THE NOTION OF o-TIGHTNESS AND C-EMBEDDED SUBSPACES OF PRODUCTS

M.G. TKAČENKO

Department of Mathematics and Mechanics, Moscow State University, 117234, Moscow, USSR

Received 18 February 1981
Revised 3 February 1982

We consider the question: when is a dense subset of a space $X$ C-embedded in $X$? We introduce the notion of o-tightness and prove that if each finite subproduct of a product $X = \prod_{a \in A} X_a$ has a countable o-tightness and $Y$ is a subset of $X$ such that $\pi_B(Y) = \prod_{a \in B} X_a$ for every countable $B \subseteq A$, then $Y$ is C-embedded in $X$. This result generalizes some of Noble and Ulmer's results on C-embedding.

AMS Subj. Class.: Primary 54C45, 54B10, 54A25; Secondary 54C05

The aim of this paper is to find conditions under which a dense subspace $Y$ of $X$ is C-embedded in $X$, i.e., each continuous real-valued function on $Y$ is extendable continuously over $X$. For this purpose we introduce the new cardinal function $o-tightness$ which we name after the open tightness or shortly o-tightness (see Definition 1). Proposition 1 determines the place the function $o-tightness$ occupies among some of the 'old' well-investigated cardinal functions. In Proposition 3 and Lemmas 1-4 we study the behaviour of o-tightness under the usual topological operations.

Lemmas 4 and 5 are used for the proof of our main result: if every finite subproduct of a product $X = \prod_{a \in A} X_a$ has countable o-tightness and $Y$ is a subspace of $X$ with the property $\pi_B(Y) = X_B = \prod_{a \in B} X_a$ for each countable $B \subseteq A$, then $Y$ is C-embedded in $X$ (Theorem 1). Theorem 1 implies some well-known results on C-embedded subspaces of products (see [1]; [2, Theorem 2.2(ii)]).

In what follows we will say that a subspace $Y \subseteq X$ is $G_\delta$-dense in $X$ if $Y$ intersects every non-empty $G_\delta$-set of $X$. We prove here the following result: each $G_\delta$-dense subspace of perfectly $\kappa$-normal space $X$ is C-embedded in $X$ (Theorem 2). Recall that a space $X$ is said to be perfectly $\kappa$-normal if the closure of each open subset of $X$ is a $G_\delta$-set in $X$.

This fact implies Corollary 3 which generalizes Scepin's result on continuous images of $\kappa$-metrizable compacts ([3, Theorem 11]).
Let \( X = \prod_{a \in A} X_a \) and \( B \subseteq A \). Then \( \pi_B \) stands for the natural projection of \( X \) onto \( X_B = \prod_{a \in B} X_a \). If \( O \) is an arbitrary standard open subset of \( X \), then we put \( k(O) = \{ \alpha \in A : \pi_\alpha(O) \neq X_\alpha \} \). Finally \( c(X) \), \( t(X) \) and \( \pi\chi(X) \) denote Souslin number, tightness and \( \pi \)-character of a space \( X \), respectively.

Let \( X \) be a space and \( \tau \) be an infinite cardinal.

**Definition 1.** We say that the \( \sigma \)-tightness of \( X \) does not exceed \( \tau \), or \( \sigma t(X) \leq \tau \), iff for every family \( \gamma \) of open subsets of \( X \) and for each point \( x \in X \) with \( x \in \text{cl}(\bigcup \gamma) \) there exists a subfamily \( \mu \subseteq \gamma \) such that \( |\mu| \leq \tau \) and \( x \in \text{cl}(\bigcup \mu) \).

Propositions 1–3 below list some of the properties of the function \( \sigma t \).

**Proposition 1.** \( \sigma t(X) \leq t(X) \) and \( \sigma t(X) \leq c(X) \) for every space \( X \).

**Proposition 2.** Let \( Y \) be a dense or an open subset of \( X \). Then \( \sigma t(Y) = \sigma t(X) \).

A continuous mapping \( f : X \to Y \) is said to be \( d \)-open if for every open subset \( O \subseteq X \) there exists an open subset \( V \subseteq Y \) such that \( f(O) \subseteq V \subseteq \text{cl}(f(O)) \) (see [4, section 2]).

**Proposition 3.** Let \( f \) be a continuous mapping of \( X \) onto \( Y \). Then \( \sigma t(Y) \leq \sigma t(X) \) in each of the following cases:

(a) \( f \) is hereditarily quotient;

(b) \( f \) is \( d \)-open.

