On parallelism in Steiner systems

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Abstract

We show that in any Steiner system $S(k, k + 1, v)$ with $k \geq 4$, there are at least $[2(v + 2)/(k + 1)]$ pairwise disjoint blocks.

1. Introduction

A Steiner system $S(k, k + 1, v)$ is a pair $(Q, q)$ where $Q$ is a set of $v$ elements and $q$ is a collection of $(k + 1)$-subsets of $Q$ such that every $k$-subset of $Q$ is contained in exactly one member of the collection $q$. The elements of $Q$ are called points; those of $q$ are called blocks. It is well known that an $S(2, 3, v)$ exists if and only if $v = 1$ or $3 \pmod{6}$; an $S(3, 4, v)$ exists if and only if $v = 2$ or $4 \pmod{6}$.

A subset $\pi \subseteq q$ is called a partial parallel class provided that the blocks of $\pi$ are pairwise disjoint. If $\pi$ actually partitions $Q$, then $\pi$ is termed a parallel class.

There has been a great deal of interest in recent years in determining lower bounds on the size of partial parallel classes in arbitrary Steiner systems (see Gionfriddo [1]). We mention here the Lindner–Phelps bound [3], for Lo Faro [4–6] has recently shown that it holds for $k = 2, 3$ without the stated restriction on $v$: in any Steiner system $S(k, k + 1, v)$ with $v \geq k^4 + 3k^3 + k^2 + 1$ there is a partial parallel class containing at least $(v - k + 1)/(k + 2)$ blocks.

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Gionfriddo [1] proved that in any Steiner system $S(k, k + 1, v)$ with $k \geq 3$ there are at least $[(v + 2)/2k]$ pairwise disjoint blocks. His technique involved assuming the existence of $t$ disjoint blocks which do not cover at least $(k - 1)t + 2(k - 1)$ points, and then showing that one can in fact construct $t + 1$ disjoint blocks.

We employ Gionfriddo's method using instead a family of functions of the form $(k - r)t + (k + r) \ (0 \leq r \leq k)$ and so improve the bound to $[2(v + 2)/3(k + 1)]$.

2. The main theorem

In this section we give a proof of our main result.

**Theorem.** Let $(Q, q)$ be a Steiner system $S(k, k + 1, v)$ where $k \geq 4$. Then $(Q, q)$ contains at least $[\frac{3}{2}(v + 2)/(k + 1)]$ pairwise disjoint blocks.

**Proof.** Let $(Q, q)$ be an $S(k, k + 1, v)$ and let $\pi$ be a partial parallel class of size $t$. Let $P_\pi = \bigcup_{b \in \pi} b$. Let $r$ be any real number with $0 \leq r \leq k$ and suppose that $|Q - P_\pi| \geq (k - r)t + (k + r)$. Let $\{A_1, A_2, \ldots, A_{k-1}\}$ be a fixed $(k - 1)$-set in $Q - P_\pi$ and let

$$M = \{u \in P_\pi : \exists z \in Q - P_\pi \text{ such that } \{A_1, A_2, \ldots, A_{k-1}, u, z\} \in q\}.$$

Let there be $m_0$ blocks of $\pi$ each containing at least $[k - r + 1]$ elements of $M$, and for each $i = [r + 1], \ldots, k + 1$ let $m_i$ denote the number of blocks of $\pi$ containing exactly $k - i + 1$ elements of $M$.

We will suppose that $\pi$ is maximal, i.e. that for every block $b \in q$, $b \subseteq Q - P_\pi$. Then

$$(k + 1)m_0 + ([k - r])m_{r+1} + \cdots + 2m_{k-1} + m_k$$

$$\geq |M| = |Q - P_\pi| - (k - 1) \geq (k - r)t + (r + 1)$$

$$= (k - r)(m_0 + m_{r+1} + \cdots + m_k + m_{k+1}) + (r + 1).$$

Thus

$$m_0 \geq \left( \sum_{i=[r+1]}^{k+1} \frac{i - r - 1}{r + 1} m_i \right) + 1 \quad \text{(1)}$$

