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ORIGINAL ARTICLE

On the oscillation of a third order rational difference equation

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Abstract In this paper, we discuss the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B + Cx_n x_{n-1} x_{n-2}}, \qquad n = 0, 1, \ldots$$

where A, B, C are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers. Although we have an explicit formula for the solutions of that equation, the oscillation character is worth to be discussed.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 39A20; 39A21; 39A23; 39A30

1. Introduction

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [1–4]. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

Cinar [5,6] examined the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

and

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

He also [7] discussed the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}, \quad n = 0, 1, \ldots$$

Stević [8] showed that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

converges to zero.

In [9], H. Sedaghat determined the global behavior of all solutions of the rational difference equations

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$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \quad x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \quad n = 0, 1, \ldots$$

where $a, b > 0$.

In [10], the author investigated the global behavior and periodic character of the two difference equations

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots$$

In this paper, we discuss the global stability and periodic character of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B + Cx_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots \tag{1.1}$$

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1 \ldots \tag{1.2}$$

where $f : R^{k+1} \to R$.

Definition 1.1 [11]. An equilibrium point for Eq. (1.2) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$.

Definition 1.2 [11].

(1) An equilibrium point $\bar{x}$ for Eq. (1.2) is called locally stable if for every $\epsilon > 0, \exists \delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$ is such that $x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[, \forall n \in N$. Otherwise $\bar{x}$ is said to be unstable.
(2) The equilibrium point $\bar{x}$ of Eq. (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.
(3) An equilibrium point $\bar{x}$ for Eq. (1.2) is called global attractor if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.
(4) The equilibrium point $\bar{x}$ for Eq. (1.2) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with Eq. (1.2) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}) y_{n-i}, \quad n = 0, 1, 2, \ldots \tag{1.3}$$

the characteristic equation associated with Eq. (1.3) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}) \lambda^{k-i} = 0. \tag{1.4}$$

Theorem 1.3 [11]. *Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of Eq. (1.2). Then the following statements are true:*

(1) *If all roots of* Eq. (1.4) *lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*
(2) *If at least one root of* Eq. (1.4) *has absolute value greater than one, then $\bar{x}$ is unstable.*

The change of variables $\sqrt[3]{\frac{C}{B}} x_n = y_n$ reduces the Eq. (1.1) to the equation

$$y_{n+1} = \frac{\gamma y_{n-2}}{1 + y_n y_{n-1} y_{n-2}}, \quad n = 0, 1, \ldots \tag{1.5}$$

where $\gamma = \frac{A}{B}$.

2. Linearized stability and solutions of Eq. (1.5)

In this section we study linearized stability analysis and the solutions of the difference Eq. (1.5). It is clear that Eq. (1.5) has the equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt[3]{\gamma - 1}$. During the paper, we suppose that $\alpha = y_{-2} y_{-1} y_0$.

The following theorem describes the behavior of the equilibrium points.

Theorem 2.1. *Assume that $\alpha \neq \frac{-1}{\sum_{i=0}^{n} \gamma^i}$ for any $n \in N$. Then the following statements are true.*

(1) *If $\gamma < 1$, then $\bar{y} = 0$ is locally asymptotically stable and $\bar{y} = \sqrt[3]{\gamma - 1}$ is unstable.*
(2) *If $\gamma = 1$, then $\bar{y} = 0$ is a nonhyperbolic point.*
(3) *If $\gamma > 1$, then $\bar{y} = 0$ is a repeller and $\bar{y} = \sqrt[3]{\gamma - 1}$ is a nonhyperbolic point.*

