Some characterizations of spheres and elliptic paraboloids

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\textbf{ABSTRACT}

We establish a characterization of spheres in $\mathbb{E}^3$ with respect to a surface area property of regions with the aid of a new meaning of Gaussian curvature. Furthermore, with respect to a volume property of regions, we characterize elliptic paraboloids in arbitrary dimensional Euclidean spaces.

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\textbf{1. Introduction}

Consider a sphere $S^2(a)$ of radius $a$ in the Euclidean space $\mathbb{E}^3$. Then by an elementary calculus, it is easy to show that for any two parallel planes with distance $h$ both of which intersect $S^2(a)$, the surface area of the region of $S^2(a)$ between the planes is $2\pi ah$.

In fact, Archimedes proved the above area property of $S^2(a)$ [8, p. 78]. For a differential geometric proof, see Archimedes’ Theorem [6, pp. 116–118].
Conversely, it is natural to ask the following question:

**Question 1.** Are there any other surfaces which satisfy the above area property?

In Section 2, we prove the following:

**Theorem 2.** Let $M$ be a closed and convex surface in the 3-dimensional Euclidean space $E^3$. If $M$ satisfies the condition:

(C) for any two parallel planes with distance $h$ both of which intersect $M$, the surface area of the region of $M$ between the planes is a nonnegative function $\phi(h)$, which depends only on $h$.

Then $M$ is an Euclidean sphere.

To establish Theorem 2, first of all, using co-area formula, we prove a lemma (Lemma 6) about a new meaning of Gaussian curvature of $M$ at a point $p \in M$.

A paraboloid of rotation in the 3-dimensional Euclidean space $E^3$ has an interesting volume property which is originally due to Archimedes. Consider a region of a paraboloid of rotation cut off by a plane not necessarily perpendicular to its axis. Let $p$ be the point of contact of the tangent plane parallel to the base. The line through $p$, parallel to the axis of the paraboloid meets the base at a point $v$. Archimedes shows that the volume of the section is $3/2$ times the volume of the cone with the same base and vertex $p$ [8, Chapter 7 and Appendix A].

In fact, in a long series of propositions, Archimedes proves the following [8, p. 66 and Appendices A and B].

**Proposition 3.** The volume of such a region of a paraboloid of rotation in the 3-dimensional Euclidean space $E^3$ is proportional to $\|p - v\|^2$, where the ratio depends only on the paraboloid.

This proposition implies directly Archimedes’ results (See Remark 8).

Conversely, it is natural to ask the following question:

**Question 4.** Which surfaces satisfy the above volume property?

In Section 3, we prove the following:

**Theorem 5.** Let $M$ be a smooth convex hypersurface in the $(n + 1)$-dimensional Euclidean space $E^{n+1}$. Then $M$ is an elliptic paraboloid if and only if there exists a line $L$ for which $M$ satisfies the condition:

(L) for any point $p$ on $M$ and any hyperplane section of $M$ parallel to the tangent plane of $M$ at $p$, let $v$ denote the point where the line through $p$ parallel to $L$ meets the hyperplane. Then the volume of the region of $M$ between these two parallel hyperplanes is $a$ times $\|p - v\|^{(n+2)/2}$ for some constant $a$ which depends only on the hypersurface $M$.

To complete the proof of Theorem 5, first of all, we get a formula for the Gauss–Kronecker curvature of $M$ at a point $p \in M$ (Lemma 7).

Throughout this article, all objects are smooth and connected, otherwise mentioned.

2. Spheres

Suppose that $M$ is a closed and convex surface in the 3-dimensional Euclidean space $E^3$. Then the Gaussian curvature $K$ is non-negative. For a fixed point $p \in M$ and for a sufficiently small $h > 0$, consider a plane $\Phi$ parallel to the tangent plane $\Psi$ of $M$ at $p$ with distance $h$ which intersects $M$. We denote by $M_p(h)$ the surface area between the two planes $\Phi$ and $\Psi$.

We introduce a coordinate system $(x, y, z)$ of $E^3$ with the origin $p$, the tangent plane of $M$ at $p$ is $z = 0$, and $M = \text{graph}(f)$ for a non-negative convex function $f : \mathbb{R}^2 \to \mathbb{R}$. Then we have
\[ M_p(h) = \int \int \mathcal{X} < h \sqrt{1 + |\nabla f|^2} dX, \] (2.1)

where \( \mathcal{X} = (x, y) \), \( dX = dxdy \) and \( \nabla f \) denote the gradient vector of \( f \).

