( A \in Q_R \) we have
\[
\int_A \psi_0 \, dx \leq \frac{C}{R^v} |A|^v \leq C |A|^v \left( v/(v + 2) \right).
\]
Since $v > n/(n + 2)$, we can choose $\alpha > 0$ small enough such that $\psi_0$ still satisfies (F).

We see that $w_1, w_2$ are weak subsolutions to equations which satisfy a similar structure condition as that of (1.1) (with $v_2 = 0$). Therefore, Lemma 1.5 (with $\sigma = 1$) implies

$$\sup_{B_{\gamma}(\theta) \times \{\gamma, \lessdot\}} w \leq \text{Const} \left( 1 + \frac{1}{R^a + 2} \int_{B_{\gamma}(2R) \times \{\gamma - 2R^2, \lessdot\}} (w^+)^2 \, dx \, dt \right).$$  \hfill (1.14)

Similarly, we can show that $w_2$ satisfies the above estimate. As mentioned before, to complete the proof of H"older continuity for $u$ we need only to estimate the quantity $\sup_{B_{\gamma}(\theta) \times \{\gamma - 2R^2, \lessdot\}} w_1$, which is either $w_1$ or $w_2$, by a constant independent of $u, R$. The above shows that we need to estimate the right hand side of (1.14).

Set $I_0 := [t_0 - 4R^2, t_0 - 2R^2]$, $Q_0 = B_{\gamma}(2R) \times I_0$. We will show that we can estimate the integral in (1.14) if $w^+$ vanishes on a sufficiently large subset of $Q_0$.

**Lemma 1.7.** Let $Q^0 := \{ (x, t) \in Q_0 \mid w^+(x, t) = 0 \}$. If $|Q^0| \geq K \cdot |Q_0|$ then there exists a constant $C(K)$ such that

$$\frac{1}{R^a + 2} \int_{B_{\gamma}(2R) \times \{\gamma - 2R^2, \lessdot\}} (w^+)^2 \leq C(K).$$

**Proof.** Let $\eta(x)$ be a cut-off function for $B_{\gamma}(2R)$, i.e. $\eta(x) \equiv 1$ in $B_{\gamma}(2R)$, $\eta(x) \equiv 0$ outside $B_{\gamma}(4R)$ and $|D_x \eta| \leq 1/2R$.

We now go back to (1.11) and replace $\eta$ by $\eta^2$. Using (1.12) and (1.13), and normalizing the constants, we obtain in a standard way that

$$\frac{d}{dt} \int_{\Omega} w \eta^2 + \int_{\Omega} \eta^2 |Dw|^2 \, dx$$

$$\leq C \int_{\Omega} |Dw| \eta |D\eta| + C(\theta) \int_{\Omega} \psi_0 \eta^2 + \psi_1 |D\eta| + \psi_2 \eta^2 \eta^2 \, dx.$$  \hfill (1.15)

The Young inequality applies to the first integrand on the right hand side and the fact that $|D_x \eta| \leq 1/2R$ (assuming also that $R \leq 1$) give

$$\frac{d}{dt} \int_{\Omega} w \eta^2 \, dx + \int_{\Omega} \eta^2 |Dw|^2 \, dx \leq CR^a - 2 + \int_{\Omega} F \eta^2 \, dx,$$
where \( F := (\psi_0 + \psi_1)/N^2(u) + \psi_2/N(u) \). As before, because of \((F)\), we can choose \( \alpha > 0 \) again to have
\[
\int_{B_{R}(2R) \times [t_{0} - 4R, t_{0}]} F \, dx \, dt \leq CR^\alpha. \tag{1.16}
\]

Let \( V(t) = \int_{\Omega} w(x, t) \eta^2 \, dx \). We see that \( V(t) \geq -\log \theta \), for all \( t \in [t_{0} - 4R^2, t_{0}] \).

Let us show that there is a \( t_1 \in I_0 \) such that \( V(t_1) \leq A \) for some universal positive constant \( A = A(K) \). We set
\[
\Omega^0_t = \{ x \in B_{R}(2R) : w^+(x, t) = 0 \}, \quad \text{and} \quad m(t) = |\Omega^0_t|, \quad t \in I_0.
\]

Assume that \( V(t) \geq A > 0 \) in \( I_0 \). Using the Poincaré type inequality due to Moser ([22, Lemma 3]), we have
\[
\int_{\Omega} \eta^2 (w - V)^2 \, dx \leq 16R^2 \int_{\Omega} \eta^2 |Dw|^2 \, dx. \tag{1.17}
\]

By reducing the above integral to the smaller set \( \Omega^0_t \), where \( w \leq 0 \), we have from (1.15) that
\[
\int_{\Omega} \eta^2 \, dx \times \frac{d}{dt} V(t) + \frac{1}{R^2} V^2(t) \, m(t) \leq CR^\alpha - 2 + \int_{\Omega} F \eta^2 \, dx
\]
for all \( t \in I_0 \). So,
\[
\int_{\Omega} \eta^2 \, dx \frac{V'(t)}{V^2(t)} + \frac{1}{R^2} \, m(t) \leq CR^\alpha - 2 + \frac{1}{A^2} \int_{\Omega} F \eta^2 \, dx.
\]

Integrating over \( I_0 \) and noting (1.16) and that \( \int_{\Gamma} m(t) \, dt = |\Omega^0| \geq K |\Omega_4| \sim KR^{n+2} \) by the assumption of the lemma, we get
\[
KR^\alpha \leq \frac{1}{R^2} \int_{t_0}^t m(t) \, dt \leq \left( \frac{1}{V(t_0 - 2R^2)} - \frac{1}{V(t_0 - 4R^2)} \right) \int_{\Omega} \eta^2 \, dx + \frac{2CR^\alpha}{A^2}
\]
\[
\leq R^n \left( \frac{1}{A} + \frac{2C}{A^2} \right).
\]

By choosing \( A = A(K) \) large enough (independent of \( u, R \)), we see that the above inequality gives a contradiction. Hence, there must exist \( t_1 \in I_0 \) such that \( V(t_1) \leq A \).
Integrating (1.15) over \([t_1, t_2]\) for \(t_2 \in [t_0 - 2R^2, t_0]\) (thus, \(|t_2 - t_1| \leq 4R^2\)), we get (using (1.16))

\[
V(t_2) \int_\Omega \eta^2 \, dx + \int_{t_1}^{t_2} \int_\Omega \eta^2 \, |Dw|^2 \, dx \, dt \leq CR^n + V(t_1) \int_\Omega \eta^2 \, dx. \tag{1.18}
\]

This and the fact that \(V(t) \geq -\log \theta \) and \(V(t_1) \leq A\) imply

\[-\log \theta \leq V(t) \leq C, \forall t \in [t_0 - 2R^2, t_0]\]

and

\[
\int_{t_0 - 2R^2}^{t_0} \int_\Omega \eta^2 \, |Dw|^2 \, dx \, dt \leq CR^n
\]

for some universal constant \(C\) depends only on \(A\), and thus, on \(K\). The above and (1.17) give

\[-\log \theta \leq V(t) \leq C, \forall t \in [t_0 - 2R^2, t_0]\]

and

\[
\int_{t_0 - 2R^2}^{t_0} \int_\Omega \eta^2 (w - V(t))^2 \, dx \, dt \leq CR^{n+2}.
\]

Obviously, we have from these two estimates

\[
\frac{1}{R^{n+2}} \int_{B_{3R}(2R) \times [t_0 - 2R^2, t_0]} w^2 \, dx \, dt \leq C
\]

for some universal constant \(C\) independent of \(R\).