Theorem 2.2. *Let y_{-2}, y_{-1} and y_0 be real numbers such that $\alpha = y_{-2} y_{-1} y_0 \neq \frac{-1}{\sum_{i=0}^{n} \gamma^i}$ for any $n \in N$. Then the solutions of Eq. (1.5) are*

$$y_n = \begin{cases} y_{-2} \gamma^{\frac{n-1}{3}+1} \prod_{j=0}^{\frac{n-1}{3}} \frac{1 + \alpha \sum_{k=0}^{3j-1} \gamma^k}{1 + \alpha \sum_{k=0}^{3j} \gamma^k}, & n = 1, 4, 7, \ldots \\ y_{-1} \gamma^{\frac{n-2}{3}+1} \prod_{j=0}^{\frac{n-2}{3}} \frac{1 + \alpha \sum_{k=0}^{3j} \gamma^k}{1 + \alpha \sum_{k=0}^{3j+1} \gamma^k}, & n = 2, 5, 8, \ldots \\ y_0 \gamma^{\frac{n}{3}} \prod_{j=1}^{\frac{n}{3}} \frac{1 + \alpha \sum_{k=0}^{3j-2} \gamma^k}{1 + \alpha \sum_{k=0}^{3j-1} \gamma^k}, & n = 3, 6, 9, \ldots \end{cases} \tag{2.1}$$

Proof. We have that

$$y_1 = y_{-2} \gamma \frac{1}{1 + \alpha}, \quad y_2 = y_{-1} \gamma \frac{1 + \alpha}{1 + \alpha(1 + \gamma)} \text{ and } y_3 = y_0 \gamma \frac{1 + \alpha(1 + \gamma)}{1 + \alpha(1 + \gamma + \gamma^2)}$$

as expected by formula (2.1). Now assume that $m > 1$. Then from formula (2.1), we can write

$$y_{3m-2} = y_{-2} \gamma^m \prod_{j=0}^{m-1} \frac{1 + \alpha \sum_{k=0}^{3j-1} \gamma^k}{1 + \alpha \sum_{k=0}^{3j} \gamma^k},$$

$$y_{3m-1} = y_{-1} \gamma^m \prod_{j=0}^{m-1} \frac{1 + \alpha \sum_{k=0}^{3j} \gamma^k}{1 + \alpha \sum_{k=0}^{3j+1} \gamma^k},$$

$$y_{3m} = y_0 \gamma^m \prod_{j=1}^{m} \frac{1 + \alpha \sum_{k=0}^{3j-2} \gamma^k}{1 + \alpha \sum_{k=0}^{3j-1} \gamma^k} = y_0 \gamma^m \prod_{j=0}^{m-1} \frac{1 + \alpha \sum_{k=0}^{3j+1} \gamma^k}{1 + \alpha \sum_{k=0}^{3j+2} \gamma^k}$$

Then

$$\frac{\gamma y_{3m-2}}{1+y_{3m}x_{3m-1}y_{3m-2}}$$

$$=\frac{\gamma y_{-2}\gamma^m\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j-1}\gamma^k}{1+\alpha\sum_{k=0}^{3j}\gamma^k}}{1+y_{-2}\gamma^m\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j-1}\gamma^k}{1+\alpha\sum_{k=0}^{3j}\gamma^k}y_{-1}\gamma^m\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j}\gamma^k}{1+\alpha\sum_{k=0}^{3j+1}\gamma^k}y_0\gamma^m\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j+1}\gamma^k}{1+\alpha\sum_{k=0}^{3j+2}\gamma^k}}$$

$$=\frac{y_{-2}\gamma^{m+1}\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j-1}\gamma^k}{1+\alpha\sum_{k=0}^{3j}\gamma^k}}{1+\alpha\gamma^{3m}\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j-1}\gamma^k}{1+\alpha\sum_{k=0}^{3j+2}\gamma^k}}=\frac{y_{-2}\gamma^{m+1}\prod_{j=0}^{m-1}\frac{1+\alpha\sum_{k=0}^{3j-1}\gamma^k}{1+\alpha\sum_{k=0}^{3j}\gamma^k}}{1+\alpha\gamma^{3m}\frac{1}{1+\alpha\sum_{k=0}^{3m-1}\gamma^k}}$$

