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β-Expansions in algebraic function fields over finite fields

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Abstract

The present paper deals with an algebraic function field analogue of β-expansions of real numbers. It completely characterizes the sets with eventually periodic and finite expansions. These characterizations are unknown in the real case.

Keywords: β-Expansion; Pisot element; Greedy algorithm

1. Introduction

β-Expansions of real numbers were introduced by Rényi [16]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors (cf. for instance [1,2,7–9,15,19]). In this paper, we consider an analogue of this concept in algebraic function fields over finite fields. There are striking analogies between these digit systems and the classical β-expansions of real numbers.

In order to pursue this analogy, we recall the definition of real β-expansions and survey the problems corresponding to our results. For $\beta > 1$, the *β-transformation* $T = T_\beta$ is

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defined for $x \in [0,1)$ by $T(x) = \beta x - \lfloor \beta x \rfloor$. By iterating this map and considering its trajectory $x \xrightarrow{x_1} T(x) \xrightarrow{x_2} T^2(x) \xrightarrow{x_3} \cdots$ with $x_j = \lfloor \beta T^{j-1} x \rfloor$, we obtain an expansion

$$x = x_1\beta^{-1} + x_2\beta^{-2} + \cdots.$$

We will call the sequence $d_\beta(x) = .x_1x_2\cdots$ the *β-expansion* of x. We say that $d_\beta(x)$ is finite when $x_i = 0$ for all sufficiently large i. This is the case when there is an integer $i \geqslant 0$ such that $T^i x = 0$.

Consider the sequence $d_\beta(\beta - \lfloor \beta \rfloor) = .c_1c_2\cdots$. Then

$$1 = \lfloor \beta \rfloor \beta^{-1} + c_1\beta^{-2} - \cdots.$$

Let

$$c_i' = \begin{cases} \lfloor \beta \rfloor & \text{for } i = 1, \\ +c_{i-1} & \text{for } i > 1, \end{cases} \quad \text{and} \quad d_\beta'(1) = .c_1'c_2'\cdots.$$

The sequence $d_\beta'(1)$ is of special interest. It plays an important role in the study of D_β, the set of all β-expansions of numbers of $[0,1)$. If $d_\beta'(1)$ is eventually periodic, then β is called a *β-number*. If $d_\beta'(1)$ is finite, β is called *simple β-number*. In the case that β is a (simple) β-number, an easy argument implies that β is an algebraic integer.

It is easy to prove that an infinite sequence of nonnegative integers $(x_i)_{i \geqslant 1}$ is the β-expansion of $x \in [0,1)$ if and only if

$$x_i\beta^{-i} + x_{i+1}\beta^{-i-1} + \cdots < \beta^{-i+1} \tag{1.1}$$

for every $i \geqslant 1$.

Now let $x \geqslant 1$. Then there is an integer n such that $\beta^n \leqslant x < \beta^{n+1}$. We define in a similar manner $d_\beta(x) = x_{-n}\cdots x_{-1}x_0.x_1x_2\cdots$. Note that $d_\beta(1) \neq d_\beta'(1)$, since $d_\beta(1) = 1.$, while $d_\beta'(1) = .c_1'c_2'\cdots$.

Note that if $\beta = b \in \mathbb{Z}$, then $d_b(x)$ coincides with the ordinary b-ary expansion of x.

Let

$$\operatorname{Per}(\beta) = \{x \in [0,\infty)\colon\ d_\beta(x) \text{ is eventually periodic}\} \quad \text{and}$$
$$\operatorname{Fin}(\beta) = \{x \in [0,\infty)\colon\ d_\beta(x) \text{ is finite}\}.$$

Recall that a *Pisot number* is an algebraic integer $\beta > 1$ for which all algebraic conjugates γ with $\gamma \neq \beta$ satisfy $|\gamma| < 1$. A *Salem number* is an algebraic integer $\beta > 1$ for which all algebraic conjugates γ with $\gamma \neq \beta$ satisfy $|\gamma| \leqslant 1$ with at least one conjugate having $|\gamma| = 1$.

Theorem 1.1. (Bertrand and Schmidt [7,19].) *If β is a Pisot number, then*

$$\operatorname{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,\infty).$$

Since every rational integer $b > 1$ is a Pisot number, this result is a natural generalization of the well-known fact that $x \in \mathbb{Q}$ if and only if $d_b(x)$ is eventually periodic.

Schmidt [19] proved a partial converse of Theorem 1.1, namely if $\mathbb{Q} \cap [0, \infty) \subset \mathrm{Per}(\beta)$, then β is a Pisot or Salem number. It is conjectured, that also if β is a Salem number, then $\mathbb{Q} \cap [0, \infty) \subset \mathrm{Per}(\beta)$. In the setting of algebraic function fields, we will prove an analogue of Theorem 1.1 for Pisot and Salem elements.

A similar situation occurs in the case of finite expansions. We say that a number $\beta > 1$ has the *finiteness property* or property (F), if

$$\mathrm{Fin}(\beta) = \mathbb{Z}\left[\beta^{-1}\right] \cap [0, \infty). \tag{F}$$

This property was introduced by Frougny and Solomyak [9]. They proved that (F) implies that β is a Pisot number. Akiyama [3] proved that even the weaker condition $\mathbb{Z}_+ \subset \mathrm{Fin}(\beta)$ implies that β is a Pisot number. Several classes of Pisot numbers are known, such that (F) holds. On the other hand, there exist examples of Pisot numbers, such that (F) is not fulfilled (cf. [3,9,11]). For algebraic function fields, we will prove that no such exceptional cases exist.

