The Geometric Approach for Linear Periodic Discrete-Time Systems*

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ABSTRACT

Recently, the disturbance localization problem and the problems of designing disturbance decoupled observers and of giving a geometric characterization of invariant zeros were solved for linear periodic discrete-time systems through an extension of the geometric approach, based on the notions of controlled invariant and conditionally invariant subspaces, to periodic ones. Here the existing periodic geometric theory is supplemented with the notions of outer reachable subspace and controllability subspace, and with some further results on the previously introduced notions of inner reachable (controllable) subspace and outer controllable subspace. In addition, it is shown that any such periodic geometric theory can be restated in an equivalent time-invariant form, which is just the same theory written for a suitable time-invariant system having an extended state space.

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1. INTRODUCTION

The geometric theory based on the notions of controlled invariant subspace and conditionally invariant subspace of the state space of a linear time-invariant system is one of the main approaches to the analysis and control of such a system that have been followed in the last two decades, and a large amount of literature has been based on this geometric approach (see e.g. [1–11]).

More recently, the geometric approach was extended to periodically time-varying subspaces of the state space, in order to deal with periodic discrete-time linear systems [12–17]. Thus, with the help of the periodic notions of inner reachable (controllable) subspace, outer controllable subspace, and outer reconstructible subspace, some classical analysis and control problems were solved also for this class of discrete-time systems, namely the disturbance localization problem [12, 14], the problem of designing disturbance-decoupled observers [13], and the geometric characterization of the algebraic notion of invariant zero of a system [15, 16]. In fact, in recent years the number of contributions on linear periodic systems has been increasing [18–43], and several other problems have been solved, including, in the discrete-time case, state and output dead-beat control, eigenvalue assignment, dead-beat observer design, dead-beat regulation, and model matching [19–14, 28–32, 34, 35, 38, 39, 42]. This increasing interest in linear periodic systems is motivated by the large variety of processes which can be modeled through such systems (e.g. multirate sampled-data systems, chemical processes, periodically time-varying filters and networks, and seasonal phenomena [40, 43–49]), as well as by the relevance of periodic control to a wide range of applications, and to the stabilization and control of time-invariant linear systems and of a class of bilinear systems [18, 50–61].

In this framework, for the discrete-time case some attempts have been made to study the $\omega$-periodic system through time-invariant descriptions (see, e.g., [17, 33, 41–43, 46]). Indeed, the time-invariant system introduced by Meyer and Burrus [43] represents a powerful tool for both analysis and control purposes, since it allows one to reduce the study of some problems to the corresponding time-invariant ones [15, 16, 34]. However, it does not allow one to study the geometric concepts considered in this paper, since it provides the state response of the given periodic system only at a time distance of an integer number of periods from the initial time.

In this paper it will be shown that the time-invariant description of a periodic discrete-time system introduced in [17] and similar to those introduced by Verriest [41] and Park and Verriest [42] (see [29, 33] for other similar descriptions) allows one to deal with the periodic geometric notions
as one does with the corresponding notions relevant to a time-invariant system, and provides a time-invariant and therefore (in principle) simpler characterization of the former ones. This constitutes one of the motivations of this paper, and will be the object of Section 4. In connection with this, and in order to be able to show the kind of results obtained, after introducing some notation and recalling some preliminaries in Section 2, the existing geometric theory for periodic systems about inner reachable (controllable) subspaces and outer controllable ones will be extended with some further results and further geometric notions, and presented in a unified treatment. This constitutes a second motivation of this paper, and will be the object of Section 3, in which, in particular, the notions of outer reachable subspace and controllability subspace will be introduced, their properties and characterizations studied and compared with those of the other existing geometric concepts, and the inherent symmetries stressed. Thus, the results of Section 4 about a time-invariant description of the geometric notions will rely on a comprehensive, although concise, presentation of the existing periodic geometric theory.

2. NOTATION AND PRELIMINARIES

Consider the linear periodic discrete-time system $\Sigma$ described by

\begin{align}
    x(k+1) &= A(k)x(k) + B(k)u(k), \\
    y(k) &= C(k)x(k),
\end{align}

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n =: X$ is the state, $u(k) \in \mathbb{R}^m =: U$ is the control input, $y(k) \in \mathbb{R}^q =: Y$ is the output, and $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are real periodic matrices of period $\omega$ (briefly, $\omega$-periodic). The state transition matrix of the system $\Sigma$ is expressed by

$$
\Phi(k,k_0) := A(k-1)A(k-2)\cdots A(k_0), \quad k > k_0, \quad k, k_0 \in \mathbb{Z},
$$

$$
\Phi(k,k) := I_n, \quad k \in \mathbb{Z},
$$

where $I_n$ is the identity matrix of dimension $n$, and $\Phi^F(\cdot, \cdot)$ and $\Phi^G(\cdot, \cdot)$ are similarly defined with $A^F(k) := A(k) + F(k)F(k)$ and, respectively, $A^G(k) := A(k) + G(k)C(k)$ instead of $A(k)$, for real $\omega$-periodic matrices $F(\cdot)$ and
G(\cdot) of proper dimensions. Define also

\[ E_k := \Phi(k + \omega, k), \quad E_k^f := \Phi^f(k + \omega, k) \quad (3a) \]

\[ \Delta_k(j) := \Phi(k + \omega, k + j + 1) B(k + j), \quad j = 0, \ldots, \omega - 1, \]

\[ J_k := [\Delta_k(0) \cdots \Delta_k(\omega - 1)]. \quad J_k := \text{Im} J_k. \quad (3b) \]

As regards the role of the matrices \( E_k \) and \( J_k \), it is stressed that, for any initial time \( k_0 \in \mathbb{Z} \), the state response of the system \( \Sigma \) for \( k = k_0 + h\omega, \quad h \in \mathbb{Z}^+ \), to a given initial state \( x(k_0) \) and input function \( u(\cdot) \) can be obtained through the state equation of the time-invariant associated system of \( \Sigma \) at time \( k_0 \) [43], which is just expressed by

\[ x_{k_0}(h + 1) = E_{k_0} x_{k_0}(h) + J_{k_0} u_{k_0}(h), \]

with \( x_{k_0}(h) \in \mathbb{R}^n \), \( u_{k_0}(h) \in \mathbb{R}^{n^2} \); namely, if \( x_{k_0}(0) = x(k_0) \) and \( u_{k_0}(h) = [u(k_0 + h\omega) \cdots u(k_0 + \omega - 1 + h\omega)]^T \) for all nonnegative integers \( h \), then \( x_{k_0}(h) = x(k_0 + h\omega) \) for all \( h \in \mathbb{Z}^+ \). Therefore the subspace \( X_r(k) \) of the states of the system \( \Sigma \) which are reachable from zero at time \( k \) coincides with that of its associated system at time \( k \) [15, 16].

Now, for a given \( \omega \)-periodic subspace \( \mathcal{K}(k) \subset \mathcal{X} \), define

\[ d_k := \sum_{i=0}^{\omega-1} \dim \mathcal{K}(i) \leq n\omega. \quad (4) \]

In the following, a sequence of \( \omega \)-periodic subspaces \( \mathcal{K}^0(k), \mathcal{K}^1(k), \mathcal{K}^2(k), \ldots \) of \( \mathcal{X} \) will be said to be nonincreasing [nondcreasing] if \( \mathcal{K}^j(k) \subset \mathcal{K}^{j-1}(k) \) for all \( k \in \mathbb{Z} \) and for all \( j \in \mathbb{Z}^+ \). For a given class of \( \omega \)-periodic subspaces, the supremal (infimal) or largest (least) element of the class will mean the subspace of the class (if it exists) which contains (is contained in) every subspace of the class for all \( k \in \mathbb{Z} \).

A \( \omega \)-periodic subspace \( \mathcal{V}(k) \subset \mathcal{X} \) is said to be \( \Lambda(\cdot) \)-invariant if \( \Lambda(k) \mathcal{V}(k) \subset \mathcal{V}(k+1) \) for all \( k \in \mathbb{Z} \). The class of \( \Lambda(\cdot) \)-invariant \( \omega \)-periodic subspaces is closed under subspace addition and subspace intersection, and nonempty. The subspace \( \mathcal{X}_r(k) \) is the least \( \Lambda(\cdot) \)-invariant \( \omega \)-periodic subspace containing \( \text{Im} B(k - 1) \) for all \( k \in \mathbb{Z} \) [12]. Subspaces \( \mathcal{X}_r(k) \) and \( \mathcal{X}_r(k) \) are also referred to the pair \((\Lambda(\cdot), B(\cdot)) \), and are expressed by \([12]\)

\[ \mathcal{X}_r(k) = J_k + E_k J_k + \cdots + E_k^{n-1} J_k \quad \forall k \in \mathbb{Z}, \quad (5a) \]

\[ \mathcal{X}_r(k) = \mathcal{X}_r(k) + \text{Ker} E_k^n \quad \forall k \in \mathbb{Z}, \quad (5b) \]
since they coincide with those of the abovementioned associated system of $\Sigma$ at time $k$.

They are not altered by a linear state feedback, i.e., the subspace $X^e(k) [X^u(k)]$ of reachable [controllable] states at time $k$ of the pair $(A^e(\cdot), B(\cdot))$ coincides with $X_e(k) [X_u(k)]$. Dual statements concern the subspaces $X^e(k)$ and $X^u(k)$ of unobservable and, respectively, unreconstructible states at time $k$ [13].

A $\omega$-periodic subspace $V(k) \subset X$ is said to be $(A(\cdot), B(\cdot))$-invariant $[(C(\cdot), A(\cdot))$-invariant] if $A(k)V(k) \subset V(k+1) + \text{Im} B(k) [A(k)(V(k) \cap \text{Ker} C(k)) \subset V(k+1)]$ for all $k \in \mathbb{Z}$. Clearly, $(A(\cdot), B(\cdot))$-invariance is the dual notion of $(C(\cdot), A(\cdot))$-invariance.

PROPOSITION 1 [12, 13]. An $\omega$-periodic subspace $V(k) \subset X$ is $(A(\cdot), B(\cdot))$-invariant $[(C(\cdot), A(\cdot))$-invariant] if and only if there exists a $\omega$-periodic linear map $F(\cdot): X \to U$ $(G(\cdot): Y \to X)$ such that $A^e(k)V(k) \subset V(k+1)$ $(A^u(k)V(k) \subset V(k+1))$ for all $k \in \mathbb{Z}$.

