Commutativity and self-duality: Two tales of one equation

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Abstract

The mathematical expressions for the commutativity or self-duality of an increasing \([0,1]^2 \to [0,1]\) function \(F\) involve the transposition of its arguments. We unite both properties in a single functional equation. The solutions of this functional equation are discussed. Special attention goes to the geometrical construction of these solutions and their characterization in terms of contour lines. Furthermore, it is shown how 'rotating' the arguments of \(F\) allows to convert the results into properties for \([0,1]^2 \to [0,1]\) functions having monotone partial functions.

1. Introduction

Binary aggregation functions (AFs) are increasing \([0,1]^2 \to [0,1]\) functions \(F\) that satisfy \(F(0,0) = 0\) and \(F(1,1) = 1\) [1]. They admit to perform a two-by-two aggregation process on multiple input values, a procedure which allows to define also more general \(n\)-ary AFs. In this capacity binary AFs become indispensable tools for various sciences such as applied mathematics, computer science, economics and psychology where multiple numerical input values are to be combined into a single numerical output value. Performing the aggregation by means of a single associative and commutative binary AF \(F\) allows to interchange the input values. Ensuring that the aggregation of the complementary values (\(x \to 1 - x\)) is the complement of the original aggregation can be established by invoking so-called self-dual, binary AFs (i.e., \(1 - F(x,y) = F(1-x,1-y)\)). The resulting \(n\)-ary AF is also self-dual.

In preference modelling and multicriteria decision making, self-dual, \(n\)-ary AFs ensure that individual, reciprocal preference relations are combined into a collective, reciprocal preference relation [4,5,14]. In the literature several characterizations for these self-dual AFs have been presented [1,5,6,13]. The approach in [13] provides a general framework for the existing characterizations and comprises the results from [1,5,6]. Each of these characterizations is based on a binary AF \(F\) that satisfies

\[
F(x, y) = 1 - F(1 - y, 1 - x)
\]  

and whose graph contains an increasing (w.r.t. the three space coordinates) curve whose \(Z\)-coordinate reaches every number of \([0,1]\). As for the commutativity property, the latter equality requires a transposition of the arguments. Both properties are mathematically expressed by a single equation of the form:

\[
F(x, y) = \Phi(F(\Phi^{-1}(y), \Phi(x))),
\]
for every \((x, y) \in [0,1]^2\) and where \(F\) is a binary AF. The commutativity property is retrieved by putting \(\Phi = \Psi = \text{id}\). Eq. (1) follows by considering \(\Phi = \Psi = \cdot\cdot\cdot\), with \(\cdot\cdot\cdot\) the standard negator defined by \(\cdot\cdot\cdot(x) = 1 - x\). Note that self-duality can be understood as the combination of commutativity and Eq. (1). A more general version of self-duality pops up when characterizing uninorms (i.e., associative, commutative binary AFs \(U\) having neutral element \(e\)) that are continuous on \([0,1]^2 \setminus \{(0,1), (1,0)\}\) and with \(e \in [0,1]\). These uninorms admit Eq. (2), for every \((x, y) \in [0,1]^2 \setminus \{(0,1), (1,0)\}\), where \(F = U\) and \(\Phi = \Psi = N\), with \(N\) an involutive, decreasing \([0,1] \rightarrow [0,1]\) bijection [3]. Recall that for this choice of \(\Phi\) and \(\Psi\) no uninorm \(U\) can satisfy Eq. (2) for \((x, y) \in \{(0,1), (1,0)\}\), as necessarily \(U(0,1) = U(1,0) \in [0,1]\) [3], whereas \(F(0,1) = F(1,0) = \beta\) with \(\beta\) the unique fixpoint of \(N\) (i.e., \(N(\beta) = \beta\)).

For given monotone \([0,1] \rightarrow [0,1]\) bijections \(\Phi\) and \(\Psi\) we will solve functional equation (2). Although the increasingness of \(F\) is crucial for the results, from a mathematical point of view there is no need to assume the boundary conditions \(f(0,0) = 0\) and \(f(1,1) = 1\). These restrictions will therefore be omitted. The solutions of the equation fit one of two types only: either they are related to solutions of the commutativity equation or they correspond to solutions of Eq. (1) (Section 2). Properties such as commutativity impose symmetry on the graph of \(F\). This symmetry admits a geometrical characterization of the solutions of Eq. (1) (Section 3). Moreover, the symmetry of the graph of \(F\) enforces also a certain symmetric behaviour upon its contour lines. Sections 4 and 5 contain some preparatory results concerning contour lines and generalized inverses of monotone \([0,1] \rightarrow [0,1]\) functions. In Section 6 we describe the solutions of Eq. (2) in terms of the orthosymmetry aspects of their contour lines. Some concluding remarks can be found in Section 7. There we point out how to render our results applicable for more general \([0,1]^2 \rightarrow [0,1]\) functions having monotone partial functions only.

2. Two types of solutions

Consider two monotone \([0,1] \rightarrow [0,1]\) bijections \(\Phi\) and \(\Psi\). To distinguish the type of monotonicity we call an increasing \([0,1] \rightarrow [0,1]\) bijection an automorphism and a decreasing \([0,1] \rightarrow [0,1]\) bijection a strict negator. Due to the structure of Eq. (2) there are some restrictions on the choice of \(\Phi\) and \(\Psi\). To be compatible with the increasingness of \(F\) it is clear that \(\Phi\) and \(\Psi\) must have the same type of monotonicity. Furthermore, applying Eq. (2) twice results in \(f(x, y) = \psi(\psi(f(x, y)))\). We will strengthen this condition and require that \(\Psi\) is involutive: \(\psi \circ \psi = \text{id}\). The observation that the binary AF \(F\) referred to in the introduction (see also [13]) should reach every element of \([0,1]\) also supports this additional condition on \(\Psi\). The considerations above force us to consider Eq. (2) in the following two cases only:

1. \(\Phi\) is an automorphism \(\Phi\) and \(\Psi\) is the identity mapping \(\text{id}\).
2. \(\Phi\) is a strict negator \(M\) and \(\Psi\) is an involutive negator \(N\).

An involutive negator is an involutive strict negator. Using the above notations Eq. (2) breaks up into two equations:

\[
F(x, y) = F(\phi^{-1}(y), \phi(x))
\]

(3)

\[
F(x, y) = N(F(M^{-1}(y), M(x)))
\]

(4)

Putting \(\phi = \text{id}\) and \(M = N = \cdot\cdot\cdot\), we retrieve the standard form of the above equations:

\[
F(x, y) = F(y, x)
\]

(3')

\[
F(x, y) = 1 - F(1 - y, 1 - x)
\]

(4')

Recall that each monotone \([0,1] \rightarrow [0,1]\) bijection can be written as the composition of at most four involutive negators [2,7,11,15]. This observation supports the following definition and corollary.