Consider now the blocks in $(Q, q)$ that contain $A_1, A_2, \ldots, A_{k-1}$ together with an element of $Q - P_\pi$ and an element of $P_\pi$. We write this set of blocks as a disjoint union $M_0 \cup M_{r+1} \cup \cdots \cup M_k$, where

$$M_0 = \{\{A_1, A_2, \ldots, A_{k-1}, X^{(0)}_{\alpha, \beta}, a^{(0)}_{\alpha, \beta}\} : 1 \leq \alpha \leq m_0, 1 \leq \beta \leq \beta(\alpha)\}$$

and for each $i = [r + 1], \ldots, k$

$$M_i = \{\{A_1, A_2, \ldots, A_{k-1}, X^{(i)}_{\alpha, \beta}, a^{(i)}_{\alpha, \beta}\} : 1 \leq \alpha \leq m_i, 1 \leq \beta \leq k - i + 1\}.$$
We will adopt the convention that $X_{\alpha,\beta}^{(i)} \in Q - P_\alpha$ and $a_{\alpha,\beta}^{(i)} \in P_\alpha$; furthermore, we will agree that we have labelled the blocks in $P_\alpha$ so that the element $a_{\alpha,\beta}^{(i)}$ belongs to block number $\alpha$ among the $m_\alpha$ blocks containing $k - i + 1$ elements of $M$, while the element $a_{\alpha,\beta}^{(0)}$ belongs to block number $\alpha$ among the $m_\alpha$ blocks containing at least $|k - r + 1|$ elements of $M$. Note that in $M_0$, $\beta(\alpha)$ is a function of $\alpha$; in particular $|k - r + 1| \leq \beta(\alpha) \leq k + 1$ for all $1 \leq \alpha \leq m_0$.

We will make occasional reference to the following observation, which is a consequence of inequality (1) and the inequalities leading up to it:

$$|M_0| \geq k + 1. \quad (2)$$

We now determine conditions on $r$ under which a partial parallel class of size $\geq |\pi|$ can be constructed.

Suppose first that $m_0 = 1$, i.e. $|b_0 \cap M| \geq |k - r + 1|$ and for all $b \in \pi - \{b_0\}$, $|b \cap M| \leq |k - r|$. From inequalities (2) and (1) it follows that $|b_0 \cap M| = k + 1$ (i.e. $b_0 \subseteq M$), $m_0 = 0$ for all $|r + 2| \leq i \leq k + 1$ and $t = 1 + m_{[r+1]}$. If $m_{[r+1]} = 0$ then $t = 1$, whence $|Q - P_\alpha| \geq 2k$ and so $v \geq 3k + 1$; but this contradicts the maximality of $\pi$ (since $k \neq 2$ no block in an $S(k, k + 1, v)$ on $v \geq 3k + 1$ points can constitute a maximal partial parallel class; see theorem 3.3 of [2]). Thus it must be that $m_{[r+1]} \geq 1$ where $r$ is an integer, $0 \leq r \leq k$. Provided that $0 \leq r \leq k - 2$ we can choose a $(k - 1)$-subset $Y$ of $\{X_{1,\beta}^{(0)}: 1 \leq \beta \leq k + 1\}$ and form two disjoint blocks

$$b_1 = Y \cup \{X_{1,\beta}^{(r+1)}: x\} \quad \text{and} \quad b_2 = \{A_1, A_2, \ldots, A_{k-1}, X_{\alpha,\beta}^{(0)}, a_{\alpha,\beta}^{(r)}\},$$

where $x$ and $a_{\alpha,\beta}^{(0)}$ occur in the same block $b$ of $\pi$. We illustrate the method. If $r \leq k - 3$, then the choice $Y = \{X_{1,\beta}^{(0)}: 1 \leq \beta \leq k + 1\}$ will work, for if $x = a_{\alpha,\beta}^{(0)}$ then take $a_{\alpha,\beta}^{(0)} = a_{\alpha,\beta}^{(r)}$; if $x = a_{\alpha,\beta}^{(0)}$ where $\beta \neq k + 1$ then take $a_{\alpha,\beta}^{(0)} = a_{1,k+1}^{(0)}$; if $x = a_{1,2}^{(r+1)}$ then take $a_{\alpha,\beta}^{(0)} = a_{1,2}^{(r+1)}$; if $x = a_{\alpha,\beta}^{(r+1)}$ where $\beta \neq 2$ then take $a_{\alpha,\beta}^{(0)} = a_{1,2}^{(r+1)}$; finally, if $x$ is anything else just choose an element $a_{\alpha,\beta}^{(r+1)}(\neq a_{1,1}^{(r+1)})$ from the same block as $x$ so that $a_{\alpha,\beta}^{(0)} \neq x$. Now, if $r = k - 2$ then the third possibility for $x$ presents a problem since $a_{1,3}^{(r+1)}$ does not exist. Instead we change our choice of $Y$ to

$$Y' = \{X_{1,\beta}^{(0)}, X_{1,2}^{(0)}, \ldots, X_{1,k}^{(0)}\} \quad \text{and} \quad b_1' = Y' \cup \{X_{1,1}^{(r+1)}: x'\};$$

then $x'$ cannot equal $x$, else $b_1'$ would have exactly $k$ elements in common with $b_1$. Since $x' \neq a_{1,2}^{(r+1)}$ we can now proceed as above; rename $b_1'$ to $b_1$. Then $(\pi \cup \{b_1, b_2\}) - \{b\}$ is a partial parallel class of size $|\pi| + 1$.

