A special planar satisfiability problem and a consequence of its NP-completeness

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Received 18 April 1989; revised 13 October 1992

Abstract

We introduce a weaker but still NP-complete satisfiability problem to prove NP-completeness of recognizing several classes of intersection graphs of geometric objects in the plane, including grid intersection graphs and graphs of boxicity two.

1. Introduction

Intersection graphs of different types of geometric objects in the plane gained more attention in recent years, mainly in connection with fast development of computational geometry and computer science. Just to mention the most frequently cited classes, these are interval graphs, circular arc graphs, circle graphs, permutation graphs, etc. If we consider only connected objects (more precisely arc-connected sets) the most general class of intersection graphs are string graphs (intersection graphs of curves in the plane) which were originally introduced by Sinden [16] in the connection with thin film RC-circuits. String graphs were then considered by several authors [4, 6, 7]. In a recent paper [8], I have shown that recognition of string graphs is NP-hard and in fact, the method developed in [8] is refined in this note to obtain other NP-completeness results. It is striking that so far no finite algorithm for string graph recognition is known.

It seems that relatively simpler classes will arise if we consider straight-line segments instead of curves and furthermore, if these segments are allowed to follow only a bounded number of directions. Let us define the following classes of graphs (here $k$ is a positive integer):
- $k$-DIR = the class of intersection graphs of straight-line segments parallel with at most $k$ directions;
- PURE-$k$-DIR = the class of $k$-DIR graphs which have a representation by straight-line segments parallel with at most $k$ directions such that every two parallel segments are disjoint;
 SEG = the class of intersection graphs of straight-line segments in the plane.
In Fig. 1, we depict a 2-DIR and a PURE-3-DIR representation of $K_3$, a PURE-2-DIR representation of $K_{3,3}$ is depicted in Fig. 2.

Straightforwardly, SEG = $\bigcup_{k=1}^{\infty} k$-DIR. We have proved in [9] that also SEG = $\bigcup_{k=1}^{\infty} \text{PURE-}k$-DIR, and in fact that every SEG graph has a SEG representation in which no two segments are parallel. The complexity of the above defined classes really increases with increasing $k$:

- For every $k$, recognizing $k$-DIR is in NP, but the recognition of SEG is only known to be in PSPACE [9];
- every 2-DIR graph has a 2-DIR representation of polynomial size, while already some 3-DIR graphs require representations of exponential size, and there are SEG graphs that require representations of double exponential size [9] (for the definition of the size of a representation see [9]);
- the INDEPENDENT SET problem is polynomially solvable for PURE-2-DIR graphs, but it is NP-complete when restricted to 2-DIR and PURE-3-DIR graphs [11].

The aim of this note is to show that recognition of $k$-DIR and PURE-$k$-DIR graphs in NP-complete for every fixed $k \geq 2$. For $k > 4$, this follows also from the proof of the NP-hardness of string graph recognition in [8]. For the sake of completeness, we include the proof of this case also in here. However, the most difficult (to prove NP-completeness) is the case $k = 2$. For this, a more restricted satisfiability NP-completeness result is needed, and we introduce it in the next section. In Section 4, we pay closer attention to PURE-2-DIR graphs, called grid intersection graphs in [2]. It follows from our result that recognizing graphs of boxicity 2 is NP-complete (the concept of boxicity was developed by Roberts [15]: a graph has boxicity $\leq k$ if it is
isomorphic to the intersection graph of a family of boxes in the \( k \)-dimensional space). This determines the complexity of the last open case, since graphs of boxicity 1 (which are exactly interval graphs) are easy to recognize and Yannakakis [20] proved that for every fixed \( k \geq 3 \), recognition of graphs of boxicity \( k \) is NP-complete.

2. Planar satisfiability with bounded occurrences of variables

We consider the following problem.

4-BOUNDED PLANAR 3-CONNECTED 3-SAT (4P3C3SAT).

Instance: A formula \( \Phi \) with a set of clauses \( C \) over a set of variables \( X \) satisfying
(i) every clause contains exactly 3 distinct variables;
(ii) every variable occurs in at most 4 clauses;
(iii) the bipartite graph \( G_\Phi = (X \cup C, \{ xc \mid x \in c \in C \text{ or } \neg x \in c \in C \}) \) is planar and vertex 3-connected.

Question: Is \( \Phi \) satisfiable?

Remarks. This problem differs from ordinary PLANAR 3-SAT [14] in requiring the 3-connectedness of \( G_\Phi \) and requiring that maximum degree of \( G_\Phi \) is bounded by 4. The restriction of degrees was considered also by Tovey [17], but he does not take planarity in account.

I have used the 3-connected version (without the condition (ii)) in [8] without actually proving its NP-completeness there.

Note that the condition (iii) implies that every variable occurs in at least 3 clauses. Note also that the degree condition (ii) is the strongest possible, since requiring ‘(ii) every variable occurs in at most 3 clauses’ yields a polynomially solvable instance (every such formula is satisfiable).

Theorem 2.1. The problem 4P3C3SAT is NP-complete.

