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Multivariate fractional Ostrowski type inequalities

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Abstract

Optimal upper bounds are given for the deviation of a value of a multivariate function of a fractional space from its average, over convex and compact subsets of $\mathbb{R}^N$, $N \geq 2$. In particular we work over rectangles, balls and spherical shells. These bounds involve the supremum and L_∞ norms of related multivariate fractional derivatives of the function involved. The inequalities produced are sharp, namely they are attained. This work has been motivated by the works of Ostrowski [A. Ostrowski, Über die Absolutabweichung einer differentiebaren Funcktion von ihrem Integralmittelwert, Commentarii Mathematici Helvetici 10 (1938) 226–227], 1938, and of the author [G.A. Anastassiou, Fractional Ostrowski type inequalities, Communications in Applied Analysis 7 (2) (2003) 203–208], 2003.

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1. Background

In the following, we follow Canavati [3]. Let $g \in C([0, 1])$, $n := [\nu]$, $\nu > 0$, and $\alpha := \nu - n$ $(0 < \alpha < 1)$. Define

$$(\mathcal{J}_\nu g)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} g(t) \mathrm{d}t, \quad 0 \leq x \leq 1, \tag{1.1}$$

the *Riemann–Liouville fractional integral*, where Γ is the gamma function $\Gamma(\nu) := \int_0^\infty \mathrm{e}^{-t} t^{\nu-1} \mathrm{d}t$. We define the subspace $C^\nu([0, 1])$ of $C^n([0, 1])$:

$$C^\nu([0, 1]) := \{g \in C^n([0, 1]) : \mathcal{J}_{1-\alpha} g^{(n)} \in C^1([0, 1])\}. \tag{1.2}$$

So letting $g \in C^\nu([0, 1])$, we define the ν*-fractional derivative* of g as

$$g^{(\nu)} := (\mathcal{J}_{1-\alpha} g^{(n)})'. \tag{1.3}$$

When $\nu \geq 1$ we have Taylor's formula [3]

$$g(t) = g(0) + g'(0)t + g''(0)\frac{t^2}{2!} + \cdots + g^{(n-1)}(0)\frac{t^{n-1}}{(n-1)!} + (\mathcal{J}_\nu g^{(\nu)})(t), \quad \forall t \in [0, 1], \tag{1.4}$$

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and when $0 < \nu < 1$ we find

$$g(t) = (\mathcal{J}_\nu g^{(\nu)})(t), \quad \forall t \in [0, 1]. \tag{1.5}$$

Next we carry the above notions over to an arbitrary interval $[a, b] \subseteq \mathbb{R}$ (see Anastassiou [4]). Let $x, x_0 \in [a, b]$ such that $x \geq x_0$, x_0 is fixed. Let $f \in C([a, b])$ and define

$$(\mathcal{J}_\nu^{x_0} f)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} f(t) \mathrm{d}t, \quad x_0 \leq x \leq b, \tag{1.6}$$

the *generalized Riemann–Liouville integral*. We define the subspace $C_{x_0}^\nu([a, b])$ of $C^n([a, b])$:

$$C_{x_0}^\nu([a, b]) := \{f \in C^n([a, b]) : \mathcal{J}_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0, b])\}. \tag{1.7}$$

For $f \in C_{x_0}^\nu([a, b])$, we define the *generalized ν-fractional derivative of f over* $[x_0, b]$, as

$$D_{x_0}^\nu f := (\mathcal{J}_{1-\alpha}^{x_0} f^{(n)})'. \tag{1.8}$$

We observe that $D_{x_0}^n f = f^{(n)}$, $n \in \mathbb{N}$.

Notice that

$$\mathcal{J}_{1-\alpha}^{x_0} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f^{(n)}(t) \mathrm{d}t \tag{1.9}$$

exists for $f \in C_{x_0}^\nu([a, b])$.

We mention the following generalization of the fractional Taylor formula (see Anastassiou [4], Canavati [3]).

Theorem 1.1. *Let $f \in C_{x_0}^\nu([a, b])$, $x_0 \in [a, b]$ fixed.*

(i) *If $\nu \geq 1$, then*

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} \\ &\quad + \cdots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + (\mathcal{J}_\nu^{x_0} D_{x_0}^\nu f)(x), \quad \text{all } x \in [a, b] : x \geq x_0. \end{aligned} \tag{1.10}$$

(ii) *If $0 < \nu < 1$, we get*

$$f(x) = (\mathcal{J}_\nu^{x_0} D_{x_0}^\nu f)(x), \quad \text{all } x \in [a, b] : x \geq x_0. \tag{1.11}$$

We also mention from Anastassiou [5], the basic multivariate fractional Taylor formula.

Theorem 1.2. *Let $f \in C^1(Q)$, where Q is convex and compact $\subseteq \mathbb{R}^N$, $N \geq 2$. For fixed $x_0, z \in Q$, assume that as a function of $t \in [0, 1]$: $f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0, 1])$, all $i = 1, \ldots, N$, where $\nu \in [1, 2)$. Then*

$$f(z) = f(x_0) + \sum_{i=1}^{N} \frac{(z_i - x_{0i})}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)} \mathrm{d}t \tag{1.12}$$

where $z = (z_1, \ldots, z_N)$, $x = (x_{01}, \ldots, x_{0N})$.

The following general multivariate fractional Taylor formula comes also from Anastassiou [5].

Theorem 1.3. *Let $f \in C^n(Q)$, Q compact and convex, $\subseteq \mathbb{R}^N$, $N \geq 2$, where $\nu \geq 1$, such that $n = [\nu]$. For fixed $x_0, z \in Q$, assume that we have functions of $t \in [0, 1]$: $f_\alpha(x_0 + t(z - x_0)) \in C^{(\nu-n)}([0, 1])$, for all $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots N$; $|\alpha| := \sum_{i=1}^{N} \alpha_i = n$. Then*

(i)

$$f(z) = f(x_0) + \sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial f}{\partial x_i}(x_0)$$
$$+ \frac{\left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^2 f\right](x_0)}{2!} + \cdots + \frac{\left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^{n-1} f\right](x_0)}{(n-1)!}$$
$$+ \frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}\left\{\left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)}(x_0 + t(z - x_0))\right\}\mathrm{d}t. \tag{1.13}$$

(ii) *If all* $f_\alpha(x_0) = 0, \alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, N, |\alpha| := \sum_{i=1}^{N}\alpha_i = l, l = 1, \ldots, n-1$, *then*

$$f(z) - f(x_0) = \frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}\left\{\left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)}(x_0 + t(z - x_0))\right\}\mathrm{d}t. \tag{1.14}$$

In Anastassiou [2] we proved the following Ostrowski type results (see [1,7]).