**Definition 1.** We call two increasing \([0,1]^2 \rightarrow [0,1]\) functions \(F\) and \(G\) \(k\)-dual, with \(k \in \{1,2,3,4\}\), if there exist three finite sequences of involutive negators \((K_i)_{i \in \mathbb{C}}\), \((M_i)_{i \in \mathbb{C}}\) and \((N_i)_{i \in \mathbb{C}}\) s.t.

\[
G(x, y) = K_1 \circ \ldots \circ K_k(F(M_1 \circ \ldots \circ M_k(x), N_1 \circ \ldots \circ N_k(y)));
\]

for every \((x, y) \in [0,1]^2\). \(F\) and \(G\) are called multi-dual if there exists \(k \in \{1,2,3,4\}\) s.t. they are \(k\)-dual.

In the literature 1-dual, increasing \([0,1] \rightarrow [0,1]\) functions with \(K_1 = M_1 = N_1 = \cdot\cdot\cdot\) are also referred to as being dual [1].

**Corollary 1.** Two increasing \([0,1]^2 \rightarrow [0,1]\) functions \(F\) and \(G\) are multi-dual if and only if there exist three monotone bijections \(\Theta\), \(\Lambda\) and \(\Gamma\) having the same type of monotonicity and s.t. \(G(x, y) = \Theta(F(\Lambda(x), \Gamma(y)))\), for every \((x, y) \in [0,1]^2\).

For a pair \((F, G)\) of multi-dual increasing \([0,1]^2 \rightarrow [0,1]\) functions with \(\Phi = \Theta^{-1} = \Lambda = \Gamma\), it is also said that \(G\) is the \(\Phi\)-transform of \(F\) [1]. Whenever \(\Theta\), \(\Lambda\) and \(\Gamma\) are increasing it is clear that the graph of \(G\) is isomorphic to the graph of \(F\). In case \(\Theta\), \(\Lambda\) and \(\Gamma\) are decreasing, the graph of \(G\) gets in some sense dualized. In particular, the continuity of \(F\) gets reversed (i.e., left continuity becomes right continuity and vice versa) and the value of \(G\) on each border of its domain \([0,1]^2\) is determined by the value of \(F\) on the opposite border.

We are now able to express the solutions of Eqs. (3) and (4) in terms of those of Eqs. (3') and (4'). Dealing with multi-dual functions we adopt the notations from Corollary 1.
Theorem 1. Consider an automorphism \( \phi \). An increasing \([0,1]^2 \rightarrow [0,1]\) function \( G \) solves Eq. (3) if and only if it is multi-dual to an increasing \([0,1]^2 \rightarrow [0,1]\) function \( F \) solving Eq. (3). In particular, if \( G(x,y) = \Theta(F(A(x),\Gamma(y))) \), for every \((x,y) \in [0,1]^2\), then \( \phi = \Gamma^{-1} \circ A \).

Proof. If \( G \) satisfies Eq. (3) we define \( F \) by \( F(x,y) = \Theta^{-1}(G(A^{-1}(x),\Gamma^{-1}(y))) \), with \( \Theta \) and \( \Gamma \) two arbitrary monotone \([0,1] \rightarrow [0,1]\) bijections having the same type of monotonicity and \( A := \Gamma \circ \phi \). It follows straightforwardly that \( F \) satisfies Eq. (3). Conversely, suppose that \( F \) satisfies Eq. (3). Consider three arbitrary monotone \([0,1] \rightarrow [0,1]\) bijections \( \Theta, A \) and \( \Gamma \). Then it is easily verified that \( G \) defined by \( G(x,y) = \Theta(F(A(x),\Gamma(y))) \) satisfies Eq. (3) with \( \phi = \Gamma^{-1} \circ A \). 

A similar reasoning leads to the following theorem.

Theorem 2. Consider a strict negator \( M \) and an involutive negator \( N \). An increasing \([0,1]^2 \rightarrow [0,1]\) function \( G \) solves Eq. (4) if and only if it is multi-dual to an increasing \([0,1]^2 \rightarrow [0,1]\) function \( F \) solving Eq. (4). In particular, if \( G(x,y) = \Theta(F(A(x),\Gamma(y))) \), for every \((x,y) \in [0,1]^2\), then \( N = \Theta \circ A \circ \Theta^{-1} \) and \( M = \Gamma^{-1} \circ A \).

It is easily verified that in the above theorem \( N := \Theta \circ A \circ \Theta^{-1} \) is indeed involutive. Also the converse is true. For every involutive negator \( N \) there exists an automorphism \( \phi \) and a strict negator \( M \) s.t. \( N = \phi \circ A \circ \phi^{-1} [17] \) and \( N = M \circ A \circ M^{-1} \) (take \( M = \phi \circ N \)).

3. Geometrical interpretation

We have introduced the \( \Phi \)-inverse of a set \( A \subseteq [0,1]^2 \) in order to investigate its symmetry w.r.t. a given monotone \([0,1] \rightarrow [0,1]\) bijection \( \Phi \).

Definition 2. [12] Let \( \Phi \) be a monotone \([0,1] \rightarrow [0,1]\) bijection. The \( \Phi \)-inverse of a set \( A \subseteq [0,1]^2 \) is given by \( A^\Phi := \{(x,y) \in [0,1]^2 \mid (\Phi^{-1}(y), \Phi(x)) \in A \} \).

Geometrically, \( A^\Phi \) is the set of those vertices that constitute the fourth point of a rectangle with sides parallel to the axes, that has two opposite vertices on the graph of \( \Phi \) and that has a third vertex belonging to \( A \). Fig. 1 illustrates this procedure. It is clear that \( (A^\Phi)^\Phi = A \) and \( A^{\Phi^d} = A^{-1} \), with \( A^{-1} \) the classical inverse of \( A \) defined by \( A^{-1} := \{(x,y) \in [0,1]^2 \mid (y,x) \in A \} \).

Eq. (2) expresses that the value of \( F \) in each point \((x,y) \in [0,1]^2\) is determined by the value of \( F \) in the \( \Phi \)-inverse point \((\Phi^{-1}(y), \Phi(x)) \). This observation provides a method for constructing all possible solutions of Eq. (2) (i.e., of resp. Eqs. (3) and (4)):

1. Consider an automorphism \( \phi \). Define \( F \) on \( \{(x,y) \in [0,1]^2 \mid y \leq \phi(x) \} \) as an arbitrary increasing function. Eq. (3) can then be used to uniquely complete \( F \) on \( \{(x,y) \in [0,1] \mid \phi(x) < y \} \). The increasingness of \( F \) is easily verified.