Proof. We show that PLANAR 3-SAT \( \preceq \) 4P3C3SAT. The PLANAR 3-SAT problem is known to be NP-complete [14] (however, using the reductions developed below, we could have started just with an instance of PLANAR SAT which is NP-complete by [13]).

Let \( \Phi_0 \) be an instance of PLANAR 3-SAT. Consider the following restrictions
(a) every clause contains exactly three variables;
(b) \( G_\Phi \) is planar;
(c) \( G_\Phi \) is connected;
(d) every variable occurs in at least three clauses;
(e) every variable occurs in at most four clauses;
(f) \( G_\Phi \) is 2-connected;
(g) \( G_\Phi \) is 3-connected.
Thus we start with $\Phi_0$ which satisfies (a) and (b) and we may suppose that (c) holds as well. Then we construct formulas $\Phi_1$ (satisfying (a)–(d)), $\Phi_2$ (satisfying (a)–(e)), $\Phi_3$ (satisfying (a)–(f)) and finally $\Phi_4$ (satisfying (a)–(g)), so that for each $i = 1, 2, 3, 4$, $\Phi_{i-1}$ is satisfiable if and only if $\Phi_i$ is satisfiable. In each step, $\Phi_i$ is constructed from $\Phi_{i-1}$ by local replacements of certain type.

To describe these local replacements, we use several auxiliary constructions (formulas) called $\Psi$'s. The graphs of these formulas are depicted in Figs. 3 and 4. Variable vertices are marked by circles (○) and clause vertices by squares (□). The sign + (resp. −) along an edge (say xc) expresses that $x \in c$ (resp. $\neg x \in c$). For instance, $\Psi^1_{\Psi_1}(x)$ in Fig. 3 is the formula $(y) \land (x \lor u \lor \neg y) \land (v \lor \neg u \lor \neg y) \land (\neg u \lor \neg v \lor w) \land (\neg y \lor \neg v \lor \neg w)$. In each auxiliary formula, some elements are distinguished, and they will be used in the constructions for putting the auxiliary

![Fig. 3](image-url)
formulas together. These are either single vertices (in $\Psi_1$, $\Psi_2$, $\Psi_6$), or single edges (in $\Psi_3$), or pairs of edges (in $\Psi_4$, $\Psi_5$), or several vertices (in $\Psi_7$).

The formulas $\Psi_i$ represent operations on formulas in a natural way: If $\Phi$ is a formula and $x$ one of its variables, then $\Psi_1(\Phi, x) = \Psi_1(x) \land \Phi$, equivalently $G_{\Psi_1(\Phi, x)} = G_{\Psi_1(x)} \cup G_\Phi$, where we suppose that $\Phi$ and $\Psi(x)$ are disjoint except for the variable vertex $x$, i.e. $V(G_{\Psi_1(\Phi, x)}) \cap V(G_\Phi) = \{x\}$. We define the action of other $\Psi_i$'s in a similar way. Note here that $\Psi_2$ is not a correct formula according to our definition (the variable $y$ occurs twice in the clause $z$), but this is fixed in the formula $\Psi_6$.

The action of $\Psi_3^+(xc)$ on a formula $\Phi$ is defined only if $xc$ is a positive edge of $G_\Phi$, i.e. if $x \in c$. In the resulting formula $\Psi_3^+(\Phi, xc)$, the variable $x$ in the clause $c$ is replaced by $y$. Similarly, $\Psi_3^-$ is applied if $\neg x \in e$, and we set

$$\Psi_3^\pm(\Phi, xc) = \begin{cases} \Psi_3^+(\Phi, xc) & \text{if } x \in c, \\ \Psi_3^-(\Phi, xc) & \text{if } \neg x \in e. \end{cases}$$

In the figure, we only depict the structure of the formula $\Psi_4$, since there are four different possibilities of locating the signs along its edges. If $xc$ and $yd$ are edges of
a formula $\Phi$, the formula $\Psi_5(\Phi, xc, yd)$ is obtained from $\Psi_3(\Psi_3(\Phi, xc), yd)$ by identifying the vertices $w$ of $\Psi_3(\Phi, xc)$ and $\Psi_3(\Phi, yd)$. The formula $\Psi_5$ is obtained by iterating $\Psi_4$ in the way depicted in Fig. 4, and it is kind of a 3-connected bridge. The action of $\Psi_5$ is defined only when edges $xc, yd$ belong to the same face of a fixed drawing of $G_\Phi$.

Similarly, $\Psi_6$ is obtained by applying $\Psi_5$ on $\Psi_2$ in the way depicted in Fig. 4. Now $\Psi_6$ is already a correct formula.

The action of $\Psi_7$ is defined as follows. Let $\Phi$ be a planar formula and $x$ one of its variables. Let $c_1, c_2, \ldots, c_r$ be the clauses which contain $x$ or $\neg x$, listed in the clockwise order in which the edges $x c_i$ leave the vertex $x$ in a fixed planar drawing of $G_\Phi$. In the formula $\Psi_7(\Phi, x)$, the variable $x$ is replaced by $x_i$ in $c_i$ (respectively), and the variables $x_i$ are linked as depicted in the figure. The sign of $x c_i$ in $G_\Phi(c,\Phi,x)$ is the same as the sign of $xc_i$ in $G_\Phi$.