Theorem 1.4. *Let* $1 \le \nu < 2$ *and* $f \in C^\nu_{x_0}([\alpha, b]), \alpha \le x_0 < b, x_0$ *fixed. Then*

$$\left|\frac{1}{b - x_0}\int_{x_0}^{b} f(y)\mathrm{d}y - f(x_0)\right| \le \frac{\|D^\nu_{x_0} f\|_{\infty,[x_0,b]}}{\Gamma(\nu+2)}(b - x_0)^\nu. \tag{1.15}$$

Inequality (1.15) *is sharp, namely it is attained by* $f(x) := (x - x_0)^\nu, 1 \le \nu < 2, x \in [a, b]$.

Also in [2] we gave

Theorem 1.5. *Let* $\alpha \le x_0 < b$ *be fixed.*
Let $f \in C^\nu_{x_0}([a, b]), \nu \ge 2, n := [\nu]$. *Assume* $f^{(i)}(x_0) = 0, i = 1, \ldots, n-1$. *Then*

$$\left|\frac{1}{b - x_0}\int_{x_0}^{b} f(y)\mathrm{d}y - f(x_0)\right| \le \frac{\|D^\nu_{x_0} f\|_{\infty,[x_0,b]}}{\Gamma(\nu+2)}(b - x_0)^\nu. \tag{1.16}$$

Inequality (1.16) *is sharp, namely it is attained by*

$$f(x) := (x - x_0)^\nu, \quad \nu \ge 2, \ x \in [a, b].$$

Establishing sharpness in (1.15) and (1.16), we proved first that [2]

$$\|D^\nu_{x_0}(x - x_0)^\nu\|_{\infty,[x_0,b]} = \Gamma(\nu+1). \tag{1.17}$$

In this article, motivated by (1.15) and (1.16), we present various multivariate fractional Ostrowski type inequalities.

2. Results

We present our first result.

Theorem 2.1. *Let* $f \in C^1(Q)$, *where* Q *is convex and compact* $\subseteq \mathbb{R}^N, N \ge 2$. *For fixed* $x_0 \in Q$ *and any* $z \in Q$ *assume that we have a function of* $t \in [0, 1] : f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0, 1])$, *for all* $i = 1, \ldots, N$, *where* $\nu \in [1, 2)$. *Then*

$$\left|f(x_0) - \frac{\int_Q f(z)\mathrm{d}z}{\mathrm{Vol}(Q)}\right| \le \frac{\max_{1\le i\le N}\|(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}\|_{\infty,(t,z)\in[0,1]\times Q}}{\Gamma(\nu+1)\mathrm{Vol}(Q)}\int_Q \|z - x_0\|_{l_1}\mathrm{d}z. \tag{2.1}$$

Proof. From (1.12) we get

$$f(z)-f(x_0)=\sum_{i=1}^{N}\frac{(z_i-x_{0i})}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}(f_{x_i}(x_0+t(z-x_0)))^{(\nu-1)}\mathrm{d}t, \tag{2.2}$$

and

$$\begin{aligned}|f(z)-f(x_0)| &\le \sum_{i=1}^{N}\frac{|z_i-x_{0i}|}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}|(f_{x_i}(x_0+t(z-x_0)))^{(\nu-1)}|\mathrm{d}t\\ &\le \frac{1}{\Gamma(\nu+1)}\sum_{i=1}^{N}|x_i-x_{0_i}|\|(f_{x_i}(x_0+t(z-x_0)))^{(\nu-1)}\|_{\infty,t\in[0,1]}.\end{aligned} \tag{2.3}$$

That is,

$$|f(z)-f(x_0)|\le\frac{1}{\Gamma(\nu+1)}\|z-x_0\|_{l_1}\max_{1\le i\le N}\|(f_{x_i}(x_0+t(z-x_0)))^{(\nu-1)}\|_{\infty,(t,z)\in[0,1]\times Q}, \tag{2.4}$$

$\forall z\in Q$, $x_0\in Q$ fixed.

Hence we have

$$\begin{aligned}\left|\frac{\int_Q f(z)\mathrm{d}z}{\mathrm{Vol}(Q)}-f(x_0)\right| &= \left|\frac{\int_Q(f(z)-f(x_0))\mathrm{d}z}{\mathrm{Vol}(Q)}\right|\le\frac{1}{\mathrm{Vol}(Q)}\int_Q|f(z)-f(x_0)|\mathrm{d}z\\ &\overset{(2.4)}{\le}\frac{\max_{1\le i\le N}\|(f_{x_i}(x_0+t(z-x_0)))^{(\nu-1)}\|_{\infty,(t,z)\in[0,1]\times Q}}{\Gamma(\nu+1)\mathrm{Vol}(Q)}\int_Q\|z-x_0\|_{l_1}\mathrm{d}z,\end{aligned} \tag{2.5}$$

proving the claim. □

Next we give

Theorem 2.2. *Let $f\in C^n(Q)$, Q compact and convex, $\subseteq\mathbb{R}^N$, $N\ge 2$, where $\nu\ge 1$ such that $n=[\nu]$. For fixed $x_0\in Q$ and any $z\in Q$ assume that we have functions of $t\in[0,1]$: $f_\alpha(x_0+t(z-x_0))\in C^{\nu-n}([0,1])$, for all $\alpha:(\alpha_1,\ldots\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\ldots,N$; $|\alpha|:=\sum_{i=1}^N\alpha_i=n$. Set*

$$\|D^{\nu-n}f(x_0+t(z-x_0))\|_{\infty,(t,z)\in[0,1]\times Q}=\max_{|\alpha|=n}\|f_\alpha^{(\nu-n)}(x_0+t(z-x_0))\|_{\infty,(t,z)\in[0,1]\times Q}. \tag{2.6}$$

Then

$$\left|f(x_0)-\frac{\int_Q f(z)\mathrm{d}z}{\mathrm{Vol}(Q)}\right|\le\frac{\|D^{\nu-n}f(x_0+t(z-x_0))\|_{\infty,(t,z)\in[0,1]\times Q}}{\Gamma(\nu+1)\mathrm{Vol}(Q)}\int_Q\|z-x_0\|_{l_1}^n\mathrm{d}z. \tag{2.7}$$

Proof. From (1.14) we have

$$\begin{aligned}|f(z)-f(x_0)| &\le \frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\left|\left\{\left[\left(\sum_{i=1}^{N}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)}(x_0+t(z-x_0))\right\}\right|\mathrm{d}t\\ &\le \frac{1}{\Gamma(\nu+1)}\left\|\left\{\left[\left(\sum_{i=1}^{N}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)}(x_0+t(z-x_0))\right\}\right\|_{\infty,t\in[0,1]}\\ &\le \frac{1}{\Gamma(\nu-1)}(\|z-x_0\|_{l_1})^n\|D^{\nu-n}f(x_0+t(z-x_0))\|_{\infty,(t,z)\in[0,1]\times Q}.\end{aligned} \tag{2.8}$$

That is, we get

$$|f(z)-f(x_0)|\le\frac{(\|z-x_0\|_{l_1})^n}{\Gamma(\nu+1)}\|D^{\nu-n}f(x_0+t(z-x_0))\|_{\infty,(t,z)\in[0,1]\times Q}, \tag{2.9}$$

$\forall z\in Q$, $x_0\in Q$ fixed.