![Fig. 1. The \( \phi \)-inverse and \( N \)-inverse (dashed gray lines) of a circle (dashed black line), with \( \phi \) the automorphism (solid line) depicted in (a) and \( N \) the strict negator (solid line) depicted in (b).](image-url)
(2) Consider a strict negator $M$ and an involutive negator $N$. Denote $\beta$ the unique fixpoint of $N$. Define $F$ on
\[(x,y) \in [0,1]^2 \mid y \leq M(x)\]as an arbitrary increasing function s.t. $F(x,M(x)) = \beta$, for every $x \in [0,1]$. The latter condition
is required as $F(x,M(x)) = N(F(x,M(x)))$ must be satisfied. Eq. (4) can then be used to uniquely complete $F$ on
\[(x,y) \in [0,1] \mid M(x) < y\].

4. Contour lines

It is a common practice to describe (the properties of) a binary operation by means of some associated unary operations
such as partial functions, generators, … In this respect, the use of contour lines turned out to be very fruitful \[10\]. The contour
lines of an increasing $[0,1]^2 \to [0,1]$ function $F$ have been defined as the upper, lower, right or left limits of its horizontal
cuts, i.e., the intersections of its graph by planes parallel to the domain $[0,1]^2$.

**Definition 3.** \[10\] We associate with each increasing $[0,1]^2 \to [0,1]$ function $F$ four types of contour lines $(a \in [0,1])$:

- $C_a : [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(x,t) \leq a\}$
- $D_a : [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(x,t) \geq a\}$
- $\bar{C}_a : [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid F(t,x) \leq a\}$
- $\bar{D}_a : [0,1] \to [0,1] : x \mapsto \inf\{t \in [0,1] \mid F(t,x) \geq a\}$

The number $a \in [0,1]$ determines the height of the intersecting plane. All contour lines are decreasing \[10\]. Contour lines
of the type $C_a$ or $\bar{C}_a$ are better suited to describe left-continuous functions, while right-continuous functions are more easily
described by contour lines of the type $D_a$ or $\bar{D}_a$.

**Theorem 3** \[10\]. Consider an increasing $[0,1]^2 \to [0,1]$ function $F$. The following assertions hold:

1. $F$ is left-continuous and $F(1,0) = 0$ if and only if $F(x,y) \leq a \iff y \leq C_a(x)$ holds for every $(x,y,a) \in [0,1]^3$.
2. $F$ is right-continuous and $F(0,1) = 1$ if and only if $D_a(x) \leq y \iff F(x,y) \leq a$ holds for every $(x,y,a) \in [0,1]^3$.
3. $F$ is left-continuous and $F(0,1) = 0$ if and only if $y \leq a \iff \bar{C}_a(x)$ holds for every $(x,y,a) \in [0,1]^3$.
4. $F$ is right-continuous and $F(1,0) = 1$ if and only if $D_a(x) \leq y \iff F(y,x) \leq a$ holds for every $(x,y,a) \in [0,1]^3$.

The equivalences in this theorem can also be interpreted as Galois connections between the members of the couples:
\[(F(x,\bullet), C_a(x)), (D_a(x), F(x,\bullet)), (F(\bullet,x), \bar{C}_a(x)) \text{ and } (\bar{D}_a(x), F(\bullet,x))\]. Hereby, $C_a(x)$, $D_a(x)$, $\bar{C}_a(x)$ and $\bar{D}_a(x)$ denote the increasing
$[0,1] \to [0,1]$ functions that map $a \in [0,1]$ to resp., $C_a(x)$, $D_a(x)$, $\bar{C}_a(x)$ and $\bar{D}_a(x)$. Note that contour lines also inherit the continuity
of $F$. Contour lines of the types $C_a$ and $D_a$ are left-continuous if $F$ is left-continuous. Contour lines of the types $\bar{D}_a$ and $\bar{D}_a$
are right-continuous if $F$ is right-continuous.

There exists a tight connection between contour lines of the type $C_a$ and $\bar{D}_a$ and between contour lines of the type $\bar{C}_a$ and $D_a$. Consider three arbitrary strict negators $\Theta$, $\Lambda$ and $F$. Dualizing $F$ as in Corollary 1 transforms contour lines of the type $C_a$, resp. $\bar{C}_a$, into contour lines of the type $D_a$, resp. $\bar{D}_a$. As pointed out before such a transformation also reverses the continuity
of $F$. In the following theorem we present a more straightforward relationship between the different types of contour lines.

**Theorem 4.** Consider an increasing $[0,1]^2 \to [0,1]$ function $F$. The following assertions hold:

1. $C_a = \lim_{b \to a} D_b$, for every $a \in [0,1]$;
2. $D_a = \lim_{b \to a} C_b$, for every $a \in [0,1]$;
3. $\bar{C}_a = \lim_{b \to a} \bar{D}_b$, for every $a \in [0,1]$;
4. $\bar{D}_a = \lim_{b \to a} \bar{C}_b$, for every $a \in [0,1]$.

**Proof.** We present the proof for the first assertion, the other assertions being proved similarly. Take $a \in [0,1]$. Then the increasingness of $F$ ensures that $C_a(x) = \inf\{t \in [0,1] \mid F(x,t) \geq a\}$, for every $x \in [0,1]$. Since $\{t \in [0,1] \mid F(x,t) \geq a\} \subseteq \{t \in [0,1] \mid F(x,t) \geq b\}$, for every $b > a$, it holds that $C_a(x) \leq \inf\{t \in [0,1] \mid F(x,t) \geq b\} = D_b(x)$, for every $x \in [0,1]$ and $b \in [a,1]$. Suppose that $C_a \neq \lim_{b \to a} D_b$, then there must exist $(x,y) \in [0,1]^2$ s.t. $C_a(x) < y \leq \lim_{b \to a} D_b(x)$. From Definition 3 we obtain that $a < F(x,y) < b$, for every $b \in [a,1]$, which leads to the contradiction $a < F(x,y) \leq a = \lim_{b \to a} b$. □

To interrelate contour lines of the type $C_a$, resp. $D_a$, and contour lines of the type $\bar{C}_a$, resp. $\bar{D}_a$, some additional properties
on $F$ are required. In Section 6 we will discuss the interaction due to Eq. (2).

5. Inverting decreasing functions

The symmetry contained in Eq. (2) manifests itself in the geometry of the contour lines of $F$ (see Section 6). We adopt the
approach from \[12\] and associate to each contour line $f \in \{C_a, D_a, \bar{C}_a, \bar{D}_a\}$ a set $Q(f, \Phi)$ of $\Phi$-inverse‘ functions, where $\Phi$ is a
given monotone \([0,1] \to [0,1]\) bijection. In this section we recall the construction and main properties of \(Q(f,\Phi)\). We assume that \(f\) is an arbitrary monotone \([0,1] \to [0,1]\) function.