**Claim 1.** Let $\Phi$ be a formula satisfying (a)-(e). Then for any variable $x$ and any two edges $e, f$, the formulas $\Psi_4(\Phi, x), \Psi_5(\Phi, e, f), \Psi_7(\Phi, x)$ also satisfy (a)-(e).

**Claim 2.** The formula $\Psi_1(x)$ is satisfiable regardless a prescribed value of $x$ (hence for every $\Phi$ and every $x$, $\Phi$ is satisfiable if and only if $\Psi_1(\Phi, x)$ is satisfiable).

**Claim 3.** The formula $\Psi_2(x) - \{c\}$ is satisfiable, but $\Psi_3(c)$ is not.

**Proof.** A truth valuation $f$ with $f(x) = f(v) = true$ and $f(u) = false$ satisfies $\Psi_2(c) - \{c\}$. Suppose that $g$ satisfies $\Psi_2(c)$. Then $g(x) = false$, and it follows from $x, \beta, \gamma, \delta$ that $g(y) = g(u) = g(v) = g(w)$. Then either $\epsilon$ or $\phi$ remains unsatisfied. □

**Claim 4.** (1) The formula $\Psi_3^+(xc)$ is satisfiable with $f(x) = f(y) = true$ and both $f(w) = true$ or $f(w) = false$;

(2) the formula $\Psi_3^- (xc) - \{c\}$ is satisfiable with $f(y) = false$ and both $f(w) = true$ or $f(w) = false$;

(3) but $\Psi_3^- (xc) - \{c\}$ is not satisfiable with $f(x) = false$ and $f(y) = true$.

(1) The same is true for $\Psi_3^-(xc)$ with the values true and false reversed.

**Proof.** (1) A truth assignment $f$ with $f(x) = f(y) = true$ and both $f(w) = true$ or $f(w) = false$ satisfies $\Psi_3^+(xc)$.

(2) A truth assignment $f$ with $f(u) = true$ and $f(v) = f(y) = false$ satisfies $\Psi_3^+(xc) - \{c\}$.

(3) Suppose a valuation $f$ satisfies $\Psi_3^-(xc) - \{c\}$ with $f(y) = true$ and $f(x) = false$. Then $f(u) = f(v) - true$ and both $\delta$ and $\gamma$ should be satisfied by $w$, which is not possible. □

A straightforward corollary of the previous claim is

**Claim 5.** For every $\Phi$ and every $xc, yd \in E(G_\Phi)$, $\Phi$ is satisfiable if and only if $\Psi_4(\Phi, xc, yd)$ is satisfiable, that is if and only if $\Psi_5(\Phi, xc, yd)$ is satisfiable.
A direct consequence of Claims 3 and 5 is

**Claim 6.** The formula $\Psi_e(c) - \{c\}$ is satisfiable, but $\Psi_e(c)$ is not.

**Claim 7.** If $f$ satisfies $\Psi_\gamma(x)$ then $f(x_1) = f(x_2) = \cdots = f(x_r)$.

**Proof.** By the preceding claim, none of the clauses $x_1, x_2, \ldots, x_r$ is satisfied from the corresponding $\Psi_e(x_i)$. □

**Claim 8.** For every $\Phi$ and every $x$, $\Phi$ is satisfiable if and only if $\Psi_\gamma(\Phi, x)$ is satisfiable.

**Proof of Theorem 2.1 (conclusion).** Now we are ready to reveal the construction of $\Phi_i$, $i = 1, 2, 3, 4$. Given $\Phi_0$ satisfying (a)-(c), multiple action of $\Psi_1$ on variables which occur in less than 3 clauses yields $\Phi_1$ satisfying (a)-(d). Variables of degree greater than 4 are killed by action of $\Psi_\gamma$, in which way we obtain $\Phi_2$ satisfying (a)-(e). It follows from Claims 2 and 8 that $\Phi_2$ is satisfiable if and only if $\Phi_0$ is satisfiable.

If $G_{\Phi_2}$ is 2-connected we set $\Phi_3 = \Phi_2$. Otherwise, we choose an articulation $v \in V(G_{\Phi_2})$. Let $C_1, C_2, \ldots, C_r$ be the connected components of $G_{\Phi_2} - \{v\}$. Consider a planar drawing of $G_{\Phi_2}$ such that the components $C_1, C_2, \ldots, C_r$, meet the outerface in this clockwise order. Consider a component $C_i$. Since each vertex of $C_i$ has at least two neighbors in $C_i$, there is an edge $e_i$ with both endpoints in $C_i$ such that $e_i$ meets the outerface and $e_i$ is contained in a cycle of $C_i \cup \{v\}$. Construct a formula $\Phi'_2$ by setting $\Phi'_2 = \Psi_{\gamma}(\Psi_{\gamma}(\ldots \Psi_{\gamma}(\Phi_2, e_1, e_2), \ldots, e_2, e_3), e_1, e_2)$. None of the added vertices is an articulation and if a former vertex of $G_{\Phi_2}$ is an articulation of $G_{\Phi_2}$, it has been an articulation of $G_{\Phi_0}$ as well. Furthermore, $v$ is not an articulation anymore. Thus $G_{\Phi_2}$ has fewer articulations than $G_{\Phi_2}$. Iterating this construction, we finally obtain a formula $\Phi_3$ whose graph $G_{\Phi_3}$ is 2-connected.