Let us discuss the behaviour of the function \( \sigma t \) under the finite product operation.

**Lemma 1.** Let \( X \) be a space of countable \( \sigma \)-tightness and \( Y \) be a locally compact space of countable tightness. Then \( \sigma t(X \times Y) \leq \aleph_0 \).

**Proof.** Let \( \gamma \) be any family of open subsets of \( X \times Y \). Let us assume that there exists a point \( z \in \text{cl}(\bigcup \gamma) \) where \( \text{cl}_{\aleph_0}(\bigcup \gamma) = \bigcup(\text{cl}(\bigcup \mu))) \) where \( \mu \subseteq \gamma \) and \( \mu \) is countable.

Put \( A = \pi_X^{-1}\pi_X(z) \cap \text{cl}_{\aleph_0}(\bigcup \gamma) \). Then \( z \neq \text{cl}(A) \). Indeed, otherwise the facts that \( \pi_X^{-1}\pi_X(z) \) is homeomorphic to \( Y \) and \( t(Y) = \aleph_0 \) imply that there exists a countable subset \( B \subseteq A \) such that \( z \in \text{cl}(B) \). However for every point \( b \in B \) there exists a countable subfamily \( \mu_b \subseteq \gamma \) such that \( b \in \text{cl}(\bigcup \mu_b) \). Put \( \mu = \bigcup(\mu_b : b \in B) \). Then \( \mu \) is countable and \( B \subseteq \text{cl}(\bigcup \mu) \); hence \( z \in \text{cl}_{\aleph_0}(\bigcup \mu) \). It contradicts our assumption that \( z \notin \text{cl}_{\aleph_0}(\bigcup \gamma) \).

A space \( Y \) is locally compact; hence we can fix an open neighbourhood \( V \) of a point \( \pi_Y(z) \) such that \( \Phi = \text{cl}(V) \) is a compact and \( A \cap \pi_Y^{-1}(\Phi) = \emptyset \). Let us consider the space \( X \times \Phi \) with projections \( p_X \) and \( p_\Phi \) onto factors \( X \) and \( \Phi \), respectively.
Put $\tilde{\gamma} = \{ W \cap \pi^{-1}_Y \Phi : W \in \gamma \}$. Then $\tilde{\gamma}$ is a family of open subsets of the space $X \times \Phi$ and $z \in \text{cl}(\bigcup \tilde{\gamma}) \cap \text{cl}_{X_0}(\bigcup \tilde{\gamma})$, $\emptyset = p_X^{-1}(p_X(z) \cap \text{cl}_{X_0}(\bigcup \tilde{\gamma}))$. Since $\Phi$ is a compact, the projection $p_X$ is closed. Put $x = p_X(z)$. Then $x \in \text{cl}(p_X(\bigcup \tilde{\gamma}))$. Since the projection $p_X$ is open and $\text{ot}(X) \leq \aleph_0$, there exists a countable subfamily $\mu \subseteq \tilde{\gamma}$ such that $x \in \text{cl}(p_X(\bigcup \mu))$. However $p_X$ is a closed mapping; hence $p_X^{-1}(x) \cap \text{cl}(\bigcup \mu) \neq \emptyset$, i.e., $p_X^{-1}p_X(z) \cap \text{cl}_{X_0}(\bigcup \tilde{\gamma}) \neq \emptyset$. This contradiction completes the proof.

It should be noted that it is impossible to weaken the restrictions on a space $Y$ in Lemma 1. Indeed, let $V_\omega$ and $V_\epsilon$ be a countable ‘fan’ and a ‘c-fan’ respectively (see [5, p. 61]) where $\epsilon$ is the cardinality of the continuum.

Each of the spaces $V_\omega$, $V_\epsilon$ has a single non-isolated point. Let $O_\omega$ and $O_\epsilon$ be these points. The spaces $V_\omega$ and $V_\epsilon$ are Frechet, hence $\text{ot}(V_\omega) \leq \aleph_0$ and $\text{ot}(V_\epsilon) \leq \aleph_0$. However there exists a subset $M \subseteq (V_\omega \cup O_\omega) \times (V_\epsilon \cup O_\epsilon)$ such that $(O_\omega, O_\epsilon) \in \text{cl}(M) \cap \text{cl}_{X_0}(M)$; therefore $\text{ot}(V_\omega \times V_\epsilon) > \aleph_0$. In particular it implies that the condition of local compactness of a space $Y$ can not be omitted.