The class of all $\omega$-periodic linear maps $F(\cdot) [G(\cdot)]$ such that a given $(A(\cdot), B(\cdot))$-invariant $[(C(\cdot), A(\cdot))$-invariant] $\omega$-periodic subspace $V(\cdot)$ is $A^e(\cdot)$-invariant $[A^u(\cdot)$-invariant] will be denoted by $F(V(\cdot)) [G(V(\cdot))]$. See [12, 13] for algorithms for the computation of elements of $F(V(\cdot))$ or of $G(V(\cdot))$.

For a given $\omega$-periodic subspace $K(k) \subset X$, the class of $(A(\cdot), B(\cdot))$-invariant $[(C(\cdot), A(\cdot))$-invariant] $\omega$-periodic subspaces contained in $[\text{containing}] K(k)$ for all $k \in \mathbb{Z}$ is closed under subspace addition [intersection] and nonempty; therefore there exists the supremal [infimal] element of the class, denoted by $K^* (\cdot) [K^*_w(\cdot)]$. The subspace $K^* (\cdot) [K^*_w(\cdot)]$ can be computed in a finite number of steps through the following nonincreasing [nondecreasing] sequence of subspaces $Z^i_k(k) [W^i_k(k)]$:

$$Z^0_k(k) := K(k) \quad \forall k \in \mathbb{Z},$$

$$Z^i_k(k) := K(k) \cap A^{-1}(k) [Z^{i-1}_k(k+1) + \text{Im} B(k)] \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, \ldots;$$  \hspace{1cm} (6a)

$$W^0_k(k) := K(k) \quad \forall k \in \mathbb{Z},$$

$$W^i_k(k+1) := K(k+1) + A(k) [W^{i-1}_k(k) \cap \text{Ker} C(k)] \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, \ldots;$$  \hspace{1cm} (6b)
satisfying [12, 13]

\[ Z_{K}^{d_K^h}(\cdot) = Z_{K}^{d_K}(\cdot), \quad W_{K}^{n_0 - d_K^h}(\cdot) = W_{K}^{n_0 - d_K}(\cdot) \quad \forall h \in \mathbb{Z}^+, \quad (7a) \]

\[ Z_{K}^{d_K}(\cdot) = kV^*(\cdot), \quad W_{K}^{n_0 - d_K}(\cdot) = kV^*(\cdot). \quad (7b) \]

The notions of \((A(\cdot),B(\cdot))\)-invariant subspace and \((C(\cdot),A(\cdot))\)-invariant subspace can be combined in order to produce the notion of \((C(\cdot),A(\cdot),B(\cdot))\)-pair [14], as in the time-invariant case [8, 9], and are the basic tools for solving classical control problems [12, 14].

3. SOME SIGNIFICANT GEOMETRIC NOTIONS

Special types of \((A(\cdot),B(\cdot))\)-invariant subspaces are strictly related to the notions of controllability to zero and reachability. An inner controllable (inner reachable or reachability) subspace is an \((A(\cdot),B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) whose states at each time \(k\) are controllable to zero (reachable from zero) along a trajectory \(x(h)\) belonging to \(V(h)\) for each \(h > k\) \((h < k)\). An outer controllable [outer reachable] subspace is an \((A(\cdot),B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) such that, for each time \(k\) and for each state \(x \in X\), the state of \(\Sigma\) can be transferred from \(x(k) = x\) into \(V(k + h)\) at time \(k + h\) for some \(h \in \mathbb{Z}^+\) [the state \(x\) can be decomposed into the sum of states, each of which is reachable at time \(k\) starting from some state of \(V(k - h)\) at time \(k - h\) for some nonnegative \(h\)]. Each of these notions admits a dual one (i.e. a special type of \((C(\cdot),A(\cdot))\)-invariant subspace [13, 14]) and implies the existence of a feedback control law with a specific control property, as stated by the following proposition for inner and outer controllable subspaces. The properties corresponding to inner and outer reachable subspaces will be derived afterwards.

**Proposition 2** [12]. An \((A(\cdot),B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) is inner (outer) controllable if and only if there exist a \(F(\cdot) \in \mathbb{F}(V(\cdot))\) and a positive integer \(h \leq d_V\) \((h \leq n_0 - d_V)\) such that \(\text{Ker} \Phi^V(k + h + i,k) \supset V(k)\) \((\text{Im} \Phi^V(k + h + i,k) \subset V(k + h + i))\) for all \(k \in \mathbb{Z}\) and for all nonnegative integers \(i\), or, equivalently, such that \(\text{Ker}(E_k^V)^n \supset V(k)\) \((\text{Im}(E_k^V)^n \subset V(k))\) for all \(k \in \mathbb{Z}\).

For a given inner (outer) controllable subspace \(V(\cdot)\), a \(F(\cdot) \in \mathbb{F}(V(\cdot))\) as in Proposition 2 can be found by the algorithms in [12]. The notions of inner
and outer controllable subspaces and their characterization stated by Proposition 2 play a crucial role in the disturbance localization problem [12,14], while the notion of inner reachable (or reachability) subspace allows us to introduce a geometric notion of zero for \( \Sigma \) [15, 16] (see Remark 5). The notion of outer reachable subspace for \( \Sigma \) is a new one (for a similar time-invariant notion see, e.g., [9]). Its main properties and characterizations will be now derived and compared with those of the other notions, thus exhibiting a nice symmetry.

The first kind of characterization of all four types of \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspaces can be obtained through proper algorithms. Namely, for a given \(\omega\)-periodic subspace \(K(k) \subset X\) define the following sequences of \(\omega\)-periodic subspaces:

\[
L^0_k(k) := K(k) \quad \forall k \in \mathbb{Z},
\]

\[
L^i_k(k) := K(k) + A^{-1}(k)[L^{i-1}_k(k) + 1 \text{m } B(k)] \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, \ldots \quad (8a)
\]

\[
M^0_k(k) := K(k) \quad \forall k \in \mathbb{Z},
\]

\[
M^i_{k+1} := K(k+1) + A(k)M^{i-1}_k(k) + 1 \text{m } B(k) \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, \ldots \quad (8b)
\]

\[
N^0_k(k) := \{0\} \quad \forall k \in \mathbb{Z},
\]

\[
N^i_k(k) := K(k) \cap A^{-1}(k)[N^{i-1}_k(k) + 1 \text{m } B(k)] \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, \ldots \quad (8c)
\]

\[
P^0_k(k) := \{0\} \quad \forall k \in \mathbb{Z},
\]

\[
P^i_{k+1} := K(k+1) \cap [A(k)P^{i-1}_k(k) + 1 \text{m } B(k)] \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, \ldots \quad (8d)
\]

which are easily seen to be nondecreasing and to satisfy the following
relations (see [12, 15] for the sequences (8c) and (8d)):

\[
L_{K}^{u_0 - d_K + h}(\cdot) = L_{K}^{u_0 - d_K}(\cdot), \quad M_{K}^{u_0 - d_K + h}(\cdot) = M_{K}^{u_0 - d_K}(\cdot), \quad \forall h \in \mathbb{Z}^+
\]

\[\text{(9a)}\]

\[
N_{K}^{d_K + h}(\cdot) = N_{K}^{d_K}(\cdot), \quad P_{K}^{d_K + h}(\cdot) = P_{K}^{d_K}(\cdot), \quad \forall h \in \mathbb{Z}^+.
\]

\[\text{(9b)}\]

**Theorem 1.** For an \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subseteq X\):

(a) \(V(\cdot)\) is outer reachable (controllable) if and only if

\[
M_{V}^{u_0 - d_V}(k) = X, \quad \forall k \in \mathbb{Z} \quad \left(L_{V}^{u_0 - d_V}(k) = X \text{ for an arbitrary } k \in \mathbb{Z}\right)
\]

\[\text{(10)}\]

(b) [12, 15] \(V(\cdot)\) is inner reachable (controllable) if and only if

\[
P_{V}^{d_V}(k) = V(k), \quad \forall k \in \mathbb{Z} \quad \left(N_{V}^{d_V}(k) = V(k) \quad \forall k \in \mathbb{Z}\right).
\]

\[\text{(11)}\]

**Proof of (a).** As regards outer reachability, notice that \(M_{V}^{i}(k)\) coincides with the set of all the states that can be decomposed into a sum of states, each of which is reachable at time \(k\) starting from some state of \(V(k - h)\) at time \(k - h\) for some \(h \in [0, i]\). Therefore, (9a) proves that the former equation of (10) is the condition for \(V(\cdot)\) to be outer reachable.

As regards outer controllability, the assumed \((A(\cdot), B(\cdot))\)-invariance of \(V(\cdot)\) implies that the sequence of subspaces \(L_{V}^{i}(\cdot)\) can be computed through the following simpler relations as well:

\[
L_{V}^{0}(k) := V(k), \quad \forall k \in \mathbb{Z},
\]

\[
L_{V}^{i}(k) := A^{-1}(k) \left[ L_{V}^{i-1}(k + 1) + \text{Im} B(k) \right], \quad \forall k \in \mathbb{Z}. \quad i = 1, 2, \ldots.
\]

\[\text{(12)}\]

For the sequence (12) it was proved in [12] that the latter equation in (10) is the condition for \(V(\cdot)\) to be outer controllable.

The meaning of the subspaces of sequences (8c), (8d), and (12) is easily obtained [12, 15]. For example, \(L_{V}^{i}(k)\) is the set of all initial states at time \(k\) from which the state of \(\Sigma\) can be transferred into the \((A(\cdot), B(\cdot))\)-invariant subspace \(V(k + i)\) at time \(k + i\).
In order to state the second, equivalent characterization of the four types of \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspaces, for a \(\omega\)-periodic linear map \(F(k) : X \to U\) and a \(\omega\)-periodic subspace \(K(k) \subseteq X\), denote by \(X_{r,F,k}(k)\) \(\subseteq X\) the subspace of reachable [controllable] states at time \(k\) of the pair \((A^F(k), B^F(k))\), where \(B^F(k) := \text{Im} B(k - 1) \cap K(k)\).