The construction of $\Phi_4$ (i.e., killing two-element cut sets) is then analogous. □

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![Fig. 5. An illustration to the construction of a 2-connected $\Phi'_2$.](image-url)
3. Intersection graphs of segments

It is shown in [8] that recognizing intersection graphs of curves in the plane is NP-hard. In fact, the reduction used there can be slightly modified so that given a planar 3-connected 3-formula \( \Phi \), a graph \( G(\Phi) \) is constructed such that if \( \Phi \) is not satisfiable \( G(\Phi) \) is not representable by curves, while it is representable by straight-line segments if \( \Phi \) is satisfiable. Thus recognizing SEG graphs is NP-hard. (This modified construction is reviewed in [10].) We use the basic idea of that construction here again, some refinement is, however, necessary when considering segments in a low number of directions. Note also that we are repeatedly using the Jordan curve theorem in our considerations without explicitly announcing it. We use a slightly colloquial expression "a curve \( x \) is tied to a curve \( y \)" to describe in short the fact that there is a curve \( z \) which intersects both \( x \) and \( y \) and no other curve.

Let \( \Phi \) be an instance of 4P3C3SAT. We construct a graph \( G(\Phi) \) as follows:

1. Fix a rectilinear planar drawing \( D_\Phi \) of \( G_\Phi \) (that is, an embedding of \( G_\Phi \) in a grid such that the edges are piecewise linear and follow the grid lines, no edge passes through a vertex and no two edges cross). Such an embedding exists, since \( G_\Phi \) has no vertices of degree greater than four, and it can be constructed in polynomial time (cf. also [18]).

2. For every edge \( e = xc \in E(G_\Phi) \), let \( r(e) \) denote the number of linear pieces the drawing of \( e \) (in \( D_\Phi \)) consists of. Let \( L(e) \) (resp. \( R(e) \)) be a path of length \( r(e) - 1 \), and call its vertices \( l_1(e), l_2(e), \ldots, l_{r(e)}(e) \) (resp. \( r_1(e), r_2(e), \ldots, r_{r(e)}(e) \)). Set \( G_e = L(e) \cup R(e) \) (i.e., \( G_e \) consists of two disjoint paths). We call \( L(e) \) the left \( e \)-path and \( R(e) \) the right \( e \)-path.

3. Consider an uppermost (rightmost, bottommost, leftmost, respectively) linear segment of \( D_\Phi \) and let it form the \( i(u) \)th (\( i(r) \)th, \( i(b) \)th, \( i(l) \)th, respectively) linear piece of an edge \( e_c \) \( (e_r, e_b, e_l, \text{respectively}) \). Set \( V_1 = \{t(u), t(r), t(b), t(l), a, b, c, d\} \) and \( E_1 = \{at(u), bt(r), ct(b), dt(l), t(u)l_{i(u)}(e_a), t(r)l_{i(r)}(e_r), t(b)l_{i(b)}(e_b), t(l)l_{i(l)}(e_l), t(u)r_{i(u)}(e_a), t(r)r_{i(r)}(e_r), t(b)r_{i(b)}(e_b), t(l)r_{i(l)}(e_l)\} \).

4. Consider the graph \( F \) depicted in Fig. 6. We call it the frame. The frame involves the vertices \( a, b, c, d \) from the previous step, and it will be used to frame up the construction which follows.

5. For every variable \( x \), construct a so called variable gadget \( G_x \) as follows. Let \( x \) occur in 4 clauses and let \( c_1, c_2, c_3, c_4 \) be these clauses numbered in the clockwise order as the edge \( xc_i \) leave the vertex \( x \) in the drawing \( D_\Phi \). The variable gadget \( G_x \) is depicted in Fig. 7, where

\[
A_i = \begin{cases} 
  l_1(xc_i) \\
  r_1(xc_i)
\end{cases} \quad \text{and} \quad
B_i = \begin{cases} 
  r_1(xc_i) \\
  l_1(xc_i)
\end{cases} \quad \text{if} \quad \begin{cases} 
  x \in c_i, \\
  \neg x \in c_i.
\end{cases}
\]

(Thus the variable gadget \( G_x \) involves also the initial vertices of both left and right \( xc_i \)-paths.)
Fig. 6. The frame $F$.

If $x$ appears in 3 clauses only, the gadget is similar and is depicted in Fig. 8. Note that in this case the clauses $c_1, c_2, c_3$ are numbered so that in $D_\Phi$, the initial segments of the edges $xc_1$ and $xc_3$ are parallel (i.e., either both vertical or both horizontal).

(6) For each clause $c$, construct a clause gadget $G_c$ as follows. Let $x_1, x_2, x_3$ be the variables occurring in $c$ numbered in the clockwise order as the edges $x_ic$ leave the vertex $c$ in the drawing $D_\Phi$. The numbering is again such that the initial segments of the edges $x_1c$ and $x_3c$ are parallel. The clause gadget $G_c$ is depicted in Fig. 9. Note that it involves the end-vertices of the $x_ic$ paths.