Now let $bV_\omega$ be any Hausdorff compactification of the space $V_\omega$. Then $\text{ot}(bV_\omega) = c(bV_\omega) = |V_\omega| = \aleph_0$. The existence of the subset $M \subseteq (V_\omega \cup O_\omega) \times (V_\epsilon \cup O_\epsilon)$ which has the property mentioned above, implies that the condition $\text{ot}(Y) \leq \aleph_0$ can not be replaced by $\text{ot}(Y) \leq \aleph_0$ in Lemma 1.

**Lemma 2.** Let $\tau$ be an infinite cardinal and $X = \prod_{\alpha \in A} X_\alpha$ where $|A| \leq \tau$ and $\text{ot}(X_B) \leq \tau$ for every finite subset $B \subseteq A$. Then $\text{ot}(X) \leq \tau$.

**Proof.** Let $\mathcal{L}$ be the family of all finite subsets of $A$. Then $|\mathcal{L}| \leq \tau$. Let $\gamma$ be a family of open subsets of $X$ and $x \in \text{cl}(\bigcup \gamma)$. Let $B$ be an arbitrary member of $\mathcal{L}$. Since $\text{ot}(X_B) \leq \tau$ there exists a subfamily $\gamma_B \subseteq \gamma$ such that $|\gamma_B| \leq \tau$ and $\pi_B(x) \in \text{cl}(\bigcup \{ \pi_B(V) : V \in \gamma_B \})$.

Put $\mu = \bigcup \{ \gamma_B : B \in \mathcal{L} \}$. Then $|\mu| \leq \tau$ and $x \in \text{cl}(\bigcup \mu)$. Thus the inequality $\text{ot}(X) \leq \tau$ is proved.

**Lemma 3.** Let $\tau$ be an infinite cardinal and $X = \prod_{\alpha \in A} X_\alpha$ where $\text{ot}(X_B) \leq \tau$ for every finite subset $B \subseteq A$. Then $\text{ot}(X) \leq \tau$.

**Proof.** Let $\gamma$ be the family of open subsets of $X$ and $x \in \text{cl}(\bigcup \gamma)$. Without loss of generality one can assume that $\gamma$ consists of standard open subsets of $X$. Let $\alpha^* \in A$ and $A_0 = \{ \alpha^* \}$. Since $\text{ot}(X_{\alpha^*}) \leq \tau$ there exists a subfamily $\gamma_0 \subseteq \gamma$ such that $|\gamma_0| \leq \tau$ and $\pi_{\alpha^*}(x) \in \text{cl}(\bigcup \{ \pi_{\alpha^*}(V) : V \in \gamma_0 \})$. Let us assume that for some $i \in \omega$ we have defined a set $A_i \subseteq A$ and a family $\gamma_i \subseteq \gamma$ such that $|A_i| \leq \tau$ and $|\gamma_i| \leq \tau$. Put $A_{i+1} = A_i \cup \{ k(V) : V \in \gamma_i \}$. Obviously, $|A_{i+1}| \leq \tau$. Lemma 2 implies that there exists a subfamily $\gamma_{i+1} \subseteq \gamma$ such that $|\gamma_{i+1}| \leq \tau$ and $\pi_{A_{i+1}}(x) \in \text{cl}(\bigcup \{ \pi_{A_{i+1}}(V) : V \in \gamma_{i+1} \})$. 
Put

\[ K = \bigcup\{A_i : i \in \omega\} \quad \text{and} \quad \mu = \bigcup\{\gamma_i : i \in \omega\}. \]

Then from our construction it follows that \( \pi_K^{-1}\pi_K(V) = V \) for each \( V \in \mu \) and \( \pi_K(x) \in \text{cl}(\bigcup\{\pi_K(V) : V \in \mu\}) \). However the projection \( \pi_K \) is open; hence \( x \in \text{cl}(\bigcup\mu) \). Thus \( \text{ot}(X) \leq \tau \).

So \( \omega \)-tightness and Souslin number behave quite similarly under the infinite product operation. This shows us a way to construct non-trivial examples of spaces with a countable \( \omega \)-tightness. The following lemma gives another series of such spaces. First we recall Ščepin's notion of a \( \kappa \)-metrizable space [3].