To construct the elements of \(Q(f,\Phi)\) we apply the following procedure:

1. Adding vertical segments we complete the graph of \(f\) to a 'monotone' continuous line that reaches every element in \([0,1]\).
2. Construct the \(\Phi\)-inverse of the completion.
3. Delete from the latter all but one point from each vertical segment.

For a constant function \(f\) we need to consider its increasing as well as its decreasing completion. The latter is required as constant functions admit the two types of monotonicity. In case \(f(0) < f(1)\) only an increasing completion is possible. If \(f(0) > f(1)\) only a decreasing completion is possible. \(Q(f,\Phi)\) can be mathematically described by means of four functions

\[
\begin{align*}
f^{\phi}_1 : [0, 1] &\to [0, 1] : x \mapsto \sup \{t \in [0, 1] | f(\Phi^{-1}(t)) < \Phi(x)\}; \\
f^{\phi}_2 : [0, 1] &\to [0, 1] : x \mapsto \inf \{t \in [0, 1] | f(\Phi^{-1}(t)) > \Phi(x)\}; \\
f^{\phi}_3 : [0, 1] &\to [0, 1] : x \mapsto \sup \{t \in [0, 1] | f(\Phi^{-1}(t)) > \Phi(x)\}; \\
f^{\phi}_4 : [0, 1] &\to [0, 1] : x \mapsto \inf \{t \in [0, 1] | f(\Phi^{-1}(t)) < \Phi(x)\}.
\end{align*}
\]

Both functions \(f^{\phi}_1\) and \(f^{\phi}_2\) have the same type of monotonicity as \(\Phi\). The monotonicity of the functions \(f^{\phi}_3\) and \(f^{\phi}_4\) is opposite to the monotonicity of \(\Phi\) [12]. In case \(f(0) < f(1)\), resp. \(f(1) < f(0)\), the function \(f^{\phi}_3\), resp. \(f^{\phi}_4\), is known as the pseudo-inverse \(f^{-1}\) of \(f\) [9]. \(Q(f,\Phi)\) is constituted by those (monotone) \([0,1] \to [0,1]\) functions \(g\) specified in Table 1.

Given the limit properties that exist between the different types of contour lines (Theorem 4), the following theorem will become very useful in Section 6.

**Theorem 5.** Consider a monotone \([0,1] \to [0,1]\) bijection \(\Phi\) and two pointwisely converging sequences \((f_n)_{n \in \mathbb{N}}\) and \((g_n)_{n \in \mathbb{N}}\) of monotone \([0,1] \to [0,1]\) functions. Let \(f = \lim_{n \to \infty} f_n\) and \(g = \lim_{n \to \infty} g_n\). If \(g_n \in Q(f_n,\Phi)\), for every \(n \in \mathbb{N}\), then \(g \in Q(f,\Phi)\).

**Proof.** Suppose for instance that \(\Phi\) is increasing; the proof for a decreasing \(\Phi\) is analogous. Clearly, \(f\) and \(g\) are also monotone \([0,1] \to [0,1]\) functions. We distinguish three cases: \(f(0) < f(1)\), \(f(0) > f(1)\) and \(f(0) = f(1)\). If \(f(0) < f(1)\), then there exists a subsequence \((f_{n_i})_{i \in \mathbb{N}}\) of \((f_n)_{n \in \mathbb{N}}\) s.t. \(f_{n_i}(0) < f_{n_i}(1)\) for every \(i \in \mathbb{N}\). Now, since \(g_n \in Q(f_n,\Phi)\), for every \(n \in \mathbb{N}\), it follows from Table 1 that \(f^{\phi}_{n_i} \leq g_{n_i} \leq f_{n_i}\). Explicitly, this means that

\[
\sup \{t \in [0, 1]|f_{n_i}(\Phi^{-1}(t)) < \Phi(x)\} \leq g_{n_i}(x) \leq \inf \{t \in [0, 1]|f_{n_i}(\Phi^{-1}(t)) > \Phi(x)\},
\]

for every \(x \in [0,1]\) and every \(i \in \mathbb{N}\). The latter implies that \(\Phi(x) \leq f_{n_i}(\Phi^{-1}(t))\) whenever \(t \in [g_{n_i}(x), 1]\) and that \(f_{n_i}(\Phi^{-1}(t)) \leq \Phi(x)\) whenever \(t \in [0, g_{n_i}(x)]\). Suppose now that there exists a number \(x \in [g(x), 1]\) such that \(f(\Phi^{-1}(t)) < \Phi(x)\). Because \(\lim_{n \to \infty} g_n = g\), there exists a natural number \(k_1\) such that for every \(i \geq k_1\) it holds that \(t \in [g_n(x), 1]\). Furthermore, as \(\lim_{n \to \infty} f_n = f\) there exists a second natural number \(k_2 > k_1\) such that \(f_{n_i}(\Phi^{-1}(t)) < \Phi(x)\), for every \(i \geq k_2\). Combining both results we obtain the contradiction that \(t \in [g_n(x), 1]\), while \(f_{n_i}(\Phi^{-1}(t)) < \Phi(x)\). Consequently, it necessarily holds that \(\Phi(x) \leq f(\Phi^{-1}(t))\) whenever \(t \in [g(x), 1]\). In a similar way, it is shown that \(f(\Phi^{-1}(t)) \leq \Phi(x)\) whenever \(t \in [0, g(x)]\). Hence,

\[
\sup \{t \in [0, 1]|f(\Phi^{-1}(t)) < \Phi(x)\} \leq g(x) \leq \inf \{t \in [0, 1]|f(\Phi^{-1}(t)) > \Phi(x)\},
\]

for every \(x \in [0,1]\), or, in other words \(f^{\phi} \leq g \leq f^{\phi}\). From Table 1, it then follows that \(g \in Q(f,\Phi)\). If \(f(0) > f(1)\), then the proof is similar.

Finally, if \(f(0) = f(1)\), then it follows from Table 1 that \(f^{\phi} \leq g \leq f^{\phi}\) or \(f^{\phi} \geq g \geq f^{\phi}\), for every \(n \in \mathbb{N}\). Hence, there always exists a subsequence \((f_{n_i})_{i \in \mathbb{N}}\) of \((f_n)_{n \in \mathbb{N}}\) s.t. \(f^{\phi}_{n_i} \leq g \leq f^{\phi}_{n_i}\) is satisfied for every \(i \in \mathbb{N}\), or \(f^{\phi}_{n_i} \geq g \geq f^{\phi}_{n_i}\) is satisfied for every \(i \in \mathbb{N}\). Applying the same reasoning as above, it then follows that \(f^{\phi} \leq g \leq f^{\phi}\) or \(f^{\phi} \geq g \geq f^{\phi}\). From Table 1, we obtain that \(g \in Q(f,\Phi)\). \(\square\)

It is not true that for a given pointwisely converging sequence \((f_n)_{n \in \mathbb{N}}\) of monotone \([0,1] \to [0,1]\) functions there always exists a pointwisely converging sequence \((g_n)_{n \in \mathbb{N}}\) of monotone \([0,1] \to [0,1]\) functions s.t. \(g_n \in Q(f_n,\Phi)\), for every \(n \in \mathbb{N}\).