$$=\frac{y_{-2}\gamma^{m+1}\left(1+\alpha\sum_{k=0}^{3m-1}\gamma^k\right)\prod_{j=0}^{m-1}\left(1+\alpha\sum_{k=0}^{3j-1}\gamma^k\right)}{\left(\prod_{j=0}^{m-1}1+\alpha\sum_{k=0}^{3j}\gamma^k\right)\left(1+\alpha\sum_{k=0}^{3m-1}\gamma^k+\alpha\gamma^{3m}\right)}$$

$$=\frac{y_{-2}\gamma^{m+1}\prod_{j=0}^{m}\left(1+\alpha\sum_{k=0}^{3j-1}\gamma^k\right)}{\prod_{j=0}^{m-1}\left(1+\alpha\sum_{k=0}^{3j}\gamma^k\right)\left(1+\alpha\sum_{k=0}^{3m}\gamma^k\right)}=\frac{y_{-2}\gamma^{m+1}\prod_{j=0}^{m}\left(1+\alpha\sum_{k=0}^{3j-1}\gamma^k\right)}{\prod_{j=0}^{m}\left(1+\alpha\sum_{k=0}^{3j}\gamma^k\right)}$$

$$=y_{-2}\gamma^{m+1}\prod_{j=0}^{m}\frac{1+\alpha\sum_{k=0}^{3j-1}\gamma^k}{1+\alpha\sum_{k=0}^{3j}\gamma^k}=y_{3m+1}.$$

This completes the proof. □

Corollary 2.3. *Assume that* $\gamma=1$ *and* $\alpha=y_{-2}y_{-1}y_0\neq-1/n$ *for any* $n\in N$. *Then the solutions of* Eq. (1.5) *are*

$$y_n=\begin{cases} y_{-2}\prod_{j=0}^{\frac{n-1}{3}}\frac{1+(3j)\alpha}{1+(3j+1)\alpha} & ,n=1,4,7,\ldots\\ y_{-1}\prod_{j=0}^{\frac{n-2}{3}}\frac{1+(3j+1)\alpha}{1+(3j+2)\alpha} & ,n=2,5,8,\ldots\\ y_0\prod_{j=1}^{\frac{n}{3}}\frac{1+(3j-1)\alpha}{1+3j\alpha} & ,n=3,6,9,\ldots\end{cases}\tag{2.2}$$

3. Periodicity and global stability

Theorem 3.1. *Assume that* $\{y_n\}_{n=-2}^{\infty}$ *is a positive solution of Eq. (1.5). Then the following statements are true.*

(1) *If* $\gamma<1$, *then* $\{y_n\}_{n=-2}^{\infty}$ *converges to zero.*
(2) *If* $\gamma=1$, *then* $\{y_n\}_{n=-2}^{\infty}$ *converges to zero.*

Proof.

(1) Let $\{y_n\}_{n=-2}^{\infty}$ be a positive solution of Eq. (1.5). Then

$$y_{n+1}=\frac{y_{n-2}}{1+y_ny_{n-1}y_{n-2}}<\gamma y_{n-2},\quad n=0,1,\ldots$$

Hence we have

$$y_{3m+i}<\gamma^{m+1}y_{-3+i},\quad i=1,2,3.$$

Therefore,

$$\lim_{n\to\infty}y_n=0.$$

(2) We consider only the case $\alpha<0$. Case $\alpha>0$ is similar and will be omitted. From formula (2.1) we have

$$y_{3m+1}=y_{-2}\prod_{j=0}^{m}\frac{1+(3j)\alpha}{1+(3j+1)\alpha}$$

$$=y_{-2}\exp\left(\sum_{j=0}^{m}\ln\frac{1+3j\alpha}{1+(3j+1)\alpha}\right)$$

$$=y_{-2}\exp\left(-\sum_{j=0}^{m}\ln\frac{1+(3j+1)\alpha}{1+3j\alpha}\right)$$

$$=y_{-2}\exp\left(-\sum_{j=0}^{m}\ln\left(1+\frac{\alpha}{1+3j\alpha}\right)\right)$$

$$=y_{-2}\exp\left(-\alpha\left(\sum_{j=0}^{m}\left(\frac{1}{1+3j\alpha}+O\left(\frac{1}{j^2}\right)\right)\right)\right)\to0\quad n\to\infty,$$

since $\sum_{j=0}^{m}\frac{1}{1+3j\alpha}\to-\infty$ *as* $m\to\infty$ and $\sum_{j=0}^{m}O\left(\frac{1}{j^2}\right)$ is convergent.