**Proof of Theorem 2.** Set \(Q_u := \{(x, t) \in Q_* | u \leq m_4 + \omega_4(1 - 1/\theta)\}\).

If \(|Q_u| \geq \frac{1}{2}|Q_*| \sim R^{n+2},\) we set \(w = w_1, Q^0 = Q_u\) and notice that \(w^+ = 0\) on \(Q^0\). Also, \(|Q^0| \geq \frac{1}{2}|Q_*|\).

Otherwise, if \(|Q_u| \leq \frac{1}{2}|Q_*|\), we set \(Q^0 = \{(x, t) \in Q_* | u \geq m_4 + \omega_4(1 - 1/\theta)\}\). Since \(\theta \geq 2\), it is easy to see that \(Q^0 = \{u > m_4 + \omega_4(1 - 1/\theta)\} = Q_* \setminus Q_u\). Thus, with \(w = w_2\), we have \(w^+ = 0\) on \(Q^0\). Again, \(|Q^0| \geq \frac{1}{2}|Q_*|\).

So, Lemma 1.7 is applicable in both cases, with \(K = 1/2\), to give a bound for the right hand side of (1.14). As we already showed, this gives the estimates (1.6) and completes the proof of Theorem 2.

**Remark 1.8.** The above proofs will be used later to obtain regularity results for systems. However, there will be cases (cf. Theorem 7) when some modifications for Lemma 1.7 are needed. In the proof of Lemma 1.7,
instead of taking $I_0$ to be the interval $[t_0 - 4R^2, t_0 - 2R^2]$ we can take $I_0 = [t_0 - 2R^2, t_0]$ and assume that $|Q^0| \geq KR^{n+2}$.

We outline the necessary modifications here. First, we redefine the cylinders $Q_R$ by $B_{A_k}(iR) \times [t_0 - 2\alpha R^2, t_0]$ and the functions $w_i$ accordingly. Here, $\sigma > 0$ is a constant independent of $R$ to be determined later. It is clear that the argument before Lemma 1.6, the lemma itself and the proof of Theorem 2 are still in force to obtain Hölder continuity of $u$ if one can estimate the quantity

$$
\frac{1}{|\sigma R^{n+2}|} \int_{B_R(2R) \times [t_0 - 2\alpha R^2, t_0]} (w^+)^2.
$$

To this end, we follow the proof of Lemma 1.7 to see that the key point is to find a time $t_1$ such that $t_1 < t_0 - 2\alpha R^2$ and $V(t_1) \leq A$ for some $A = A(K)$. We now define $I_0 = [t_0 - 2R^2, t_0 - 2\alpha R^2]$, $Q_R = B_{A_k}(2R) \times I_0$ and $Q^0 = \{w^+ = 0\} \cap Q_R$. From the proof, it is easy to see that such a $t_1$ can be found if $|Q_R| \geq (K/2) R^{n+2}$. Since we are assuming $|Q_R| \geq KR^{n+2}$, $Q_0 = \{w^+ = 0\} \cap B_{A_k} \times I_0$, and because $|Q^0 \cap (Q_0 \setminus Q_\sigma)| \leq |Q_\sigma \setminus Q_\sigma| \leq 2\sigma R R^{n+2}$, with $\sigma_R$ the area of the unit sphere $S^n$, we can see that $|Q_\sigma| \geq (K/2) R^{n+2}$ if $\sigma = K/4\sigma_R$. This concludes the remark on Lemma 1.7.

In the sequel we are going to need the notion of Campanato–Morrey function spaces $L_{loc}^{p,\infty}(\Omega_T)$. We recall that, see [13, 14, 27], $L_{loc}^{p,\infty}(\Omega_T)$ consists of measurable functions $f$ such that for any $R > 0$, there is a constant, denoted by $\|f\|_{L_{loc}^{p,\infty}(\Omega_T)}$, such that

$$
\int_{Q_R} f \, dx \, dt \leq \|f\|_{L_{loc}^{p,\infty}(\Omega_T)} R^p.
$$

If $K \subset \Omega_T$ is a compact set, we set $\|f\|_{L_{loc}^{p,\infty}(K)} := \sup_{\emptyset} \|f\|_{L_{loc}^{p,\infty}(Q_R)}$, where $\emptyset$ is a finite covering of $K$ by cylinders of the form $Q(R, R^2)$. Let $R > 0$. We write $B_R = B_R(x_0)$, $I_R = [t_0 - R^2, t_0]$ and $Q_R = B_R \times I_R$. For a measurable function $u$ on $Q(R)$ we denote its mean value over $Q(R)$ by $u_R$. That is, $u_R := \frac{1}{|Q(R)|} \int_{Q(R)} u \, dx \, dt$.

We have the following result on the local $L^2$ norm of $|Du|$.

**Corollary 3.** Assume that $u$ is locally Hölder continuous. There exists $\mu_0 > n$ such that $|Du|^2 \in L_{loc}^{1,\mu_0}(\Omega_T)$. That is, there exists positive constants $C$ such that

$$
\int_{Q_R} |Du|^2 \, dx \, dt \leq C \rho^{\mu_0}, \quad \text{for small } \rho.
$$

(1.19)
Proof. Let $\rho > 0$ and $\eta$ be a cutoff function for $Q_{2\rho}$, that is, $\eta \equiv 1$ in $Q_{\rho}$, $\eta$ vanishes outside $Q_{2\rho}$, and $|D\eta| \leq 1/\rho$ and $0 \leq \eta \leq 1/\rho^2$. Multiply the equation (1.1) by $(u - u_\rho) \eta^2$ and use the structure condition and Young inequality to get

$$
\sup \int_{B_{\rho}} (u - u_\rho)^2 \eta^2 \, dx + \int_{Q_{2\rho}} |Du|^2 \eta^2 \, dx \, dt \\
\leq \int_{Q_{2\rho}} |Du| |u - u_\rho| \eta |D\eta| \, dx \, dt \\
+ \int_{Q_{2\rho}} |Du|^2 |u - u_\rho| \eta^2 + (\psi_0 + \psi_1^2 + \psi_2^2) \eta^2 \\
+ (u - u_\rho)^2 |D\eta|^2 + |u - u_\rho|^2 \eta \, dx \, dt. 
$$

(1.20)

By the choice of $\eta$ and the fact that $u$ is Hölder continuous, there are positive constants $C$, $\alpha$ such that $|u(x, t) - u_\rho| \leq C \rho^\alpha$ for $(x, t) \in Q_{2\rho}$. We have then

$$
\int_{Q_{2\rho}} |u - u_\rho|^2 \eta \, dx \, dt
$$

and

$$
\int_{Q_{2\rho}} |u - u_\rho|^2 \eta \, dx \, dt \leq C \rho^{2\alpha} \frac{1}{\rho^2} |Q_{2\rho}| \leq C \rho^{n + 2\alpha}.
$$

By (F), the integral of $(\psi_0 + \psi_1^2 + \psi_2^2) \eta^2$ can be majorized by $C \rho^{n+\gamma}$ for some $\gamma > 0$. If $R > 0$ and sufficiently small, the integral of $|Du|^2 |u - u_\rho| \eta^2$ can be absorbed into that of $|Du|^2 \eta^2$ on the left of (1.20). Using the Young inequality and the above estimate we have

$$
\int_{Q_{2\rho}} |Du| |u - u_\rho| \eta |D\eta| \, dx \, dt \\
\leq \varepsilon \int_{Q_{2\rho}} |Du|^2 \eta^2 \, dx \, dt + C(\varepsilon) \int_{Q_{2\rho}} (u - u_\rho)^2 |D\eta|^2 \, dx \, dt.
$$

The last term is bounded by $C \rho^{n+\gamma}$. By choosing $\varepsilon$ small and combining these estimates we obtain (1.19) $\mu_0 = \min \{n + \alpha, n + \gamma\}$ and complete our proof. \qed
The above corollary and the cross-diffusion systems considered in Section 2 inspire us to consider a class of parabolic equation of the form

$$\frac{\partial u}{\partial t} = \text{div}(a(x, t, u) Du) + f_0 + \text{div} F, \quad (x, t) \in \Omega_T,$$

(1.21)

with the data $f_0$, $F$ belong to some Campanato-Morrey spaces. We shall impose the following condition on $f_0$ and $F = (f_1, \ldots, f_n)$.