**Example 1.** Let \(\Phi = \text{id}\) and define the sequence \((f_n)_{n \in \mathbb{N}}\) as

\[
f_n(x) = \begin{cases} 
-\frac{1}{2^{n+1}} + \frac{1}{2}, & \text{if } n \text{ is even}, \\
-\frac{1}{2^n} + \frac{1}{2}, & \text{if } n \text{ is odd}.
\end{cases}
\]

<table>
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<tr>
<th>(Q(f,\Phi))</th>
<th>(f(0) &lt; f(1))</th>
<th>(f(0) &gt; f(1))</th>
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<tr>
<td>(\Phi) is increasing</td>
<td>(f^{\phi} \leq g \leq f^{\phi})</td>
<td>(f^{\phi} \leq g \leq f^{\phi})</td>
<td>(f^{\phi} \leq g \leq f^{\phi}) or (f^{\phi} \geq g \geq f^{\phi})</td>
</tr>
<tr>
<td>(\Phi) is decreasing</td>
<td>(f^{\phi} \leq g \leq f^{\phi})</td>
<td>(f^{\phi} \leq g \leq f^{\phi})</td>
<td>(f^{\phi} \geq g \leq f^{\phi}) or (f^{\phi} \leq g \geq f^{\phi})</td>
</tr>
</tbody>
</table>
for every \( x \in [0,1] \) and \( n \in \mathbb{N} \). Clearly, \( \lim_{n \to \infty} f_n = \frac{1}{2} \). Since all functions \( f_n \) are non-constant and decreasing it follows from Table 1 that every \( g_n \in Q(f_n, \text{id}) \) must satisfy \( f_n^\text{id} \leq g_n \leq f_n \). In particular,

\[
 g_n(\frac{1}{2}) = \begin{cases} 
 f_n^\text{id}(\frac{1}{2}), & \text{if } n \text{ is even}, \\
 f_n^\text{id}(\frac{1}{2}), & \text{if } n \text{ is odd}, 
\end{cases}
\]

Consequently, the sequence \((g_n(\frac{1}{2}))_{n \in \mathbb{N}}\) does not converge from which we conclude that also \((g_n)_{n \in \mathbb{N}}\) is not convergent.

Our definition of the set \( Q(f, \Phi) \) largely generalizes the work of Schweizer and Sklar [16] who associate to each increasing \([0,1] \to [0,1]\) function \( f \) a set of \( \text{id} \)-inverse functions. Some additional results for monotone functions are due to Klement et al. [8,9]. The following theorem shows the close relationship between sets of \( \text{id} \)-inverse functions and sets of \( \Phi \)-inverse functions.

**Theorem 6.** Consider a monotone \([0,1] \to [0,1]\) bijection \( \Phi \) and two monotone \([0,1] \to [0,1]\) functions \( f \) and \( g \). Then the following assertions are equivalent

1. \( g \in Q(f, \Phi) \);
2. \( g \in Q(\Phi^{-1} \circ f \circ \Phi^{-1}, \text{id}) \);
3. \( \Phi^{-1} \circ g \circ \Phi^{-1} \in Q(f, \text{id}) \);
4. \( f \in Q(g, \Phi) \);
5. \( f \in Q(\Phi^{-1} \circ g \circ \Phi^{-1}, \text{id}) \);
6. \( \Phi^{-1} \circ f \circ \Phi^{-1} \in Q(g, \text{id}) \).

**Proof.** It has been proven in [12] that assertion 1 is equivalent with assertions 3 and 4. Hence, it suffices to prove the equivalence between assertions 1 and 2. Denoting \( \Phi^{-1} \circ f \circ \Phi^{-1} \) as \( F \) it holds that \( g \in Q(F, \text{id}) \iff F \in Q(g, \text{id}) \). In combination with the equivalence between assertions 1, 4 and 6 this finishes the proof. \( \square \)

The functions \( f^\Phi, \Phi \circ f, f^\Phi \text{id} \) and \( f_\Phi \) do not only constitute the boundaries of the set \( Q(f, \Phi) \). They can also be sieved out of \( Q(f, \Phi) \) based on their continuity properties. The results differ depending on the type of monotonicity of both \( \Phi \) and \( f \). A full study on the topic can be found in [12]. In view of Theorem 6 and since we intend to study the symmetry aspects of contour lines, we recall the results only for \( \Phi = \text{id} \) and decreasing functions \( f \).

**Theorem 7** [12]. Consider a decreasing \([0,1] \to [0,1]\) function \( f \) s.t. \( f \neq \{0,1\} \). Then

1. \( f_\text{id} \) is the only left-continuous member of \( Q(f, \text{id}) \) that maps 0 to 1;
2. \( f^\Phi \text{id} \) is the only right-continuous member of \( Q(f, \text{id}) \) that maps 1 to 0.

Furthermore,

3. \( f_\text{id} \) is the only left-continuous member of \( Q(1, \text{id}) \) that maps 0 to 1;
4. \( f^\Phi \text{id} \) is the only right-continuous member of \( Q(0, \text{id}) \) that maps 1 to 0.

6. Solutions in terms of contour lines

The results from Sections 4 and 5 allow to characterize the solutions of Eq. (2) in terms of contour lines. Following the argumentation in Section 2 it suffices to solve Eqs. (3) and (4). Note that one can also solve the ‘standard’ functional Eqs. (3) and (4) and then apply Theorem 1 to transform the results. However, this second procedure does not yield any substantial reduction of the proofs and is therefore omitted.

6.1. Solving Eq. (3)

First of all, the symmetry of \( F \) embodied by Eq. (3) bounds up contour lines of the type \( C_a, \) resp. \( D_a, \) with those of the type \( \bar{C}_a, \) resp. \( \bar{D}_a. \)

**Theorem 8.** Consider an automorphism \( \phi \). For each increasing \([0,1] \to [0,1]\) function \( F \) satisfying Eq. (3) the following assertions hold:

1. \( C_a = \phi \circ \bar{C}_a \circ \phi, \) for every \( a \in [0,1] \).
2. \( D_a = \phi \circ \bar{D}_a \circ \phi, \) for every \( a \in [0,1] \).