This paper is organized as follows. In Section 2, we will define $\mathbb{F}((x^{-1}))$, the field of pole like formal Laurent series about ∞ as well as the analogues to Pisot and Salem numbers in $\mathbb{F}((x^{-1}))$. Furthermore, we will provide a simple algorithm to compute the coefficients of Pisot and Salem elements in $\mathbb{F}((x^{-1}))$. In Section 3, we will define the β-expansion algorithm for $\mathbb{F}((x^{-1}))$ and prove that there are no dependencies between consecutive digits. Section 4 is devoted to periodic expansions. We will prove an extended analogue of Theorem 1.1 together with its converse. In Section 5, we will give a complete characterization of all bases, which give rise to finite expansions. Such a characterization seems to be very hard to achieve in the real case.

While preparing this paper, the author was informed that similar results have been proved in a forthcoming paper by Hbaib and Mkaouar [10]. They considered the analogue of β-numbers in algebraic function fields. We will mention their results in Remarks 4.6 and 5.7 below.

2. Pisot and Salem elements in the field of formal Laurent series over a finite field

Let $\mathbb{F}$ be a finite field, $\mathbb{F}[x]$ the ring of polynomials, $\mathbb{F}(x)$ the field of rational functions. Let $\mathbb{F}((x^{-1}))$ be the field of formal Laurent series of the form

$$z = \sum_{k=-\infty}^{\ell} z_k x^k, \quad z_k \in \mathbb{F}, \tag{2.1}$$

where

$$\ell = \deg z := \begin{cases} \max\{k\colon\ z_k \neq 0\} & \text{for } z \neq 0, \\ -\infty & \text{for } z = 0. \end{cases}$$

Remark 2.1. $v(z) := -\deg z$ is an exponential valuation of $\mathbb{F}((x^{-1}))$.

Define the absolute value

$$|z| = \begin{cases} h^{\deg z} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases}$$

where $h > 1$ is an arbitrary but fixed real number. Note that the set of possible values of $|\cdot|$ is a discrete set. Then $\mathbb{F}((x^{-1}))$ is the completion of $\mathbb{F}(x)$ with respect to $|\cdot|$.

Thus $\mathbb{F}[x]$ is the analogue to $\mathbb{Z}$, $\mathbb{F}(x)$ is the analogue to $\mathbb{Q}$ and $\mathbb{F}((x^{-1}))$ is the analogue to $\mathbb{R}$. If $\beta \in \mathbb{F}((x^{-1}))$, then $\mathbb{F}(x, \beta)$ is the analogue to $\mathbb{Q}(\beta)$, $\mathbb{F}[x, \beta]$ is the analogue to $\mathbb{Z}[\beta]$ and $\mathbb{F}[x, \beta^{-1}]$ is the analogue to $\mathbb{Z}[\beta^{-1}]$.

Since $|\cdot|$ is not archimedean, $|\cdot|$ fulfills the strict triangle inequality

$$|z + w| \leqslant \max\bigl(|z|, |w|\bigr) \quad \text{and} \tag{2.2}$$

$$|z + w| = \max\bigl(|z|, |w|\bigr) \quad \text{if } |z| \neq |w|. \tag{2.3}$$

For $a \in \mathbb{F}((x^{-1}))$ and $r \in \mathbb{R}_+$, set $D(a, r) = \{z \in \mathbb{F}((x^{-1}))\colon |z - a| < r\}$. Let z be as in (2.1). Define the integer (polynomial) part $\lfloor z \rfloor = \sum_{k=0}^{\ell} z_k x^k$ where the empty sum, as usual, is defined to be zero. Therefore $\lfloor z \rfloor \in \mathbb{F}[x]$ and $z - \lfloor z \rfloor \in D(0, 1)$ for all $z \in \mathbb{F}((x^{-1}))$. Note that $\lfloor z + w \rfloor = \lfloor z \rfloor + \lfloor w \rfloor$, $\lfloor -z \rfloor = -\lfloor z \rfloor$ and $|\lfloor z \rfloor| \leqslant |z|$.

For a good reference on function field arithmetic, we refer to Rosen [17]. The following definitions are from [6, Chapter 12].

Definition 2.2. An element $\beta = \beta_1 \in \mathbb{F}((x^{-1}))$ is called Pisot element if it is an algebraic integer over $\mathbb{F}[x]$, $|\beta| > 1$ and $|\beta_j| < 1$ for all Galois conjugates β_j.

Definition 2.3. An element $\beta = \beta_1 \in \mathbb{F}((x^{-1}))$ is called Salem element if it is an algebraic integer over $\mathbb{F}[x]$, $|\beta| > 1$, $|\beta_j| \leqslant 1$ for all Galois conjugates β_j, and there exists at least one Galois conjugate β_k such that $|\beta_k| = 1$.

In general, β and its Galois conjugates are hard to compute. Therefore, the conditions in Definitions 2.2 and 2.3 are difficult to verify. By considering the Newton polygon (cf. [13,14]) of the minimal polynomial, the following, more useful equivalences [6, Theorem 12.1.1] can be derived.

Theorem 2.4. *Let $\beta \in \mathbb{F}((x^{-1}))$ be an algebraic integer over $\mathbb{F}[x]$ and*

$$p(y) = y^n - a_1 y^{n-1} - \cdots - a_n, \quad a_i \in \mathbb{F}[x], \tag{2.4}$$

be its minimal polynomial. Then

(i) *β is a Pisot element if and only if $\deg a_1 > 0$ and $\deg a_1 > \max_{j=2}^{n} \deg a_j$;*
(ii) *β is a Salem element if and only if $\deg a_1 > 0$ and $\deg a_1 = \max_{j=2}^{n} \deg a_j$.*

In both cases β is a single zero with $|\beta| = |a_1|$.

In Theorem 2.6, a method to compute the coefficients of Pisot or Salem elements is given. For the proof, we will need the following auxiliary result.