(7) Finally, set $G(\Phi) = (V, E)$, where

$$V = V(F) \cup \bigcup_{e \in E(G_\Phi)} V(G_e) \cup V_1 \cup \bigcup_{x \in X} V(G_x) \cup \bigcup_{c \in C} V(G_c)$$

and

$$E = E(F) \cup \bigcup_{e \in E(G_\Phi)} E(G_e) \cup E_1 \cup \bigcup_{x \in X} E(G_x) \cup \bigcup_{c \in C} E(G_c).$$
We claim that $\Phi$ is satisfiable if and only if $G(\Phi) \in 2$-DIR. Furthermore, if $\Phi$ is satisfiable then $G(\Phi) \in$ PURE-2-DIR. To prove this, we first investigate some properties of the gadgets.

**Claim 1.** The frame $F$ is in PURE-2-DIR. In every SEG representation of $F$, there is exactly one region (let us call it $\Omega$) which meets all four segments $a, b, c, d$. This region is convex and it is bounded by these segments. (In particular, all four segments $a, b, c, d$ cannot reach the outerface of the representation simultaneously.)

**Proof.** A PURE-2-DIR representation of $F$ is depicted in Fig. 6 (right).

Suppose $R$ is a representation of $F$ by curves such that any two curves share at most one common point. (Since $F$ is bipartite, every SEG representation of $F$ determines such an $R$.) Consider the curves $a, b, c, d$. They form a 4-cycle in the plane, thus dividing the plane into two regions, say $\Omega_1$ and $\Omega_2$. Each curve is divided by the
intersections with the neighboring curves into the inner part and two ends. Since
F = \{a, b, c, d\} is connected, the curves e, f, g, h must lie all inside the same region, say
\Omega_1. These curves are joined by curves representing vertices of degree two, forming
a cycle in \Omega_1. They are also tied to the curves a, b, c, d. One can check that they must
be tied to the ends of the curves a, b, c, d. Hence the region \Omega_1 is met by all ends of the
curves a, b, c, d (cf. Fig. 10 left).

When the curves a, b, c, d are straight, this is possible only if \Omega_1 is the outerface.
(One can argue as follows: Denote the crossing points of the curves a, b, c, d by
A, B, C, D. Thus the inner parts of the curves bound a quadrilateral ABCD, which
cannot have more than one angle of size greater than \pi, and only at such angles the
extensions of the sides point into the interior of ABCD. Hence the inner region
bounded by a, b, c, d contains at most two end points of these segments.) A SEG
representation then looks like that depicted in Fig. 6 right or Fig. 10 right.

Claim 2. Let us call C(x) the cycle \{A_1, B_1, ..., A_4, B_4\}. Suppose R_x is a SEG
representation of the variable gadget G_x such that there is a region \Omega_x which contains
all intersecting points of the curves and such that for every i = 1, 2, 3, (4), the segments A_i, B_i
leave the boundary of \Omega_x next to each other, and the pairs \{A_i, B_i\}, i = 1, 2, 3, (4) leave
the boundary in this clockwise order. Then the segments A_1, B_1 leave the boundary
of \Omega_x either in the clockwise order A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4, or in the order
A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4.

Proof. The realization of the cycle C(x) is either faithful with its orientation, in which
case the crossing points of A_1, B_2 and A_3, B_4 are inside this cycle and the situation
looks as in Fig. 11 left, or (as in Fig. 11 right) its orientation is reversed and the
crossing points of A_1, B_2 and A_3, B_4 are outside. (It may, however, happen that the
crossing points of A_2, B_1 and A_4, B_3 drop outside (resp. inside) the cycle).

Claim 3. Consider a clause gadget G_c. For the sake of simplicity let us write
l_i = l_{r(x_i)}(x_i) and r_i = r_{l(x_i)}(x_i). Let R_c be a 2-DIR representation of G_c, such that there

Fig. 10.
exists a region $Q_c$ which contains all intersecting points of $R_c$ and the segments $r_i, l_i$ meet the boundary of $Q_c$. Then they cannot meet it in the (clockwise) order $r_1, l_1, r_2, l_2, r_3, l_3$. Conversely, whenever at least one pair $(r_i, l_i)$ is permuted, $G_c$ has a PURE-2-DIR representation respecting this order.

**Proof.** The clause gadget $G_c$ contains a copy of the frame $F$, which, by Claim 1, has only two essentially different representations (depicted in Fig. 6 and 10 right). Since in that in Fig. 10 right, three of the segments $e, f, g, h$ do not meet the outerface, $R_c$ must contain the frame realized as that in Fig. 6 right. There are two regions which meet the segments $a, b, c$, these are $Q$ (with the boundary $abed$) and $Q'$ (with the boundary $abcf$ and the short segments connecting $a$ and $c$ to $f$). Since the union of the segments $w, u_1, v_2, u_3$ is connected and $w$ crosses only $u_1$ and $u_3$, the segment $w$ must lie in $Q \cup Q'$. However, both the segments $u_2, v_2$ cross $b$, and thus the intersecting points of the segments $u_1, v_1$ and $v_2$ lie inside $Q$. Therefore $w$ and the intersecting points of $v_3, u_3$ with $u_2, v_2$ lie inside $Q$ as well. The segments $u_1, v_1, u_2, v_2, u_3, v_3$ meet the boundary of $Q$ in the same order as the segments $r_1, l_1, r_2, l_2, r_3, l_3$ meet the boundary of $Q_c$.