Therefore as before in (2.5) we have that

$$\left|\frac{\int_Q f(z)\mathrm{d}z}{\mathrm{Vol}(Q)} - f(x_0)\right| \le \frac{1}{\mathrm{Vol}(Q)}\int_Q |f(z) - f(x_0)|\mathrm{d}z$$

$$\overset{(2.9)}{\le} \frac{\|D^{\nu-n} f(x_0 + t(z - x_0))\|_{\infty,(t,z)\in[0,1]\times Q}}{\Gamma(\nu+1)\mathrm{Vol}(Q)} \int_Q (\|z - x_0\|_{l_1})^n \mathrm{d}z, \tag{2.10}$$

proving the claim. □

We continue with

Theorem 2.3. *Let* $Q := [x_0, b] \times [c, d]$, $x_0 \in [a, b)$, *and* $f \in C([a, b] \times [c, d])$.

Let $1 \le \nu < 2$ *and* $\frac{\partial^\nu_{x_0} f}{\partial x^\nu} \in C^\nu_{x_0}([a, b])$, $y_0 \in [a, b]$.

Then

$$\left|\frac{1}{(b - x_0)(d - c)}\int_Q f(x, y)\mathrm{d}x\mathrm{d}y - f(x_0, y_0)\right|$$

$$\le \frac{1}{d - c}\int_c^d |f(x_0, y) - f(x_0, y_0)|\mathrm{d}y + \frac{(b - x_0)^\nu}{\Gamma(\nu+2)}\left\|\frac{\partial^\nu_{x_0} f}{\partial x^\nu}\right\|_{\infty,Q}. \tag{2.11}$$

Proof. By (1.10) we have

$$f(x, y) - f(x_0, y) = \frac{1}{\Gamma(\nu)}\int_{x_0}^x (x - t)^{\nu-1}\frac{\partial^\nu_{x_0} f}{\partial x^\nu}(t, y)\mathrm{d}t, \tag{2.12}$$

$x \ge x_0$, all $y \in [c, d]$.

That is,

$$|f(x, y) - f(x_0, y)| \le \frac{1}{\Gamma(\nu)}\left\|\frac{\partial^\nu_{x_0} f}{\partial x^\nu}\right\|_{\infty,Q}\int_{x_0}^x (x - t)^{\nu-1}\mathrm{d}t, \tag{2.13}$$

and

$$|f(x, y) - f(x_0, y)| \le \frac{(x - x_0)^\nu}{\Gamma(\nu+1)}\left\|\frac{\partial^\nu_{x_0} f}{\partial x^\nu}\right\|_{\infty,Q}, \tag{2.14}$$

for all $x \ge x_0$, all $y \in [c, d]$.

However it holds that

$$|f(x, y) - f(x_0, y_0)| \le |f(x, y) - f(x_0, y)| + |f(x_0, y) - f(x_0, y_0)|$$

$$\le |f(x_0, y) - f(x_0, y_0)| + \frac{1}{\Gamma(\nu+1)}\left\|\frac{\partial^\nu_{x_0} f}{\partial x^\nu}\right\|_{\infty,Q}(x - x_0)^\nu, \tag{2.15}$$

$\forall x \ge x_0, \forall y \in [c, d]$.

Consequently we obtain

$$\left|\frac{1}{(b - x_0)(d - c)}\int_{[x_0,b]\times[c,d]} f(x, y)\mathrm{d}x\mathrm{d}y - f(x_0, y_0)\right|$$

$$= \frac{1}{(b - x_0)(d - c)}\left|\int_{[x_0,b]\times[c,d]} (f(x, y) - f(x_0, y_0))\mathrm{d}x\mathrm{d}y\right|$$

$$\le \frac{1}{(b - x_0)(d - c)}\int_{[x_0,b]\times[c,d]} |f(x, y) - f(x_0, y_0)|\mathrm{d}x\mathrm{d}y \tag{2.16}$$

$$\overset{(2.15)}{\le} \frac{1}{(b - x_0)(d - c)}\left[(b - x_0)\int_c^d |f(x_0, y) - f(x_0, y_0)|\mathrm{d}y + \frac{(d - c)}{\Gamma(\nu+1)}\left\|\frac{\partial^\nu_{x_0} f}{\partial x^\nu}\right\|_{\infty,Q}\int_{x_0}^b (x - x_0)^\nu \mathrm{d}x\right]$$

$$= \frac{1}{(b-x_0)(d-c)} \left[(b-x_0) \int_c^d |f(x_0,y) - f(x_0,y_0)| \mathrm{d}y + \frac{(b-x_0)^{\nu+1}(d-c)}{\Gamma(\nu+2)} \left\| \frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty,Q} \right] \tag{2.17}$$

$$= \frac{1}{d-c} \int_c^d |f(x_0,y) - f(x_0,y_0)| \mathrm{d}y + \frac{(b-x_0)^{\nu}}{\Gamma(\nu+2)} \left\| \frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty,Q}, \tag{2.18}$$

proving the claim. □

We further have

Theorem 2.4. *Let* $Q := [x_0, b] \times [c, d]$, $x_0 \in [a, b)$, *and* $f \in C^n([a, b] \times [c, d])$. *Let* $\nu \geq 2$ *be such that* $n = [\nu]$ *and* $\frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}} \in C_{x_0}^{\nu}([a, b])$, $y_0 \in [a, b]$. *We further assume that* $\frac{\partial^j f(x_0,y)}{\partial x^j} = 0$, $j = 1, \ldots, n-1$.

Then

$$\left| \frac{1}{(b-x_0)(d-c)} \int_Q f(x,y) \mathrm{d}x\mathrm{d}y - f(x_0,y_0) \right|$$
$$\leq \frac{1}{d-c} \int_c^d |f(x_0,y) - f(x_0,y_0)| \mathrm{d}y + \frac{(b-x_0)^{\nu}}{\Gamma(\nu+2)} \left\| \frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty,Q}. \tag{2.19}$$

Proof. By (1.10) we get again

$$f(x,y) - f(x_0,y) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} \frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}}(t,y) \mathrm{d}t, \tag{2.20}$$

$x \geq x_0, \forall y \in [c, d]$.