Lemma 2.5. *If $z, w \in \mathbb{F}((x^{-1}))$ with $|z| = |w|$, then $|z^n - w^n| \leqslant |z-w||z|^{n-1}$ for all $n \in \mathbb{Z}$.*

Proof. The statement is trivial for $n = 0$. For $n > 0$, we have

$$\begin{aligned}\left|z^n - w^n\right| &= |z-w|\left|z^{n-1} + z^{n-2}w + \cdots + w^{n-1}\right| \\ &\leqslant |z-w| \max_{j=0}^{n-1}\left|z^{n-1-j}w^j\right| = |z-w||z|^{n-1}\end{aligned}$$

and

$$\begin{aligned}\left|z^{-n} - w^{-n}\right| &= \left|w^n - z^n\right|\left|z^{-n}w^{-n}\right| \leqslant |w-z||w|^{n-1}\left|z^{-n}w^{-n}\right| \\ &= |z-w||z|^{-n-1}. \qquad \square\end{aligned}$$

Theorem 2.6. *Let β be a Pisot or Salem element and* (2.4) *be its minimal polynomial. Then $\deg \beta = \deg a_1$ and the recurrence*

$$y_1 = a_1, \qquad y_{k+1} = a_1 + \frac{a_2}{y_k} + \cdots + \frac{a_n}{y_k^{n-1}} \quad \textit{for } k \geqslant 1$$

fulfills

$$\lim_{k\to\infty} y_k = \beta.$$

Proof. First we prove by induction that $|y_k| = |a_1|$ for all $k \geqslant 1$. For $k = 1$ this assertion is trivial.

Let $|y_k| = |a_1|$ or equivalently, $\deg y_k = \deg a_1$. For $j = 2, \ldots, n$, it follows from $\deg a_1 > 0$ and $\deg a_1 \geqslant \deg a_j$ that

$$\deg a_j / y_k^{j-1} = \deg a_j - (j-1)\deg y_k \leqslant \deg a_1 - 1 \deg a_1 = 0 < \deg a_1.$$

Thus $\deg y_{k+1} = \deg a_1$ or equivalently $|y_{k+1}| = |a_1|$. From Lemma 2.5, we get

$$\begin{aligned}|y_{k+1} - y_k| &= \left|a_2\left(\frac{1}{y_k} - \frac{1}{y_{k-1}}\right) + \cdots + a_n\left(\frac{1}{y_k^{n-1}} - \frac{1}{y_{k-1}^{n-1}}\right)\right| \\ &\leqslant \max\left(\frac{|a_2|}{|a_1|^2}, \ldots, \frac{|a_n|}{|a_1|^n}\right)|y_k - y_{k-1}|.\end{aligned}$$

Since $|a_1| > 1$ and $|a_1| \geqslant |a_j|$, the left factor is constant and less than 1. Thus, the sequence converges to a limit α with $|\alpha| = |a_1|$. Since β is the only zero of (2.4) with $|\beta| = |\alpha|$ and $\alpha = a_1 + a_2/\alpha + \cdots + a_n/\alpha^{n-1}$, we have $\alpha = \beta$. $\square$

Example 2.7. Let $p(y) = y^2 + xy + x$ over $\mathbb{Z}_2$. Since $\deg a_1 = \deg a_2$, its zero must be a Salem element. Then the above sequence converges to

$$\beta = \lim_{k\to\infty} y_k = x + \sum_{k=0}^{\infty} \frac{1}{x^{2^k-1}}.$$

Since $p(y)$ is a quadratic polynomial, we have

$$\beta_2 = a_1 - \beta = \sum_{k=0}^{\infty} \frac{1}{x^{2^k-1}}.$$

3. β-Expansions in $\mathbb{F}((x^{-1}))$

Let $\beta, z \in \mathbb{F}((x^{-1}))$ with $|\beta| > 1$, $|z| < 1$. A *representation in base* β (*or* β-*representation*) of z is an infinite sequence $(d_i)_{i\geqslant 1}$, $d_i \in \mathbb{F}[x]$, such that

$$z = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}. \tag{3.1}$$

A particular β-representation—called the β-*expansion*—can be computed by a greedy algorithm.

This algorithm works as follows. Set $r^{(0)} = z$ and let $d_j = \lfloor \beta r^{(j-1)} \rfloor$, $r^{(j)} = \beta r^{(j-1)} - d_j$ for $j \geqslant 1$. This procedure yields a representation of z of the form (3.1). Note that $|d_j| < |\beta|$ and $|r^{(j)}| < 1$ for all j. The β-expansion of z will be denoted by

$$d_\beta(z) = .d_1 d_2 \cdots.$$

Note that

$$r^{(k)} = \beta^k \left(z - \sum_{\ell=1}^{k} d_\ell \beta^{-\ell} \right).$$

An equivalent definition of the β-expansion is obtained by using the β-transformation $T = T_\beta$ on $D(0,1)$ which is given by the mapping

$$T : D(0,1) \to D(0,1), \quad z \mapsto \beta z - \lfloor \beta z \rfloor.$$

Then $d_\beta(z) = (d_i)_{i=1}^{\infty}$ if and only if $d_i = \lfloor \beta T^{i-1}(z) \rfloor$. Note that $d_\beta(z)$ is finite if and only if there is a $k \geqslant 0$ such that $T^k(z) = 0$.

Analogously to the real case, let $d_\beta(\beta - \lfloor \beta \rfloor) = .c_1 c_2 \cdots$,

$$c_i' = \begin{cases} \lfloor \beta \rfloor & \text{for } i = 1, \\ +c_{i-1} & \text{for } i > 1, \end{cases} \quad \text{and} \quad d_\beta'(1) = .c_1' c_2' \cdots.$$

If $d'_\beta(1)$ is eventually periodic, then β is called a *β-element*. If $d'_\beta(1)$ is finite, β is called *simple β-element*.