In such a representation, the segments $v_1, v_2, v_3$ divide $Q$ into two regions, $\alpha$ with the boundary $dav_1v_2v_3c$ and $\beta$ with the boundary $bcv_3v_2v_1a$ (cf. Fig. 12). If the segments $u_i, v_i$ meet the boundary of $Q$ in the order $u_1, v_1, u_2, v_2, u_3, v_3$, it is clear that $u_1 \subset \alpha, u_3 \subset \beta$ and inserting the segment $w$ which should join $u_1$ and $u_3$ is impossible.

If the order of $u_1$ and $v_1$ is reversed, both $u_1$ and $u_3$ lie in $\beta$ and $w$ can be added there. If $u_3$ and $v_3$ are reversed, $w$ can be inserted inside $\alpha$. In the last case, when the order is $u_1, v_1, u_2, v_2, u_3, v_3$, we realize $w$ inside $\alpha$. All these cases are illustrated in Fig. 13.

Now we are ready to prove the statement on $G(\Phi)$. Suppose first that $G(\Phi)$ has a 2-DIR representation $R$. Denote by $R_1$ the representation of the frame $F$ induced by $R, R_2$ the four tying segments $t(a), t(r), t(b), t(l)$, and $R_3 = R \setminus (R_1 \cup R_2)$. Thus $R_3$ is a representation of the graph $G_0(\Phi) = \bigcup_{e \in E(G_0)} G_e \cup \bigcup_{x \in X} G_x \cup \bigcup_{c \in C} G_c$. 


Let $\Omega$ be the inner region of $R_1$, as in Claim 1. Since $G_0(\Phi)$ is connected, we have either $R_3 \subseteq \Omega$ or $R_3 \cap \Omega = \emptyset$. But the segments of $R_1$ tie $R_3$ to all four segments $a, b, c, d$ and hence by Claim 1, $R_3 \subseteq \Omega$.

Consider a variable $x$ and the representation of the cycle $C(x)$ in $G_x$ induced by $R$. Let us call the region bounded by this cycle $\Omega'$. Since $G_0(\Phi) - G_x$ is connected, the segments corresponding to the vertices of $G_0(\Phi) - G_x$ lie either all inside $\Omega'$, or all of them outside $\Omega'$. Since at least one of them is tied to the frame, they must lie outside $\Omega'$. Similarly for the representation of the clause gadgets.

It follows that for every vertex $v$ of $G_\Phi$, there exists a region $\Omega_v$ such that all intersections of the segments of the corresponding gadget $G_v$ lie inside $\Omega_v$. Let these regions be chosen small enough to be pairwise disjoint.

Contract each region $\Omega_v$ into a single point $v$. A planar drawing $D'_\Phi$ of $G_\Phi$ is obtained, in which the edges are doubled (each edge $e$ is realized by the representation of the left and right $e$-path). Since $G_\Phi$ is 3-connected, it has a unique (from the topological point of view) drawing in the plane. Since the $e_{a\cdot}, e_{b\cdot}, e_{c\cdot},$ and $e_{d\cdot}$-paths are tied to the frame, it is uniquely determined which face of $D'_\Phi$ is the outerface. Therefore we may suppose without loss of generality that $D'_\Phi$ coincides with $D_\Phi$, the rectilinear drawing of $G_\Phi$ we started the construction of $G(\Phi)$ with. In particular, for every variable $x$, the segments $A_i, B_i$ leave the boundary of $\Omega_x$ next to each other and the pairs $(A_1, B_1), (A_2, B_2), (A_3, B_3), (A_4, B_4)$ leave the boundary of $\Omega_x$ in this clockwise
order. Similarly, for every clause $c$, the segments $r_i, l_i$ arrive to the boundary of $\Omega_c$ next to each other, and the pairs $(r_1, l_1), (r_2, l_2), (r_3, l_3)$ arrive to the boundary of $\Omega_c$ in this order.

By Claim 2, there are just two ways in which the variable gadgets may be realized. Check every variable and set $f(x) = true$ if $G_x$ is realized as in the second variant offered by Claim 2 (i.e., the connecting segments leave $\Omega_x$ as in Fig. 11 right), and set $f(x) = false$ otherwise. Due to the construction of $G(\Phi)$ we get

**Claim 5.** A variable $x$ receives the value true in the clause $c$ if and only if the left and right $xc$-paths arrive to the boundary of $\Omega_c$ in this clockwise order (cf. Fig. 14).

Having this, we can reformulate Claim 3 as follows: $G_c$ is 2-DIR representable in $\Omega_c$ if and only if at least one variable receives the value true in $c$. It follows that $\Phi$ is satisfied by $f$.