(F') There is $\mu > n$ such that $f_0^\mu \in L^{1, n-\lambda}(\Omega_T)$, $f_i^\mu \in L^{1, n}(\Omega_T)$.

Obviously, (F') define a larger class than that of (F). We will show that

**Theorem 4.** Assume (F') and that $a = (a_{ij})$ is continuous with respect to $x$, $t$, $u$ and satisfies

$$\left| a_{ij}(x, t, u) \zeta_i \zeta_j \right| \leq \lambda_0 \| \zeta \|^2 \quad (1.22)$$

for some positive constants $\lambda_0$, $\lambda_1$ and for any vector $\zeta = (\zeta_i)_{i=1}^n \in \mathbb{R}^n$. Then every bounded solution to (1.21) is Hölder continuous.

The proof bases on the perturbation method of [13] and imbedding theorems of Campanato-Morrey spaces. We will show

**Proposition 5.** There are positive constants $C$, $\gamma$ such that, for any $R > 0$,

$$\| Du \|^2_{L^2_{Q_R}} \leq C(K + 1) R^{n+\gamma}, \quad (1.23)$$

where $K = \| f_0^\mu \|_{L^{1, n-\lambda} (Q_R)} + \sum \| f_i^\mu \|_{L^{1, n} (Q_R)}$.

First, let $V \in C(I_R, H^1_0(B_R))$ be the solution to the problem

$$\begin{cases}
\frac{\partial V}{\partial t} = \text{div}(a(x, t, u)DV)) + f_0 + \text{div} F & \text{in } Q_R, \\
V(x, t) = 0 & \text{on } \partial B_R \times I_R, \\
V(x, t_0 - R^2) = 0.
\end{cases} \quad (1.24)$$

Multiplying the equation of $V$ by $V$ and integrating over $Q_R$, we easily obtain

$$\int_{B_R} V^2(t_0, \bullet) \, dx + \int_{Q_R} |DV|^2 \, dx \, dt \leq \| f_0 \|_{2, Q_R} \| V \|_{2, Q_R} + \| f_i \|_{2, Q_R} \| V_x \|_{2, Q_R}.$$
Using the Young inequality and Poincaré inequality \((V = 0\ on\ \partial B_R)\), we have
\[
\int_{B_R} V^2(t_0, \cdot) \ dx + \int_{Q_R} |DV|^2 \ dx \ dt \leq R^2 \, \|f_0\|_{L^2(Q_R)}^2 + \sum_{\tau} \|f_{\tau}\|_{L^2(Q_\tau)}^2 \leq KR^\mu. \tag{1.25}
\]
Again, since \(V = 0\ on\ \partial B_R\), the Poincaré inequality and (1.25) show that
\[
\int_{Q_R} |V| \ dx \ dt \leq \frac{C}{R^\nu} \int_{Q_R} |DV|^2 \ dx \ dt \leq o(R). \tag{1.26}
\]
Now let \(W(x, t) = u(x, t) - V(x, t)\ on\ Q_R\). We see that \(W\) satisfies
\[
\frac{\partial W}{\partial t} = \text{div}(a(x, t, u) \, DW), \quad (x, t) \in Q_R. \tag{1.27}
\]
By (1.22) the above parabolic equation is regular. Since \(W = u\), which is bounded, on \(\partial Q_R\), it is well known that \(W\) is also bounded and therefore Hölder continuous. Moreover, from (1.26) and the continuity of \(W\) we see that \(\int_{Q_R} |u - u_R|^2 = o(R)\). Thus, \(\int_{Q_R} \|a(x, t, u) - a(x_0, t_0, u_0)\| = o(R)\) (see [13]). This fact allow us to apply a perturbation argument as in [5, 13, 14] to conclude that there are constants \(C\) and \(\mu_0 > n\) such that \(W\) satisfies the estimate
\[
\|DW\|_{L^2(Q_\rho)} \leq C \left(\frac{\rho}{R}\right)^{\mu_0} \left(\|DW\|_{L^2(Q_R)} + R^{\mu_0}\right), \quad 0 < \rho < R. \tag{1.28}
\]
With these preparations we now go back to

**Proof of Proposition 5.** Using (1.25) and (1.28), we have
\[
\|Du\|_{L^2(Q_\rho)} \leq 2\|DW\|_{L^2(Q_R)} + \|DV\|_{L^2(Q_\rho)} \leq C \left(\frac{\rho}{R}\right)^{\mu_0} \|DW\|_{L^2(Q_R)} + CR^{\mu_0} + 2\|DV\|_{L^2(Q_\rho)} \leq C \left(\frac{\rho}{R}\right)^{\mu_0} \|DW\|_{L^2(Q_R)} + CR^{\mu_0} + KR^{\mu_0}.
\]
We can assume that \(R < 1\) so that (using \(\|DW\|_{L^2(Q_\rho)} \leq 2(\|Du\|_{L^2(Q_\rho)} + \|DV\|_{L^2(Q_\rho)})\))
\[
\|Du\|_{L^2(Q_\rho)} \leq C \left(\frac{\rho}{R}\right)^{\mu_0} \|Du\|_{L^2(Q_R)} + C(K + 1) \, R^4 \tag{1.29}
\]
for any $\lambda < \min\{\mu_0, \mu\}$. Applying the iteration technique as in [13] we obtain

$$|Du|^2_{2, Q} \leq C(K+1)\rho^4, \quad 0 < \rho < R.$$  

This gives the estimate (1.23) of our proposition.

Proof of Theorem 4. Using the Poincaré inequality and the imbedding theorems of Campanato–Morrey spaces (see [13]) we see that (1.23) implies $u$ is Hölder continuous. □

Remark 1.9. Theorem 4 holds for systems of the form

$$\frac{\partial u^i}{\partial t} = \text{div}(a^i(x, t, u) Du^i) + f^i_0 + \text{div}F^i, \quad i = 1, ..., m,$$

where $\bar{u} = (u^1, ..., u^m)$ and $f^i_0, F^i$ satisfying (F'). The proof for the vector case follows exactly the same lines. Moreover, we can relax the assumption on $F^i = (f^i_1, ..., f^i_n)$ in (F') by allowing $F^i$ to depend on $Du$ and assume that

$$\sum_{i=1}^m \sum_{j=1}^n \|f^i_j\|^2_{2, Q} \leq KR^n + \varepsilon \|Du\|_{2, Q} \tag{1.30}$$

for some $\varepsilon > 0$ sufficiently small. Indeed, we define $V^i, W^i$ accordingly and see that, under the condition (1.30), (1.25) now implies

$$\int_{Q} |DV^i|^2 \, dx \, dt \leq R^2 \|f^i_0\|^2_{2, Q} + \sum_{i=1}^m \|f^i\|^2_{2, Q} \leq KR^n + \varepsilon \|Du\|_{2, Q}.$$  

Accordingly, (1.29) becomes

$$\|Du\|^2_{2, Q} \leq C \left( \frac{\rho}{R} \right)^{\frac{n}{2}} + \varepsilon \|Du\|_{2, Q} + C(K+1)R^2, \quad \forall i.$$  

If $\varepsilon$ is sufficiently small (in terms of $C, \mu_0, \lambda$) the iteration technique in [13] still applies and gives the proposition.