**Theorem 2.** For an \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subseteq X\):

(a) \(V(\cdot)\) is outer reachable (controllable) if and only if

\[
V(k) + X_r(k) = X \quad \forall k \in \mathbb{Z}
\]

or

\[
(V(k) + X_r(k) = X \quad \text{for an arbitrary } k \in \mathbb{Z}).
\] (13)

(b) \([12, 15]\) \(V(\cdot)\) is inner reachable (controllable) if and only if, for an arbitrary \(F(\cdot) \in \mathbb{F}(V(\cdot))\),

\[
V(k) = X_{r,F,v}(k) \quad \forall k \in \mathbb{Z}
\]

or

\[
(V(k) = X_{r,F,v}(k) + \text{Ker} (E^F_k)^n \cap V(k) \quad \forall k \in \mathbb{Z}).
\] (14)

**Remark 1.** The \((A(\cdot), B(\cdot))\)-invariance of \(V(\cdot)\) and the property that \(F(\cdot) \in \mathbb{F}(V(\cdot))\) are implied by the former equation of (14). Therefore the former property could be removed as hypothesis from the “inner reachable” part of Theorem 2(b), while the words “for an arbitrary \(F(\cdot) \in \mathbb{F}(V(\cdot))\)” could be replaced with “there exists an \(\omega\)-periodic linear map \(F(\cdot)\) such that” in the same part of the theorem.

**Proof of Theorem 2(a).** As regards outer reachability, the meaning of \(M_{\omega,v}^{\omega}d_{V}(k)\) [see the proof of Theorem 1(a)] and the assumed \((A(\cdot), B(\cdot))\)-invariance of \(V(\cdot)\) imply that

\[
M_{\omega,v}^{\omega}d_{V}(k) = V(k) + X_r(k) \quad \forall k \in \mathbb{Z}.
\] (15)

Therefore, the outer reachability condition in (10) can be rewritten as in (13).
The proof of the outer controllability condition in (13) was given in [12], by showing that

$$L^w_{V^{-1}}(V(k) + X_c(k)) \quad \forall k \in \mathbb{Z}. \quad \blacksquare \quad (16)$$

The relations (5b), (13), and (14) yield the following corollary.

**Corollary 1.** If an \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) is outer (inner) reachable, it is outer (inner) controllable.

In order to state the specific feedback properties of outer and inner reachable subspaces, note that for any \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) and for any \(F(\cdot) \in \mathbb{F}(V(\cdot))\), we have \(E_k^F V(k) \subset V(k + \omega) = V(k)\) for all \(k \in \mathbb{Z}\), whence the standard notation for restriction and induced map in the quotient space can be applied to \(E_k^F\) and \(V(k)\). Denoting by \(\mu(k)\) the dimension of \(V(k)\), and by \(\mu_m, \mu_M\) its minimum (maximum), the spectrum of \(E_k^F|V(k)\) is the union of a set of \(\mu(k) - \mu_m\) zero elements, irrespective of \(F(\cdot)\), and another set of \(\mu_m\) elements, called its core spectrum, which is independent of \(k\). It is the object of the following theorem, whose proof is in the Appendix.

**Theorem 3.** If an \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) is inner (outer) reachable, then, for each set of \(\mu_m, (n - \mu_M)\) complex numbers, subject to the requirement that nonreal elements appear in conjugate pairs, there exists an \(F(\cdot) \in \mathbb{F}(V(\cdot))\) such that the core spectrum of \(E_k^F|V(k)\) coincides with this set.

**Remark 2.** Theorem 3 constitutes the counterpart of Proposition 2 for inner and outer reachable subspaces. However, it is easily seen through counterexamples that the necessary condition given by Theorem 3 for inner (outer) reachability is not sufficient.

For a given inner (or outer) reachable subspace \(V(\cdot)\), an \(F(\cdot) \in \mathbb{F}(V(\cdot))\) satisfying the requirements of this theorem can be found by the procedure sketched in the proof, with the help of the design technique given in [12] for the computation of an \(F(\cdot) \in \mathbb{F}(V(\cdot))\) and the design algorithm given in [31, 32] for the eigenvalues assignment for a "reachable pair" with possibly time-varying dimensions, and the dynamic matrix possibly nonsquare.

Now, coming back to the characterizations of all the four special types of \((A(\cdot), B(\cdot))\)-invariant subspaces stated by Theorem 2, it seems useful to restate part (b) of that theorem in the following different form, in which the
equivalence between the condition in part (a) and the former equation of (14) can be shown just like the corresponding time-invariant result on the characterizations of the Wonham's controllability subspace notion [1], and a remark similar to Remark 1 applies to part (α).

**Proposition 3.** For an \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\):

(α) \(V(\cdot)\) is inner reachable if and only if, for an arbitrary \(F(\cdot) \in \mathcal{F}(V(\cdot))\), there exists an \(\omega\)-periodic linear map \(H(k): U \rightarrow U\) such that

\[
V(k) = X^{F,H}(k) \quad \forall k \in \mathbb{Z}, \quad H(k) := \text{Im}[B(k-1)H(k-1)].\tag{17}
\]

(β) [12] \(V(\cdot)\) is inner controllable if and only if, for an arbitrary \(F(\cdot) \in \mathcal{F}(V(\cdot))\), the following condition is satisfied:

\[
V(k) \subset X^{F,V}(k) \quad \forall k \in \mathbb{Z}.\tag{18}
\]

By (17) the notion of inner reachable subspace is just the extension to the \(\omega\)-periodic discrete-time system \(\Sigma\) of the controllability-subspace notion defined by Wonham [1] for a time-invariant continuous-time system. This justifies the name “reachability subspace” given first in [15, 16] to this notion. On the contrary, by Corollary 1 the notion of inner controllable subspace is a more general one (indeed, for \(F(\cdot) \in \mathcal{F}(V(\cdot)), X^{F,V}(k) \subset V(k)\) for all \(k \in \mathbb{Z}\) [12]), even though it coincides with the notion of inner reachable subspace in the continuous-time time-invariant case considered by Wonham, since in this case \(X^{F,V} = X_{c}^{F,V}\) [see (14) and (18)].

A different extension of Wonham's controllability-subspace notion can be based on controllability (to zero) instead of reachability. Namely, an \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset X\) is said to be a controllability subspace if there exist \(\omega\)-periodic linear maps \(F(\cdot): X \rightarrow U\) and \(H(k): U \rightarrow U\) such that

\[
V(k) = X^{F,H}(k) \quad \forall k \in \mathbb{Z}, \quad H(k) := \text{Im}[B(k-1)H(k-1)].\tag{19}
\]

Therefore, since \(X_{c}(k)\) has a time-invariant dimension [27], a controllability subspace has a dimension independent of time. Notice that (19) [as well as the former equations of (14) and (17)] implies the \((A(\cdot), B(\cdot))\)-invariance of \(V(\cdot)\) and the property that \(F(\cdot) \in \mathcal{F}(V(\cdot))\), whence the former property could be removed from the above definition. For any \(\omega\)-periodic \(F(\cdot), \text{Ker}(E_{k})^{n}\) is a simple example of a controllability subspace.
The following theorem gives some equivalent characterizations of controllability subspaces, which turn out to be a special type of inner controllable subspace, as well as reachability (or inner reachable) subspaces.

**Theorem 4.** An \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(\cdot) \subset X\) is a controllability subspace if and only if any of the following equivalent conditions is satisfied:

(i) there exists an \(\omega\)-periodic linear map \(F(k): X \to U\) such that 

\[
V(k) = X^{f,k}(k) \quad \forall k \in \mathbb{Z};
\]  

(ii) \(V(\cdot)\) is an inner controllable subspace, and there exists a \(F(\cdot) \in \mathbb{F}(V(\cdot))\) such that 

\[
\ker(E_k^f)^n \subset V(k) \quad \forall k \in \mathbb{Z},
\]  

(iii) \(V(\cdot)\) is an inner controllable subspace with a dimension independent of \(k\), and 

\[
X_r(k) + V(k) = X_r(k) \quad \forall k \in \mathbb{Z}.
\]  

**Proof.** Because of (5b), the relations (19) and (20) can be rewritten, respectively, as follows:

\[
V(k) = X^{f,k}(k) + \ker(E_k^f)^n \quad \forall k \in \mathbb{Z},
\]

\[
H(k) := \text{Im}[B(k-1)H(k-1)].
\]

Now, if (23a) holds for some real \(\omega\)-periodic \(F(\cdot)\) and \(H(\cdot)\), then \(H(k) \subset V(k)\), whence \(H(k) \subset V(k) \cap \text{Im} B(k-1)\). Therefore \(X^{f,k}(k) \subset X^{f,k}(k)\) for all \(k \in \mathbb{Z}\). This, together with (23a) and the relation \(X^{f,k}(k) \subset V(k)\) for all \(k \in \mathbb{Z}\) [implied by the \((A(\cdot), B(\cdot))\)-invariance of \(V(\cdot)\)], yields

\[
V(k) \subset X^{f,k}(k) + \ker(E_k^f)^n \subset V(k) \quad \forall k \in \mathbb{Z},
\]  

which implies (23b). Vice versa, if (23b) holds, one can easily find a
\( \omega \)-periodic linear map \( H(k): U \rightarrow U \) such that \( \text{Im}[B(k)H(k)] = \text{Im} B(k) \cap V(k+1) \) for all \( k \in \mathbb{Z} \), proving (23a). Therefore, (i) is necessary and sufficient for \( V(\cdot) \) to be a controllability subspace.

Notice that \( F(\cdot) \) in (i) certainly belongs to \( \mathcal{F}(V(\cdot)) \), because of the \( \Delta^F(\cdot)-\text{invariance of } X^F(\cdot) \).

Now, if (i) holds, by (20) and Proposition 3(b) \( V(\cdot) \) is an inner controllable subspace and has a dimension independent of \( k \) [27]. In addition, by (5b), (20), and the relations \( X^F(\cdot) = X_r(\cdot) \) and \( X^F(\cdot) = X_p(\cdot) \), Equation (23b) and the following relation hold:

\[
X_r(k) + V(k) = X^F_r(k) + X^F_rV(k) + \text{Ker}\left( E^F_k \right)^n = X_r(k) \quad \forall k \in \mathbb{Z},
\]

proving the necessity of (ii) and (iii).

If (ii) holds, then the latter equations of (14) and (21) imply (23b), whence (20), proving the sufficiency of (ii).

The proof of the sufficiency of (iii) is given in the Appendix.

**Remark 3.** By (20), the \( \omega \)-periodic linear map \( F(\cdot) \) of condition (i) belongs to \( \mathcal{F}(V(\cdot)) \), and by the proof of Theorem 4 it satisfies (21), and, vice versa, the linear map \( F(\cdot) \in \mathcal{F}(V(\cdot)) \) of condition (ii) satisfies (20). However, for a controllability subspace \( V(\cdot) \) in general the class of linear maps \( F(\cdot) \in \mathcal{F}(V(\cdot)) \) satisfying (20) and (21) is a proper subclass of \( \mathcal{F}(V(\cdot)) \); in other words, in general a \( F(\cdot) \in \mathcal{F}(V(\cdot)) \) does not satisfy (21), even if \( V(\cdot) \) is a controllability subspace. The sufficiency proof of condition (iii) provides a design procedure for a \( F(\cdot) \in \mathcal{F}(V(\cdot)) \) satisfying (20) and (21).