**Proof.** We prove assertion 1. Assertion 2 follows from this assertion by invoking Theorem 4 and the observation that \( D_0 = 0 = \phi \circ \bar{0} \circ \phi = \phi \circ \bar{D}_0 \circ \phi. \)
If $F$ satisfies Eq. (3) then, by definition, we obtain that
\[
C_a(x) = \sup \{t \in [0,1]| F(x,t) \leq a \} = \sup \{t \in [0,1]| F(\phi^{-1}(t),\phi(x)) \leq a \} = \phi(\sup \{s \in [0,1]| F(s,\phi(x)) \leq a \}) = \phi(\tilde{C}(\phi(x)));
\]
for every $(x,a) \in [0,1]^2$. □

Secondly, Eq. (3) requires that the contour lines $C_a$ and $D_a$, resp. $\tilde{C}_a$ and $\tilde{D}_a$, are $\phi$-orthosymmetrical, resp. $\phi^{-1}$-orthosymmetrical.

**Definition 4.** [12] Let $\Phi$ be a monotone $[0,1] \rightarrow [0,1]$ bijection. A monotone $[0,1] \rightarrow [0,1]$ function $f$ is called $\Phi$-orthosymmetrical if $f \in \Phi(f, \Phi)$.

**Theorem 9.** Consider an automorphism $\phi$. For each increasing $[0,1]^2 \rightarrow [0,1]$ function $F$ satisfying Eq. (3) the following assertions hold:

1. $C_a \in Q(C_a, \phi)$, for every $a \in [0,1]$.
2. $D_a \in Q(D_a, \phi)$, for every $a \in [0,1]$.
3. $C_a \in Q(C_a, \phi^{-1})$, for every $a \in [0,1]$.
4. $D_a \in Q(D_a, \phi^{-1})$, for every $a \in [0,1]$.

**Proof.** If $F$ satisfies Eq. (3) it always holds that $\tilde{C}_a = \phi^{-1} \circ C_a \circ \phi^{-1}$ and $\tilde{D}_a = \phi^{-1} \circ D_a \circ \phi^{-1}$ (Theorem 8). Invoking Theorem 6, assertion 3 amounts to $C_a \in Q(C_a, \phi)$ and assertion 4 amounts to $D_a \in Q(D_a, \phi)$. It is therefore sufficient to focus on assertions 1 and 2 only. We will present the proof of assertion 1. Assertion 2 follows from this assertion by considering Theorems 4, 5 and the observation that $D_0 = 0 \in Q(0, \phi) = Q(D_0, \phi)$.

Take arbitrary $a \in [0,1]$. By definition and due to the decreasingness of $C_a$ it holds that
\[
C_a(x) = \sup \{t \in [0,1]| F(x,t) \leq a \},
\]
and
\[
C_a(x) = \sup \{t \in [0,1]| F(\phi^{-1}(t),\phi(x)) \leq a \} = \phi(\sup \{s \in [0,1]| F(s,\phi(x)) \leq a \}) = \phi(\tilde{C}(\phi(x)));
\]
for every $(x,a) \in [0,1]^2$. □

Next, we investigate to which extent the necessary conditions in the above theorems also become sufficient conditions for Eq. (3) to hold. Without additional continuity conditions, the inversion of Theorem 8 is not guaranteed. For example, define $F$ on $[0,1]^2 \cup \{(1,0)\}$ as $F(x,y) = 0$ and put $F(x,y) = 1$ elsewhere. Then $F$ is not left-continuous and $C_a = C_a$, for every $a \in [0,1]$. As $C_a(x) = 1$, for every $x \in [0,1]$, and $C_a(1) \in [0,1]$ it clearly holds that $C_a = \phi \circ C_a \circ \phi$. However, $F(0,1) = 0 < F(0,1) = 1$ such that Eq. (3) is not satisfied.

**Theorem 10.** Consider an automorphism $\phi$ and an increasing $[0,1]^2 \rightarrow [0,1]$ function $F$. If $F$ is left- or right-continuous, then the following assertions are equivalent:

1. $F$ satisfies Eq. (3).
2. $C_a = \phi \circ C_a \circ \phi$, for every $a \in [0,1]$.
3. $D_a = \phi \circ D_a \circ \phi$, for every $a \in [0,1]$.

**Proof.** We will present the proof for a left-continuous, increasing $[0,1]^2 \rightarrow [0,1]$ function $F$. The equivalence between assertions 2 and 3 follows from Theorem 4 and the observations that $C_1 = 1 = \phi \circ 1 \circ \phi = \phi \circ \tilde{C}_1 \circ \phi$ and $D_0 = 0 = \phi \circ 0 \circ \phi = \phi \circ D_0 \circ \phi$. If $F$ satisfies Eq. (3), then assertion 2 follows immediately from Theorem 8. Conversely, take $F$ such that assertion 2 holds and suppose that $F(x,y) < F(\phi^{-1}(y),\phi(x))$, for some $(x,y) \in [0,1]^2$. Clearly, $0 < x$ or $0 < y$. It follows from Definition 3 that $C_{F(x,y)}(\phi^{-1}(y)) = \phi(x)$, if $0 < x$, and that $C_{F(x,y)}(\phi(x)) < \phi^{-1}(y)$, if $0 < y$. Since $C_{F(x,y)} = \phi \circ C_{F(x,y)} \circ \phi$, this leads to $\tilde{C}_{F(x,y)}(y) < x$, if $0 < x$, and $\tilde{C}_{F(x,y)}(x) < y$, if $0 < y$. Invoking Definition 3 a second time, we obtain in both cases the contradiction $F(x,y) < F(x,y)$. Hence, $F(\phi^{-1}(y),\phi(x)) < F(x,y)$, for every $(x,y) \in [0,1]^2$. From the observation that $F(\phi^{-1}(y),\phi(x)) < F(x,y)$ can be reformulated as $F(x,y) < F(\phi^{-1}(y),\phi(u))$, with $u = \phi^{-1}(y)$ and $v = \phi(x)$, we conclude that $F(x,y) = F(\phi^{-1}(y),\phi(x))$ is fulfilled for every $(x,y) \in [0,1]^2$. □

To invert Theorem 9 continuity conditions alone are not restrictive enough. For example, if $F(x,0) = 0$, for all $x \in [0,1]$, and $F(x,y) = 1$, elsewhere, then $F$ is left-continuous but does not fulfill Eq. (3) ($F(1,0) = 0 < 1 = F(0,1)$). It is easily verified that in this example all contour lines $C_a$ and $D_a$, resp. $\tilde{C}_a$ and $\tilde{D}_a$, are $\phi$-orthosymmetrical, resp. $\phi^{-1}$-orthosymmetrical. To solve this problem we need to impose on $F$ also some additional boundary conditions.
Theorem 11. Consider an automorphism $\phi$ and an increasing $[0,1]^2 \to [0,1]$ function $F$. If $F$ is left-continuous and satisfies $F(0,1) = F(1,0) = 0$ or $F$ is right-continuous and satisfies $F(0,1) = F(1,0) = 1$, then the following assertions are equivalent:

1. $F$ satisfies Eq. (3).
2. $C_\alpha \in Q(C_\alpha \phi)$, for every $\alpha \in [0,1]$.
3. $C_\alpha \in Q(C_\alpha \phi^{-1})$, for every $\alpha \in [0,1]$.
4. $D_\alpha \in Q(D_\alpha \phi)$, for every $\alpha \in [0,1]$.
5. $D_\alpha \in Q(D_\alpha \phi^{-1})$, for every $\alpha \in [0,1]$.