2. HÖLDER REGULARITY FOR PARABOLIC SYSTEMS

In this section we study bounded solutions to some reaction-diffusion systems with strong coupling in the diffusion terms. We first consider the case of triangular systems and then the case of fully coupled systems which satisfy certain special structure.
2.1. A Triangular System

We consider the following system

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \text{div}(a_1(x, t, u, v, Du)) + \phi(x, t, u, v) \cdot Dv + f(x, t, u, v), \\
\frac{\partial v}{\partial t} &= \text{div}(a_2(x, t, u, v, Dv)) + g(x, t, u, v), \quad t > 0, \quad x \in \Omega.
\end{aligned}
\]

(2.1)

We assume that the flux vectors \(a_i(x, t, u, v, Du), \ i = 1, 2\) satisfy the structure condition (1.2) of Section 1. Because the coupling occurs only in the first equation, (2.1) is a triangular quasilinear parabolic system, in the terminology of [2]. General form of (2.1) and regularity was investigated in [2]. We would like to show that the proof in previous section can be directly carried over to this case.

Assuming that we already know a priori \(L^\infty\) estimates for \(u, v\). We are going to show that \(u, v\) are Hölder continuous and their norms are bounded in terms of the \(L^\infty\) norms of \(u, v\). We have the following theorem.

**Theorem 6.** Assume that the flux vectors \(a_i, i = 1, 2\), satisfy the structure condition (1.2) and \(\phi, f, g\) are bounded on any compact set of their arguments \((x, t, u, v)\). If \(u, v\) are bounded solution to (2.1) then \(u, v\) are Hölder continuous.

**Proof.** Consider the equation for \(v\), since \(u, v\) are bounded we see that \(g(x, t, u, v)\) is bounded so that Theorem 2 implies \(v \in C^\ast\) for some positive \(x\). By Corollary 3, we see that \(Dv \in L^1_{loc}\) for some \(\mu > n\).

Rewrite the equation for \(u\) in the form

\[
\frac{\partial u}{\partial t} = \text{div}(a(x, t, u, Du)) + f_0 + \text{div}F,
\]

with \(f_0 = f(x, t, u, v)\) and \(F = -\phi(x, t, u, v) \cdot Dv\) we see that \(f_0, F\) satifies the condition (F') so that Hölder continuity of \(u\) follows from Theorem 4.

**Remark 2.1.** Special forms of (2.1) have been studied recently in the context of population dynamics. For example, in [6, 18, 17, 23, 25, 32] the authors used semigroup theory to study global existence of solutions (and existence of global attractor, see [25]) of the following problem

\[
\begin{aligned}
u_t &= A\{d_1 + cv\} u + uf(u, v), \\
v_t &= Av +vg(u, v).
\end{aligned}
\]

The semigroup techniques for semilinear systems and imbedding theorems in these papers cannot apply to a quasilinear system of the form
On the other hand, by a simple induction argument, one can see that the above proof also applies to triangular systems of more than two equations as well.

### 2.2. A Strongly Coupled System

Next, we consider the following strongly coupled system

\[
\frac{\partial u'}{\partial t} = \text{div} \cdot \mathbf{a}'(x, t, u, Du) + f'(x, t, u, Du, \mathbf{a}'(x, t, u, Du)) + c'(x, t, u) \text{DH},
\]

where \( u = (u', \ldots, u^m) \) and the flux vector \( \mathbf{a}' \) is given by

\[
\mathbf{a}' = a'(x, t, u, Du) + r'(x, t, u) + c'(x, t, u) \text{DH}.
\]

Here \( H = H(x, t, u) \) is some \( C^2 \)-function. For simplicity, we will consider only the case where \( H \) depends only on \( u \). The general form of \( H \) will introduce some extra terms in our calculation below but they cause no new difficulty.

We shall impose the following structure conditions on (2.1).

(H.1) The norms of the matrix-valued functions \( a' = (a'_i) \), \( r' = (r'_i) \), \( c' = (c'_i) \) and the partial derivatives \( H_{ui}, H_{ui uj} \) are uniformly bounded in terms of that of \( u \). That is

\[
|a'_i(x, t, u)|, |r'_i(x, t, u)|, |c'_i(x, t, u)|, |H_{ui}(x, t, u)|, |H_{ui uj}(x, t, u)| \leq C(|u|),
\]

\( \forall x \in \Omega, t > 0, u \in \mathbb{R}^m \). Here, \( C \) is some nonnegative continuous function.

(H.2) There exists a matrix \( a = (a_{ig}(x, t, u)) \) such that if \( d_{ig} = a_{ig} + c'_{ig} H_{ui} \) and \( e_{ig} = a'_{ig} - a_{ig} \) then there are positive constants \( \lambda, \lambda_0 \) such that

\[
\lambda |\zeta|^2 \leq d_{ig} e'_{ig} \zeta^2, \quad \sum_{i=1}^{m} |c'_{i}(x, t, u)| = \sum_{i=1}^{m} |a'_{ig}(\bullet) - a_{ig}(\bullet)| < \lambda_0, \quad (2.2)
\]

\[
\lambda |M|^2 \leq a'_{ig} H'_{ui uj} M_i M_j \quad (2.3)
\]

for any real vector \( \zeta \in \mathbb{R}^n \) and any real \( m \times n \) matrix \( M = (M_i)^{s} \).

(H.3) There are positive constant \( \varepsilon_0 < \lambda \) and positive function \( K(u) \) such that for all \( (x, t, u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R} \) and \( p \in \mathbb{R}^m \) (recall that we are assuming \( u \) is bounded)

\[
H_{ui} f'(x, t, u, p) \leq \varepsilon_0 |p|^2 + K(u). \quad (2.4)
\]

The following result was then proved by Wiegner in [30].
Theorem 7. Assume (H.1)–(H.3) and \( \lambda_0 = 0 \), that is, the \( a' \) are identical. Then bounded solutions to (2.1) are Hölder continuous. Moreover, for any \( \alpha \in (0,1) \) the \( C^{\alpha,\alpha/2} \) norm of \( u \) is bounded in terms of the supremum norm of \( u \) and the data.

We shall give not only a slightly different proof (for simplicity we restrict to the interior regularity part) but are able to relax the conditions by assuming (H) below. In [30], it was also assumed that the self-diffusion matrices \( (a'_i) \) are identical. If the function \( H \) is linear in \( u \) and (2.1) satisfies some additional structural conditions we can allow the self-diffusion matrices \( a' \) to be different. In particular, we consider the following condition.

(H) Assume (H.1) and the ellipticity condition (2.2) of (H.2). Moreover, we assume that \( H(u) \) is linear so that \( H_{x_i x_j} \equiv 0, i,j=1,\ldots,m \).

Finally, assume that either

(i) the functions \( f' \) are independent of \( Du \), or,

(ii) condition (2.4) and, in addition, \( |H_{,u}| \geq K_0 \) with \( \varepsilon_0 \sup_{Q_{1R}} H(u(x,t)) < K_0^2 \) for some constant \( K_0 > 0 \).