Now let $z \in \mathbb{F}((x^{-1}))$ be an element with $|z| \geqslant 1$. Then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leqslant |z| < |\beta|^{k+1}$. Hence $|z/\beta^{k+1}| < 1$ and we can represent z by shifting $d_\beta(z/\beta^{k+1})$ by k digits to the left. Therefore, if $d_\beta(z) = .d_1 d_2 d_3 \ldots$, then $d_\beta(\beta z) := d_1.d_2 d_3 \ldots$. In the sequel, we will use the following notations:

$$\mathrm{Per}(\beta) = \bigl\{z \in \mathbb{F}\bigl(\bigl(x^{-1}\bigr)\bigr)\colon\ d_\beta(z) \text{ is eventually periodic}\bigr\} \quad \text{and}$$
$$\mathrm{Fin}(\beta) = \bigl\{z \in \mathbb{F}\bigl(\bigl(x^{-1}\bigr)\bigr)\colon\ d_\beta(z) \text{ is finite}\bigr\}.$$

The following theorem provides an analogue to the condition mentioned in (1.1).

Theorem 3.1. *An infinite sequence $(d_j)_{j\geqslant 1}$ is the β-expansion of $z \in D(0,1)$ if and only if $|d_j| < |\beta|$ for $j \geqslant 1$. Therefore, consecutive digits of z are independent.*

Proof. The proof runs nearly along the same lines as in the real case. □

Remark 3.2. In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if $z, w \in \mathbb{F}((x^{-1}))$, we have $d_\beta(z + w) = d_\beta(z) + d_\beta(w)$ digitwise.

Example 3.3. Take $p(y)$ from Example 2.7. Since $\beta^2 + x\beta + x = 0$ and $1 = -1$ in $\mathbb{Z}_2$, we obtain

$$x = \frac{\beta^2}{\beta + 1} = \beta + 1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots.$$

Thus $d_\beta(x) = 11.11\ldots$. Therefore, $x \in \mathrm{Per}(\beta)$ but $x \notin \mathrm{Fin}(\beta)$.

4. Periodic expansions

The aim of the current section is to study the set of elements with eventually periodic expansions. In the case of Pisot elements, Theorem 4.1 provides an analogue to the classical result of Bertrand and Schmidt [7,19] mentioned in the introduction. Theorem 4.4 contains the converse of Theorem 4.1.

Theorem 4.1. *Let β be a Pisot or Salem element. Then*

$$\mathrm{Per}(\beta) = \mathbb{F}(x, \beta).$$

Proof. The proof for $\mathrm{Per}(\beta) \subset \mathbb{F}(x,\beta)$ is trivial. To prove $\mathbb{F}(x,\beta) \subset \mathrm{Per}(\beta)$, we mainly follow [12, Proposition 7.2.19].

It is sufficient to prove the result for $\mathbb{F}(x,\beta)\cap D(0,1)$. Let $z\in\mathbb{F}(x,\beta)\cap D(0,1)$. Then

$$z = q^{-1}\sum_{i=0}^{n-1} p_i\beta^i$$

with $q, p_i \in \mathbb{F}[x]$ and $\deg q$ as small as possible. Let $(d_k)_{k\geqslant 1}$ be the β-expansion of z, let β_j, $j=2,\dots,n$, be the Galois conjugates of $\beta=\beta_1$ and

$$r_j^{(k)} = \beta_j^k\left(q^{-1}\sum_{i=0}^{n-1} p_i\beta_j^i - \sum_{\ell=1}^{k} d_\ell\beta_j^{-\ell}\right) \tag{4.1}$$

for $j=1,\dots,n$. Therefore, $r_1^{(k)}=r^{(k)}$ and $r_j^{(k)}$, $j=2,\dots,n$, are the conjugates of $r^{(k)}$.

We have $|r_1^{(k)}|=|r^{(k)}|<1$ for all k. For $j=2,\dots,n$, from $|\beta_j|\leqslant 1$ and $|d_\ell|<|\beta|$ follows

$$\left|r_j^{(k)}\right| \leqslant \max\Bigl(|\beta_j|^k\left|r_j^{(0)}\right|, \max_{\ell=1}^{k}\bigl(\left|d_\ell\beta_j^{k-\ell}\right|\bigr)\Bigr) \leqslant \max\bigl(\left|r_j^{(0)}\right|, |\beta|\bigr) < \infty. \tag{4.2}$$

In (4.2), the strict triangle inequality (2.2) has been applied (this is the crucial step which does not work for real β-expansions by Salem numbers). Therefore, $|r_j^{(k)}|$ is bounded for all k and j. We need a technical result.

Lemma 4.2. *Let $R^{(k)}=(r_1^{(k)},\dots,r_n^{(k)})$ and $B=(\beta_j^{-i})_{1\leqslant i,j\leqslant n}$. Then for every $k\geqslant 0$, there exists a unique n-tuple $W^{(k)}=(w_1^{(k)},\dots,w_n^{(k)})\in\mathbb{F}[x]^n$ such that $R^{(k)}=q^{-1}W^{(k)}B$.*

Proof. The proof runs along the same lines as the proof of [12, Lemma 7.2.20]. Thus, we will skip it. □

Now we proceed with the proof of Theorem 4.1. Let $H^{(k)}=qR^{(k)}$. Since $|r_j^{(k)}|$ is bounded for every j, the sequence $\|H^{(k)}\|$ is bounded. As the matrix B is invertible, for every $k\geqslant 1$,

$$\left\|H^{(k)}B^{-1}\right\| = \left\|W^{(k)}\right\| = \left\|\bigl(w_1^{(k)},\dots,w_n^{(k)}\bigr)\right\| = \max_{1\leqslant j\leqslant n}\left|w_j^{(k)}\right| < \infty.$$

Thus there exist p and m such that $W^{(m+p)}=W^{(m)}$, and therefore, $r^{(m+p)}=r^{(m)}$ which implies that the β-expansion of z is eventually periodic. □

In order to prove the converse of Theorem 4.1, we will need the following

Lemma 4.3. *Let $z,w\in\mathbb{F}((x^{-1}))$, $z\neq w$ and $|z|>1$. Then for every $k>0$ there exists some $n\geqslant 0$ with $|z^n-w^n|>k$.*

Proof. Let $k > 0$. We distinguish three cases.