Although in general an \((A(\cdot), B(\cdot))\)-invariant subspace \( V(\cdot) \) and a \( F(\cdot) \in \mathcal{F}(V(\cdot)) \) do not satisfy either of (14) nor (20) [e.g., if \( X_r(k) \neq X \), then \( V(k) = X \) does not satisfy either of (14) or (20) for any \( F(\cdot) \), inner reachable, inner controllable, and controllability subspaces can be easily obtained by making use of the following theorem, as well as outer reachable and outer controllable subspaces by making use of Theorem 2(a).

**Theorem 5.**

(a) Given an arbitrary \( \omega \)-periodic subspace \( K(k) \subset X \) and an arbitrary \( \omega \)-periodic linear map \( F(k): X \rightarrow U \), the \((A(\cdot), B(\cdot))\)-invariant \( \omega \)-periodic subspace \( X^F_cK(k) \) \([X^F_cK(k)]\) is a controllability \( [\text{an inner reachable}] \) subspace \( V(\cdot) \), for which \( F(\cdot) \) belongs to \( \mathcal{F}(V(\cdot)) \) and (20) and (21) are [i.e., the former equation of (14) is] satisfied by \( F(\cdot) \).
(b) Given an arbitrary \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(K(k) \subset X\) and an arbitrary \(F(\cdot) \in \mathcal{F}(K(\cdot))\), the subspace \(V(k) \subset X\) defined by \(V(k) := X^E_K(k) + \text{Ker}(E_k^E)^n \cap K(k)\) is an inner controllable subspace, for which \(F(\cdot) \in \mathcal{F}(V(\cdot))\), i.e., the latter equation of (14) is satisfied.

Proof. (a): For \(V(k) := X^E_K(k)\) \([V(k) := X^E_K(k)]\), \(V(\cdot)\) is \(A^F(\cdot)\)-invariant. The subspace \(V_\ell(k) := X^E_k V(k)\) \([V_\ell(k) := X^E_k V(k)]\) is the least \(A^F(\cdot)\)-invariant \(\omega\)-periodic subspace containing \(\text{Im} B(k - 1) \cap V(k) + \text{Ker}(E_k^E)^n [\text{Im} B(k - 1) \cap V(k)]\) for all \(k \in \mathbb{Z}\). Since \(V(k)\) certainly contains such a subspace for all \(k \in \mathbb{Z}\) and is \(A^F(\cdot)\)-invariant and \(\omega\)-periodic, then \(V_\ell(k) \subset V(k)\) for all \(k \in \mathbb{Z}\). Since, by (5), \(V(k)\) contains \(\text{Im} B(k - 1) \cap K(k)\), then \(\text{Im} B(k - 1) \cap K(k) \subset \text{Im} B(k - 1) \cap V(k)\), whence \(V(k) \subset V_\ell(k)\) for all \(k \in \mathbb{Z}\). Therefore \(V_\ell(\cdot) = V(\cdot)\), and (20) \([\text{the former equation of (14)}]\) is satisfied by \(F(\cdot)\).

(b): Since the subspaces \(X^E_K(k), K(k), \text{ and Ker}(E_k^E)^n\) are \(\omega\)-periodic and \(A^F(\cdot)\)-invariant \([12]\), then \(V(\cdot)\) is \(A^F(\cdot)\)-invariant, whence \((A(\cdot), B(\cdot))\)-invariant. Define \(V_\ell(k) := X^E_k V(k)\), \(V_2(k) := \text{Ker}(E_k^E)^n \cap V(k)\), \(V_2(k) := V_\ell(k) + V_2(k)\). Then reasoning similar to the proof of part (a) shows that \(V_\ell(k) \subset V(k)\), whence \(V_2(k) \subset V(k)\) for all \(k \in \mathbb{Z}\), and that \(X^E_K(k) \subset V_\ell(k)\) for all \(k \in \mathbb{Z}\). Since \(\text{Ker}(E_k^E)^n \cap K(k) \subset V_2(k)\), this proves that \(V(k) \subset V_2(k)\) for all \(k \in \mathbb{Z}\), whence \(V_2(\cdot) = V(\cdot)\), i.e. the latter equation of (14) is satisfied.

Now, for each special type of \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace studied so far, consider the subclass of those subspaces which are contained in a given \(\omega\)-periodic subspace \(K(k) \subset X\) for all \(k \in \mathbb{Z}\).

Remark 4. As far as controllability subspaces are concerned, by condition (iii) of Theorem 4 the abovementioned subclass can be empty (as can be seen through counterexamples \([62]\), unless \(X_\ell(\cdot) = X_\ell(\cdot)\), since in this case the constant inner reachable subspace \(0\) belongs to the subclass for any \(K(\cdot)\), by the same condition (iii) of Theorem 4. In general, such a subclass is nonempty iff there exists a (possibly non-inner-controllable) \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subset K(k)\) with a dimension independent of \(k\) satisfying (22) \([\text{since this hypothesis implies the existence of an } F(\cdot) \in \mathcal{F}(V(\cdot))\] such that (21) holds, where \(\text{Ker}(E_k^E)^n\) is a controllability subspace\), which is guaranteed \([62]\) if the dimension of \(X_\ell V^*_k(k)\) is independent of \(k\) and it satisfies

\[
X_\ell(k) + K^\ast(k) \supset X_\ell(k) \quad \forall k \in \mathbb{Z}.
\]
Even in the time-invariant case (i.e. when $\omega = 1$), the class of the controllability subspaces contained in a given subspace $K \subset X$ can be empty, and it is nonempty if and only if the time-invariant version of (26) holds [62]. In this case, there exists the supremal element of the class, since, by condition (iii) of Theorem 4, the class is closed under subspace addition (since the class of inner controllable subspaces is).

If $\omega > 1$ (i.e. in the periodic nonconstant case), the class of the controllability subspaces contained in a given $\omega$-periodic subspace $K(k) \subset X$, if nonempty, by the time invariance of their dimensions, is nonclosed in general under subspace addition (as can be easily seen by counterexamples [62]); then the supremal element $K V_c^*(\cdot)$ of this class may not exist for $\omega > 1$ [62]. However, it can be shown that if the dimension of $K V^*(k)$ is independent of $k$ and (26) holds, then there exists the supremal element $K V_c^*(\cdot)$, and it is expressed by [62]

$$K V_c^*(k) = X c \cdot K V^*(k) \quad \forall k \in \mathbb{Z}$$

for an arbitrary $F(\cdot) \in F(K V^*(\cdot))$ such that (21) holds with $V(\cdot) = K V^*(\cdot)$ (whose existence can be shown to be guaranteed by (26) and the constant dimensionality of $K V^*(k)$ [62]).

On the contrary, it is easily seen that, for a given $\omega$-periodic subspace $K(k) \subset X$, the class of inner reachable [controllable] subspaces contained in $K(k)$ for all $k \in \mathbb{Z}$ is closed under subspace addition and nonempty, and therefore there always exists the supremal element $K V_i^*(k) [K V_{i c}^*(k)]$ of the class. Also, the class of outer reachable [controllable] subspaces contained in $K(k)$ for all $k \in \mathbb{Z}$ is closed under subspace addition, but in general can be empty; therefore, if it is nonempty, there certainly exists the supremal element $K V_{o r}^*(k) [K V_{o c}^*(k)]$ of the class. Then Theorem 2(a) yields part (a) of the following theorem.

**Theorem 6.**

(a) For a given $\omega$-periodic subspace $K(k) \subset X$, the class of outer reachable (controllable) subspaces contained in $K(k)$ for all $k \in \mathbb{Z}$ is nonempty if and only if

$$K V^*(k) + X_c(k) = X \quad \forall k \in \mathbb{Z}$$

$$(K V^*(k) + X_c(k) = X \text{ for an arbitrary } k \in \mathbb{Z}), \quad (28)$$

in which case $K V_{o r}^*(\cdot) = K V^*(\cdot)$ ($K V_{o c}^*(\cdot) = K V^*(\cdot)$).
(b) [12, 15, 16] For a given $\omega$-periodic subspace $K(k) \subset X$, an arbitrary $F(\cdot) \in \mathbb{F}(\mathbb{K}V^*(\cdot))$ satisfies the relations $F(\cdot) \in \mathbb{F}(\mathbb{K}V^*_i(\cdot))$, $F(\cdot) \in \mathbb{F}(\mathbb{K}V^*_r(\cdot))$, and

\[ kV^*_i(k) = X^F_r \cdot kV^*(k) \subset kV^*_r(k) \]

\[ = kV^*_i(k) + \text{Ker}(E^F_r)^n \cap kV^*(k) \subset kV^*(k) \quad \forall k \in \mathbb{Z}, \quad (29a) \]

\[ kV^*_i(k) = \mathbb{P}^d_{kV^*}(k) \quad \forall k \in \mathbb{Z}, \quad (29b) \]

\[ kV^*_c(k) = \mathbb{N}^d_{kV^*}(k) = \mathbb{N}^d_k(k) \quad \forall k \in \mathbb{Z}. \quad (29c) \]

**Remark 5.** It seems useful to recall that, defining $C(k) := \text{Ker} C(k)$, the subspaces $\mathbb{C}V^*(\cdot)$, $\mathbb{C}V^*_i(\cdot)$, $\mathbb{C}V^*_r(\cdot)$, and $X_j(\cdot)$ are the ingredients through which it is possible to express the existence conditions for a solution of the disturbance localization problem, with and without an additional requirement of output or state dead-beat control, for when an unmeasurable and unknown additive disturbance affects Equation (1a), $y(k)$ plays the role of the output to be controlled, and the state of $\Sigma$ is measurable [12]. If, more generally, the measured output does not coincide either with the state of $\Sigma$ or with the output to be controlled, the existence conditions of a solution involve also $M \mathbb{V}^*(\cdot)$ and the least outer reconstructible subspace (the notion dual to that of inner controllable subspace) containing the subspace $M(k) := \text{Im} M(k - 1)$, where $M(k)$ is the matrix through which the disturbance affects Equation (1a) [14]; the same least outer reconstructible subspace allows one to express the existence conditions for a disturbance-decoupled dead-beat estimator of an output of $\Sigma$ different from $y(k)$ [13]. In addition, $V^*(\cdot)$ and $V^*_i(\cdot)$ allow one to introduce for $\Sigma$ a geometric notion of zero, similar to the time-invariant one [15, 16]; the corresponding structure at infinity allows one to express the existence conditions for a solution of the model-matching problem for $\Sigma$ [39].