Thus our main result is the following

Theorem 8. Given (H), if \( \lambda_0 = 0 \) then the conclusion of Theorem 7 holds. If (i) of (H) holds and if \( \lambda_0 \) sufficiently small (that is, the \( a' \) are slightly different) then bounded solutions are also Hölder continuous.

The main idea of the proof of Theorem 8 is to show that \( H(u(x,t)) \) is Hölder continuous and then apply Theorem 4, if \( f' \) are independent of \( Du \), to each equation of (2.1) to obtain the regularity for each component of the solution. When \( f' \) depend on \( Du \), the regularity of \( H \) also implies estimates for integral of \( |Du|^2 \) so that one can apply imbedding theorems of Campanato and Morrey spaces to obtain Hölder continuity for \( u \). The fact that \( H(u(x,t)) \) is regular is proven by using similar logarithmic functions for \( H \) and following exactly the techniques in 1.

However, the above argument can only apply to the case when \( H \) is linear. For nonlinear \( H \), the logarithmic function \( w_2 \) (see (2.5) below) is not a weak subsolution of some scalar parabolic equation so that the argument in Section 1 can not apply directly. To prove Theorem 7 we have to borrow some ideas of Wiegner in [30] and combine with the logarithmic function technique.

First, we introduce the logarithmic functions and derive some equations and inequalities which will be used later.

Let \( R > 0 \) be given. In the cylinder \( Q_{1R} \) (see Section 1) we define

\[
M_i = \sup_{Q_{1R}} H_i(u(x,t)), \quad m_i = \inf_{Q_{1R}} H_i(u(x,t)), \quad \text{and} \quad \omega_i = M_i - m_i,
\]

where \( H_i(u(x,t)) \) are the \( i \)-th components of \( H(u(x,t)) \).
and consider the following functions

\[ w_1(x, t) = \log \left( \frac{\omega_k + R^*}{N_1(u)} \right), \quad w_2(x, t) = \log \left( \frac{\omega_k + R^*}{N_2(u)} \right), \tag{2.5} \]

where \( N_1(u) = \theta(M_k - H(u)) + R^* \), \( N_2(u) = \theta(H(u) - M_k) + R^* \), for some constants \( \theta > 2 \) and \( \alpha > 0 \) to be determined later.

Denote by \( w := w_i, N(u) := N_i(u) \) and let \( \gamma = 1 \) if \( i = 1 \) and \( \gamma = -1 \) if \( i = 2 \). For each \( i = 1, \ldots, m \), consider the function \( \psi^i = \gamma(\eta/N) H_i \), with \( \eta > 0 \), and note that

\[ \frac{\partial w}{\partial x} = \gamma \frac{\partial H}{\partial x}, \quad \frac{\partial w}{\partial t} = \gamma \frac{\partial H}{\partial t}, \quad \frac{\partial \psi^i}{\partial x} = \gamma \frac{\theta H_i}{N} \frac{\partial u^i}{\partial x}, \quad \frac{\partial \psi^i}{\partial t} = \gamma \frac{\theta H_i}{N} \frac{\partial u^i}{\partial t}, \tag{2.6} \]

and

\[ \frac{\partial w}{\partial x} = \gamma \frac{\partial H}{\partial x} + \eta \left( \frac{\theta}{N^2} H_i \frac{\partial u^i}{\partial x} + \frac{1}{N} H_i \frac{\partial^2 u^i}{\partial x^2} \right). \]

From (2.6), we have

\[ (\alpha_{i\beta} u^i_{xx} + \gamma^i_{\beta} H_i) \frac{\partial H}{\partial x} \eta_{x} = \frac{1}{\theta} d_{\alpha\beta} w_{x} \eta_{x}, \tag{2.7} \]

and

\[ \frac{\partial H_i}{N^2} \left( \alpha_{i\beta} u^i_{xx} + \gamma^i_{\beta} H_i \right) = \frac{1}{\theta} d_{\alpha\beta} w_{x} u^i_{x}. \tag{2.8} \]

Therefore, by testing the equation of \( u^i \) by \( \psi^i \), integrating over \( \Omega \), summing over \( i \) and using the above identities we obtain

\[ \left\{ \begin{array}{l}
\int_{\Omega} \frac{\partial w}{\partial t} \eta + \int_{\Omega} (d_{\alpha\beta} w_{x} + \phi \beta) \eta_{x} \, dx \\
+ \int_{\Omega} \eta \left( d_{\alpha\beta} w_{x} w_{x} + \gamma \frac{\partial u_{x} H_{x}}{N} u^i_{x} u^i_{x} \right) \, dx \\
- \int_{\Omega} \eta \left( \gamma H_{x} \frac{\partial H}{H} u_{x} + \theta R_{x} u_{x} \right) \, dx + \int_{\Omega} \frac{\gamma}{N} \frac{\partial u_{x}}{\partial x} \eta \, dx \end{array} \right. \tag{2.9} \]

where \( \phi := \gamma(\theta H_i) d_{x} \) and \( R_{x} := \gamma \frac{H_{x} H_{x}}{N^2} + \gamma H_{x}^2 \).
Next, by testing the equation of \( u' \) by \( H_u \) and proceeding similarly as above (see also [30, Eq. (17)]) we easily get

\[
\int Q \frac{\partial H}{\partial t} \eta \, dx + \int Q (d_{xH} H_u \eta_{x^i} + \varepsilon_1 |Du|^2 \eta) \, dx \leq C \int Q (1 + |DH|^2) \eta + |D\eta| \, dx.
\]

(2.10)

**Proof of Theorem 7.** As before, we denote \( I_0 := [t_0 - 2R^2, t_0] \), \( Q_0 := B_{x_0}(2R) \times I_0 \). Let \( Q_0^\prime = \{(x, t) \in Q_0 | w = 0\} \). We consider two alternatives.

(A) There is \( R_0 > 0 \) such that

\[
|Q_0^\prime| > \frac{1}{\theta} R^{n+2}, \quad \forall R \in (0, R_0).
\]

(2.11)

(B) There is a sequence \( \{R_k\} \), \( R_k \to 0 \), such that

\[
|Q_0^\prime| < \frac{1}{\theta} R^{n+2}, \quad \text{for} \quad R = R_k.
\]

(2.12)

If (B) holds then for any \( \varepsilon > 0 \), we will show that there exists \( \theta = \theta(\varepsilon) \) such that

\[
\lim_{R \to 0} \frac{1}{R^n} \int_{Q_R} |Du|^2 \, dx \, dt < \varepsilon, \quad Q_R := B(x_0, R) \times [t_0 - R^2, t_0].
\]

(2.13)

Otherwise, if (A) holds then it will be shown that \( H_u(x, t) \) is Hölder continuous. The proof of Corollary 3 can be applied to show that \( |DH(u)|^2 \) and \( |Du|^2 \) belong to \( L^1_{\text{loc}} \), for some \( \mu > n \). We then obtain again (2.13).

Thanks to the Poincaré type inequality (see [14, Prop. 3.1])

\[
\int_{Q_R} |u - u_R|^2 \, dx \, dt \leq cR^2 \int_{Q_R} |Du|^2 \, dx \, dt,
\]

(2.14)

we see that (2.13) implies \( \liminf_{R \to 0} \int_{Q_R} |u - u_R|^2 \, dx \, dt < \varepsilon \). Since \( \varepsilon > 0 \) can be arbitrarily small, the Hölder continuity of \( u \) follows from [14, Theorem 3.1]. Moreover, provided \( \varepsilon \) is taken sufficiently small, the proof (see [14, pages 445-446]) also shows that \( \int_{Q_R} |u - u_R|^2 \, dx \, dt \leq CR^\alpha \), for any \( \alpha \in (0, 1) \) and \( R > 0 \) with the constant \( C \) depends uniformly on \( \alpha, \varepsilon \) the data and the supremum norm of \( u \). As it is well known, this implies the desired estimate for the \( C^{n+\alpha} \) norm of \( u \) to conclude the proof of Theorem 7.