Similarly $y_{3m+2}\to0$ *as* $m\to\infty$ and $y_{3m+3}\to0$ *as* $m\to\infty$.This completes the proof. □

Theorem 3.2. Eq. (1.5) *has period-3 solutions* $\left\{\varphi_1,\varphi_2,\frac{\gamma-1}{\varphi_1\varphi_2},\varphi_1,\varphi_2,\frac{\gamma-1}{\varphi_1\varphi_2},\ldots\right\}$ *with* $\varphi_1\varphi_2\varphi_3=\gamma-1$ *when* $\gamma\neq1$, *and* $\{\varphi_1,\varphi_2,\varphi_3,\varphi_1,\varphi_2,\varphi_3,\ldots\}$ *with* $\varphi_1\varphi_2\varphi_3=\alpha=0$ *when* $\gamma=1$.

Proof. Case $\gamma\neq1$

It is clear that $\left\{\varphi_1,\varphi_2,\frac{\gamma-1}{\varphi_1\varphi_2},\varphi_1,\varphi_2,\frac{\gamma-1}{\varphi_1\varphi_2},\ldots\right\}$ are period-3 solutions of Eq. (1.5). Now let $\{\ldots,\varphi_1,\varphi_2,\varphi_3,\varphi_1,\varphi_2,\varphi_3,\ldots\}$ be a period-3 solution of Eq. (1.5). Then

$$\varphi_1=\frac{\gamma\varphi_1}{1+\varphi_1\varphi_2\varphi_3},\quad\varphi_2=\frac{\gamma\varphi_2}{1+\varphi_1\varphi_2\varphi_3},\quad\varphi_3=\frac{\gamma\varphi_3}{1+\varphi_1\varphi_2\varphi_3}.$$

As $\gamma\neq1$, we have that $\varphi_1\varphi_2\varphi_3=\gamma-1$.

Case $\gamma=1$

Let $\alpha=0$. Using formula (2.1) it is sufficient to see that

$$y_n=\begin{cases}y_{-2} & ,n=1,4,7,\ldots\\ y_{-1} & ,n=2,5,8,\ldots\\ y_0 & ,n=3,6,9,\ldots\end{cases}$$

therefore, we have

$$y_{3m}=y_0,\quad y_{3m+1}=y_{-1}\quad\text{and}\quad y_{3m+2}=y_{-2},\quad n=0,1,\ldots$$

Now suppose that $y_{-2}=\varphi_1,y_{-1}=\varphi_2,y_0=\varphi_3$. It follows that $\{\ldots,\varphi_1,\varphi_2,\varphi_3,\varphi_1,\varphi_2,\varphi_3,\ldots\}$ is a period-3 solution with $\varphi_1\varphi_2\varphi_3=\alpha=0$. This completes the proof. □

4. Oscillation behavior

Let $\xi_{3j+i-1}=\frac{1+\alpha\sum_{k=0}^{3j+i-2}\gamma^k}{1+\alpha\sum_{k=0}^{3j+i-1}\gamma^k}$, $\quad i=1,2,3$ and $j\geqslant0$.