The apparent asymmetry between the former and the latter equation of (28), as well as (10) and (13), seems to perturb the strong symmetry of Proposition 2, Theorems 1, 2, 3, and 6, and Corollary 1. The following result, whose proof is given in the Appendix, clarifies why the subspaces $kV^*(k) + X_c(k)$ in the latter equation of (28) and $V(k) + X_c(k)$ in the latter equation of
Proposition 4. For any \((A(\cdot), B(\cdot))-\text{invariant } \omega\)-periodic subspace \(V(k) \subset X\), the subspace \(V(k) + X_c(k)\) has a dimension which is independent of \(k\).

Before concluding this section, it seems useful to point out a connection between the notion of \((A(\cdot), B(\cdot))-\text{invariant subspace and that of } (C(\cdot), A(\cdot))-\text{invariant subspace (whose definition is the only one of a dual notion explicitly recalled here). It is expressed by the following theorem, whose proof is given in the Appendix.}

Theorem 7. Defining \(B(k) := \text{Im} B(k - 1)\) and \(C(k) := \text{Ker} C(k)\), the following relation holds:

\[
cV_r(k) = cV^*(k) \cap bV_*(k) \quad \forall k \in \mathbb{Z}.
\]  

4. A TIME-ININVARIANT CHARACTERIZATION OF THE PERIODIC GEOMETRIC NOTIONS

The theory developed or recalled in the previous sections obviously remains valid also in the case when \(\omega = 1\). The case \(\omega > 1\) differs from the time-invariant case simply in that for \(\omega > 1\) all the \(\omega\)-periodic subspaces involved [see for example (5), (6), and (8)] must be computed at \(\omega\) different time instants \(k\) over a period (e.g., \(k = 0, 1, \ldots, \omega - 1\)) in order to be known, while all the relations between them [e.g., (10), (11), (13), (14), and (17)-(22)] must be checked at \(\omega\) different values of \(k\) over a period, with the sole exceptions of the latter equations of (10), (13), and (28) among the relations here reported. Therefore, the aim of this section is just to show that this whole theory can be restated in a different but equivalent form, which is just the same theory written for a suitable time-invariant system of dimension \(n\omega\), similar to those introduced in [41, 42]. In this way, although the overall dimension of any analysis problem here considered for \(\Sigma\) remains unchanged, the “conceptual time-invariance” of such a problem (implied by the periodicity of \(\Sigma\), which is a form of stationarity) will be strongly exhibited by relations involving time-invariant subspaces of \(\mathbb{R}^{n\omega}\) which express the solution of just the same problem for a time-invariant system of dimension \(n\omega\).
For this purpose, for any \( k \in \mathbb{Z} \) define the following matrices:

\[
R := \begin{bmatrix} 0 & I_{(\omega-1)n} \\ I_n & 0 \end{bmatrix},
\]

\[
\mathcal{A}_k := \text{diag}\{A(k), A(k+1), \ldots, A(k+\omega-1)\}, \quad \tilde{\mathcal{A}}_k := R^{-1} \mathcal{A}_k,
\]

\[
\mathcal{B}_k := \text{diag}\{B(k), B(k+1), \ldots, B(k+\omega-1)\}, \quad \tilde{\mathcal{B}}_k := R^{-1} \mathcal{B}_k,
\]

\[
\mathcal{C}_k := \text{diag}\{C(k), C(k+1), \ldots, C(k+\omega-1)\},
\]

and, for real \( \omega \)-periodic linear maps \( F(h): X \to U \), \( G(h): Y \to X \), and \( H(h): U \to U \),

\[
\mathcal{F}_k := \text{diag}\{F(k), F(k+1), \ldots, F(k+\omega-1)\},
\]

\[
\mathcal{G}_k := \text{diag}\{G(k), G(k+1), \ldots, G(k+\omega-1)\}, \quad \tilde{\mathcal{G}}_k := R^{-1} \mathcal{G}_k,
\]

\[
\mathcal{H}_k := \text{diag}\{H(k), H(k+1), \ldots, H(k+\omega-1)\},
\]

\[
\mathcal{A}^F_k := \text{diag}\{A^F(k), A^F(k+1), \ldots, A^F(k+\omega-1)\} = \mathcal{A}_k + \mathcal{B}_k \mathcal{F}_k,
\]

\[
\tilde{\mathcal{A}}^F_k := \tilde{\mathcal{A}}_k + \tilde{\mathcal{B}}_k \tilde{\mathcal{F}}_k = R^{-1} \mathcal{A}^F_k,
\]

\[
\mathcal{G}^F_k := \text{diag}\{A^F_G(k), A^F_G(k+1), \ldots, A^F_G(k+\omega-1)\} = \mathcal{A}_k + \mathcal{G}_k \mathcal{G}_k,
\]

\[
\tilde{\mathcal{G}}^F_k := \tilde{\mathcal{A}}_k + \tilde{\mathcal{G}}_k \tilde{\mathcal{G}}_k = R^{-1} \mathcal{G}^F_k.
\]

Define also the following operator \( S_k: \mathcal{V}_n^{\omega} \to \mathcal{V}_n^{\omega} \), where \( \mathcal{V}_n^{\omega} \) is the set of all \( \omega \)-periodic subspaces \( \mathcal{V}(\cdot) \) contained in \( \mathbb{R}^n = X \) at each time \( h \), while \( \mathcal{V}_n^{\omega} \) is the set of all the subspaces of \( \mathbb{R}^{n\omega} \):

\[
S_k \mathcal{V}(\cdot) := \{ w \in \mathbb{R}^{n\omega} : w = [x_0' \quad x_1' \quad \cdots \quad x_{\omega-1}'] \\
\text{with } x_i \in \mathcal{V}(k+i), \ i = 0, 1, \ldots, \omega-1 \}.
\]

and the following time-invariant system, which will be called the "extended"
system $\Sigma_k^\varepsilon$ of the system $\Sigma$ at time $k$:

$$z(h + 1) = \mathcal{A}_k z(h) + \mathcal{B}_k v(h), \quad (34a)$$

$$w(h) = \mathcal{L}_k z(h), \quad (34b)$$

with $z(h) \in \mathbb{R}^{n_\omega}$, $v(h) \in \mathbb{R}^{p_\omega}$, $w(h) \in \mathbb{R}^{q_\omega}$. Finally, restrict the linear state feedback, output injection, and input feedforward for $\Sigma_k^\varepsilon$ to be characterized, respectively, by real matrices $\mathcal{F}_k$, $\mathcal{G}_k := R^{-1} \mathcal{I}_k$ and $\mathcal{H}_k$, with $\mathcal{F}_k$, $\mathcal{G}_k$, and $\mathcal{H}_k$ having the block-diagonal structure described by (32a), (32b) and (32c). The classes of such matrices $\mathcal{F}_k$, $\mathcal{G}_k$, and $\mathcal{H}_k$ will be denoted, respectively, by $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$.

**Remark 6.** Notice that, by (31),

$$\begin{bmatrix}
0 & 0 & \cdots & 0 & A(k + \omega - 1) \\
A(k) & 0 & \cdots & 0 & 0 \\
0 & A(k + 1) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A(k + \omega - 2) & 0
\end{bmatrix}$$

$$\begin{bmatrix}
0 & 0 & \cdots & 0 & B(k + \omega - 1) \\
B(k) & 0 & \cdots & 0 & 0 \\
0 & B(k + 1) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B(k + \omega - 2) & 0
\end{bmatrix}$$

Therefore, denoting by $\delta$ the one-step forward shift operator satisfying $\delta x(h) = x(h + 1)$, for a given $k$ the matrix $[\mathcal{A}_k \cdot \text{diag}(\delta \omega I_n, I_{(\omega - 1)n}), \mathcal{B}_k]$ allows us to compute the state response of $\Sigma$ to the input function $u(\cdot)$ from the initial state $x(k)$ at any time $k_1 > k$, as the matrix $[A - \delta I_n, B]$ does for time-invariant systems.

By (33) the subspace $S_k V(\cdot)$, which is a subspace of $\mathbb{R}^{n_\omega}$ (i.e. of the state space of $\Sigma_k^\varepsilon$), could be called the “stacked form” at time $k$ of the $\omega$-periodic subspace $V(h)$ of $X$. It will now be shown that:

(i) $V(\cdot)$ has any of the properties here considered in the previous sections with respect to $\Sigma$ iff $S_k V(\cdot)$ has the same property with respect to the time-invariant system $\Sigma_k^\varepsilon$, under the mentioned restriction that the matrices $\mathcal{F}_k$, $\mathcal{G}_k$, and $\mathcal{H}_k$ belong to $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$, respectively.
(ii) It is possible to obtain the expression in stacked form for any \( \omega \)-periodic subspace of \( X \) of interest for \( \Sigma \) [e.g. \( X_r(h), X_c(h), \) and \( \kappa V^*(h) \) for some \( \omega \)-periodic \( K(h) \subset X \)], or to translate into stacked form any relation between \( \omega \)-periodic subspaces of \( X \) of interest for \( \Sigma \) [e.g. any of (10), (11), (13), (14), and (17)–(22)], in two different ways which lead exactly to the same expression for the stacked form of the subspace, or of the two members of the relation, just as in a commutative diagram. The first way is to apply the operator \( S_k \) to each term of the expression for the subspace, or of the expressions in the two members of the relation, with the help of some properties enjoyed by \( S_k \) for the \( \omega \)-periodic subspaces of the state space of \( \Sigma \). The second way is simply to write the expression for the subspace of \( \mathbb{R}^{\omega \omega} \), or to put into the two members of the relation, in stacked form, the subspaces of \( \mathbb{R}^{\omega \omega} \) having the same meaning for \( \Sigma \) that the \( \omega \)-periodic subspace of \( X \) has for the system \( \Sigma \), or that the subspaces of \( X \) appearing in the given relation have for the system \( \Sigma \).