First of all, we prove

Lemma 6. If the Gaussian curvature \( K(p) \) of \( M \) at \( p \) is positive, then we have

\[ M_p'(0) = \frac{2\pi}{\sqrt{K(p)}}. \]

Proof. Consider the Taylor expansion of \( f(X) \) as follows:

\[ f(X) = \mathcal{X}^t A \mathcal{X} + f_3(X), \] (2.2)

where \( A \) is a symmetric \( 2 \times 2 \) matrix and \( f_3(X) \) is an \( O(|X|^3) \) function. Then the Hessian matrix of \( f \) at the origin is given by

\[ D^2 f(0) = 2A. \]

Hence we see that

\[ K(p) = \det D^2 f(0) = 4 \det A. \] (2.3)

Since \( K(p) > 0 \) and \( f \) is non-negative, we see that the matrix \( A \) is positive definite. Thus there exists a non-singular matrix \( B \) satisfying

\[ A = B^t B, \] (2.4)

where \( B^t \) denotes the transpose of \( B \). Therefore we obtain

\[ f(X) = |BX|^2 + f_3(X). \] (2.5)

In order to compute \( M_p'(0) \), we use the decomposition of \( M_p(h) \) as follows:

\[ M_p(h) = Q(h) + N(h), \]

\[ Q(h) = \int \int_{f(X) < h} 1 dX, \]

\[ N(h) = \int \int_{f(X) < h} (\sqrt{1 + |\nabla f|^2} - 1) dX. \] (2.6)

Then we have

\[ N(h) \leq \int \int_{f(X) < h} |\nabla f| dX. \]

Hence, by the co-area formula [2, p. 86] we get

\[ \frac{N(h)}{h} \leq \frac{1}{h} \int_{t=0}^{h} \left( \int_{f^{-1}(t)} |\nabla f| \right) dt. \] (2.7)
where $ds_t$ denotes the line element of the curve $f^{-1}(t)$, which shows that the integrand is nothing but the length $L(t)$ of $f^{-1}(t)$. By the fundamental theorem of calculus, we see that

$$N'(0) = \lim_{h \to 0} \frac{N(h)}{h} = L(0) = 0. \tag{2.8}$$

Now we let $h = \epsilon^2$ and $X = \epsilon Y$, then (2.6) gives

$$Q(h) = \frac{1}{h} \int \int_{f(X) < h} 1\,dX = \int \int_{|BY|^2 + \epsilon g_3(Y) < 1} 1\,dY, \tag{2.9}$$

where $g_3(Y)$ is an $O(|Y|^3)$ function. As $\epsilon \to 0$, it follows from (2.9) that

$$Q'(0) = \int \int_{|BY|^2 < 1} 1\,dY. \tag{2.10}$$

If we let $W = BY$, then from (2.10) we get

$$Q'(0) = \frac{1}{\det B} \int \int_{|W| < 1} 1\,dW = \frac{\pi}{\det B}. \tag{2.11}$$

Hence it follows from (2.3) and (2.4) that

$$Q'(0) = \frac{2\pi}{\sqrt{K(p)}}. \tag{2.12}$$

Thus together with (2.8) and (2.12), (2.6) completes the proof of Lemma 6.

Now we give a proof of Theorem 2. Since $M$ is closed, there exists a point $p$ where $K(p) > 0$. Hence we see that $U = \{ p \in M|K(p) > 0 \}$ is nonempty. Together with Condition (C), Lemma 6 implies that at every point $p \in U$, we have $K(p) = 4\pi^2/\phi'(0)^2$, which is independent of $p \in M$. Thus, continuity of $K$ shows that $U = M$, and hence we have $K = 4\pi^2/\phi'(0)^2$ on $M$. This completes the proof of Theorem 2. $\square$

3. Elliptic paraboloids

Suppose that $M$ is a smooth convex hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For a fixed point $p \in M$ and for a sufficiently small $t > 0$, consider a hyperplane $\Phi$ parallel to the tangent hyperplane $\Psi$ of $M$ at $p$ with distance $t$ which intersects $M$.

We denote by $S_p(t)$ (respectively, $R_p(t)$) the volume of the region bounded by the hypersurface and the hyperplane $\Phi$ (respectively, of the cylinder with base $\Phi \cap M$ and height $t$). Then $R_p(t)$ is $(n + 1)$ times the volume of the cone with the same base and the vertex $p$.