We then consider first the alternative (A).
Assume (A). By assumption (2.2) and (2.3), there is some positive \( \lambda \) such that
\[
d_{ab}w_{x_\alpha}w_{y_\beta} \geq \lambda |Dw|^2,
\]
\[
\frac{a_{ab}H^{2}_{ab}u^\gamma_{x_\alpha}u^\gamma_{y_\beta}}{N} \geq \lambda \frac{|Du|^2}{N}.
\]

On the other hand, because \( u \), and therefore, \( r, H, H^2 \) are bounded so that we can use the Young inequality to get
\[
\left| \frac{\theta H_{x_\alpha}}{N} \right| = \left| \frac{H^2_{ab}c_{ab}w_{x_\alpha}u^\gamma_{y_\beta}}{N} \right| \leq \varepsilon N |Dw|^2 + C(\varepsilon) \frac{|Du|^2}{N},
\]
\[
|\theta R_{i}^\gamma_{x_\alpha}u^\gamma_{y_\beta}| \leq \varepsilon \frac{|Du|^2}{N} + C(\varepsilon) N(\theta R_{i}^2)^2.
\]

Moreover, from (2.4) of (H.3), we have
\[
\theta \frac{\int H_{x_\alpha}}{N} \leq \theta \varepsilon_N \frac{|Du|^2}{N} + \frac{\theta K(u)}{N}.
\]

Let \( R < R_0 \) and \( w = w_1 \), hence \( \gamma = 1 \). Using these inequalities in (2.9) results in the occurrence of integrals of \( N |Dw|^2 \) and \( |Du|^2/N \) on the right hand side. However, by choosing \( \varepsilon \) small (\( \varepsilon N \leq \lambda \)) and then \( \theta \) large (\( C(\varepsilon) \leq (\lambda - \varepsilon)\theta \), noting the condition on \( \varepsilon_0 \) in (H.3)), we see that the integrals of \( N |Dw|^2 \) and \( |Du|^2/N \) can be absorbed to the third integral on the left hand side of (2.9). So, we have shown that \( w \) satisfies the following inequality
\[
\int_{\Omega} \frac{\partial w}{\partial t} \eta + \int_{\Omega} (d_{ab}w_{x_\alpha} + \phi_{ab}) \eta_{x_\alpha} dx 
\leq C \int_{\Omega} \left( \frac{K(u)}{N} + \frac{(r^2H_{ab}H_{x_\alpha})^2}{N^3} + \frac{(r^2H^2_{ab})^2}{N^3} \right) \eta dx. \tag{2.15}
\]

The above is similar to (1.9) of Lemma 1.6 and satisfies the same structure condition assumed in that lemma. So, we can proceed exactly as in the proof of Theorem 2 to show that \( w \) is bounded from above by a universal constant.

To this end, we go back to (2.9) and replace \( \eta \) by \( \eta^2 \). We keep only the first and third integrals on the left and move the rest to the right hand side. As before, by choosing \( \varepsilon \) small and \( \theta \) large appropriately, we obtain
\[
\int_{\Omega} \frac{\partial w}{\partial t} \eta^2 + \varepsilon_1 \int_{\Omega} (|Dw|^2 + |Du|^2) \eta^2 dx 
\leq C \int_{\Omega} |Dw| \eta |D\eta| + \int_{\Omega} \frac{C}{N} \eta dx \tag{2.16}
\]
for some positive constants $C, \varepsilon_1$. By the Young inequality and an appropriate choice of $\alpha$ in the definition of $N$, we see that $w$ satisfies an inequality of the form (1.15) and (1.16) in the proof of Lemma 1.7. Now, with the assumption (A) and Remark 1.8, Lemma 1.7 (with $K = 1/\theta$) implies that, for any $R < R_0$, $w_j$ is bounded by a universal constant depending on $\theta$ and, therefore, $(x, t) \rightarrow H(u(x, t))$ is Hölder continuous.

Replacing $\eta$ in (2.10) by $(H(u) - \inf_{Q(R, R)} H(u)) \eta^2$, with $\eta$ is a cutoff function for $Q(R, R^2)$, and using the fact that, for some positive $\varepsilon_2$, $|H(u) - \inf_{Q(R, R)} H(u)| \leq CR^*\varepsilon_2$ we easily adapt the proof of Corollary 3 to show that $|DH(u)|^2 \in L^1_{loc}$ for some $\mu > n$.

Once such a regularity of $H(u)$ is shown, it is easy to show the Hölder continuity of $u$. Especially, let us consider first the case $f^i$ are independent of $Du$. In this case, we can rewrite the equation for $u'$ as

$$\frac{\partial u'}{\partial t} = \text{div}(a'(x, t, u) Du') + f'(x, t, u) + \text{div}(F')$$

with $F' = r'(x, t, u) + c'(x, t, u) D H$. Since $|DH(u)|^2 \in L^1_{loc}$ for some $\mu > n$, condition (F') is verified. Applying Theorem 4 we conclude that the $u'$ are also Hölder continuous.

Finally, if the $f^i$ depend on $Du$ we will seek for an estimate for the integral of $|Du|^2$ over $Q(R, R^2)$ as follows. Let $H(u) = H(u) - H(u)_R$, where $H(u)_R$ denotes the mean value of $H(u(x))$ over $Q(R, R^2)$. From (2.10), we derive

$$C \int_{Q} \left[ (1 + |DH|^2) \eta + |D\eta| - d_{ap} H_{x\eta} \eta_{x\gamma} + H \frac{\partial \eta}{\partial t} \right] dx.$$

Let $\eta$ be a cutoff function for $Q(R, R^2)$ and satisfy: $\eta \equiv 1$ in $Q(R, R^2)$, $\eta(x, t) \equiv 0$ outside the cylinder $Q^*$ given by $B_{R_0}(2R) \times [t_0 - 2R^2, t_0 + R^2]$. In addition, $|D\eta| \leq 1/R$ and $|\partial \eta/\partial t| \leq 1/R^2$. With this choice of $\eta$, we integrate the above inequality with respect to $t$ over the interval $[t_0 - 2R^2, t_0 + R^2]$ and obtain

$$C \int_{Q} \left[ (1 + |DH|^2 + |D\eta| + |D\eta| |DH| |D\eta| + H \frac{\partial \eta}{\partial t} \right] dx dt.$$
Using the definition of $\eta$ and the facts that $|H| \leq CR^s$ (since $H(u(x,t))$ is Hölder continuous), we can majorize the terms on the right as follows

$$
\int_Q \left( 1 + |DH|^2 + |D\eta| + \frac{\|\eta\|}{\sqrt{t}} \right) dx \, dt
\leq C(R^{n+2} + R^s + R^{s+1} + R^{s+2}),
$$

$$
\int_Q |DH| \, dx \, dt \leq \frac{|Q'|^{1/2}}{R} \left( \int_Q |DH|^2 \, dx \, dt \right)^{1/2} \leq CR^{(s+\mu)/2}.
$$

Since $\mu > n$, we have $(n + \mu)/2 > n$. From these estimates, we obtain

$$
\varepsilon_1 \int_{Q_{2R}^1} |Du|^2 \, dx \, dt \leq \varepsilon_1 \int_{Q_r} |D\eta|^2 \, \eta \, dx \, dt \leq CR^{s+\gamma},
$$

for some $\gamma > 0$. Obviously, the above implies (2.13).