In order to prove these assertions, notice that \( S_k \) enjoys the following properties for \( \omega \)-periodic subspaces \( V_1(h), V_2(h), V(h) \) of \( X \):

\[
V_1(h) = V_2(h) \quad \forall h \in [k, k + \omega - 1] \quad \Leftrightarrow \quad S_k V_1(\cdot) = S_k V_2(\cdot), \quad (36a)
\]

\[
V(h) = V_1(h) + V_2(h) \quad \forall h \in [k, k + \omega - 1] \quad \Leftrightarrow \quad S_k V(\cdot) = S_k V_1(\cdot) + S_k V_2(\cdot), \quad (36b)
\]

\[
V(h) = V_1(h) \cap V_2(h) \quad \forall h \in [k, k + \omega - 1] \quad \Leftrightarrow \quad S_k V(\cdot) = S_k V_1(\cdot) \cap S_k V_2(\cdot), \quad (36c)
\]

\[
V_1(h) \subset V_2(h) \quad \forall h \in [k, k + \omega - 1] \quad \Leftrightarrow \quad S_k V_1(\cdot) \subset S_k V_2(\cdot), \quad (36d)
\]

\[
S_k \delta V(\cdot) = S_{k+1} V(\cdot) = RS_k V(\cdot), \quad (36e)
\]

\[
V_2(h + 1) = A(h) V_1(h) \quad \forall h \in [k, k + \omega - 1] \quad \Leftrightarrow \quad S_{k+1} V_2(\cdot) = \mathcal{A}_k S_k V_1(\cdot)
\]

\[
\Leftrightarrow \quad S_k V_2(\cdot) = \mathcal{A}_k S_k V_1(\cdot), \quad (36f)
\]

\[
V_1(h) = A^{-1}(h) V_2(h + 1) \quad \forall h \in [k, k + \omega - 1] \quad \Leftrightarrow \quad S_k V_1(\cdot) = \mathcal{A}_k^{-1} S_k V_2(\cdot)
\]

\[
\Leftrightarrow \quad S_k V_1(\cdot) = \mathcal{A}_k^{-1} S_k V_2(\cdot), \quad (36g)
\]

\[
\dim S_k V(\cdot) = d_V, \quad (36h)
\]
where the symbol \( \delta \) denotes the one-step forward shift operator, satisfying 
\[
\delta V(h) = V(h + 1) \quad \text{(as in Remark 6)}.
\]
Notice also that a sequence of \( \omega \)-periodic subspaces \( K^0(h), K^1(h), K^2(h), \ldots \) of \( X \) is nonincreasing (nondecreasing) if and 
only if the sequence \( \sum_k K^0(\cdot), \sum K^1(\cdot), \sum K^2(\cdot), \ldots \) is nonincreasing (nondecreasing).

From the above relations, the following theorem and lemma follow.

**Theorem 8.**

(a) An \( \omega \)-periodic subspace \( V(h) \subset X \) is \( A(\cdot) \)-invariant if and only if, for 
an arbitrary \( k \in \mathbb{Z} \), \( S_k V(\cdot) \) is \( \mathcal{A}^k \)-invariant, i.e.,

\[
\mathcal{A}^k S_k V(\cdot) \subset S_k V(\cdot). \tag{37}
\]

It is \((A(\cdot), B(\cdot))\)-invariant \(((C(\cdot), A(\cdot))\)-invariant) if and only if, for an
arbitrary \( k \in \mathbb{Z} \), any of the following equivalent conditions is satisfied:

(i) \( S_k V(\cdot) \) is \((\mathcal{A}^k, \mathcal{B}_k)\)-invariant \(((\mathcal{C}^k, \mathcal{A}^k)\)-invariant), i.e.

\[
(\mathcal{A}^k S_k V(\cdot) + \text{Im} \mathcal{B}_k) \left( \mathcal{A}^k (S_k V(\cdot) \cap \text{Ker} \mathcal{C}_k) \subset S_k V(\cdot) \right). \tag{38}
\]

(ii) There exists \( \mathcal{F}_k \in \mathcal{F} \) \((\mathcal{F}_k \in \mathcal{G})\) such that

\[
(\mathcal{A}^k S_k V(\cdot) \subset S_k V(\cdot)) \quad \left( \mathcal{F}_k \mathcal{A}^k S_k V(\cdot) \subset S_k V(\cdot) \right). \tag{39}
\]

(b) Given any \( \omega \)-periodic subspace \( K(k) \subset X \), for an arbitrary \( k \in \mathbb{Z} \),
\( S_h V(\cdot) \) coincides with the largest \((\mathcal{A}^h, \mathcal{B}_h)\)-invariant (the least 
\((\mathcal{C}^h, \mathcal{A}^h)\)-invariant) subspace \( \mathcal{K}^* \) \((\mathcal{K}^*_*)\) contained in (containing) \( K := S_h K(\cdot) \).

**Proof.** By (36), \( V(\cdot) \) is \( A(\cdot) \)-invariant, or \((A(\cdot), B(\cdot))\)-invariant 
\(((C(\cdot), A(\cdot))\)-invariant) if, for an arbitrary \( k \in \mathbb{Z} \),

\[
\mathcal{A}^k S_k V(\cdot) \subset R S_k V(\cdot), \quad \tag{40}
\]

or, respectively,

\[
(\mathcal{A}^k S_k V(\cdot) \subset R S_k V(\cdot) + \text{Im} \mathcal{B}_k) \left[ \mathcal{A}^k (S_k V(\cdot) \cap \text{Ker} \mathcal{C}_k) \subset R S_k V(\cdot) \right], \quad \tag{41}
\]

which are equivalent to (37) and, respectively, to (38). The proof of (a) is
completed with the help of Proposition 1 and (40) written with $\mathcal{A}_k$ substituted by $\mathcal{A}_k^f$. Part (b) follows by noting that, by (6), (31), and (36),

$$S_hZ^0_k(\cdot) = K = Z^0_K,$$

$$S_hZ^i_k(\cdot) = K \cap \mathcal{A}_h^{-1}(Z^i_K + \text{Im } \mathcal{B}_h) = Z^i_K, \quad i = 1, 2, \ldots, (42a)$$

$$S_hW^0_k(\cdot) = K = W^0_K,$$

$$S_hW^i_k(\cdot) = K + \mathcal{A}_h(W^i_K + \text{Ker } \mathcal{C}_h) = W^i_K, \quad i = 1, 2, \ldots, (42b)$$

which, by (7), (36a), (36h), and the relations similar to (7) and involving $Z^i_K$, $W^i_K$, $K^*$, and $K^*$ instead of $Z^i(\cdot)$, $W^i(\cdot)$, $K^*$, and $K^*$, imply that

$$S_hK^*(\cdot) = Z^d_K = K^*, (43a)$$

$$S_hK^*(\cdot) = W^{n\omega-d}_K = K^*.$$

**Remarks.** The algorithms for the computation of $S_hK^*(\cdot)$ and $S_hK^*(\cdot)$ are expressed by (42) and (43). Notice that the application of $S_h$ to the class of $(A(\cdot), B(\cdot))$-invariant [(C(\cdot), A(\cdot))]-invariant $\omega$-periodic subspaces contained in [containing] $K(k)$ for all $k \in \mathbb{Z}$ allows one to obtain only a subclass, in general, of the class of all $(\mathcal{A}_h, \mathcal{B}_h)$-invariant [(C_h, A_h)-invariant] subspaces contained in [containing] $K$, as can be easily seen through counterexamples. Therefore Theorem 8(b) is not a trivial implication of (36a)-(36d) and Theorem 8(a). Notice also that condition (ii) of Theorem 8(a) is not a simple restatement of condition (i) (obtained from the application of Proposition 1 to system $\Sigma_k^f$), since it consists in the existence of some $\mathcal{A}_k [\mathcal{F}_k]$ in the special class $\mathcal{F} [\mathcal{G}]$.

**Lemma 1.** For an arbitrary $h \in \mathbb{Z}$, the following relations hold:

$$S_hX_r(\cdot) = \text{Im } \mathcal{B}_h + \mathcal{I}_h + \cdots + \mathcal{I}_h^{\omega-1} \text{ Im } \mathcal{B}_h =: X_r, (44a)$$

$$S_hX_c(\cdot) = X_r + \text{Ker } \mathcal{I}_h^{\omega} =: X_c. (44b)$$

**Proof.** Firstly notice that (3) and (35) imply the following relations:

$$\text{diag}\{E_h, E_{h+1}, \ldots, E_{h+\omega-1}\} = \mathcal{A}_h^\omega, (45a)$$

$$\text{Im diag}\{J_h, J_{h+1}, \ldots, J_{h+\omega-1}\} = \text{Im } \left[ \begin{array}{ccc} \mathcal{B}_h & \mathcal{I}_h \mathcal{B}_h & \cdots & \mathcal{I}_h^{\omega-1} \mathcal{B}_h \end{array} \right]. (45b)$$
Then the relation (44a) follows from (5a) written for \( k = h, h + 1, \ldots, h + \omega - 1 \), (31), (33), (36b), and (45). The relation (44b) follows from (5b), (36b), and (44a), taking into account that, by (45a),

\[
V(k) = \text{Ker} \, E_k^0 \quad \forall k \in [h, h + \omega - 1] \iff S_h V(\cdot) = \text{Ker} \, \mathcal{A}_h^{\omega}.
\]  

\[ \blacksquare \quad (46) \]

**Corollary 2.** For any \( \omega \)-periodic subspace \( V(k) \subset X \), for an arbitrary \( h \in \mathbb{Z} \), for \( \mathcal{V} := S_h V(\cdot) \), and for an arbitrary \( \omega \)-periodic linear map \( F(k): X \to U \) and the corresponding \( \mathcal{F}_h \subset \mathcal{F} \) defined by (32a), the following relations hold:

\[
S_h X_{\mathcal{V}}^{F, V}(\cdot) = \text{Im} \, \mathcal{A}_h \cap \mathcal{V} + \mathcal{A}_h^{\omega} \left( \text{Im} \, \mathcal{A}_h \cap \mathcal{V} \right) + \cdots
\]

\[
+ \left( \mathcal{A}_h^{\omega} \right)^{n_\omega - 1} (\text{Im} \, \mathcal{A}_h \cap \mathcal{V}) = X_{\mathcal{F}_h}^{\mathcal{V}}.
\]

\[
S_h X_{\mathcal{V}}^{F, V}(\cdot) = X_{\mathcal{F}_h}^{\mathcal{V}} + \text{Ker} \left( \mathcal{A}_h^{\omega} \right)^{n_\omega} = X_{\mathcal{F}_h}^{\mathcal{V}}.
\]  

(47a)

(47b)