Hence (2.1) can be written as

$$y_n=\begin{cases} y_{-2}\gamma^{\frac{n-1}{3}+1}\prod_{j=0}^{\frac{n-1}{3}}\xi_{3j} & ,n=1,4,7,\dots\\ y_{-1}\gamma^{\frac{n-2}{3}+1}\prod_{j=0}^{\frac{n-2}{3}}\xi_{3j+1} & ,n=2,5,8,\dots\\ y_{0}\gamma^{\frac{n}{3}}\prod_{j=0}^{\frac{n}{3}-1}\xi_{3j+2} & ,n=3,6,9,\dots\end{cases} \tag{4.1}$$

Lemma 4.1. *Assume that either* $\alpha=y_{-2}y_{-1}y_0>0$, *or* $\alpha=y_{-2}y_{-1}y_0<0$ *and* $1-\gamma+\alpha\geqslant 0$. *Then*

$$sgn(y_{3m+i})=sgn(y_{-3+i}),\ i=1,2,3 \text{ and } m=-1,0,1,\dots$$

Proof. Assume that $\alpha=y_{-2}y_{-1}y_0>0$. Then we have that $\xi_{3j+i-1}>0,\ j=0,1,\dots,\ i=1,2,3$.

Therefore, $sgn(y_{3m+i})=sgn(y_{-3+i}\gamma^{m+1}\prod_{j=0}^{m}\xi_{3j+i-1})=sgn(y_{-3+i}),\ i=1,2,3$ and $m=-1,0,1,\dots$.

Now assume that $\alpha=y_{-2}y_{-1}y_0<0$ and $1-\gamma+\alpha\geqslant 0$. Then

(1) If $1-\gamma+\alpha=0$, then $\xi_{3j+i-1}=\frac{1+\alpha\sum_{k=0}^{3j+i-2}\gamma^k}{1+\alpha\sum_{k=0}^{3j+i-1}\gamma^k}=\frac{1-\gamma+\alpha-\alpha\gamma^{3j+i-1}}{1-\gamma+\alpha-\alpha\gamma^{3j+i}}=\frac{1}{\gamma}>0.$

(2) If $1-\gamma+\alpha>0$, $\xi_{3j+i-1}=\frac{1+\alpha\sum_{k=0}^{3j+i-2}\gamma^k}{1+\alpha\sum_{k=0}^{3j+i-1}\gamma^k}=\frac{1-\gamma+\alpha-\alpha\gamma^{3j+i-1}}{1-\gamma+\alpha-\alpha\gamma^{3j+i}}>0.$

This implies that $sgn(y_{3m+i})=sgn(y_{-3+i}),\ i=1,2,3$ and $m=-1,0,1,\dots$ ☐

Theorem 4.2. *Assume that* $\{y_n\}_{n=-2}^{\infty}$ *be a solution of* Eq. (1.5). *Then the following statements are true:*

(1) If $\alpha=y_{-2}y_{-1}y_0>0$, then $\{y_n\}_{n=-2}^{\infty}$ is positive or except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y}=0$ with negative semicycles of length two and positive semicycles of length one.
(2) If $\alpha=y_{-2}y_{-1}y_0<0,\ 1-\gamma+\alpha\geqslant 0$, then $\{y_n\}_{n=-2}^{\infty}$ is negative or except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y}=0$ with negative semicycles of length one and positive semicycles of length two.

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (1.5).