Assume (B). This was considered in [30]. We combine the arguments in [14, 30] to give a little bit shorter proof. It is clear that (B) implies

$$
|\{(x,t) \in Q_* | H \leq (1 - \rho) M_4^s\}| < \rho R^s, \quad \text{with} \quad \rho = 1/\theta. \quad (2.17)
$$

Following [30, pages 716–717], we then substitute $(H - k)^+\eta^2$, with $k \in \mathbb{R}$ and $\eta$ a cut-off function on $Q_{2R}$ with respect to $Q_{R}$, into the places of $\eta$ in (2.10). Since we can choose $\eta$ such that $|D\eta|^2 + |\eta| \leq cR^{-2}$, standard estimates give

$$
\varepsilon_1 \int_{Q_{2R}^1} |D\eta|^2 + (H - k)^+ \, dx \, dt + \int_{Q_R} |DH|^2 \, \eta^2 \, dx \, dt
\leq C \int_{Q_{2R}} \left\{(H - k)^+ \left( R^{-2} + 1 \right) + |DH|^2 \right\} \, dx \, dt + CR^{n+2}.
$$

By the choice of $k := (1 - 2\rho) M_4^s$, $(H - k)^+ \leq 2\rho M_4$ on $Q_{2R}$ so that if $\rho$ is small ($2C\rho M_4 \leq \lambda$) then

$$
\int_{Q_{2R}} |Du|^2 \, (H - k)^+ \, dx \, dt \leq C(\rho M_4)^2 \, R^s + CR^{n+2}.
$$
Let $A_0 := \{(x, t) \in Q_R \mid H \geq (1 - \rho) M_4 \}$. Then $(H - k)^+ \geq \rho M_4$ on $A_0$.

So,

$$\int_{A_0} |Du|^2 \, dx \, dt \leq CR^n \left( \rho M_4 + \frac{R^2}{\rho M_4} \right) \leq 2\rho M_4 R^n, \quad \forall R < \rho M_4 \tag{2.18}$$

Also, by substituting $e^{\mu t} \eta^2$ into places of $\eta$ in (2.10) with $s > 0$ sufficiently large, it is standard to show

$$\int_{Q_R} |Du|^2 \, dx \, dt \leq CR^n. \tag{2.19}$$

Using a simple invariant of Poincaré and Sobolev–Poincaré inequalities as in [14, p. 443] we easily show that

$$\int_{Q_R} |Du|^2 \, dx \, dt \leq C(\varepsilon) \left( \int_{Q_R} |Du|^q \, dx \, dt \right)^{2/q} + \varepsilon \int_{Q_R} |Du|^2 \, dx \, dt, \quad \forall \varepsilon > 0, \tag{2.20}$$

with $q = 2n/(n+2)$. We are going to estimate the integral of $|Du|^q$. By Hölder inequality, we have for any subset $A$ of $Q_R$

$$\int_A |Du|^q \, dx \, dt \leq \left( \int_A |Du|^2 \, dx \, dt \right)^{q/(n+2)} |A|^{2/(n+2)}.$$  

Taking $A = A_0$ and using (2.18), the above gives

$$\int_{A_0} |Du|^q \, dx \, dt \leq (2\rho M_4 R^n)^{n/(n+2)} R^2 = (2\rho M_4)^{n/(n+2)} R^n + (4(n+2)). \quad \forall R < \rho M_4.$$  

Similarly, we take $A = Q_R \setminus A_0$ in (2.21). Using (2.19) and also the fact that $|A| \leq \rho R^{n+2}$ by assumption (B), we have

$$\int_{Q_R \setminus A_0} |Du|^q \, dx \, dt \leq (CR^n)^{n/(n+2)} (\rho M_4)^{2(n+2)} R^2 = C(\rho M_4)^{2n+2} R^n + (4(n+2)).$$
Since \(2(n+4)/(n+2) - n - 2/q = -2\). The above estimates give
\[
\left( \iint_Q |Du|^q \, dx \, dt \right)^{2/q} \leq \omega(\rho) R^{-2}, \quad \forall R < \rho M_4. \tag{2.22}
\]
with \(\omega(\rho) \to 0\) as \(\rho \to 0\). Finally, using (2.19), (2.22) in (2.20) to estimate the integrals on the right hand side and multiplying through by \(R^2\) we obtain
\[
\frac{1}{R^2} \iint_{Q_R} |Du|^2 \, dx \, dt \leq C(\varepsilon) \omega(\rho) + C\varepsilon, \quad \forall R < \rho M_4.
\]
Obviously, we can make the right hand side arbitrarily small by choosing \(\varepsilon\) and then \(\rho = \rho(\varepsilon)\), sufficiently small. We have shown (2.13), given (B). Our proof is now complete.

Proof of Theorem 8. We first assume that \(\lambda_0 = 0\), that is \(\alpha'\) are identical. In this case, we do not have to consider the two alternatives as in the previous proof. Since \(H')_{u, u} \equiv 0\), the functions \(w_i\) satisfy inequalities similar to (2.10), (2.16). To see this, we need only to estimate the terms on the right hand side of (2.9) as follows
\[
|\partial R^i_{u, u}^{t, t}| = \left| \partial^i_{u, u} \frac{H_{u, u}^{t, t}}{N^2} u_{s, s} \right| = \frac{r^i_{u, u} H_{u}}{N} w_{s, s} \leq \varepsilon |Dw|^2 + C(\varepsilon) \left( \frac{r^i_{u, u} H_{u}}{N^2} \right)^2, \quad \forall \varepsilon > 0.
\]
If \(f_i\) depends on \(Du\), as in (H-ii), we assume that \(|H_u| \geq K_0\) so that
\[
\frac{\partial f^i_{u, u}}{N} \leq \frac{\theta |Du|^2}{N} + \frac{\theta K(u)}{N} \leq \frac{\varepsilon_0 N}{K_0^2} |Dw|^2 + \frac{\theta K(u)}{N}.
\]
Since \(N(u) \leq \theta \sup_{\partial B} H + R^*\) and \(\varepsilon_0 \sup_{\partial B} H < K_0^2\), as we assumed in (H-ii), we see that \(\sup_{\partial B}(\varepsilon_0 N/\theta) < \lambda\) if \(R\) is small. Therefore, the resulting integrals of \(|Dw|^2\) on the right of (2.9) can be absorbed into that of \(|Du|^2\) on the left giving inequalities similar to (2.15), (2.16).

From this point, we need only to repeat the same argument, which is now much simpler, after (2.15) of the proof for the alternative (A) of Theorem 7 to conclude our theorem.