Now we may introduce a coordinate system $(x, z) = (x_1, x_2, \ldots, x_n, z)$ of $\mathbb{R}^{n+1}$ with the origin $p$, the tangent plane of $M$ at $p$ is $z = 0$. Furthermore, we may assume that $M$ is locally the graph of a non-negative convex function $f: \mathbb{R}^n \to \mathbb{R}$. Then we have

$$R_p(t) = t \int_{f(x) < t} 1\,dx, \tag{3.1}$$

$$S_p(t) = \int_{f(x) < t} \{t - f(x)\}dx.$$
where \( dx = dx_1 dx_2 \cdots dx_n \).

Note that we also have

\[
S_p(t) = \int_{f(x) < t} \{t - f(x)\} dx = \int_{z=0}^{t} \{ \int_{f(x) < z} dx \} dz.
\]

Hence together with the fundamental theorem of calculus, (3.2) shows that

\[
tS'_p(t) = t \int_{f(x) < t} 1dx = R_p(t).
\]

First of all, we prove the following.

**Lemma 7.** If the Gauss–Kronecker curvature \( K(p) \) of \( M \) at \( p \) is positive, then we have

\[
\lim_{t \to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \frac{2^{n/2} \omega_n}{\sqrt{K(p)}},
\]

where \( \omega_n \) denotes the volume of the \( n \)-dimensional unit ball.

**Proof.** Consider the Taylor expansion of \( f(x) \) as follows:

\[
f(x) = x^t Ax + f_3(x),
\]

where \( x \) denotes the column vector \((x_1, x_2, \ldots, x_n)^t\), \( A \) is a symmetric \( n \times n \) matrix, and \( f_3(x) \) is an \( O(|x|^3) \) function.

Then the Hessian matrix of \( f \) at the origin is given by

\[
D^2 f(0) = 2A.
\]

Hence we see that

\[
K(p) = \det D^2 f(0) = 2^n \det A.
\]

Since \( K(p) > 0 \) and \( f \) is non-negative, we see that the matrix \( A \) is positive definite. Thus there exists a nonsingular matrix \( B \) satisfying

\[
A = B^t B,
\]

where \( B^t \) denotes the transpose of \( B \). Therefore we obtain

\[
f(x) = |Bx|^2 + f_3(x).
\]

Now we let \( t = \epsilon^2 \) and \( x = \epsilon y \). Then (3.3) gives

\[
\frac{1}{t^{(n+2)/2}} R_p(t) = \frac{1}{t^{n/2}} \int_{f(x) < t} 1dx = \int_{|By|^2 + \epsilon g_3(y) < 1} 1dy,
\]

where \( g_3(y) \) is an \( O(|y|^3) \) function. As \( \epsilon \to 0 \), it follows from (3.9) that

\[
\lim_{t \to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \int_{|By|^2 < 1} 1dy.
\]
If we let \( w = By \), then from (3.10) we get
\[
\lim_{t \to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \frac{1}{|\det B|} \int_{|w| < 1} \frac{\omega_n}{|\det B|},
\]
(3.11)
where \( \omega_n \) denotes the volume of the \( n \)-dimensional unit ball. Hence it follows from (3.6) and (3.7) that
\[
\lim_{t \to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \frac{2^{n/2} \omega_n}{\sqrt{K(p)}}.
\]
(3.12)
This completes the proof of Lemma 7.

Now we give a proof of the if part of Theorem 5. We may assume that the line \( L \) is the \( z \)-axis and the convex hypersurface \( M \) is given locally by \( z = f(x) \) for some convex function \( f : \mathbb{R}^n \to \mathbb{R} \). For a fixed point \( p = (x, f(x)) \) in \( M \) with positive Gauss–Kronecker curvature \( K(p) \) and \( t > 0 \), consider the region of \( M \) cut off by the hyperplane parallel to the tangent hyperplane to \( M \) at \( p \) with distance \( t > 0 \). Then the hypothesis shows that \( S_p(t) = a\|p - v\|(n+2)/2 \) for all \( t \in \mathbb{R} \), where \( a \) is a constant.

If we let
\[
N = \frac{1}{W} (-f_{x_1}, -f_{x_2}, \ldots, -f_{x_n}, 1),
\]
(3.13)
where \( W = \{1 + f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2\}^{1/2} \), then \( N \) is an upward unit normal vector. Let \( \theta \) be the angle between \( N \) and \( \frac{\partial f}{\partial x} \), then we have \( \cos \theta = 1/W \).