(i) If $|z| \neq |w|$, then $|z^n| \neq |w^n|$ for $n \neq 0$. Thus

$$\left|z^n - w^n\right| = \max\left(\left|z^n\right|, \left|w^n\right|\right) \geqslant |z|^n.$$

Since $|z| > 1$, there exists some $n \geqslant 0$ with $|z|^n > k$.

(ii) Let $|z| = |w|$ and $|z - w| > 1$. If $p > 0$ is the characteristic of $\mathbb{F}$, then for $m \geqslant 0$ we have $z^{pm} - w^{pm} = (z - w)^{pm}$ and thus $|z^{pm} - w^{pm}| = |z - w|^{pm}$. Since $|z - w| > 1$, there exists some $m \geqslant 0$ with $|z - w|^{pm} > k$.

(iii) Let $|z| = |w|$ and $|z - w| \leqslant 1$. Then

$$\begin{aligned}\left|z^{pm+1} - w^{pm+1}\right| &= \left|z^{pm}(z - w) + \left(z^{pm} - w^{pm}\right)w\right| \\ &\leqslant \max\left(|z|^{pm}|z - w|, |z - w|^{pm}|w|\right).\end{aligned}$$

Note that $|z - w|^{pm}|w|$ is bounded for $m \geqslant 0$. From (2.3) and $|z| > 1$ follows that there exists some $m_0 \geqslant 0$ such that

$$\left|z^{pm+1} - w^{pm+1}\right| = |z|^{pm}|z - w|$$

holds for all $m \geqslant m_0$. Now the statement follows easily. □

Theorem 4.4. *Let* $\mathbb{F}[x] \subset \mathrm{Per}(\beta)$. *Then* β *is a Pisot or Salem element.*

Proof. From $|\beta - \lfloor\beta\rfloor| < 1$ we obtain

$$\lfloor\beta\rfloor = \beta + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots, \quad \text{where } |d_i| < |\beta|.$$

Since $\lfloor\beta\rfloor \in \mathbb{F}[x]$, the expansion of $\lfloor\beta\rfloor$ must be eventually periodic. Therefore,

$$\lfloor\beta\rfloor = \beta + \frac{d_1}{\beta} + \cdots + \frac{d_k}{\beta^k} + \frac{d_{k+1}}{\beta^{k+1}} + \cdots + \frac{d_{k+p}}{\beta^{k+p}} + \frac{d_{k+1}}{\beta^{k+p+1}} + \cdots + \frac{d_{k+p}}{\beta^{k+2p}} + \cdots.$$

Thus

$$\beta^k\left(\lfloor\beta\rfloor - \beta - \frac{d_1}{\beta} - \cdots - \frac{d_k}{\beta^k}\right) = \beta^{k+p}\left(\lfloor\beta\rfloor - \beta - \frac{d_1}{\beta} - \cdots - \frac{d_{k+p}}{\beta^{k+p}}\right).$$

Therefore, β is an integral element of $\overline{\mathbb{F}(x)}$. Note that if the β-expansion of $\lfloor\beta\rfloor$ is finite, the right-hand side of this equation is zero.

Suppose that β has a Galois conjugate $\beta_j \neq \beta$ with $|\beta_j| > 1$. By Lemma 4.3, we can choose m with

$$\left|\beta^m - \beta_j^m\right| > \max\left(1, \left|\frac{\beta}{\beta_j}\right|\right). \tag{4.3}$$

Since $|\beta^m - \lfloor\beta^m\rfloor| < 1$, we have

$$\lfloor\beta^m\rfloor = \beta^m + \frac{e_1}{\beta} + \frac{e_2}{\beta^2} + \cdots, \quad \text{where } |e_i| < 1.$$

Since $\lfloor\beta^m\rfloor \in \mathbb{F}[x]$, the expansion must be eventually periodic. Consider the β-expansion of $z = \lfloor\beta^m\rfloor - \beta^m \in D(0,1)$. Let $r^{(k)}, r_j^{(k)}$ be as in the proof of Theorem 4.1. Equation (4.1) yields

$$\lfloor\beta^m\rfloor = \beta^m + \sum_{\ell=1}^{k} \frac{e_\ell}{\beta^\ell} + \frac{r^{(k)}}{\beta^k} \quad \text{for } k \geqslant 0.$$

Since $\lfloor\beta^m\rfloor \in \mathbb{F}[x]$ and β_j is a Galois conjugate of β, it must fulfill the equation

$$\lfloor\beta^m\rfloor = \beta_j^m + \sum_{\ell=1}^{k} \frac{e_\ell}{\beta_j^\ell} + \frac{r_j^{(k)}}{\beta_j^k} \quad \text{for } k \geqslant 0.$$

Since the expansion of z is eventually periodic, the $r^{(k)}$ take only finitely many values. Thus, the same is true for $r_j^{(k)}$. Hence, $\lim_{k\to\infty} r^{(k)}/\beta^k = 0$. If there exists β_j with $|\beta_j| > 1$, then $\lim_{k\to\infty} r_j^{(k)}/\beta_j^k = 0$. Therefore,

$$\beta^m - \beta_j^m + \sum_{\ell=1}^{\infty} e_\ell\left(\frac{1}{\beta^\ell} - \frac{1}{\beta_j^\ell}\right) = 0.$$

From

$$\left|\sum_{\ell=1}^{\infty} e_\ell\left(\frac{1}{\beta^\ell} - \frac{1}{\beta_j^\ell}\right)\right| \leqslant \max\left(\max_{\ell=1}^{\infty}\left|\frac{e_\ell}{\beta^\ell}\right|, \max_{\ell=1}^{\infty}\left|\frac{e_\ell}{\beta_j^\ell}\right|\right) < \max\left(1, \left|\frac{\beta}{\beta_j}\right|\right)$$

we get a contradiction to (4.3). □

We can combine Theorems 4.1 and 4.4 to obtain

Corollary 4.5. *An element $\beta \in \mathbb{F}((x^{-1}))$ is a Pisot or Salem element if and only if*

$$\mathbb{F}[x] \subset \mathrm{Per}(\beta).$$

Remark 4.6. Hbaib and Mkaouar [10] proofed a slightly stronger result: in Theorem 4.4, already the condition $d_\beta(\beta - \lfloor\beta\rfloor) \in \mathrm{Per}(\beta)$ implies that β is Pisot or Salem. Thus β is Pisot or Salem, if and only if it is a β-element.