Conversely, the construction of a PURE-2-DIR representation of $G(\Phi)$ based on a satisfying truth assignment $f$ to $\Phi$ is now at hand. Thus we have proved the following theorem.

**Theorem 3.1.** Recognition of 2-DIR and PURE-2-DIR graphs are NP-complete problems.

We can use a very similar construction for $k$-DIR graphs when $k \geq 3$. In this case, we start with a rectilinear drawing $D_0$ in which for every clause vertex $c$, the edges incident with $c$ leave it upwards, to the right and downwards. The construction of $G(\Phi)$ then differs only in taking another clause gadget, depicted in Fig. 15. We have

**Claim 3'.** The clause gadget $G_c$ has a SEG representation in $\Omega_c$ if and only if at least one variable receives the value true in $c$. In such a case, $G_c$ has a PURE-3-DIR representation.

**Proof.** Suppose $G_c$ has a SEG representation respecting the order $r_1, l_1, r_2, l_2, r_3, l_3$. Then there is a region $\Omega'_c \subset \Omega_c$ such that the segments $u_i, v_i$ lie in $\Omega'_c$ and meet its boundary in the order $u_1, v_1, u_2, v_2, u_3, v_3$.

One can check that even if the segments are not required to be straight (we only require that any two of them share at most one common point), the representation is unique (from the topological point of view), depicted in Fig. 16. It is then an easy exercise to show that all six segments cannot be stretched at once.

PURE-3-DIR representations of the cases when at least one variable receives the value true in $c$ are depicted in Fig. 17. 0

Thus we have proved the following theorem.
**Theorem 3.2.** For every \( k \geq 3 \), recognition of \( k\)-DIR and PURE-\( k\)-DIR graphs are NP-complete problems.

Note that we have only proved NP-hardness of the recognition problems. Belonging to the class NP is straightforward for the recognition of 2-DIR and PURE-2-DIR graphs (cf. Fig. 18). For \( k > 2 \), it is proved in [9].
4. PURE-2-DIR representations with preordered segments

We have just seen that even recognizing PURE-2-DIR graphs (which are bipartite) in NP-complete. With J. Nešetřil, we considered possibilities of imposing further restrictions which could define polynomially solvable problems. One of such restrictions – ordering the segments of the representation – is considered in this section. It results in one trivially polynomial case, and one open question.
Throughout this section, $G$ is a bipartite graph with color classes $V$ and $H$, which are linearly ordered, say $V = \{v_1, v_2, \ldots, v_n\}$ and $H = \{u_1, u_2, \ldots, u_m\}$. It will be supposed that vertices of $V$ (resp. $H$) are represented by vertical (resp. horizontal) segments. In a representation, every vertical segment $v$ is described by a triple $(x(v), y_1(v), y_2(v))$, where $[x(v), y_1(v)]$ and $[x(v), y_2(v)]$ are the endpoints of the segment. Similarly, every horizontal segment $u$ is described by a triple $(x_1(u), x_2(u), y(u))$.

Since in a PURE-2-DIR representation segments do not overlap, we will consider without loss of generality only representations such that no two vertical (resp. horizontal) segments lie on the same line. Such a representation will be called a grid representation. A grid representation of $G$ is called ordered if $x(v_1) < x(v_2) < \cdots < x(v_n)$ and $y(u_1) < y(u_2) < \cdots < y(u_m)$.

**Observation 4.1.** A bipartite graph $G = (V \cup H, E)$ with ordered color classes has an ordered grid representation if and only if there are no six indices $a < b < c, i < j < k$ such that $v_bu_i, v_bu_j, v_bu_k \in E$ (we call such a configuration, illustrated also in Fig. 19, a volkswagen).

**Proof.** Suppose $G$ has an ordered grid representation. We may suppose that $x(v_i)$ (resp. $y(u_j)$) are consecutive integers, say $x(v_i) = i, i = 1, 2, \ldots, n$ and $y(u_j) = j, j = 1, 2, \ldots, m$. In order to realize all desirable intersections, it has to be $y_1(v_i) < j < y_2(v_i)$ and $x_1(u_j) < i < x_2(u_j)$ whenever $v_i, u_j \in E$. Hence we set

\[
y_1(v_i) = \min \{ j \mid v_iu_j \in E \} - \varepsilon,
\]

\[
y_2(v_i) = \max \{ j \mid v_iu_j \in E \} + \varepsilon,
\]

\[
x_1(u_j) = \min \{ i \mid v_iu_j \in E \} - \varepsilon,
\]

\[
x_2(u_j) = \max \{ i \mid v_iu_j \in E \} + \varepsilon,
\]
with $0 < \varepsilon < 1$. All desirable intersections are thus realized, and undesirable crossings correspond exactly to volkswagen configurations.

**Corollary 4.2.** A bipartite graph is in PURE-2-DIR iff its color classes admit an ordering without volkswagens.

The preceding considerations can be reformulated in terms of matrices. The following approach which we discuss in [12] was suggested to me by V. Chvátal, and considered independently by Zelikovski and Gorpinevich [1990, personal communication] and by Hartman et al. in [2]. It is proved in [2] that every planar bipartite graph is a grid intersection (i.e., PURE-2-DIR) graph.