Suppose now that $m_0 = 2$. Let $Y$ be a $(k - 1)$-subset of the set

$$\{X_{\alpha,\beta}^{(0)}: 1 \leq \alpha \leq m_0, 1 \leq \beta \leq \beta(\alpha)\}.$$ 

Such a set exists from inequality (2). Let $\Gamma(Y)$ be the collection of blocks in $q$ of
the form \( Y \cup \{X^{(0)}_{\alpha, \beta}, x\} \) where \( x \in P_\pi \). Then

\[
|\Gamma(Y)| = |M| - (k - 1) - \sum_{i=r+1}^{k} (k - i + 1)m_i
\]

\[
= |Q - P_\pi| - 2(k - 1) - \sum_{i=r+1}^{k} (k - i + 1)m_i
\]

\[
\geq (k - r)t + (k + r) - 2(k - 1) - \sum_{i=r+1}^{k} (k - i + 1)m_i
\]

\[
= (k - r)m_0 + (k + r) - 2(k - 1) + \sum_{i=r+1}^{k+1} (i - r - 1)m_i. \quad (3)
\]

Let \( L \) be the following subset of \( P_\pi \):

\[
L = \{a^{(k)}_{\alpha, 1}; 1 \leq \alpha \leq m_k\} \cup \bigcup_{b \in \pi, b \cap M = \emptyset} b.
\]

Then clearly

\[
|L| = m_k + (k + 1)m_{k+1}. \quad (4)
\]

Let \( b' \) and \( b'' \) be blocks in \( \pi \) so that \( |b' \cap M| \geq |k - r + 1| \) and \( |b'' \cap M| \geq |k - r + 1| \), and suppose that \( b' \) and \( b'' \) are the largest such blocks in \( \pi \). We will take

\[
Y = \{X^{(0)}_{1,1}, X^{(0)}_{1,2}, \ldots, X^{(0)}_{1,k-r+1}, X^{(0)}_{2,1}, X^{(0)}_{2,2}, \ldots, X^{(0)}_{2,r+1}\}
\]

where the corresponding \( a^{(0)}_{1, \beta} \) are in \( b' \) and the \( a^{(0)}_{2, \beta} \) are in \( b'' \). (We are assuming of course that \( [r] \leq [k - r + 1] \), but we shall soon see that this is forced.) Consider now the difference \( |\Gamma(Y)| - |L| \). Substituting the right-hand side of inequality (1) for \( m_0 \) in inequality (3) and then subtracting equation (4) we obtain

\[
|\Gamma(Y)| - |L| \geq \sum_{i=r+1}^{k+1} \frac{(k + 1)(i - r - 1)}{r + 1} m_i + \left[ \frac{(k + 1)(k - r - 1) - (r - 1)}{r + 1} \right] m_{k+1} + 2.
\]

Thus, provided that \( 0 \leq r \leq (k - 1)/2 \) we get

\[
|\Gamma(Y)| - |L| \geq 2. \quad (5)
\]

Now if equality holds in (5), then equality holds in (3) and therefore in (1); but this implies that \( |b' \cap M| = |b'' \cap M| = k + 1 \). Provided that \( r > 0 \) it is then a simple matter to produce a partial parallel class of size \( |\pi| + 1 \); we just choose a block \( b_1 = Y \cup \{X^{(0)}_{\alpha, \beta}, x\} \) from \( \Gamma(Y) \) such that \( x \notin L \), together with a disjoint block

\[
b_2 = \{A_1, A_2, \ldots, A_{k-1}, X^{(i)}_{\alpha, \beta}, a^{(i)}_{\alpha, \beta}\}
\]

where \( a^{(i)}_{\alpha, \beta} \) and \( x \) occur in the same block \( b \) of \( \pi \)—then form the class \( (\pi \cup \{b_1, b_2\}) - \{b\} \).
We henceforth assume then that

$$|\Gamma(Y)| - |L| \geq 3.$$  \hspace{1cm} (6)

The remainder of the proof is divided into two cases.