And again

$$|f(x,y) - f(x_0,y)| \leq \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} \left\| \frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty,Q}, \tag{2.21}$$

$\forall x \geq x_0, \forall y \in [c, d]$.

Also, it holds again that

$$|f(x,y) - f(x_0,y_0)| \leq |f(x_0,y) - f(x_0,y_0)| + \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} \left\| \frac{\partial_{x_0}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty,Q}, \tag{2.22}$$

$\forall x \geq x_0, \forall y \in [c, d]$.

Integrating (2.22) over Q we prove (2.19). □

One can prove similarly to (2.11) and (2.19) inequalities in more than two variables. Next we study fractional Ostrowski type inequalities over balls and spherical shells.

For that we make:

Remark 2.1. We define the ball $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N, N \geq 2, R > 0$, and the sphere $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, where $|\cdot|$ is the Euclidean norm.

Let $\mathrm{d}\omega$ be the element of surface measure on S^{N-1} and let $\omega_N = \int_{S^{N-1}} \mathrm{d}\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)}$. For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}, |\omega| = 1$. Note that $\int_{B(0,R)} \mathrm{d}y = \frac{\omega_N R^N}{N}$ is the Lebesgue measure of the ball.

For $F \in C(\overline{B(0,R)})$ we have $\int_{B(0,R)} F(x)\mathrm{d}x = \int_{S^{N-1}} \left(\int_0^R F(r\omega) r^{N-1} \mathrm{d}r \right) \mathrm{d}\omega$; we use this formula a lot.

The function $f : \overline{B(0,R)} \to \mathbb{R}$ is *radial* if there exists a function g such that $f(x) = g(r)$, where $r = |x|, r \in [0, R], \forall x \in \overline{B(0,R)}$. Here we assume that $g \in C_0^{\nu}([0, R]), 1 \leq \nu < 2$.

By (1.10) we get

$$g(s) - g(0) = \frac{1}{\Gamma(\nu)} \int_0^s (s-w)^{\nu-1} (D_0^{\nu} g)(w) \mathrm{d}w, \tag{2.23}$$

$\forall s \in [0, R]$.

Hence

$$|g(s)-g(0)| \leq \frac{s^{\nu}}{\Gamma(\nu+1)}\|D_0^{\nu}g\|_{\infty,[0,R]}, \quad \forall s \in [0,R]. \tag{2.24}$$

Next we observe that

$$\left|f(0)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{\mathrm{Vol}(B(0,R))}\right| = \left|g(0)-\frac{\int_{S^{N-1}}(\int_0^R g(s)s^{N-1}\mathrm{d}s)\mathrm{d}\omega}{\int_{S^{N-1}}(\int_0^R s^{N-1}\mathrm{d}s)\mathrm{d}\omega}\right| \tag{2.25}$$

$$= \left|g(0)-\frac{N}{R^N}\int_0^R g(s)s^{N-1}\mathrm{d}s\right| = \frac{N}{R^N}\left|\int_0^R s^{N-1}(g(0)-g(s))\mathrm{d}s\right| \tag{2.26}$$

$$\leq \frac{N}{R^N}\int_0^R s^{N-1}|g(s)-g(0)|\mathrm{d}s \overset{(2.24)}{\leq} \frac{\|D_0^{\nu}g\|_{\infty}}{\Gamma(\nu+1)}\frac{N}{R^N}\int_0^R s^{\nu+N-1}\mathrm{d}s = \frac{\|D_0^{\nu}g\|_{\infty}NR^{\nu}}{\Gamma(\nu+1)(\nu+N)}. \tag{2.27}$$

That is, we have proved that

$$\left|f(0)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{\mathrm{Vol}(B(0,R))}\right| = \left|g(0)-\frac{N}{R^N}\int_0^R g(s)s^{N-1}\mathrm{d}s\right| \leq \frac{\|D_0^{\nu}g\|_{\infty}NR^{\nu}}{\Gamma(\nu+1)(\nu+N)}. \tag{2.28}$$

The last inequality (2.28) is sharp, namely it is attained by $g(r)=r^{\nu}$, $1\leq\nu<2$, $r\in[0,R]$. Indeed by (1.17) we get $\|D_0^{\nu}x^{\nu}\|_{\infty,[0,R]}=\Gamma(\nu+1)$.

Notice also that

$$\text{LHS (2.28)} = \frac{N}{R^N}\int_0^R s^{\nu+N-1}\mathrm{d}s = \frac{NR^{\nu}}{\nu+N} = \text{RHS (2.28)}, \tag{2.29}$$

proving optimality.

We have established

Theorem 2.5. *Let $f:\overline{B(0,R)}\to\mathbb{R}$, which is radial, i.e. there exists g such that $f(x)=g(r)$, $r=|x|$, $\forall x\in\overline{B(0,R)}$. Assume that $g\in C_0^{\nu}([0,R])$, $1\leq\nu<2$. Then*

$$\left|f(0)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{\mathrm{Vol}(B(0,R))}\right| = \left|g(0)-\frac{N}{R^N}\int_0^R g(s)s^{N-1}\mathrm{d}s\right| \leq \frac{\|D_0^{\nu}g\|_{\infty}NR^{\nu}}{\Gamma(\nu+1)(\nu+N)}. \tag{2.30}$$

Inequality (2.30) is sharp, that is attained by $g(r)=r^{\nu}$.

We continue on from the previous remark.

Remark 2.2. We treat here the general, not necessarily radial case of $f\in C(\overline{B(0,R)})$. For any fixed $\omega\in S^{N-1}$ the function $f(\cdot\omega)$ is radial on $[0,R]$. We assume that $f(\cdot\omega)\in C_0^{\nu}([0,R])$, $1\leq\nu<2$.

That is, $\exists\frac{\partial_0^{\nu}f(r\omega)}{\partial r^{\nu}}$ and it is continuous in $r\in[0,R]$, for any $\omega\in S^{N-1}$. Here we have

$$\frac{\partial_0^{\nu}f(r\omega)}{\partial r^{\nu}} = \frac{\partial}{\partial r}\left(\mathcal{J}_{2-\nu}\left(\frac{\partial f}{\partial r}(\cdot\omega)\right)\right)(r) = \frac{1}{\Gamma(2-\nu)}\frac{\partial}{\partial r}\left(\int_0^r (r-t)^{1-\nu}\frac{\partial f}{\partial r}(t\omega)\mathrm{d}t\right). \tag{2.31}$$

For $x\neq0$, i.e. $x=r\omega$, $r>0$, $\omega\in S^{N-1}$, the *fractional radial derivative* $\frac{\partial_0^{\nu}f(x)}{\partial r^{\nu}}$ is defined as in (2.31). Clearly $\frac{\partial_0^{\nu}f(x)}{\partial r^{\nu}}|_{x=0}$ is not defined.