5. Finite expansions

In the present section, we will study $\mathrm{Fin}(\beta)$, the set of finite expansions. Contrary to the case of real β-expansions, we can prove a complete characterization result in our setting. We will need the following

Lemma 5.1. *Let β be an arbitrary element of $\mathbb{F}((x^{-1}))$ with $\deg\beta > 0$, and let $z \in \mathbb{F}[x, \beta^{-1}]$ have purely periodic β-expansion with period n. Then $z \in \mathbb{F}[x, \beta]$.*

Proof. Assume $z \in \mathbb{F}[x, \beta^{-1}]$ is purely periodic with period n. Let $d_\beta(z) = .d_1 d_2 \ldots.$ Since $z \in \mathbb{F}[x, \beta^{-1}]$, there is an m such that $\beta^{mn} z \in \mathbb{F}[x, \beta]$. Therefore,

$$z = \beta^{mn} z - d_1 \beta^{mn-1} - \cdots - d_{mn} \in \mathbb{F}[x, \beta]. \qquad \square$$

Theorem 5.2. *Let $\beta \in \mathbb{F}((x^{-1}))$ be a Pisot element. Then*

$$\mathrm{Fin}(\beta) = \mathbb{F}\big[x, \beta^{-1}\big]. \tag{F}$$

Remark 5.3. Note that (F) is true if and only if for every $z \in \mathbb{F}[x, \beta^{-1}]$, there is $k \geqslant 0$ such that $T^k(z) = 0$.

Proof. Since it is trivial that $\mathrm{Fin}(\beta) \subset \mathbb{F}[x, \beta^{-1}]$, we need to prove only the opposite inclusion. Let

$$\beta^n - a_1 \beta^{n-1} - \cdots - a_n = 0 \tag{5.1}$$

with $\deg a_1 > \deg a_j$ for $j > 1$.

From Theorem 4.1 it follows that $\mathbb{F}[x, \beta^{-1}] \subset \mathbb{F}(x, \beta) \subset \mathrm{Per}(\beta)$. Thus $z \in \mathbb{F}[x, \beta^{-1}]$ has an eventually periodic expansion. Since addition is performed digitwise, z can be decomposed into $z = z_f + z_p$, where $d_\beta(z_f)$ is finite and $d_\beta(z_p)$ is purely periodic. Hence, by Lemma 5.1, $z_p \in \mathbb{F}[x, \beta]$. Since

$$r^{(k)} = \beta^k (z_f + z_p) - \sum_{\ell=1}^{k} d_\ell \beta_j^{k-\ell},$$

there is an integer k such that $r^{(k)} \in \mathbb{F}[x, \beta]$, and we can restrict our attention to $\mathbb{F}[x, \beta]$.

Let $B = \{1, \beta, \ldots, \beta^{n-1}\}$ and $V = \{v_1, \ldots, v_n\}$ where

$$v_i = \beta^{i-1} - a_1 \beta^{i-2} - \cdots - a_{i-1} \tag{5.2}$$

$$= \frac{a_i}{\beta} + \cdots + \frac{a_n}{\beta^{n-i+1}}. \tag{5.3}$$

Note that $v_1 = 1$. Then B and V are bases of $\mathbb{F}[x, \beta]$ considered as lattice over $\mathbb{F}[x]$. Using (5.2), the coordinates with respect to V can be computed from the coordinates with respect to B by a linear system of equations. Hence, for every $z \in \mathbb{F}[x, \beta]$, there are $z_1, \ldots, z_n \in \mathbb{F}[x]$ such that $z = z_1 v_1 + \cdots + z_n v_n$. Denote by $\varphi(z) = \mathbf{z}^\top = (z_1, \ldots, z_n)^\top$, the (transposed) vector of coordinates with respect to V. If $\mathbf{v} = (v_1, \ldots, v_n)^\top$, then $z = \mathbf{z} \cdot \mathbf{v}$.

In base V, multiplication by β is represented by the matrix

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Define the vectors $\mathbf{e} = (1, 0, \ldots, 0)^\top$ and $\mathbf{0} = (0, \ldots, 0)^\top$. We consider the greedy algorithm for $z \in \mathbb{F}[x, \beta]$ with respect to V. Since $T(z) = \beta z - \lfloor \beta z \rfloor$, the β-transformation with respect to V takes the form $\tau_n : \mathbb{F}[x]^n \to \mathbb{F}[x]^n$ with

$$\mathbf{z} \mapsto M\mathbf{z} - \lfloor M\mathbf{z} \cdot \mathbf{v} \rfloor \mathbf{e}. \tag{5.4}$$

Furthermore, $\varphi(T(z)) = \tau_n(\varphi(z))$, which shows that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{F}[x, \beta] & \xrightarrow{\;T\;} & \mathbb{F}[x, \beta] \\ \varphi \downarrow & & \downarrow \varphi \\ \mathbb{F}[x]^n & \xrightarrow[\tau_n]{} & \mathbb{F}[x]^n. \end{array}$$

Substituting M, $\mathbf{v}$ and $\mathbf{e}$ into (5.4), we can express τ_n as follows:

$$\tau_n : (z_1, z_2, \ldots, z_n)^\top \mapsto \left(-\lfloor z_1 v_2 + \cdots + z_{n-1} v_n \rfloor, z_1, \ldots, z_{n-1}\right)^\top. \tag{5.5}$$

Thus, $\mathbb{F}[x]^n$ together with τ_n provides an analogue to the so-called *shift radix system* defined in [4,5]. However, due to our notation, the indices here are in the reverse direction as in [4,5].