Proof. We prove the theorem for a left-continuous, increasing $[0,1]^2 \to [0,1]$ function $F$ satisfying $F(0,1) = F(1,0) = 0$. Theorems 4 and 5, together with the observations

$$
C_1 = \overline{C}_1 = 1 \in Q(1, \phi) = Q(C_1, \phi) = Q(\overline{C}_1, \phi)
$$

$$
D_0 = \overline{D}_0 = 0 \in Q(0, \phi) = Q(D_0, \phi) = Q(\overline{D}_0, \phi).
$$

ensure the equivalence between assertions 2 and 4 and between assertions 3 and 5. From Theorem 9 we know that assertion 1 implies assertions 2 and 3. Hence, it suffices to prove that each of the assertions 2 and 3 imply assertion 1. We illustrate only the first implication, the proof of the second one being similar.

Assume that $C_\alpha \in Q(C_\alpha \phi)$, for every $\alpha \in [0,1]$. Then, equivalently, $\phi^{-1} \circ C_\alpha \circ \phi^{-1} \in Q(C_\alpha \text{id})$, for every $\alpha \in [0,1]$ (Theorem 6).

The left continuity of $F$ ensures that every $C_\alpha$ and thus also every $\phi^{-1} \circ C_\alpha \circ \phi^{-1}$ is left-continuous. Due to the boundary condition $F(0,1) = 0$ it holds that $C_\alpha(0) = 1$ and $\phi^{-1}(C_\alpha(\phi^{-1}(0))) = \phi^{-1}(C_\alpha(0)) = \phi^{-1}(1) = 1$. Invoking Theorem 7 these considerations lead to $\phi^{-1} \circ C_\alpha \circ \phi^{-1} = \overline{C}_\alpha$. From the decreasingness of $C_\alpha$ and Theorem 3, we obtain the following chain of equalities:

$$
\phi^{-1}(C_\alpha(\phi^{-1}(x))) = \inf\{t \in [0,1]|C_\alpha(t) < x\} = \sup\{t \in [0,1]|C_\alpha(t) \geq x\} = \sup\{t \in [0,1]|F(t, x) < a\} = \overline{C}_\alpha(x),
$$

for every $(x,a) \in [0,1]^2$. We conclude that $\phi^{-1} \circ C_\alpha \circ \phi^{-1} = \overline{C}_\alpha$, for every $\alpha \in [0,1]$, and thus $F$ satisfies Eq. (3) (Theorem 10). □

6.2. Solving Eq. (4)

Similarly to the previous subsection we lay bare the influence Eq. (4) has upon the contour lines of its solutions.

Theorem 12. Consider a strict negator $M$ and an involutive negator $N$ with unique fixpoint $\beta$. For each increasing $[0,1]^2 \to [0,1]$ function $F$ satisfying Eq. (4) the following assertions hold:

1. $C_{N(\alpha)} = M \circ D_\alpha \circ M$, for every $\alpha \in [0,\beta]$.
2. $D_{N(\alpha)} = M \circ \overline{C}_\alpha \circ M$, for every $\alpha \in [0,\beta]$.
3. $\overline{C}_{N(\alpha)} = M^{-1} \circ D_\alpha \circ M^{-1}$, for every $\alpha \in [0,\beta]$.
4. $\overline{D}_{N(\alpha)} = M^{-1} \circ \overline{C}_\alpha \circ M^{-1}$, for every $\alpha \in [0,\beta]$.

Proof. We prove assertions 1 and 4. Assertions 2 and 3 follow from these assertions by invoking Theorem 4 and the observation that $C_1 = 1 = M^{-1} \circ \overline{C}_1 = M^{-1} \circ D_0 = M^{-1}$.

If $F$ satisfies Eq. (4) then, by definition, we obtain that

$$
C_{N(\alpha)}(x) = \sup\{t \in [0,1]|F(x, t) \leq N(\alpha)\}
$$

$$
= \sup\{t \in [0,1]|F(M^{-1}(t), M(x)) \leq N(\alpha)\}
$$

$$
= M(\inf\{s \in [0,1]|F(s, M(x)) \geq \alpha\})
$$

$$
= M(D_\alpha(M(x))),
$$

for every $(x,a) \in [0,1]^2$. □

As indicated in the proof of the theorem, it is possible to merge the first and last assertion and the second and third assertion: $C_{N(\alpha)} = M \circ D_\alpha \circ M$ and $D_{N(\alpha)} = M \circ \overline{C}_\alpha \circ M$ hold for every $\alpha \in [0,1]$. However, in contrast to Eq. (3), the involutive negator $N$ in Eq. (4) allows us to consider four assertions (Theorem 12) instead of two (Theorem 8). Each of these assertions will turn out to be sufficient for Eq. (4) to hold that $F$ is continuous (see Theorem 14).

The following theorem states that whenever $F$ satisfies Eq. (4), $C_{N(\alpha)}$ can be understood as some $M$-inverse function of $D_\alpha$ and $\overline{C}_{N(\alpha)}$ as some kind of $M^{-1}$-inverse function of $\overline{D}_\alpha$.

Theorem 13. Consider a strict negator $M$ and an involutive negator $N$ with unique fixpoint $\beta$. For each increasing $[0,1]^2 \to [0,1]$ function $F$ satisfying Eq. (4) the following assertions hold:

1. $C_{N(\alpha)} \in Q(D_\alpha M)$, for every $\alpha \in [0,\beta]$.
2. $D_{N(\alpha)} \in Q(C_\alpha M)$, for every $\alpha \in [0,\beta]$.
3. $\overline{C}_{N(\alpha)} \in Q(D_\alpha M^{-1})$, for every $\alpha \in [0,\beta]$.
4. $\overline{D}_{N(\alpha)} \in Q(C_\alpha M^{-1})$, for every $\alpha \in [0,\beta]$.
Proof. From Theorem 6 we know that $D_{(N_0)} \in Q(C_0,M)$ is equivalent with $C_0 \in Q(D_{(N_0)}M)$ and that $\bar{D}_{(N_0)} \in Q(\bar{C}_2,M)$ is equivalent with $\bar{C}_2 \in Q(\bar{D}_{(N_0)},M)$, for every $a \in [0,1]$. Hence, combining assertion 1 with assertion 2 and assertion 3 with assertion 4, it suffices to prove that $C_{(N_0)} \in Q(D_{(N_0)}M)$ and $\bar{C}_{(N_0)} \in Q(\bar{D}_{(N_0)},M^{-1})$, for every $a \in [0,1]$. If $F$ satisfies Eq. (4) it always holds that $\bar{C}_{(N_0)} = M^{-1} \circ D_a \circ M^{-1}$ and $D_a = M^{-1} \circ C_{(N_0)} \circ M^{-1}$ (Theorem 12). Invoking Theorem 6, $\bar{C}_{(N_0)} \in Q(\bar{D}_{(N_0)},M^{-1})$ amounts to $C_{(N_0)} \in Q(D_{(N_0)},M)$, for every $a \in [0,1]$.