(1) Suppose that $\alpha=y_{-2}y_{-1}y_0>0$. From lemma (4.1), we have that $sgn(y_{3m+i})=sgn(y_{-3+i}),\ i=1,2,3$ and $m=-1,0,1,\dots$. That is, each subsequence $\{y_{3m+i}\}_{m=-1}^{\infty},\ i=1,2,3$ preserves sign. It follows that, if $y_{-3+i}>0,\ i=1,2,3$, then $\{y_n\}_{n=-2}^{\infty}$ is positive.Otherwise, there exists $i_0\in\{1,2,3\}$ such that $y_{-3+i_0}>0$ and $y_{-3+i}<0,\ i\in\{1,2,3\}\setminus\{i_0\}$. Therefore, except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y}=0$ with negative semicycles of length two and positive semicycles of length one.
(2) Suppose that $\alpha=y_{-2}y_{-1}y_0<0,\ 1-\gamma+\alpha\geqslant 0$. Again from lemma (4.1), we have that $sgn(y_{3m+i})=sgn(y_{-3+i}),\ i=1,2,3$ and $m=-1,0,1,\dots$. That is, each subsequence $\{y_{3m+i}\}_{m=-1}^{\infty},\ i=1,2,3$ preserves sign. It follows that, if $y_{-3+i}<0,\ i=1,2,3$, then $\{y_n\}_{n=-2}^{\infty}$ is negative.Otherwise, there exists $i_0\in\{1,2,3\}$ such that $y_{-3+i_0}<0$ and $y_{-3+i}>0,\ i\in\{1,2,3\}\setminus\{i_0\}$. Therefore, except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y}=0$ with negative semicycles of length one and positive semicycles of length two. ☐

Lemma 4.3. *Assume that* $\alpha=y_{-2}y_{-1}y_0<0,\ 1-\gamma+\alpha<0$ *and let* $\theta=\frac{\ln(1-\gamma+\alpha/\alpha)}{\ln\gamma}$. *Then*

(1) If $\gamma=1$, then $\xi_{3j+i-1}<0$ when $-\frac{1}{3}\left(\frac{1}{\alpha}+i\right)<j<-\frac{1}{3}\left(\frac{1}{\alpha}+i-1\right),\ i=1,2,3,$
(2) If $\gamma\neq 1$, then $\xi_{3j+i-1}<0$ when $\frac{\theta-i}{3}<j<\frac{\theta-i+1}{3},\ i=1,2,3.$

Proof. Assume that $\alpha=y_{-2}y_{-1}y_0<0,\ 1-\gamma+\alpha<0.$

(1) If $\gamma=1$, then $\xi_{3j+i-1}=\frac{1+\alpha\sum_{k=0}^{3j+i-2}\gamma^k}{1+\alpha\sum_{k=0}^{3j+i-1}\gamma^k}=\frac{1+\alpha(3j+i-1)}{1+(3j+i)\alpha}$. It is clear that $\xi_{3j+i-1}>0$ if $j\in]-\infty,-\frac{1}{3}\left(\frac{1}{\alpha}+i\right)[\cup]-\frac{1}{3}\left(\frac{1}{\alpha}+i-1\right),\infty[$. Therefore, if $-\frac{1}{3}\left(\frac{1}{\alpha}+1\right)<j<-\frac{1}{3\alpha}$, we have that $\xi_{3j+i-1}<0,\ i=1,2,3.$
(2) If $\gamma\neq 1$, then we have two cases:
   - If $\gamma<1$, then $\theta=\frac{\ln((1-\gamma+\alpha)/\alpha)}{\ln\gamma}>0$. Now set $\xi_{3j+i-1}=\frac{1-\gamma+\alpha-\alpha\gamma^{3j+i-1}}{1-\gamma+\alpha-\alpha\gamma^{3j+i}}=\frac{I}{II}$. As $\alpha=y_{-2}y_{-1}y_0<0$, we have that $I>II$. But $I>0\Longleftrightarrow 1-\gamma+\alpha>\alpha\gamma^{3j+i-1}$ $\Longleftrightarrow(1-\gamma+\alpha)/\alpha<\gamma^{3j+i-1}\Longleftrightarrow\ln((1-\gamma+\alpha)/\alpha)<(3j+i-1)\ln\gamma\Longleftrightarrow\frac{\ln((1-\gamma+\alpha)/\alpha)}{\ln\gamma}=\theta>3j+i-1$ $\Longleftrightarrow j<\frac{\theta-i+1}{3}$. Also $II<0\Longleftrightarrow j>\frac{\theta-i}{3}$. Therefore, if $\frac{\theta-i+1}{3}<j<\frac{\theta-i}{3}$, we have $\xi_{3j+i-1}<0,\ i=1,2,3.$
   - case $\gamma>1$ is similar and will be omitted. ☐

Lemma 4.4. *Assume that* $\alpha\neq\frac{-1}{\sum_{i=0}^{n}\gamma^i}$ *for any* $n\in N$. *Let* $\alpha=y_{-2}y_{-1}y_0<0,\ \gamma\neq 1$ *and* $1-\gamma+\alpha<0$. *Then* $\theta=\frac{\ln((1-\gamma+\alpha)/\alpha)}{\ln\gamma}\neq n$, *for any* $n\in\mathbb{N}$.