Since the image of τ_n in (5.5) does not depend of z_n, we can confine ourselves to the mapping $\tau_{n-1} : \mathbb{F}[x]^{n-1} \to \mathbb{F}[x]^{n-1}$ defined by

$$\tau_{n-1} : (z_1, z_2, \ldots, z_{n-1})^\top \mapsto \left(-\lfloor z_1 v_2 + \cdots + z_{n-1} v_n \rfloor, z_1, \ldots, z_{n-2}\right)^\top. \tag{5.6}$$

Thus, (F) is true if and only if for every $\mathbf{z}^{(0)} \in \mathbb{F}[x]^{n-1}$ with $\mathbf{z}^{(0)} \neq \mathbf{0}$, there is a $k \geqslant 0$, such that $\mathbf{z}^{(k)} = \tau_{n-1}^k(\mathbf{z}^{(0)}) = \mathbf{0}$. Therefore, if $\mathbf{z}^{(k)} = (z_1^{(k)}, \ldots, z_{n-1}^{(k)})$, then (F) is true if and only if $\max_{i=1}^{n-1} \deg z_i^{(k)} = -\infty$.

For $i = 2, \dots, n$ it follows that

$$\begin{aligned}\deg v_i &= \deg\left(\frac{a_i}{\beta} + \cdots + \frac{a_n}{\beta^{n-i+1}}\right)\\ &\leqslant \max_{j=i}^{n}\bigl(\deg a_j - (j-i+1)\deg \beta\bigr) < \deg a_1 - 1\ \deg a_1 = 0.\end{aligned}$$

Hence, for $(z_1^{(0)}, \dots, z_{n-1}^{(0)})^\top \neq \mathbf{0}$, we obtain

$$\deg\bigl(-\lfloor z_1^{(0)} v_2 + \cdots + z_{n-1}^{(0)} v_n\rfloor\bigr) \leqslant \max_{i=1}^{n-1} \deg\bigl(z_i^{(0)} v_{i+1}\bigr) < \max_{i=1}^{n-1} \deg z_i^{(0)}. \tag{5.7}$$

Note that the left-hand side of (5.7) is either nonnegative or $-\infty$. It follows from (5.6) and (5.7) that

$$\deg z_1^{(1)} \leqslant \max_{i=1}^{n-1}\bigl(\deg z_i^{(0)} - 1\bigr).$$

From $(z_1^{(1)}, \dots, z_{n-1}^{(1)})^\top = (z_1^{(1)}, z_1^{(0)}, \dots, z_{n-2}^{(0)})^\top$, we obtain

$$\begin{aligned}\deg z_1^{(2)} &\leqslant \max_{i=1}^{n-1}\bigl(\deg z_i^{(1)} - 1\bigr)\\ &= \max\Bigl(\max_{i=1}^{n-1} \deg z_i^{(0)} - 2, \max_{i=1}^{n-2} \deg z_i^{(0)} - 1\Bigr)\\ &= \max\bigl(\deg z_1^{(0)} - 1, \dots, \deg z_{n-2}^{(0)} - 1, \deg z_{n-1}^{(0)} - 2\bigr).\end{aligned}$$

Analogously follows from $(z_1^{(2)}, \dots, z_{n-1}^{(2)})^\top = (z_1^{(2)}, z_1^{(1)}, z_1^{(0)}, \dots, z_{n-3}^{(0)})^\top$, that

$$\begin{aligned}\deg z_1^{(3)} &\leqslant \max_{i=1}^{n-1}\bigl(\deg z_i^{(2)} - 1\bigr)\\ &= \max\bigl(\deg z_1^{(0)} - 1, \dots, \deg z_{n-3}^{(0)} - 1, \deg z_{n-2}^{(0)} - 2, \deg z_{n-1}^{(0)} - 2\bigr).\end{aligned}$$

After $n-1$ such steps we have

$$\deg z_1^{(n-1)} \leqslant \max\bigl(\deg z_1^{(0)} - 1, \deg z_2^{(0)} - 2, \dots, \deg z_{n-1}^{(0)} - 2\bigr).$$

Therefore, since $(z_1^{(n-1)}, \dots, z_{n-1}^{(n-1)})^\top = (z_1^{(n-1)}, z_1^{(n-2)}, \dots, z_1^{(1)})^\top$, it follows that

$$\max_{i=1}^{n-1} \deg z_i^{(n-1)} = \max_{i=1}^{n-1} \deg z_1^{(n-i)} \leqslant \max_{i=1}^{n-1}\bigl(\deg z_i^{(0)} - 1\bigr).$$

Going on this way, we will find a number k such that $\max_{i=1}^{n-1} \deg z_i^{(k)} = -\infty$. □

The following theorem forms the converse of Theorem 5.2.

Theorem 5.4. *If* $\mathbb{F}[x, \beta^{-1}] \subset \mathrm{Fin}(\beta)$*, then* β *is a Pisot element.*

Proof. Applying arguments from complex analysis, the proof of the corresponding result for real β-expansions [9, Lemma 1(b)] is rather short. Unfortunately, this technique does not work in our context. Therefore, we will adapt the idea of [18, Lemma 2.4].