We mention

Lemma 2.1. *Everything is as in Remark 2.2. The function $\frac{\partial_0^{\nu}f(x)}{\partial r^{\nu}}$ is measurable over $\overline{B(0,R)}-\{0\}$.*

Proof. For each $n\in\mathbb{N}$ define

$$g_n(r,\omega) := n\left[f\left(\left(r-\frac{1}{n}\right)\omega\right)-f(r\omega)\right] = \frac{f\left(\left(r-\frac{1}{n}\right)\omega\right)-f(r\omega)}{\frac{1}{n}}$$

and note that each g_n is jointly measurable in (r, ω) since it is jointly continuous in (r, ω) by $f \in C(\overline{B(0, R)})$, here $r \in (0, R]$ and $\omega \in S^{N-1}$. In view of $g_n(r, \omega) \to \frac{\partial f(r\omega)}{\partial r}$ as $n \to \infty$, we get that $\frac{\partial f(r\omega)}{\partial r}$ is jointly measurable in $(r, \omega) \in (0, R] \times s^{N-1} = \overline{B(0, R)} - \{0\}$.

Then $\frac{\partial f}{\partial r}(r\cdot)$ is measurable in $\omega \in S^{N-1}, \forall r \in (0, R]$. Thus the integral $I_\varepsilon(r, \omega) = \int_0^{r-\varepsilon} (r-t)^{1-\nu} \frac{\partial f}{\partial r}(t\omega)\mathrm{d}t$, $r \in (0, R], \omega \in S^{N-1}, \epsilon > 0$ small, because it is a limit of Riemann sums, is measurable in $\omega \in S^{N-1}$. Since $(r-t)^{1-\nu}\frac{\partial f}{\partial r}(t\omega)$ is integrable over $[0, r]$, we get that $I_\varepsilon(r, \omega)$ is continuous in $r - \varepsilon$, $\forall \varepsilon > 0$ small. Hence

$$\lim_{\varepsilon \to 0} I_\varepsilon(r, \omega) = I(r, \omega) := \int_0^r (r-t)^{1-\nu} \frac{\partial f}{\partial r}(t\omega)\mathrm{d}t,$$

proving $I(r, \omega)$ measurable in $\omega \in s^{N-1}, \forall r \in (0, R]$.

But, by the assumption that $f(\cdot\omega) \in C_0^\nu([0, R])$, we have that $I(r, \omega)$ is continuous in $r \in [0, R], \forall \omega \in s^{N-1}$. Therefore by the Caratheodory theorem, see [6], p. 156, we get that $I(r, \omega)$ is jointly measurable in $(r, \omega) \in (0, R] \times S^{N-1}$, as it is a Caratheodory function. Since

$$\frac{\partial I(r, \omega)}{\partial r} = \lim_{n \to \infty} n \left[I\left(r - \frac{1}{n}, \omega\right) - I(r, \omega) \right]$$

and also $I(r - \frac{1}{n}, \omega)$ is jointly measurable in $(r, \omega) \in (0, R] \times S^{n-1}$, we get that $\frac{\partial I(r,\omega)}{\partial r}$ is jointly measurable in $(r, \omega) \in \overline{B(0, R)} - \{0\}$, proving the claim. □

We need

Lemma 2.2. *All is as in Remark 2.2.*

Additionally assume that $\frac{\partial_0^\nu f(x)}{\partial r^\nu}$ *is continuous on* $\overline{B(0, R)} - \{0\}$*, and*

$$K := \left\| \frac{\partial_0^\nu f(x)}{\partial r^\nu} \right\|_{L_\infty(B(0,R))} = \operatorname{esssup} \left| \frac{\partial_0^\nu f(x)}{\partial r^\nu} \right|_{B(0,R)} < \infty.$$

Then

$$\left\| \frac{\partial_0^\nu f(r\omega)}{\partial r^\nu} \right\|_{\infty,(r \in [0,R])} \le K, \quad \forall \omega \in s^{N-1}. \tag{2.32}$$

Proof. In the radial case (2.32) is obvious. Also it is obvious if $\left\| \frac{\partial_0^\nu f(r\omega)}{\partial r^\nu} \right\|_{\infty,[0,R]} = \left| \frac{\partial_0^\nu f(r_0\omega)}{\partial r^\nu} \right|$, for some $r_0 \in (0, R]$. The only difficulty comes here if for specific $\omega_0 \in S^{N-1}$ we have that

$$\left\| \frac{\partial_0^\nu f(r\omega_0)}{\partial r^\nu} \right\|_{\infty,[0,R]} = \left| \frac{\partial_0^\nu f(0)}{\partial r^\nu} \right|.$$

Then it is evident, for very small $r^* > 0$, that by continuity of $\frac{\partial_0^\nu f(\cdot\omega_0)}{\partial r^\nu}$ we have

$$\left| \frac{\partial_0^\nu f(r^*\omega_0)}{\partial r^\nu} \right| \approx \left| \frac{\partial_0^\nu f(0)}{\partial r^\nu} \right|.$$

If

$$\left| \frac{\partial_0^\nu f(0)}{\partial r^\nu} \right| > \operatorname{esssup} \left| \frac{\partial_0^\nu f(x)}{\partial r^\nu} \right|_{B(0,R)},$$

then

$$\left| \frac{\partial_0^\nu f(r^*\omega_0)}{\partial r^\nu} \right| > \operatorname{esssup} \left| \frac{\partial_0^\nu f(x)}{\partial r^\nu} \right|_{B(0,R)} = \left\| \frac{\partial_0^\nu f(x)}{\partial r^\nu} \right\|_{\infty,\overline{B(0,R)}-\{0\}},$$

a contradiction. □

Remark 2.2 (*Continuation*). By (2.30) we get

$$\left|f(0)-\frac{N}{R^N}\int_0^R f(s\omega)s^{N-1}\mathrm{d}s\right|\le\frac{\left\|\frac{\partial_0^\nu f(r\omega)}{\partial r^\nu}\right\|_{\infty,(r\in[0,R])}NR^\nu}{\Gamma(\nu+1)(\nu+N)}\le\frac{KNR^\nu}{\Gamma(\nu+1)(\nu+N)}.$$

Consequently we find

$$\left|f(0)-\frac{N}{\omega_N R^N}\int_{S^{N-1}}\left(\int_0^R f(s\omega)s^{N-1}\mathrm{d}s\right)\mathrm{d}\omega\right|\le\frac{KNR^\nu}{\Gamma(\nu+1)(\nu+N)}.$$