Proof. Assume that $\alpha=y_{-2}y_{-1}y_0<0,\ 1-\gamma+\alpha<0$. Then from lemma (4.3), we have that $\theta=\frac{\ln((1-\gamma+\alpha)/\alpha)}{\ln\gamma}>0.$

Now let $\theta=\frac{\ln((1-\gamma+\alpha)/\alpha)}{\ln\gamma}=n,\ n\in\mathbb{N}$. This implies that $\ln((1-\gamma+\alpha)/\alpha))=n\ln\gamma\Longleftrightarrow 1-\gamma+\alpha=\alpha\gamma^n\Longleftrightarrow\alpha=-\frac{1-\alpha}{1-\alpha^n}=-\frac{1}{\sum_{l=0}^{n-1}\gamma^l}$, which is a contradiction, as $\alpha\neq\frac{-1}{\sum_{i=0}^{n}\gamma^i}$ for any $n\in N$. ☐

Now consider the two situations,

S_1: There is no natural number $j_0\in\mathbb{N}$ with $|j_0-c|<\frac{1}{6}$ and

S_2: There is a natural number $j_0\in\mathbb{N}$ with $|j_0-c|<\frac{1}{6}$, where

$$c=\begin{cases}-\frac{1}{6}\left(\frac{2}{\alpha}+2i-1\right), & \gamma=1,\\ \frac{1}{6}(2\theta-2i+1), & \gamma\neq 1.\end{cases}$$

Lemma 4.5. *Assume that* $\alpha=y_{-2}y_{-1}y_0<0,\ \xi_j>0$ *for each* $j\in\mathbb{N}$, *and let* $\{y_n\}_{n=-2}^{\infty}$ *be a solution of* Eq. (1.5). *Then* $\{y_n\}_{n=-2}^{\infty}$ *is negative or except (possibly) for the first semicycle,* $\{y_n\}_{n=-2}^{\infty}$ *oscillates about* $\bar{y}=0$ *with negative semicycles of length one and positive semicycles of length two.*

Proof. Assume that $\alpha = y_{-2}y_{-1}y_0 < 0$. Then

$sgn(y_{3m+i}) = sgn(y_{-3+i}\gamma^{m+1}\prod_{j=0}^{m}\xi_{3j+i-1}) = sgn(y_{3+i}),\ i = 1,2,3$ and $m = -1,0,1,\ldots$.

That is, each subsequence $\{y_{3m+i}\}_{m=-1}^{\infty},\ i = 1,2,3$ preserves sign. It follows that, if $y_{-3+i} < 0,\ i = 1,2,3$, then $\{y_n\}_{n=-2}^{\infty}$ is negative.

Otherwise, there exists $i_0 \in \{1,2,3\}$ such that $y_{-3+i_0} < 0$ and $y_{-3+i} > 0,\ i \in \{1,2,3\} \setminus \{i_0\}$. Therefore, except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length one and positive semicycles of length two. □

Theorem 4.6. *Assume that* $\alpha = y_{-2}y_{-1}y_0 < 0,\ 1 - \gamma + \alpha < 0$, *and let* $\{y_n\}_{n=-2}^{\infty}$ *be a solution of* Eq. (1.5). *Then one of the following statements will be satisfied.*

(1) $\{y_n\}_{n=-2}^{\infty}$ is negative or except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length one and positive semicycles of length two.
(2) There exists a natural number L_0 such that $\{y_n\}_{n=-2}^{L_0}$ is negative and $\{y_n\}_{n=L_0+1}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length two and positive semicycles of length one or except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{L_0}$ oscillates about $\bar{y} = 0$ with negative semicycles of length one and positive semicycles of length two and $\{y_n\}_{n=L_0+1}^{\infty}$ is either positive or oscillates about $\bar{y} = 0$ with negative semicycles of length two and positive semicycles of length one.