Suppose that

$$\max_{j=2}^{n} \deg a_j \geqslant \deg a_1.$$

We will construct an element $z \in \mathbb{F}[x, \beta]$ which does not have a finite representation. Define

$$i_0 := \max\Big\{i \in \{1, \dots, n\}\colon \deg a_i = \max_{j=1}^{n} \deg a_j\Big\}$$

and

$$j_0\big((z_1, \dots, z_n)^\top\big) := \begin{cases} \min\{i \in \{1, \dots, n\}\colon \deg z_i = \max_{j=1}^{n} \deg z_j\} & \text{if } \max_{j=1}^{n} \deg z_j > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Select $\mathbf{z}^{(0)} := (z_1^{(0)}, \dots, z_n^{(0)})^\top = (x, 0, \dots, 0)^\top$. Then

$$j_0\big(\mathbf{z}^{(0)}\big) + 1 \leqslant i_0. \tag{5.8}$$

Thus $1 < i_0 \leqslant n$ and $1 \leqslant j_0(\mathbf{z}^{(0)}) < n$. Let $\mathbf{z}^{(k)} = (z_1^{(k)}, \dots, z_n^{(k)})^\top$. We will show that $\mathbf{z}^{(0)}$ has an infinite representation by proving that

$$j_0\big(\mathbf{z}^{(k)}\big) + 1 \leqslant i_0 \quad \text{for all } k \geqslant 0. \tag{5.9}$$

This implies that $\max_{j=1}^{n} \deg z_j^{(k)} > 0$ and hence $\mathbf{z}^{(k)} \neq 0$.

We will prove (5.9) by induction. Since (5.9) holds for $k = 0$ by (5.8), we can proceed to the induction step. Suppose that (5.9) holds for a certain k and note that

$$\mathbf{z}^{(k+1)} = \big(z_1^{(k+1)}, \dots, z_n^{(k+1)}\big)^\top = \big(z_1^{(k+1)}, z_1^{(k)}, \dots, z_{n-1}^{(k)}\big)^\top. \tag{5.10}$$

We distinguish two cases.

Case 1. $j_0 := j_0(\mathbf{z}^{(k)}) < i_0 - 1$. By (5.10) and because $j_0 < n$, we have

$$\begin{aligned} \max_{j=1}^{n} \deg z_j^{(k+1)} &= \max\big(\deg z_{j_0}^{(k)}, \deg z_1^{(k+1)}\big) \\ &= \max\big(\deg z_{j_0+1}^{(k+1)}, \deg z_1^{(k+1)}\big) \quad \text{(by the definition of } j_0) \\ &> 0. \end{aligned}$$

Thus $j_0(\mathbf{z}^{(k+1)}) = 1$ or $j_0(\mathbf{z}^{(k+1)}) = j_0 + 1$. Both of these inequalities imply that

$$j_0\big(\mathbf{z}^{(k+1)}\big) \leqslant i_0 - 1$$

and we are done.

Case 2. $j_0 := j_0(\mathbf{z}^{(k)}) = i_0 - 1$. Let v_i be as in (5.2). The definitions of i_0 and j_0 imply that

$$\deg v_{i_0} \geqslant 0, \qquad \deg v_i < \deg v_{i_0} \quad \text{for } i > i_0,$$
$$\deg z_{i_0-1}^{(k)} > 0, \qquad \deg z_j^{(k)} < \deg z_{i_0-1}^{(k)} \quad \text{for } j < i_0 - 1.$$

Thus

$$\deg\big(z_{i_0-1}^{(k)} v_{i_0}\big) > \deg\big(z_{i-1}^{(k)} v_i\big) \quad \text{for } i \neq i_0.$$

This implies that no cancellations occur in the highest power of x in the sum

$$z_1^{(k)} v_2 + \cdots + z_{n-1}^{(k)} v_n.$$

Hence,

$$\deg\big(z_1^{(k)} v_2 + \cdots + z_{n-1}^{(k)} v_n\big) = \deg z_{i_0-1}^{(k)} + \deg v_{i_0} > 0,$$

and therefore

$$\begin{aligned}&\deg\big(-\big\lfloor z_1^{(k)} v_2 + \cdots + z_{n-1}^{(k)} v_n\big\rfloor\big)\\ &\quad = \deg\big(z_1^{(k)} v_2 + \cdots + z_{n-1}^{(k)} v_n\big) = \deg\big(z_{i_0-1}^{(k)} v_{i_0}\big) \geqslant \deg z_{i_0-1}^{(k)} = \max_{j=1}^{n-1} \deg z_j^{(k)}.\end{aligned}$$

This implies that

$$\deg z_1^{(k+1)} \geqslant \max_{j=2}^{n} \deg z_j^{(k+1)}.$$

Thus,

$$j_0\big(\mathbf{z}^{(k+1)}\big) = 1 \leqslant i_0 - 1$$

and we are done also in this case. □

It turns out that condition (F) is equivalent to a seemingly weaker condition.

Theorem 5.5. $\mathbb{F}[x, \beta^{-1}] \subset \mathrm{Fin}(\beta)$ *if and only if* $\mathbb{F}[x] \subset \mathrm{Fin}(\beta)$.

Proof. Of course, if $\mathbb{F}[x,\beta^{-1}] \subset \mathrm{Fin}(\beta)$, then $\mathbb{F}[x] \subset \mathrm{Fin}(\beta)$. To prove the converse, consider an element

$$z = z_0 + \frac{z_1}{\beta} + \cdots + \frac{z_\ell}{\beta^\ell} \in \mathbb{F}\big[x,\beta^{-1}\big], \quad \text{where } z_i \in \mathbb{F}[x].$$

There exist finite expansions $z_i = \sum_j d_{ij}/\beta^j$. Therefore $z_i/\beta^i = \sum_j d_{ij}/\beta^{i+j}$. Adding up the corresponding digits, we obtain the β-expansion of z, which is again finite. □

We can combine Theorem 5.2 with Theorems 5.4 and 5.5 to obtain

Corollary 5.6. *An element $\beta \in \mathbb{F}((x^{-1}))$ is a Pisot element if and only if*

$$\mathbb{F}[x] \subset \mathrm{Fin}(\beta).$$

Remark 5.7. In [10] it was proved that β is a Pisot element if and only if it is a simple β-element.

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