Given a bipartite graph $G = (V \cup H, E)$ with color classes $V = \{v_1, v_2, \ldots, v_n\}$ and $H = \{u_1, u_2, \ldots, u_m\}$, define its adjacency matrix $A_G = (A_{ij})_{i=1,\ldots,n, j=1,\ldots,m}$ by

$$A_{ij} = \begin{cases} 0 & \text{if } u_iv_j \notin E, \\ 1 & \text{if } u_iv_j \in E. \end{cases}$$

A volkswagen configuration corresponds to a cross in the adjacency matrix:

$$
\begin{pmatrix}
\vdots \\
1 \\
\vdots \\
\cdots 1 \cdots 0 \cdots 1 \cdots \\
\vdots \\
1 \\
\vdots
\end{pmatrix}
$$

Following [2], let us call a 0-1 matrix cross-free if it has no crosses, and cross-free-able if it can be turned into a cross-free matrix by premutations of rows and columns. Hence we have the following corollary.

**Corollary 4.3** (Kratochvíl and Nešetřil [12] and Ben-Arroyo Hartman et al. [2]).

A bipartite graph with ordered color classes is an ordered grid intersection graph iff its adjacency matrix is cross-free.

Obviously, one can decide in polynomial time ($O(nm)$) whether a given $n \times m$ matrix is cross-free, and consequently whether a given bipartite graph has an ordered grid.
representation. On the other hand, we have proved in the preceding section that deciding whether a given bipartite graph is a grid intersection graph is NP-complete, and thus deciding whether a given 0-1 matrix can be turned into a cross-free matrix by suitable permutations of columns and rows in NP-complete. We proposed the following relaxation in [12]: Call a grid representation of \( G \) *vertically ordered* if \( x(v_1) < x(v_2) < \cdots < x(v_n) \). Then \( G \) has a vertically ordered grid representation iff its adjacency matrix can be turned into a cross-free matrix by a suitable permutation of its rows. The computational complexity of this question is left as an open problem:

**Problem (Kratochvil and Nešetřil [12]).** Is there a polynomial algorithm for recognition of vertically ordered grid intersection graphs? Equivalently, is there a polynomial algorithm which, given a 0-1 matrix decides whether it can be transformed into a cross-free matrix by a suitable row permutation?

Zelikovski and Gorpinevich suggested [personal communication, cf. [12]] the following construction: Given a 0-1 matrix \( A \), define a graph \( G(A) \) whose vertices are pairs \((i, j)\) such that \( A_{ij} = 0 \), and \((i, j)(i', j')\) is an edge if \( A_{ij} = A_{i'j'} = 1 \). This graph is invariant under row and column permutations of \( A \), and thus if the chromatic number \( \chi(G(A)) \geq 5 \), \( A \) is not cross-free-able. Zelikovski asked whether the converse is also true. We have observed with O. Zýka that this is not the case. A simple counterexample is the graph \( K_{2,3} \) (i.e., \( K_{2,3} \) with each edge subdivided by a new extra vertex). It is well known that \( K_{2,3} \) cannot be represented as the intersection graph of curves in the plane [4, 8], hence it is not a grid-intersection graph and \( A_{K_{2,3}} \) is not cross-free-able. On the other hand, \( \chi(G(A_{K_{2,3}})) \leq 4 \).

5. Isothetic rectangles and boxicity two

The notion of boxicity was introduced by Roberts [15], setting \( \text{box}(G) = \min \{k \mid G \text{ is isomorphic to the intersection graph of a family of boxes in the } k \text{-dimensional space} \} \) (here boxes are \( k \)-dimensional intervals, i.e., cartesian products of closed intervals). Note that it is not quite obvious at first sight that every graph has finite boxicity. Boxicity of graphs was then studied by other authors. Cozzens [3] proved that determining the boxicity of graphs is NP-hard, and Yannakakis [20] proved that for every fixed \( k \geq 3 \), recognizing graphs of boxicity at most \( k \) is NP-complete. Since graphs of boxicity one are polynomially recognizable (these are interval graphs), the only case for which the complexity of the recognition problem was unresolved were graphs of boxicity two [5]. We remark that these graphs were also considered by Wood [19], called *intersection graphs of isothetic rectangles*. In particular, he asks the question on the complexity of their recognition in [19, Problem 3.8]. This problem is answered by the following Corollary.

**Corollary 5.1.** Recognizing intersection graphs of isothetic rectangles in the plane (i.e. recognizing graphs of boxicity two) is NP-complete.
Proof. One can check directly following the proof of Theorem 2.1 that if a formula $\Phi$ is not satisfiable then the graph $G(\Phi)$ cannot be represented as the intersection graph of isothetic rectangles in the plane. However, in a so far unpublished paper [1], Hartman et al. proved that every bipartite graph of boxicity two is a grid intersection graph. Thus the statement follows directly from Theorem 2. □

Acknowledgment

This research was partially supported by EC Cooperative Action IC-1000 (project ALTEC: Algorithms for Future Technologies) and by Charles University Research grants GAUK 361 and GAUK 351.

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