A real-valued function \( f(x, C) \equiv 0 \) with \( x \in X \) and a regular closed subset \( C \subseteq X \) is a \( \kappa \)-metric on a space \( X \) provided that

1. \( \rho(x, C) = 0 \) iff \( x \in C \);
2. If \( C \subseteq C' \), then \( \rho(x, C) \geq \rho(x, C') \) for any \( x \in X \);
3. \( \rho(x, C) \) is a continuous function on \( X \) whenever \( C \) is fixed;
4. \( \rho(x, \text{cl}(\bigcup C)) = \inf_\alpha \rho(x, C_\alpha) \) for any increasing sequence \( \{C_\alpha\} \) and \( x \in X \).

Lemma 4. Let \( X \) be a \( \kappa \)-metrizable space. Then \( \text{ot}(X) \leq \aleph_0 \).

Proof. Let us fix a \( \kappa \)-metric \( \rho \) on a space \( X \). Let \( \gamma \) be a family of open subsets of \( X \) and \( x \in \text{cl}(\bigcup \gamma) \). Then for each \( \varepsilon > 0 \) there exists a finite subfamily \( \gamma_x \subseteq \gamma \) such that \( \rho(x, \text{cl}(\bigcup \gamma_x)) < \varepsilon \) (see [3, Lemma 4]). Put \( \mu = \bigcup\{\gamma_{1/n} : n \in \omega\} \). Then \( |\mu| \leq \aleph_0 \) and \( \rho(x, \text{cl}(\bigcup \mu)) = 0 \), i.e., \( x \in \text{cl}(\bigcup \mu) \). Consequently \( \text{ot}(X) \leq \aleph_0 \).

Now we consider the following general question. Let \( Y \) be a dense subspace of \( X \). What additional conditions on \( X \) and \( Y \) must be satisfied for a space \( Y \) to be \( C \)-embedded in \( X \)? If a space \( X \) is completely regular, then \( Y \) should be \( G_\delta \)-dense in \( X \), i.e., \( Y \) must intersect every nonempty \( G_\delta \)-set of \( X \). Therefore this condition will occur in the formulations of our results on \( C \)-embedding.

For the sequel we need the following well-known result [8].

Lemma 5. Let \( Y \) be a \( G_\delta \)-dense subspace of \( X \) and \( f \) be a continuous real-valued function on \( Y \). Then \( f \) is continuously extendable over \( X \) iff \( \text{cl}_X(f^{-1}F) \cap \text{cl}_X(f^{-1}\Phi) = \emptyset \) for every pair of disjoint closed subsets \( F, \Phi \) of the reals.

The following theorem is one of the main results of the paper.

Theorem 1. Let \( Y \) be a subspace of a product \( X = \prod_{a \in A} X_a \) where \( \text{ot}(X_B) \leq \aleph_0 \) for every finite subset \( B \subseteq A \). If \( \pi_B(Y) = X_B \) for every countable \( B \subseteq A \), then \( Y \) is \( C \)-embedded in \( X \).
Proof. Lemma 3 implies that $\alpha t(X) \leq N_0$. Since $\pi_B(Y) = X_B$ for each countable $B \subseteq A$ we conclude that $Y$ is $G_\delta$-dense in $X$. Let $f$ be a continuous function on $Y$. We must show that $\text{cl}_X(f^{-1}F) \cap \text{cl}_X(f^{-1}\Phi) = \emptyset$ for every pair of disjoint closed subsets $F, \Phi \subseteq \mathbb{R}$ (Lemma 5). Let us assume the contrary. Then we fix open neighbourhoods $U, V$ of $F$ and $\Phi$ respectively such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Then $\text{cl}_Y(f^{-1}V) \cap \text{cl}_Y(f^{-1}V) = \emptyset$. Let $U'$ and $V'$ be open subsets of $X$ such that $U' \cap Y = f^{-1}U$ and $V' \cap Y = f^{-1}V$. Then $\text{cl}_X(U') \cap \text{cl}_X(V') \neq \emptyset$. We choose a point $x \in \text{cl}_X(U') \cap \text{cl}_X(V')$. Let $\gamma_U$ and $\gamma_V$ be the families of standard open subsets of $X$ such that $U' = \bigcup \gamma_U$ and $V' = \bigcup \gamma_V$. Since $\alpha t(X) \leq N_0$, there exist countable subfamilies $\mu_U \subseteq \gamma_U$ and $\mu_V \subseteq \gamma_V$ such that $x \in \text{cl}_X(\bigcup \mu_U) \cap \text{cl}_X(\bigcup \mu_V)$. Put $B = \bigcup \{k(O) : O \in \mu_U \cup \mu_V\}$. Then $B$ is a countable subset of $A$ and $O = \pi_B^{-1}\pi_B(O)$ for every $O \in \mu_U \cup \mu_V$. Since the set $B$ is countable, the conditions of the theorem imply that there exists a point $y \in Y$ such that $\pi_B(y) = \pi_B(x)$. Since $\pi_B$ is an open mapping, we conclude that $y \in \text{cl}_X(\bigcup \mu_U) \cap \text{cl}_X(\bigcup \mu_V)$. However $\bigcup \mu_U \subseteq U'$ and $\bigcup \mu_V \subseteq V'$; hence $y \in \text{cl}_X(U') \cap \text{cl}_X(V')$. But $f^{-1}U$ is dense in $U'$ and $f^{-1}V$ is dense in $V'$, therefore $y \in \text{cl}_Y(f^{-1}U) \cap \text{cl}_Y(f^{-1}V)$. This contradiction completes the proof.