Hence we get \( \|p - v\| = tW \), which shows that
\[
S_p(t) = aW^{(n+2)/2}t^{(n+2)/2}.
\]
(3.14)
Thus (3.3) yields
\[
R_p(t) = \frac{n + 2}{2} a W^{(n+2)/2} t^{(n+2)/2}.
\]
(3.15)
Therefore, Lemma 7 shows that
\[
K(p) = \frac{2^{n+2} \omega_n^2}{(n + 2)^2 a^2 W^{n+2}}.
\]
(3.16)
Since the Gauss–Kronecker curvature \( K(p) \) of \( M \) at \( p \) is given by [9, p. 93]
\[
K(p) = \frac{\det D^2f(x)}{W^{n+2}},
\]
(3.17)
it follows from (3.16) that the determinant \( \det D^2f(x) \) of the Hessian of \( f(x) \) is a positive constant. The continuity of \( \det D^2f(x) \) shows that it is a positive constant on the whole space \( \mathbb{R}^n \). Thus \( f(x) \) is a globally defined quadratic polynomial [3, 5]. This completes the proof of the if part of Theorem 5.

Finally, consider an elliptic paraboloid \( M : z = \Sigma_{i=1}^n a_i^2 x_i^2, a_i > 0 \), a hyperplane \( \Sigma \) intersecting \( M \), a point \( p \in M \) where the tangent plane of \( M \) is parallel to \( \Sigma \), and a point \( v \) where the line through \( p \) parallel to the \( z \)-axis meets \( \Sigma \). Then the linear mapping
\[
T_1(x_1, x_2, \ldots, x_n, z) = (a_1x_1, a_2x_2, \ldots, a_nx_n, z)
\]
(3.18)
transforms \( M \) onto a paraboloid of revolution \( M' : z = x_1^2 + x_2^2 + \cdots + x_n^2, \Sigma \) to a hyperplane \( \Sigma' \), \( p \in M \) to a point of tangency \( p' \in M' \), and \( v \) to a point \( v' \) where the line through \( p' \) parallel to the \( z \)-axis meets \( \Sigma' \) (cf. [8, Appendix A]).
Let’s consider the affine mapping defined by

\[ x'_1 = tx_1 + h_1, \ldots, x'_n = tx_n + h_n, \]
\[ z' = 2th_1x_1 + \cdots + 2th_nx_n + t^2z + h_1^2 + \cdots + h_n^2. \] (3.19)

Then for any constants, \( t, h_1, \ldots, h_n \), where \( t \) is not 0, the affine mapping takes the paraboloid \( M' \) into itself.

Suppose that the equation of \( \Sigma' \) is given by

\[ z'' = p_1x'_1 + p_2x'_2 + \cdots + p_nx'_n + d. \] (3.20)

Then we denote by \( T_2 \) the affine mapping defined by (3.19) with

\[ h_1 = p_1/2, \ldots, h_n = p_n/2, t = \sqrt{(p_1^2 + \cdots + p_n^2 + 2d)/2}. \] (3.21)

The inverse mapping \( T_2^{-1} \) of \( T_2 \) takes \( \Sigma' \) into \( \Sigma'' : z'' = 1, p' to p'' = 0, \) the origin, and \( v' \) to \( v'' = (0, 0, \ldots, 0, 1) \).

It follows from (3.18) and (3.19) that \( T_1 \) (resp., \( T_2^{-1} \)) magnifies volumes by the factor \( a_1a_2\cdots a_n \) (resp., \( t^{-(n+2)} \)) and segments parallel to the \( z \)-axis by the factor 1 (resp., \( t^{-2} \)). Hence, if we denote by \( VS(M) \) the volume of the region of \( M \) cut off by \( \Sigma \) and so on, we obtain

\[
\frac{VS(M)}{\|p - v\|^{(n+2)/2}} = \frac{VS(M')}{\|p' - v'\|^{(n+2)/2}} = \frac{a_1a_2\cdots a_n\|p' - v'\|^{(n+2)/2}}{t^{n+2}VS(M')} = \frac{2\sigma_{n-1}}{n(n + 2)a_1a_2\cdots a_n},
\] (3.22)

where \( \sigma_{n-1} \) denotes the surface area of the \((n - 1)\)-dimensional unit sphere. This completes the proof of the only if part of Theorem 5. \( \square \)

**Remark 8.** Consider a region of an elliptic paraboloid \( M \) cut off by a hyperplane \( \Sigma \). Then with the same notations as above, (3.22) shows that \( VS(M) = c\|p - v\|^{(n+2)/2} \). Since \( \|p - v\| = tW \), it follows from (3.3) that \( R_p(t) = \frac{n+2}{2(n+1)}VS(M) \). Therefore the volume \( VC(M) \) of the cone with the vertex \( p \) becomes \( \frac{n+2}{2(n+1)}VS(M) \). For \( n = 2 \), this gives Archimedes’ results.

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**References**