Next, we then use a perturbation argument to deal with the case when \(\alpha'\) are slightly different. Since the idea is similar to that of Theorem 4, we will only sketch the main points here.
Let \( (x_0, t_0) \in \Omega_T \). We consider the solution \( V = (V^1, \ldots, V^m) \) to
\[
\begin{cases}
\frac{\partial V}{\partial t} = \text{div}(a(x, t, u) \, DV^i + c_i(u) \, DH(V)) + f^i(x, t, u), & \text{in } Q_R, \\
V(x, t) = 0, & \text{on } \partial B_R \times I_R, \\
V(x, t_0 - R^2) = 0.
\end{cases}
\] (2.23)

Let \( H(u) = \sum b_i u_i \). We have \( H(u) = H(V) = b_i \) so that (2.23) satisfies (H). As we already shown, \( V(x, t) \) is Hölder continuous. Adapting the proof of Corollary 3 as in the proof of Theorem 7 we see that \( |DV|^2 \in L^1_{loc} \) for some \( \mu_0 > n \). In addition, we also have
\[
\|DV\|_{L^2_{Q_\delta}} \leq C \left( \left( \frac{\rho}{R} \right)^{\mu_0} \|DV\|_{L^2_{Q_\delta}} + R^{\mu_0} \right), \quad 0 < \rho < R. \tag{2.24}
\]

Let \( W = u - V \) which satisfies (using the fact that \( H \) is linear)
\[
\frac{\partial W}{\partial t} = \text{div}(aDW^i + (a_i - a) \, Du^i + c_i \, DH(W)). \tag{2.25}
\]
Multiplying (2.25) by \( b_i(H(W) - H(W)_R) \eta^2 \), where \( \eta \) is the usual cutoff function for \( Q_\delta \) and \( Q_{2R} \), and \( H(W)_R \) is the mean value of \( H(W) \) over \( Q_\delta \), and integrating over \( Q_{2R} \), we deduce easily
\[
\int_{Q_\delta} d_x \|DH(W)\|^2 \eta^2 \, dx \, dt \\
\leq \int_{Q_\delta} b_i \|a' - a\| \|Du\| \|DH(W)\| \eta^2 \, dx \, dt \\
+ \int_{Q_\delta} b_i \|a' - a\| \|Du\| \|H(W) - H(W)_R\| \eta \|D\eta\| \, dx \, dt \\
\leq C(\varepsilon) \int_{Q_\delta} \|a' - a\|^2 \|Du\|^2 \eta^2 \, dx \, dt \\
+ \varepsilon \int_{Q_\delta} \|DH(W)\|^2 \eta^2 + (H(W) - H(W)_R)^2 \|D\eta\|^2 \, dx \, dt, \tag{2.26}
\]
where we have used the Young inequality as usual. Since \( |D\eta| \leq 2/R \), by means of Poincaré inequality, the last integral can be majorized by a multiple
of the integral of $|DH(W)|^2 q^2$. Thus, by choosing $\epsilon$ sufficiently small this integral can be absorbed into the left (using the ellipticity of (H.1)). So,

$$
\int_{Q_\alpha} |DH(W)|^2 \, dx \, dt \leq C \int_{Q_\alpha} |a' - a|^2 \, |Du|^2 \, dx \, dt
$$

$$
\leq C\delta_0^2 \int_{Q_\alpha} |Du|^2 \, dx \, dt.
$$

(2.27)

We now write the equation of $u'$ as

$$
\frac{\partial u'}{\partial t} = \text{div}(a'Du') + f' + \text{div}(F')
$$

with $F' = c'DH(V) + c'DH(W)$. Thanks to (2.24) and (2.27), if $\delta_0$ is sufficiently small, one can see that Remark 1.9 (in particular, (1.30)) in the previous section applies here to conclude that $u$ is Hölder continuous.

We conclude our paper by some simple applications of the above theorems.

**Example 2.2.** Let us consider the following system

$$
\frac{\partial u}{\partial t} = \text{div}(a_1(x, t, u, v) \, Du) + b_{12} \, Au + f(x, t, u, v),
$$

$$
\frac{\partial v}{\partial t} = \text{div}(a_2(x, t, u, v) \, Dv) + b_{21} \, Au + g(x, t, u, v),
$$

(2.28)

with $b_{12}, b_{21} > 0$. We assume that there are positive constants $a_{11}, a_{22}$ such that

$$
a_1(x, t, u, v) \geq a_{11}, \quad a_2(x, t, u, v) \geq a_{22},
\forall (x, t, u, v) \in \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}.
$$

(2.28)

We assume that $a_{11} a_{22} > b_{12} b_{21}$. That is, the self diffusions dominate the cross diffusions.

We claim that there exist positive constants $k, l, x, \beta,$ and $A$ such that

$$
k \alpha < a_{11}, \quad k \beta = b_{12}, \quad l \beta < a_{22}, \quad lx = b_{21}.
$$

and

$$
A = a_{11} - k \alpha = a_{22} - l \beta > 0.
$$
To see this, one need to pick any $\gamma \in (b_{21} / a_{22}, a_{11} / b_{12})$, any $\beta > 0$, and then define $\alpha = \gamma \beta$, $k = b_{12} / \beta$ and $l = b_{21} / (\gamma \beta)$. It is easy to check that $k, l, \alpha, \beta$ verify (2.30). For (2.31), we need to choose $\gamma$ such that $a_{11} - b_{12} \gamma = a_{22} - b_{21} / \gamma$. This equation is equivalent to $b_{12} \gamma^2 - (a_{11} - a_{22}) \gamma - b_{21} = 0$ which has two real roots $\gamma_1 < \gamma_2$. Using the fact that $a_{11} a_{22} > b_{12} b_{21}$ one can easily check that $\gamma_2 \in (b_{21} / a_{22}, a_{11} / b_{12})$. So, we take $\gamma = \gamma_2$ to fulfill (2.31).

With such a choice of parameters, we set $H(u, v) = \alpha u + \beta v$ and rewrite (2.28) in the form

$$
\frac{\partial u}{\partial t} = \text{div}(A_1(x, t, u, v) \, Du + k DH) + f(x, t, u, v),
$$
$$
\frac{\partial v}{\partial t} = \text{div}(A_2(x, t, u, v) \, Dv + l DH) + g(x, t, u, v),
$$

where $A_1(x, t, u, v) = a_{11}(x, t, u, v) - k \alpha$, $A_2(x, t, u, v) = a_{22}(x, t, u, v) - l \beta$. By the choice of $k, l, \alpha, \beta$, we can see that the ellipticity conditions in (H.2) are fulfilled. If $a_i(x, t, u, v)$ are slightly different from the constants $a_i$ then the functions $A_i$ are also slightly different from $A_i$. In this case, the above system satisfies the structure condition (H). We can apply Theorem 8 to assert that bounded solutions to (2.28) are Hölder continuous.

**Example 2.3.** In the same spirit, one can also apply Theorem 7 to systems of the form

$$
\frac{\partial u}{\partial t} = \text{div}(a_1(x, t, u, v) \, Du) + \lambda (a_{11} u^2 + a_{12} uv + a_{22} v^2) + f(x, t, u, v),
$$
$$
\frac{\partial v}{\partial t} = \text{div}(a_2(x, t, u, v) \, Dv) + \lambda (b_{11} u^2 + b_{12} uv + b_{22} v^2) + g(x, t, u, v),
$$

with $a_{22}, b_{22} > 0$. A possible candidate for $H(u, v)$ could be of a positive definite quadratic form $\alpha u^2 + 2 \beta uv + \gamma v^2$ with $\alpha, \beta, \gamma$ to be determined. However, the general conditions on $a_{ij}, b_{i, j}, i, j = 1, 2$, to guarantee the existence of such a function $H$ and to verify (H.1), (H.2) for (2.32) will not be expressed nicely as those for (2.28) in the previous example.

Finally, other examples of systems to which Theorem 7 is applicable can be found in the works of Küffner [19, 20] where $L^p$ norms of the solutions were also derived as a consequence of his results on invariant regions for the systems.
ACKNOWLEDGMENTS

The author would like to express his thanks to professor M. Wiegner for pointing out the errors in the proof of Theorem 7 of the earlier version and his encouragement.

REFERENCES