That proves

$$\left|f(0)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{\mathrm{Vol}(B(0,R))}\right|\le\frac{KNR^\nu}{\Gamma(\nu+1)(\nu+N)}. \tag{2.33}$$

We have established

Theorem 2.6. *Let $f\in C(\overline{B(0,R)})$, not necessarily radial, and assume that $f(\cdot\omega)\in C_0^\nu([0,R])$, $1\le\nu<2$, for any $\omega\in S^{N-1}$. Suppose also $\frac{\partial_0^\nu f(x)}{\partial r^\nu}$ to be continuous on $\overline{B(0,R)}-\{0\}$, and that $\left\|\frac{\partial_0^\nu f(x)}{\partial r^\nu}\right\|_{L_\infty(B(0,R))}<\infty$.*

Then

$$\left|f(0)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{\mathrm{Vol}(B(0,R))}\right|\le\frac{NR^\nu}{\Gamma(\nu+1)(\nu+N)}\left\|\frac{\partial_0^\nu f(x)}{\partial r^\nu}\right\|_{L_\infty(B(0,R))}. \tag{2.34}$$

We make

Remark 2.3. Let the spherical shell $A:=B(0,R_2)-\overline{B(0,R_1)}$, $0<R_1<R_2$, $A\subseteq\mathbb{R}^N$, $N\ge 2$, $x\in\bar{A}$. Consider $f\in C^1(\bar{A})$ and assume that there exists $\frac{\partial_{R_1}^\nu f(x)}{\partial r^\nu}\in C(\bar{A})$, $1\le\nu<2$; $x=r\omega$, $r\in[R_1,R_2]$, $\omega\in S^{N-1}$; where $\frac{\partial_{R_1}^\nu f(x)}{\partial r^\nu}=\frac{1}{\Gamma(2-\nu)}\frac{\partial}{\partial r}(\int_{R_1}^r(r-t)^{1-\nu}\frac{\partial f}{\partial r}(t\omega)\mathrm{d}t)$. Clearly here $f(r\omega)\in C^1([R_1,R_2])$ and $\frac{\partial_{R_1}^\nu f(r\omega)}{\partial r^\nu}\in C([R_1,R_2])$, $\forall\omega\in S^{N-1}$. For $F\in C(\bar{A})$ it holds $\int_A F(x)\mathrm{d}x=\int_{S^{N-1}}(\int_{R_1}^{R_2}F(r\omega)r^{N-1}\mathrm{d}r)\mathrm{d}\omega$; we exploit this formula a lot here.

Initially we assume that f is *radial*, i.e., there exists g such that $f(x)=g(r)$. Here $\mathrm{Vol}(A)=\frac{\omega_N(R_2^N-R_1^N)}{N}$. Then we get via the polar method that

$$\begin{aligned}\left|f(R_1\omega)-\frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)}\right|&=\left|g(R_1)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)s^{N-1}\mathrm{d}s\right|\\&=\left(\frac{N}{R_2^N-R_1^N}\right)\left|\int_{R_1}^{R_2}(g(R_1)-g(s))\,s^{N-1}\mathrm{d}s\right|\\&\le\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}|g(R_1)-g(s)|s^{N-1}\mathrm{d}s=:(\star)\end{aligned} \tag{2.35}$$

Here by (1.10) we get for $s\ge R_1$,

$$g(s)-g(R_1)=\frac{1}{\Gamma(\nu)}\int_{R_1}^s(s-w)^{\nu-1}(D_{R_1}^\nu g)(w)\mathrm{d}w. \tag{2.36}$$

Thus

$$|g(s)-g(R_1)|\le\frac{\|D_{R_1}^\nu g\|_{\infty,[R_1,R_2]}}{\Gamma(\nu+1)}(s-R_1)^\nu, \tag{2.37}$$

$\forall s\ge R_1$.

Consequently it holds that

$$(\star) \le \left(\frac{N(\|D^{\nu}_{R_1} g\|_{\infty,[R_1,R_2]})}{(R_2^N - R_1^N)\Gamma(\nu+1)}\right) \int_{R_1}^{R_2} (s-R_1)^{\nu} s^{N-1} \mathrm{d}s$$

$$= \frac{N\|D^{\nu}_{R_1} g\|_{\infty,[R_1,R_2]}}{(R_2^N - R_1^N)\Gamma(\nu+1)} I =: (\star\star). \tag{2.38}$$

Here

$$I := \int_{R_1}^{R_2} (s-R_1)^{\nu} s^{N-1} \mathrm{d}s = (-1)^{N-1} \int_{R_1}^{R_2} (-s)^{N-1} (s-R_1)^{\nu} \mathrm{d}s$$

$$= (-1)^{N-1} \int_{R_1}^{R_2} (-R_2 + R_2 - s)^{N-1} (s-R_1)^{\nu} \mathrm{d}s \tag{2.39}$$

$$= (-1)^{N-1} \int_{R_1}^{R_2} \left[\sum_{k=0}^{N-1} \binom{N-1}{k} (-R_2)^{N-1-k} (R_2-s)^k\right] (s-R_1)^{\nu} \mathrm{d}s$$

$$= ((-1)^{N-1})^2 \left(\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^{-k} R_2^{N-k-1} \int_{R_1}^{R_2} (R_2-s)^{(k+1)-1} (s-R_1)^{(\nu+1)-1} \mathrm{d}s\right)$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k R_2^{N-k-1} \frac{\Gamma(k+1)\Gamma(\nu+1)}{\Gamma(k+\nu+2)} (R_2-R_1)^{k+\nu+1}$$

$$= \Gamma(\nu+1)(N-1)! \sum_{k=0}^{N-1} \frac{(-1)^k}{(n-k-1)!} R_2^{n-k-1} \frac{(R_2-R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}. \tag{2.40}$$

That is,

$$I = \int_{R_1}^{R_2} (s-R_1)^{\nu} s^{N-1} \mathrm{d}s = \Gamma(\nu+1)(N-1)! \sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2-R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}. \tag{2.41}$$

Continuing with (2.38) via (2.41), we have

$$(\star\star) = \left(\frac{N!\|D^{\nu}_{R_1} g\|_{\infty,[R_1,R_2]}}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2-R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right). \tag{2.42}$$

Hence in the radial case we have proved

$$\left| f(R_1\omega) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| \le \left(\frac{N!\|D^{\nu}_{R_1} g\|_{\infty,[R_1,R_2]}}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2-R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right). \tag{2.43}$$

Inequality (2.43) is attained by $g(s) := (s-R_1)^{\nu}$, $1 \le \nu < 2$, $s \in [R_1, R_2]$.