Proof. Case $\gamma \neq 1$. Suppose that the situation S_1 is satisfied. In this case $\xi_{3j+i-1} > 0$ for each $j \in \mathbb{N}$ and $i = 1,2,3$. It follows from lemma (4.1) that $sgn(y_{3m+i}) = sgn(y_{-3+i}),\ i = 1,2,3$ and $m = -1,0,1,\ldots$, and from lemma (4.5) the result in (1) follows.

Now suppose that the situation S_2 is satisfied for some $i_0 \in \{1,2,3\}$. In this case

$$\xi_{3j+i_0-1}\begin{cases} < 0 & ,j = j_0, \\ > 0 & ,otherwise. \end{cases}, \quad i = 1,2,3.$$

This implies that

- $sgn(y_{3(j_0-1)+i_0}) = sgn\left(y_{-3+i_0}\gamma^{j_0}\prod_{j=0}^{j_0-1}\xi_{3j+i_0-1}\right) = sgn(y_{-3+i_0})$,
- $sgn(y_{3j_0+i_0}) = sgn(y_{-3+i_0}\gamma^{j_0+1}\prod_{j=0}^{j_0}\xi_{3j+i_0-1}) = sgn(y_{-3+i_0}\gamma^{j_0+1}$ $\xi_{3j_0+i_0-1}\prod_{j=0}^{j_0-1}\xi_{3j+i_0-1}) = -sgn(y_{-3+i_0})$,
- $sgn(y_{3m+i_0}) = sgn\left(y_{-3+i_0}\gamma^{m+1}\prod_{j=0}^{m}\xi_{3j+i_0-1}\right) = -sgn(y_{-3+i_0})$, $m \geqslant j_0$.
- $sgn(y_{3m+i}) = sgn\left(y_{-3+i}\gamma^{m+1}\prod_{j=0}^{m}\xi_{3j+i-1}\right) = sgn(y_{-3+i})$, for $i_0 \in \{1,2,3\} \setminus \{i_0\}$ $m = -1,0,1,\ldots$.

Hence $\{y_{3m+i_0}\}_{m=-1}^{j_0-1}$, has the sign of y_{-3+i_0}, and $\{y_{3m+i_0}\}_{m=j_0}^{\infty}$, has the opposite sign of y_{-3+i_0}.

Now assume that $i_0 = 1$. Then

$$\xi_{3j}\begin{cases} < 0 & ,j = j_0, \\ > 0 & ,otherwise. \end{cases} \quad \text{and } y_{3m+1} = y_{-2}\gamma^{m+1}\prod_{j=0}^{m}\xi_{3j}.$$

Also there exists $L_0 = 3j_0 \in \mathbb{N}$ such that we have the following:

- If $y_{-i} < 0, i = 1,2,3$, then $\{y_n\}_{n=-2}^{L_0}$ is negative and $\{y_n\}_{n=L_0+1}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length two and positive semicycles of length one.
- Otherwise, except (possibly) for the first semicycle, $\{y_n\}_{n=-2}^{L_0}$ oscillates about $\bar{y} = 0$ with negative semicycles of length one and positive semicycles of length two and $\{y_n\}_{n=L_0+1}^{\infty}$ is either positive or oscillates about $\bar{y} = 0$ with negative semicycles of length two and positive semicycles of length one.

By similar way, we can show that the last assertion is satisfied for $i_0 = 2,3$ and will be omitted.

Case $\gamma = 1$ is similar and will be omitted. □

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