**Case A:** \(|b' \cap M| \geq [k - r + 2]\) (whence \(r > 0\)).

Choose a block \(b_1 = Y \cup \{X_{\alpha, \beta}^{(0)}, x\}\) from \(\Gamma(Y)\) such that \(x \notin L\). Inequality (6) asserts that there are at least three such blocks. If there is such a block where \(x \notin b''\) then producing a partial parallel class of size \(|\pi| + 1\) is a simple matter, as above. Otherwise, use the provision \(0 < r \leq \frac{1}{2}(k-1)\) to observe that \([k - r + 1] \geq [r] + 1\). There is therefore an element \(X_{2, [r] + 1}^{(0)} \notin Y\). Since we have three choices for the element \(x \in b''\), at least one of the choices \(b_1 = Y \cup \{X_{\alpha, \beta}^{(0)}, x\}\) will have the properties

\(x \neq a_{2, [r] + 1}^{(0)}\) and \(X_{\alpha, \beta}^{(0)} \neq X_{2, [r] + 1}^{(0)}\).

Now set

\(b_2 = \{A_1, A_2, \ldots, A_{k-1}, X_{2, [r] + 1}^{(0)} \cup \{X_{\alpha, \beta}^{(0)}, x\}\}\)

and form the class \((\pi \cup \{b_1, b_2\}) - \{b''\}\).

**Case B:** \(|b' \cap M| = [k - r + 1]|b'' \cap M| = [k - r + 1]\).

Here we are supposing that for all \(b \in \pi\), \(|b \cap M| \geq [k - r + 1]\); we will need a slightly stronger inequality than (6) here. For this we return to the derivation of inequality (1), replacing the term \((k + 1)m_0\) by the term \([(k - r + 1)]m_0\); we obtain

\[ m_0 \geq \left( \sum_{i=[r+1]}^{k+1} (i - r - 1)m_i \right) + (r + 1). \]

Substituting the right-hand side of this inequality into inequality (3) and then subtracting equation (4) we obtain

\[ |\Gamma(Y)| - |L| \geq \sum_{i=[r+1]}^{k-1} (k - r + 1)(i - r - 1)m_i + [(k - r + 1)(k - r - 1) - 1]m_k \]

\[ + [(k - r + 1)(k - 1) - (k + 1)]m_{k+1} + r(k - r) + 2. \]

Using the provision \(1 \leq r \leq (k - 1)/2\) and the fact that \(k \geq 4\) the above implies

\[ |\Gamma(Y)| - |L| > 5. \hspace{1cm} (7) \]

Now we can proceed in analogous fashion to Case A, this time noting that the 'tricky' cases occur when \(x \in b' \cup b''\). (Since there are (at least) five choices for \(x\), one of them will correspond to a block \(b_1 = Y \cup \{X_{\alpha, \beta}^{(0)}, x\}\) where \(X_{\alpha, \beta}^{(0)}, x\) is disjoint from one of

\(\{X_{1, [k-1], \alpha, \beta}^{(0)}, \{k-r\}, \{X_{2, [r] + 1}, a_{2, [r] + 1})\}\).\)

What we have shown then is that if \(r\) is any real number with \(1 \leq r \leq \frac{1}{2}(k - 1)\) and \(\pi\) is a partial parallel class of size \(t\) in a Steiner system \(S(k, k + 1, v)\) \((k \geq 4)\)
on $v \geq (2k - r + 1)t + (k + r)$ points then one can construct a partial parallel class of size $t + 1$. It follows then that an $S(k, k + 1, v)$ has a partial parallel class containing at least

$$\left\lceil \frac{v - k - r}{2k - r + 1} \right\rceil + 1 = \left\lceil \frac{v + k - 2r + 1}{2k - r + 1} \right\rceil$$

blocks. Now $(v + k - 2r + 1)/(2k - r + 1)$ is an increasing function of $r$ when $v > 3k + 1$, and so, subject to our restriction on $r$, attains its maximum value at $r = \frac{1}{2}(k - 1)$. The theorem follows. □

3. Concluding remarks

It is perhaps worth pointing out that it seems unlikely that a stronger result can be obtained without a significant modification of Gionfriddo’s method. This is because once $P_x$ becomes twice as big as $Q - P_x$, the set $M$ becomes too ‘small’, allowing for too many blocks $b \in \pi$ to be disjoint from $M$.

References