Corollary 1. Let $X = \prod_{\alpha \in A} X_\alpha$ where $X_\alpha$ is a space of point-countable type and $t(X_\alpha) \leq N_0$ for every $\alpha \in A$. Let $Y$ be a subset of $X$ such that $\pi_B(Y) = X_B$ for each countable subset $B \subseteq A$. Then $Y$ is $C$-embedded in $X$.

Proof. The class of spaces of point-countable type with countable tightness is closed under the finite product operation (see [7]). Hence the conclusion of Corollary 1 follows from the inequality $\alpha t(Z) \leq t(Z)$ which holds for every space $Z$.

Theorem 2. Let $X$ be a perfectly $\kappa$-normal space and $Y$ be a $G_\delta$-dense subset of $X$. Then $Y$ is $C$-embedded in $X$.

Proof. Let $f$ be a continuous function on $Y$ and $F, \Phi$ be disjoint closed subsets of $\mathbb{R}$. We assume that $\text{cl}_X(f^{-1}F) \cap \text{cl}_X(f^{-1}\Phi) \neq \emptyset$. Let $U$ and $V$ be open neighbourhoods of $F$ and $\Phi$ respectively such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Then the closures of the sets $f^{-1}U$ and $f^{-1}V$ in $Y$ are disjoint. Let $U'$ and $V'$ be open subsets of $X$ such that $U' \cap Y = f^{-1}U$ and $V' \cap Y = f^{-1}V$. Then $f^{-1}U$ is dense in $U'$ and $f^{-1}V$ is dense in $V'$; hence $Y \cap \text{cl}_X(U') = \text{cl}_Y(f^{-1}U)$ and $Y \cap \text{cl}_X(V') = \text{cl}_Y(f^{-1}V)$. Consequently

$$\emptyset = Y \cap \text{cl}_X(U') \cap \text{cl}_X(V').$$

Further the assumption that

$$\text{cl}_X(f^{-1}F) \cap \text{cl}_X(f^{-1}\Phi) \neq \emptyset$$

implies that

$$T = \text{cl}_X(U') \cap \text{cl}_X(V') \neq \emptyset.$$
However a space $X$ is perfectly $\kappa$-normal; hence $T$ is a $G_\delta$-set in $X$ and $T \cap Y \neq \emptyset$, which is a contradiction. Thus Lemma 5 implies that $Y$ is $C$-embedded in $X$.

Since each $\kappa$-metrizable space is perfectly $\kappa$-normal [3], we deduce the following.

**Corollary 2.** Let $Y$ be a $G_\delta$ dense subset of a $\kappa$-metrizable space $X$. Then $Y$ is $C$-embedded in $X$.

Our last result is an illustration of corollary 2.

**Corollary 3.** Let $S$ be a $G_\delta$-dense subspace of a $\kappa$-metrizable compact $X$ and $f$ be a continuous mapping of $S$ onto a completely regular space $Y$ with a dense subset of points of countable $\pi$-character. Then $Y$ is compact metric.

**Proof.** Corollary 2 implies that $S$ is $C$-embedded in $X$. Hence $X$ is the Čech–Stone compactification of its dense subset $S$. Thus there exists a continuous mapping $\bar{f}$ of $X$ to $\beta Y$, the Čech–Stone compactification of $Y$, which extends $f$. Obviously, $f(X) = \beta Y$. However $\pi_X(y, \beta Y) = \pi_Y(y, Y)$ for every point $y \in Y$, therefore a compact $\beta Y$ has a dense subset of points of countable $\pi$-character. So [3, Theorem 11] implies that $\beta Y$ is compact metric. Hence $Y = \beta Y$ which completes the proof.

**References**