Indeed, we observe that

$$\text{LHS (2.43)} = \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} (s-R_1)^{\nu} s^{N-1} \mathrm{d}s$$

$$= \frac{\Gamma(\nu+1)N!}{(R_2^N - R_1^N)} \left(\sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2-R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right) = \text{RHS (2.43)}, \tag{2.44}$$

and by (1.17) that says

$$\|D^{\nu}_{R_1} (s-R_1)^{\nu}\|_{\infty,[R_1,R_2]} = \Gamma(\nu+1). \tag{2.45}$$

We have established

Theorem 2.7. *Let* $A := B(0, R_2) - \overline{B(0, R_1)}, 0 < R_1 < R_2, A \subseteq \mathbb{R}^N, N \geq 2$. *Consider* $f : \bar{A} \to \mathbb{R}$ *that is* radial*, i.e., there exists* g *such that* $f(x) = g(r), x = r\omega, r \in [R_1, R_2], \omega \in S^{N-1}, x \in \bar{A}$. *Assume that* $g \in C^{\nu}_{R_1}([R_1, R_2]), 1 \leq \nu < 2$.

Then

$$\left| f(R_1\omega) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| = \left| g(R_1) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s) s^{N-1} \mathrm{d}s \right|$$
$$\leq \left(\frac{N! \|D^{\nu}_{R_1} g\|_{\infty,[R_1,R_2]}}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2 - R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right). \tag{2.46}$$

Inequality (2.46) *is sharp, namely attained by* $g(s) = (s - R_1)^{\nu}, s \in [R_1, R_2]$.

We continue on from the last remark.

Remark 2.4. We treat the non-radial case here. For fixed $\omega \in S^{N-1}$ the function $f(r\omega)$ is radial over $[R_1, R_2]$. We apply (2.46) for $g = f(\cdot\omega)$ to get

$$\left| f(R_1\omega) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} f(s\omega) s^{N-1} \mathrm{d}s \right|$$
$$\leq \left(\frac{N! \left\| \frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}}}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2 - R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right). \tag{2.47}$$

Hence it holds that

$$\left| \frac{\int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{\omega_N} - \frac{N}{(R_2^N - R_1^N)\omega_N} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} \mathrm{d}s\right) \mathrm{d}\omega \right|$$
$$\leq \left(\frac{N! \left\| \frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}}}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k-1)!} R_2^{N-k-1} \frac{(R_2 - R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right)$$
$$=: C \left\| \frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}}. \tag{2.48}$$

That is, we have proved that

$$\left| \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{2\pi^{N/2}} - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| \leq C \left\| \frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}}. \tag{2.49}$$

However we have for $x \in \bar{A}$

$$\left| f(x) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{2\pi^{N/2}} \right| + C \left\| \frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}}. \tag{2.50}$$

We have established the following result.

Theorem 2.8. *Consider* $f \in C^1(\bar{A})$ *such that there exists* $\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}} \in C(\bar{A}), 1 \leq \nu < 2, x \in (\bar{A})$. *Then*

$$\left|f(x) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)}\right| \le \left|f(x) - \frac{\Gamma\left(\frac{N}{2}\right)\int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{2\pi^{N/2}}\right| + \left(\frac{N!}{R_2^N - R_1^N}\right)\left(\sum_{k=0}^{N-1}\frac{(-1)^k}{(N-k-1)!}R_2^{N-k-1}\frac{(R_2-R_1)^{k+\nu+1}}{\Gamma(k+\nu+2)}\right)\left\|\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}\right\|_{\infty,\bar{A}}. \tag{2.51}$$

We make

Remark 2.5. This continues Remarks 2.3 and 2.4. Here we establish higher order multivariate fractional Ostrowski type inequalities over spherical shells.

Here $\nu \ge 2, n := [\nu] \ge 2, \alpha := \nu - n$. Consider $f \in C^n(\bar{A})$, which implies that $f(r\omega) \in C^n([R_1, R_2]), \forall\omega \in S^{N-1}$. Furthermore assume that there exists $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A}), x \in \bar{A}; x = r\omega, r \in [R_1, R_2], \omega \in S^{N-1}$, where $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial r}(\int_{R_1}^{r}(r-t)^{-\alpha}\frac{\partial^n f(t,\omega)}{\partial r^n}\mathrm{d}t)$. The last part implies $\frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \in C([R_1, R_2]), \forall\omega \in S^{N-1}$. We start again with f being radial, i.e. $\exists g : f(x) = g(r), r \in [R_1, R_2], x \in \bar{A}$.

We have

$$\left|f(R_1\omega) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)}\right| = \left(\frac{N}{R_2^N - R_1^N}\right)\left|\int_{R_1}^{R_2}(g(s) - g(R_1))s^{N-1}\mathrm{d}s\right| =: (\star). \tag{2.52}$$

By (1.10) we have

$$g(s) - g(R_1) = \sum_{k=1}^{n-1} g^{(k)}(R_1)\frac{(s-R_1)^k}{k!} + \frac{1}{\Gamma(\nu)}\int_{R_1}^{s}(s-w)^{\nu-1}(D_{R_1}^{\nu}g)(w)\mathrm{d}w, \tag{2.53}$$

all $s \ge R_1$.

Consequently it holds that

$$(\star) = \left(\frac{N}{R_2^N - R_1^N}\right)\left[\sum_{k=1}^{n-1}\frac{|g^{(k)}(R_1)|}{k!}\left|\int_{R_1}^{R_2} s^{N-1}(s-R_1)^k\mathrm{d}s\right| + \frac{1}{\Gamma(\nu)}\int_{R_1}^{R_2} s^{N-1}\left|\int_{R_1}^{s}(s-w)^{\nu-1}(D_{R_1}^{\nu}g)(w)\mathrm{d}w\right|\mathrm{d}s\right] \text{ by (2.41)} \tag{2.54}$$

$$\le \left(\frac{N}{R_2^N - R_1^N}\right)\left[(N-1)!\sum_{k=1}^{n-1}|g^{(k)}(R_1)|\left|\sum_{\lambda=0}^{N-1}\frac{(-1)^{\lambda}}{(N-\lambda-1)!}R_2^{N-\lambda-1}\frac{(R_2-R_1)^{\lambda+k+1}}{(\lambda+k+1)!}\right| + \frac{\|D_{R_1}^{\nu}g\|_{\infty,[R_1,R_2]}}{\Gamma(\nu+1)}\left(\int_{R_1}^{R_2}(s-R_1)^{\nu}s^{N-1}\mathrm{d}s\right)\right]$$

$$\overset{(2.41)}{=} \left(\frac{N!}{R_2^N - R_1^N}\right)\left[\sum_{k=1}^{n-1}|g^{(k)}(R_1)|\left|\sum_{\lambda=0}^{N-1}\frac{(-1)^{\lambda}}{(N-\lambda-1)!}R_2^{N-\lambda-1}\frac{(R_2-R_1)^{\lambda+k+1}}{(\lambda+k+1)!}\right| + (\|D_{R_1}^{\nu}g\|_{\infty,[R_1,R_2]})\left(\sum_{\lambda=0}^{N-1}\frac{(-1)^{\lambda}}{(N-\lambda-1)!}R_2^{N-\lambda-1}\frac{(R_2-R_1)^{\lambda+\nu+1}}{\Gamma(\lambda+\nu+2)}\right)\right]. \tag{2.55}$$

We have established the following result.