Now, given any \( \omega \)-periodic subspace \( K(k) \subset X \), for an arbitrary \( h \in \mathbb{Z} \) apply the operator \( S_h \) to (8). By (31), (33), and (36) and the properties of the sequences \( L^i_k(\cdot), M^i_k(\cdot), N^i_k(\cdot), \) and \( P^i_k(\cdot) \) \((i = 0, 1, 2, \ldots)\), the sequences \( S_h L^i_k(\cdot), S_h M^i_k(\cdot), S_h N^i_k(\cdot), \) and \( S_h P^i_k(\cdot) \) \((i = 0, 1, 2, \ldots)\) are nondecreasing and coincide, respectively, with the sequences of subspaces \( L^i_k, M^i_k, N^i_k, \) and \( P^i_k \) \((i = 0, 1, 2, \ldots)\) of \( \mathbb{R}^{n_\omega} \) defined in the same way for the time-invariant system \( \Sigma^*_h \) and for the subspace \( K := S_h K(\cdot) \) of \( \mathbb{R}^{n_\omega} \). Namely,

\[
S_h L^0_k(\cdot) = S_h K(\cdot) =: K =: L^0_k.
\]

\[
S_h L^i_k(\cdot) = K + \mathcal{A}_h^{i-1} \left( L^i_k -1 + \text{Im} \, \mathcal{A}_h \right) =: L^i_k, \quad i = 1, 2, \ldots,
\]

\[
S_h M^0_k(\cdot) = S_h K(\cdot) =: K =: M^0_k.
\]

\[
S_h M^i_k(\cdot) = K + \mathcal{A}_h M^{i-1}_k + \text{Im} \, \mathcal{A}_h =: M^i_k \quad i = 1, 2, \ldots,
\]

\[
S_h N^0_k(\cdot) = \{0\} =: N^0_k.
\]

\[
S_h N^i_k(\cdot) = K \cap \mathcal{A}_h^{i-1} \left( N^i_k -1 + \text{Im} \, \mathcal{A}_h \right) =: N^i_k \quad i = 1, 2, \ldots,
\]

\[
S_h P^0_k(\cdot) = \{0\} =: P^0_k.
\]

\[
S_h P^i_k(\cdot) = K \cap \left( \mathcal{A}_h P^{i-1}_k + \text{Im} \, \mathcal{A}_h \right) =: P^i_k \quad i = 1, 2, \ldots
\]  

(48a)

(48b)

(48c)

(48d)
Now, the central part of the theory developed or recalled in the previous section can be restated as follows, using the notation introduced in Theorem 8, Lemma 1, and Corollary 2.

**Theorem 9.** An \((A(\cdot), B(\cdot))\)-invariant \(\omega\)-periodic subspace \(V(k) \subseteq X\) is outer reachable (controllable) or inner reachable (controllable) for the system \(\Sigma\) if and only if, for an arbitrary \(h \in \mathbb{Z}\), \(V := S_h V(\cdot)\) is outer reachable (controllable) or, respectively, inner reachable (controllable) for the time-invariant system \(\Sigma^c_h\); equivalently, if and only if, for an arbitrary \(h \in \mathbb{Z}\), any of the following equivalent conditions is satisfied, for the outer reachability (controllability) property:

(i) one has

\[
M^n_{V}^{dV} = \mathbb{R}^{n_o} \quad (L_{V}^{dV} = \mathbb{R}^{n_o});
\]

(ii) one has

\[
V + X_r = \mathbb{R}^{n_o} \quad (V + X_c = \mathbb{R}^{n_o});
\]

—or, respectively, for the inner reachability (controllability) property:

(iii) one has

\[
P_{V}^{dV} = V \quad (N_{V}^{dV} = V);
\]

(iv) an arbitrary \(\mathcal{F}_h \in X\) such that \(V\) is \(\mathcal{F}_h\)-invariant satisfies the relation

\[
V = X_r^{\mathcal{F}_h} V \quad (V = X_c^{\mathcal{F}_h} V + \ker(\mathcal{F}_h)^{n_o} \cap V),
\]

or, equivalently, for some \(\mathcal{H}_h \in \mathcal{H}\), the relation

\[
V = X_r^{\mathcal{F}_h} H, \quad H := \text{Im} \mathcal{B}_h \mathcal{H}_h \quad (V \subseteq X_c^{\mathcal{F}_h} V).
\]

**Proof:** The first statement, together with conditions (i) and (iii), follows from Theorem 1, (36a), (36b), (48), and the application of Theorem 1 to \(V\) and \(\Sigma^c_h\). Condition (ii) follows from (36a), (36b), Theorem 2(a), and Lemma 1, or simply from the first statement, together with the application of Theorem 2(a) to \(V\) and \(\Sigma^c_h\). Condition (iv) follows from (3a), (31b), (32d),
Proposition 5. An \((A(\cdot), B(\cdot))\) invariant \(\omega\)-periodic subspace \(V(k) \subset X\) is inner (outer) controllable for \(\Sigma\) if and only if, for an arbitrary \(h \in \mathbb{Z}\), there exists \(T_h \in \mathcal{G}\) such that \(V := S_h V(\cdot)\) is \(\mathcal{A}_{T_h}\)-invariant and

\[
V \subseteq \text{Ker} \left( \mathcal{A}_{T_h} \right)^n \cap \left( \text{Im} \left( \mathcal{A}_{T_h} \right)^n \cap V \right).
\]  

\[(53)\]

Proof. It is easily obtained from Proposition 2, making use of \((3a), (31b), (32d), (33), (36a), (36d), (45a), (46),\) and Theorem B(a).

Theorem 10.

(a) Given any \(\omega\)-periodic subspace \(K(k) \subset X\), the class of outer reachable (controllable) subspaces contained in \(K(k)\) for all \(k \in \mathbb{Z}\) is nonempty if and only if, for an arbitrary \(h \in \mathbb{Z}\), in the time invariant system \(\Sigma_h^e\) the class of outer reachable (controllable) subspaces contained in \(K := S_h K(\cdot)\) is nonempty, i.e., if and only if

\[
\mathcal{A}_h V^* + X_r = \mathbb{R}^{n\omega} \quad \left( \mathcal{A}_h V^* + X_r = \mathbb{R}^{n\omega} \right),
\]  

\[(54)\]

in which case \(S_h V^*_{ir} (\cdot) = K V^* (S_h V^*_{oc} (\cdot) = K V^*)\), where \(K V^* = Z_k^{d_k}\).

(b) Given any \(\omega\)-periodic subspace \(K(k) \subset X\), for an arbitrary \(h \in \mathbb{Z}\) the subspace \(S_h V^*_{ir} (\cdot) (S_h V^*_{ic} (\cdot))\) coincides with the largest inner reachable (controllable) subspace \(K V^*_{ir} (K V^*_{ic} (\cdot))\) of \(\Sigma_h^e\) contained in \(K := S_h K(\cdot)\), i.e.,

\[
S_h V^*_{ir} (\cdot) = R_{K K}^{d_k} V^* \quad \left( S_h V^*_{ic} (\cdot) = N_{K K}^{d_k} V^* = N_{K K}^{d_k} \right),
\]  

\[(55a)\]

\[
S_h V^*_{ir} (\cdot) = X_r^{S_{T_h} K V^*} \quad \left( S_h V^*_{ic} (\cdot) = X_r^{S_{T_h} K V^*} + \text{Ker} \left( \mathcal{A}_{T_h} \right)n \cap K V^* \right)
\]  

\[(55b)\]

for an arbitrary \(T_h \in \mathcal{G}\) such that \(K V^*\) is \(\mathcal{A}_{T_h}\)-invariant.

Proof. Part (a) follows from Theorems 6(a) and 8(b), Lemma 1, \((36a), (36b), (43a),\) and the application of Theorem 6(a) to \(K\) and \(\Sigma_h^e\).
Part (b) follows from Theorems 6(b) and 8(b), Corollary 2, (3a), (31b), (32d), (36a), (36b), (36c), (36h), (46), (48), and the application of (29b) and (29c) to $K$ and $\Sigma_K$.

**Remark 8.** A comment similar to Remark 7 applies to the classes of $\omega$-periodic subspaces $V(k)$ contained in $K(k)$ considered in Theorem 10 and the classes of subspaces of $\mathbb{R}^{\omega m}$ with similar meaning, each of which in general is larger than the class of subspaces $S_h V(\cdot)$, with $V(\cdot)$ ranging in the corresponding class of $\omega$-periodic subspaces of $X$; Theorem 10 holds in spite of this. Notice also that Proposition 5 expresses a characterization of inner (outer) controllability of $V$ for $2; which is stronger than a simple restate-ment for $\omega = 1$ of the corresponding condition given by Proposition 2, since it consists of the existence of a matrix $\mathcal{F}_h$ belonging to the special class $\mathcal{F}$ of block-diagonal $\mathcal{F}_h$.

Although statements corresponding to Theorems 3, 4, 5, and 7 and Proposition 4 can be easily given in terms of $S_h V(\cdot), S_h K(\cdot), X, X_v$, and the subspaces of $\mathbb{R}^{\omega m}$ corresponding to subspaces in (30), they seem not to be really significant and are omitted.

5. CONCLUSIONS

Some developments of the geometric theory for a linear periodic discrete-time system $\Sigma$, mostly about the notions of outer reachable subspace and controllability subspace, have been presented within a unifying frame-work. It has been shown that the whole geometric theory can be stated in an equivalent form as the same geometric theory for a time-invariant system having an extended state space. However, although the effective dimension of any analysis problem remains unchanged, its treatment in the extended state space can cause obvious numerical difficulties, due to the increasing sizes of the subspaces involved. Therefore, the results reported in the last section seem to mainly provide a conceptual framework for studying the linear periodic system $\Sigma$, and a powerful tool for solving further control problems for $\Sigma$ through the geometric approach.

APPENDIX

**Proof of Theorem 3.** For any $F(\cdot) \in \mathcal{F}(V(\cdot))$, in a $\omega$-periodic basis of $X$ containing a $\omega$-periodic basis of $V(k)$ at each $k \in \mathbb{Z}$, the matrices $A^k(k)$,
GEOMETRIC APPROACH FOR PERIODIC SYSTEMS

$B(k)$, and $E^F_k$ can take the form

$$A^F(k) = \begin{bmatrix} A^F_{11}(k) & A^F_{12}(k) \\ 0 & A^F_{22}(k) \end{bmatrix}, \quad B(k) = \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix}, \quad (A.1a)$$

$$E^F_k = \begin{bmatrix} E^F_{11}(k) & E^F_{12}(k) \\ 0 & E^F_{22}(k) \end{bmatrix}, \quad (A.1b)$$

where $A^F_{11}(k) \in \mathbb{R}^{\mu(k+1) \times \mu(k)}$, and $E^F_k[E^F_k]$ is defined like $E^F_k$ but with $A^F_{11}(k) [A^F_{22}(k)]$ instead of $A^F(k)$.