Theorem 2.9. *Here $\nu \ge 2, n := [\nu]$. Suppose that f is* radial, *i.e. $f(x) = g(r), r \in [R_1, R_2], \forall x \in \bar{A}$. Assume that $g \in C_{R_1}^{\nu}([R_1, R_2])$.*

Then

$$E := \left| f(R_1\omega) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| = \left| g(R_1) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s)s^{N-1}\mathrm{d}s \right|$$
$$\le \left(\frac{N!}{R_2^N - R_1^N}\right) \left[\sum_{k=1}^{n-1} |g^{(k)}(R_1)| \left| \sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+k+1}}{(\lambda+k+1)!} \right| \right.$$
$$\left. + \, (\|D_{R_1}^\nu g\|_{\infty,[R_1,R_2]}) \left(\sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{\Gamma(\lambda+\nu+2)} \right) \right]. \tag{2.56}$$

Inequality (2.56) is sharp, namely it is attained by $g^*(s) = (s-R_1)^\nu$, $s \in [R_1, R_2]$.

Proof of Sharpness. Again by (1.17) we get

$$\|D_{R_1}^\nu g^*\|_{\infty,[R_1,R_2]} = \Gamma(\nu+1). \tag{2.57}$$

Also it holds that $g^{*(k)}(R_1) = 0, k = 1, \ldots, n-1$.

Thus

$$\text{LHS (2.56)} = \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} (s-R_1)^\nu s^{N-1}\mathrm{d}s$$
$$= \left(\frac{N!\Gamma(\nu+1)}{R_2^N - R_1^N}\right) \left[\sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{\Gamma(\lambda+\nu+2)} \right] = \text{RHS (2.56)}. \quad \square \tag{2.58}$$

We give

Corollary 2.1. *With the terms and assumptions of Theorem 2.9, additionally assume that* $g^{(k)}(R_1) = 0, k = 1, \ldots, n-1$.

Then

$$E \le \left(\frac{N!}{R_2^N - R_1^N}\right) \left(\sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{\Gamma(\lambda+\nu+2)} \right) \cdot \|D_{R_1}^\nu g\|_{\infty,[R_1,R_2]}. \tag{2.59}$$

We continue from Remark 2.5 with

Remark 2.6. We treat here the general, not necessarily radial, case of f. We apply (2.56) to $f(r\omega)$, ω fixed, $r \in [R_1, R_2]$. We then have

$$\left| f(R_1\omega) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} f(s\omega)s^{N-1}\mathrm{d}s \right| \le \left(\frac{N!}{R_2^N - R_1^N}\right)$$
$$\times \left[\sum_{k=1}^{n-1} \left| \frac{\partial^k f}{\partial r^k}(R_1\omega) \right| \left| \sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+k+1}}{(\lambda+k+1)!} \right| \right.$$
$$\left. + \left\| \frac{\partial_{R_1}^\nu f}{\partial r^\nu} \right\|_{\infty,\bar{A}} \left(\sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{\Gamma(\lambda+\nu+2)} \right) \right]. \tag{2.60}$$

Therefore

$$\left| \frac{\int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{\omega_N} - \frac{N}{(R_2^N - R_1^N)\omega_N} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)s^{N-1}\mathrm{d}s \right) \mathrm{d}\omega \right|$$
$$\le \left(\frac{N!}{R_2^N - R_1^N}\right) \left[\sum_{k=1}^{n-1} \left(\frac{\int_{S^{N-1}} \left| \frac{\partial^k f}{\partial r^k}(R_1\omega) \right| \mathrm{d}\omega}{\omega_N} \right) \left| \sum_{\lambda=0}^{N-1} \frac{(-1)^\lambda}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+k+1}}{(\lambda+k+1)!} \right| \right.$$

$$+ \left\| \frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}} \left(\sum_{\lambda=0}^{N-1} \frac{(-1)^{\lambda}}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{(\lambda+\nu+2)} \right) \Bigg] =: \delta. \tag{2.61}$$

That is,

$$\left| \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{2\pi^{N/2}} - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| \leq \delta. \tag{2.62}$$

Consequently it holds for $x \in \bar{A}$ that

$$\left| f(x) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{2\pi^{N/2}} \right| + \delta. \tag{2.63}$$

We have established the next result.

Theorem 2.10. *Here $\nu \geq 2, n := [\nu]$. Consider $f \in C^n(\bar{A})$ and assume that there exists $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A}), x \in \bar{A}$. Then*

$$\begin{aligned} M &:= \left| f(x) - \frac{\int_A f(y)\mathrm{d}y}{\mathrm{Vol}(A)} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega}{2\pi^{N/2}} \right| \\ &\quad + \left(\frac{N!}{R_2^N - R_1^N} \right) \left[\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}} \sum_{k=1}^{n-1} \left(\int_{S^{N-1}} \left| \frac{\partial^k f}{\partial r^k}(R_1\omega) \right| \mathrm{d}\omega \right) \right. \\ &\quad \times \left| \sum_{\lambda=0}^{N-1} \frac{(-1)^{\lambda}}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+k+1}}{(\lambda+k+1)!} \right| \\ &\quad \left. + \left\| \frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}} \left(\sum_{\lambda=0}^{N-1} \frac{(-1)^{\lambda}}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{(\lambda+\nu+2)} \right) \right]. \end{aligned} \tag{2.64}$$

We finish with

Corollary 2.2. *With the terms and assumptions of Theorem 2.10, additionally assume that $\frac{\partial^k f}{\partial r^k}, k = 1, \ldots, n-1$, vanish on $\partial B(0, R_1)$. Then*

$$\begin{aligned} M &\leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}} \int_{S^{N-1}} f(R_1\omega)\mathrm{d}\omega \right| \\ &\quad + \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{\lambda=0}^{N-1} \frac{(-1)^{\lambda}}{(N-\lambda-1)!} R_2^{N-\lambda-1} \frac{(R_2-R_1)^{\lambda+\nu+1}}{\Gamma(\lambda+\nu+2)} \right) \left\| \frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}} \right\|_{\infty,\bar{A}}. \end{aligned} \tag{2.65}$$

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