Now, if $V(\cdot)$ is inner reachable, take an $\omega$-periodic basis of $U$ containing a $\omega$-periodic basis of $U_0(k):= B^{-1}(k)[\text{Im} B(k) \cap V(k + 1)]$ at each time $k$; in such a basis, $B(k)$ in (A.1a) can be specified in form as follows:

$$B(k) = \begin{bmatrix} B_{1a}(k) & B_{1b}(k) \\ 0 & B_{2a}(k) \end{bmatrix} = \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix}, \quad (A.2)$$

where $\text{Ker} B_{2b}(k) = \{0\} \forall k \in \mathbb{Z}$, and, by (5a), (A.1), and the former equation of (14), $(A^F_{11}(\cdot), B_{1a}(\cdot))$ is a “reachable pair” [with possibly time-varying dimensions, and $A^F_{11}(k)$ possibly nonsquare] in the sense of [31,32]. Therefore, defining $\hat{F}(k) = F(k) + F_0(k)$, if $F_0(\cdot)$ is $\omega$-periodic and, in the mentioned bases of $X$ and $U$ [partitioning it in accordance with partitions (A.1a) and (A.2)], it exhibits the form

$$F_0(k) = \begin{bmatrix} F_a(k) & 0 \\ 0 & 0 \end{bmatrix}, \quad (A.3)$$

then $\hat{F}(\cdot) \in \mathbb{F}(V(\cdot))$, and

$$E^F_k = \begin{bmatrix} E^F_{11}(k) & E^F_{12}(k) \\ 0 & E^F_{22}(k) \end{bmatrix}, \quad (A.4)$$

where $E^F_{11}(k)$ is defined like $E^F_{11}(k)$ but with $A^F_{11}(k) + B_{1a}(k) F_a(k)$ instead of $A^F_{11}(k)$. Since $(A^F_{11}(\cdot), B_{1a}(\cdot))$ is a “reachable pair,” there certainly exists $F_a(\cdot)$ such that the core spectrum of $E^F_{11}(k)$ coincides with an arbitrarily assigned set of $\mu_m$ complex numbers (subject to the complex-conjugate-pairs requirement) [31,32]. Since, in a coordinate-free notation, $E^F_{11}(k)$ is written as $E^F_k|V(k)$, this proves the “inner reachable” part of the theorem.
If, vice versa, \( V(\cdot) \) is outer reachable, then (5a), (A.1), and the former equation of (13), taking into account that \( X^f(k) = X_r(k) \), imply that \( (A^f(\cdot), B(\cdot)) \) is a "reachable pair" in the sense of [31, 32]. Now, the "outer reachable" part of the proof can be completed through a reasoning similar to the "inner reachable" part, by considering an \( \omega \)-periodic \( F^2_0(\cdot) \) of the following form [in the chosen basis of \( X \), if it is partitioned in accordance with (A.1a)]:

\[
F_0(k) = \begin{bmatrix}
0 & F_2^0(k)
\end{bmatrix}
\]  \hspace{1cm} (A.5)

and by noting that \( E^F_{22}(k) \) is written as \( E^F_k(\mod V(k)) \) in a coordinate-free notation, where \( F(\cdot) \) is defined as before.

Proof of the sufficiency of condition (iii) of Theorem 4. The proof will consist in showing that, if the dimension of \( V(k) \) is independent of \( k \) and (22) holds, then there exists a \( F(\cdot) \in F(V(\cdot)) \) such as to satisfy (20), thus proving that (iii) implies (ii).

In fact, choose an arbitrary \( F(\cdot) \in F(V(\cdot)) \), and a \( \omega \)-periodic basis which is canonical, in the sense of Kalman [63], with respect to both the \( A^f(\cdot) \)-invariant subspaces \( V(\cdot) \) and \( X_r(\cdot) \), given by the union of bases of four \( \omega \)-periodic subspaces \( X_i(\cdot) \) \((i = 1, \ldots, 4)\) such that

\[
X_1(k) := V(k) \cap X_r(k) \quad \forall k \in \mathbb{Z}, \hspace{1cm} (A.6a)
\]

\[
X_1(k) \oplus X_2(k) = V(k) \quad \forall k \in \mathbb{Z}, \hspace{1cm} (A.6b)
\]

\[
X_1(k) \oplus X_3(k) = X_r(k) \quad \forall k \in \mathbb{Z}, \hspace{1cm} (A.6c)
\]

\[
X_1(k) \oplus X_2(k) \oplus X_3(k) + X_4(k) = X \quad \forall k \in \mathbb{Z}. \hspace{1cm} (A.6d)
\]

Then, by (22), the following relation is satisfied:

\[
X_1(k) \oplus X_2(k) \oplus X_3(k) = X_c(k) \quad \forall k \in \mathbb{Z}, \hspace{1cm} (A.7)
\]

and therefore, in the mentioned basis, \( A^f(k) \) and \( B(k) \) take the following
form:

$$A^E(k) = \begin{bmatrix}
A_{11}^E(k) & A_{12}^E(k) & A_{13}^E(k) & A_{14}^E(k) \\
0 & A_{22}^E(k) & 0 & A_{24}^E(k) \\
0 & 0 & A_{33}^E(k) & A_{34}^E(k) \\
0 & 0 & 0 & A_{44}^E(k)
\end{bmatrix} \quad \text{and} \quad B(k) = \begin{bmatrix}
B_1(k) \\
0 \\
B_3(k) \\
0
\end{bmatrix},$$

(A.8)

where $A_{44}^E(k)$ is square and nonsingular for all $k \in \mathbb{Z}$ [27], and, by hypothesis (iii) and the constant dimensionality of $X_r(k)$ [27], $A_{33}^E(k)$ is square for all $k \in \mathbb{Z}$ (and has a constant dimension), and $(A_{33}^E(k), B_3(k))$ describes a system reachable at all times $k$. Hence,

$$\text{rank}\left[ A_{33}^E(k) \quad B_3(k) \right] = \dim A_{33}^E(k) \quad \forall k \in \mathbb{Z},$$

(A.9)

which implies the existence of an $\omega$-periodic matrix $F_{03}(k)$ such that $A_{33}^E(k) + B_3(k)F_{03}(k)$ is nonsingular for all $k \in \mathbb{Z}$. Therefore, defining

$$F_0(k) := F(k) + \begin{bmatrix} 0 & 0 & F_{03}(k) & 0 \end{bmatrix},$$

(A.10)

we have $F_0(\cdot) \in \mathbb{F}(V(\cdot))$ and $\text{Ker}(E_k^{F_0})^n \subset V(k)$ for all $k \in \mathbb{Z}$, i.e., (21) is satisfied.

Proof of Proposition 4. Choose an arbitrary $F(\cdot) \in \mathbb{F}(V(\cdot))$ and a $\omega$-periodic basis as in the sufficiency proof of condition (iii) of Theorem 4, but with $X_c(k)$ instead of $X_r(k)$ in (A.6). Therefore the canonical form (A.8) of $A^E(k)$ and $B(k)$ follows, where the blocks $A_{22}^E(k)$ and $A_{44}^E(k)$ are square and nonsingular for all $k \in \mathbb{Z}$, since otherwise the square matrix

$$A_{uc}(k) := \begin{bmatrix}
A_{22}^E(k) & A_{24}^E(k) \\
0 & A_{44}^E(k)
\end{bmatrix}$$

(A.11)

could not be nonsingular for all $k \in \mathbb{Z}$, as it is implied by the choice of the basis [27]. Hence $\dim[X_1(k) + X_2(k) + X_3(k)] = \text{constant}$, i.e. the thesis.
Proof of Theorem 7. By (7), (9b), and (29b), Equation (30) will be proved by showing that, for each integer \( i \geq 1 \),

\[
P^i_{cV^*}(k) = cV^*(k) \cap W_{B}^{-1}(k) \quad \forall k \in \mathbb{Z}. \quad (A.12)
\]

Now, (A.12) trivially holds for \( i = 1 \). For \( i > 1 \) the proof is obtained by induction. Indeed, if (A.12) holds, then, by (6a), (7), and (8d),

\[
P^{i+1}_{cV^*}(k + 1) = cV^*(k + 1) \cap \left\{ A(k) \left[ cV^*(k) \cap W_{B}^{-1}(k) \right] + \text{Im} B(k) \right\}
\]

\[
= cV^*(k + 1) \cap \left\{ A(k) \left[ cV^*(k) \cap W_{B}^{-1}(k) \cap A^{-1}(k) \left[ cV^*(k + 1) + \text{Im} B(k) \right] \right) \right\}
\]

\[
+ \text{Im} B(k) \right\} \quad \forall k \in \mathbb{Z}. \quad (A.13)
\]

Now, by the modular distributive rule [1],

\[
\left[ \text{Ker} A(k) + C(k) \cap W_{B}^{-1}(k) \right] \cap A^{-1}(k) \left[ cV^*(k + 1) + \text{Im} B(k) \right]
\]

\[
= \text{Ker} A(k) + C(k) \cap W_{B}^{-1}(k) \cap A^{-1}(k) \left[ cV^*(k + 1) + \text{Im} B(k) \right] \quad \forall k \in \mathbb{Z}, \quad (A.14)
\]

which yields [1]

\[
\left\{ C(k) \cap W_{B}^{-1}(k) + A^{-1}(k) \left[ cV^*(k + 1) + \text{Im} B(k) \right] \right\} \cap \text{Ker} A(k)
\]

\[
= C(k) \cap W_{B}^{-1}(k) \cap \text{Ker} A(k) + \text{Ker} A(k) \quad \forall k \in \mathbb{Z}. \quad (A.15)
\]
Then (6b), (A.13), (A.15), and the modular distributive rule imply that

\[
P_{cV}^{i+1}(k+1) = cV^*(k+1) \cap \left[ [cV^*(k+1) + \text{Im} B(k)] \right.
\]
\[\cap A(k) \left( C(k) \cap W^{-1}_B(k) \right) + \text{Im} B(k)]
\[= cV^*(k+1) \cap \left( [A(k) \left( C(k) \cap W^{-1}_B(k) \right) + \text{Im} B(k)] \right.
\]
\[\cap [cV^*(k+1) + \text{Im} B(k)] \right)
\[= cV^*(k+1) \cap W_B^i(k+1) \quad \forall k \in \mathbb{Z}, \quad (A.16)
\]

that is, (A.12) written for \( i + 1 \). This completes the proof. \( \square \)

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