Take arbitrary $a \in [0,1]$. By definition it holds that

$$
\begin{align*}
\mathcal{D}_{a,M}(x) &= \sup\{t \in [0,1] | D_a(M^{-1}(t)) < M(x)\}, \\
\mathcal{C}_{N_0}(x) &= \sup\{t \in [0,1] | F(x,t) \leq N(a)\}, \\
\mathcal{D}_{a,M}(x) &= \sup\{t \in [0,1] | D_a(M^{-1}(t)) \leq M(x)\}.
\end{align*}
$$

Eq. (4) guarantees that

$$
D_a(M^{-1}(t)) < M(x) \Rightarrow N(F(x,t)) = F(M^{-1}(t),M(x)) \geq a \Rightarrow D_a(M^{-1}(t)) \leq M(x),
$$

which leads to $\mathcal{D}_{a,M} \leq \mathcal{C}_{N_0} \leq \mathcal{D}_{a,M}$. As $M$ is decreasing, it follows from Table 1 that $C_{(N_0)} \in Q(D_{(N_0)},M)$. □

Unfortunately, the assertions of Theorems 12 and 13 are again not sufficient for Eq. (4) to hold. For example, take an arbitrary strict negator $K$ s.t. $K < M$ and $K < M^{-1}$. Define $F$ as follows

$$
F(x,y) := \begin{cases} 
0, & \text{if } y < K(x), \\
\beta, & \text{if } K(x) < y \leq M(K^{-1}(M(x))), \\
1, & \text{if } M(K^{-1}(M(x))) < y.
\end{cases}
$$

Then $F$ is left-continuous and satisfies $F(x,M(x)) = \beta$, which is necessary for Eq. (4) to hold. The contour lines of $F$ can be found in Table 2. Clearly, these contour lines satisfy the assertions in Theorems 12 and 13. However, $F$ can never satisfy Eq. (4) as $F(x,K(x)) = 0 < \beta = N(F(M^{-1}(K(x)),M(x)))$. Also, in this case some additional continuity conditions are required to retrieve Eq. (4). In contrast to Eq. (3), the use of the strict negator $M$ in Eq. (4) enforces full continuity upon $F$ whenever $F$ is left- or right-continuous.

**Theorem 14.** Consider a strict negator $M$, an involutive negator $N$ with unique fixpoint $\beta$ and an increasing $[0,1]^2 \to [0,1]$ function $F$. If $F$ is continuous, then the following assertions are equivalent:

1. $F$ satisfies Eq. (4).
2. $G_{(N_0)} = M \circ D_a \circ M$, for every $a \in [0,\beta]$.
3. $G_{(N_0)} = M \circ C_a \circ M$, for every $a \in [0,\beta]$.
4. $G_{(N_0)} = M^{-1} \circ D_a \circ M^{-1}$, for every $a \in [0,\beta]$.
5. $G_{(N_0)} = M^{-1} \circ C_a \circ M^{-1}$, for every $a \in [0,\beta]$.

Proof. Due to Theorem 12 we know that assertion 1 implies assertions 2, 3, 4, 5. Invoking Theorem 4 and $C_1 = 1 = M \circ 0 \circ M = M \circ D_0 \circ M$, it follows that assertion 3 implies assertion 2. Similarly, Theorem 4 together with $C_1 = 1 = M \circ 0 \circ M = M \circ D_0 \circ M$ yields that assertion 5 implies assertion 4. This leaves us to prove that each of the assertions 2 and 4 implies assertion 1. We prove the first implication, the proof of the second one being similar.

Take $F$ such that $C_{(N_0)} = M \circ D_a \circ M$ holds for every $a \in [0,\beta]$. Take arbitrary $(x,y) \in [0,1]^2$ s.t. $F(x,y) \leq \beta$. Since $0 < x, y < 1$, we obtain from Definition 3, the continuity of $F$ and assertion 2 that

$$
N(F(x,y)) < F(M^{-1}(y),M(x)) \Rightarrow C_{N,F(x,y)}(M^{-1}(y)) < M(x) \Rightarrow x < \bar{D}_{F(x,y)}(y) \Leftrightarrow F(x,y) < F(x,y).
$$

Hence, $F(M^{-1}(y),M(x)) \leq N(F(x,y))$. Furthermore,

$$
F(M^{-1}(y),M(x)) < \beta \Leftrightarrow M^{-1}(y) < \bar{D}_{\beta}(M(x)) \Leftrightarrow C_{\beta}(x) < y \Leftrightarrow \beta < F(x,y)
$$

implies that $\beta \leq F(M^{-1}(y),M(x))$ such that

$$
F(x,y) < N(F(M^{-1}(y),M(x)))
\Leftrightarrow x < \bar{D}_{N,F(M^{-1}(y),M(x))}(y)
\Leftrightarrow C_{F(M^{-1}(y),M(x))}(M^{-1}(y)) < M(x)
\Leftrightarrow F(M^{-1}(y),M(x)) < F(M^{-1}(y),M(x)).
$$

<table>
<thead>
<tr>
<th>$a = 0$</th>
<th>$a \in [0,\beta]$</th>
<th>$a = \beta$</th>
<th>$a \in ]\beta,1[$</th>
<th>$a = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_a$</td>
<td>$K$</td>
<td>$K$</td>
<td>$M \circ K^{-1} \circ M$</td>
<td>$M \circ K^{-1} \circ M$</td>
</tr>
<tr>
<td>$C_a$</td>
<td>$K^{-1}$</td>
<td>$K^{-1}$</td>
<td>$M^{-1} \circ K \circ M^{-1}$</td>
<td>$M^{-1} \circ K \circ M^{-1}$</td>
</tr>
<tr>
<td>$D_a$</td>
<td>$0$</td>
<td>$K$</td>
<td>$M \circ K^{-1} \circ M$</td>
<td>$M \circ K^{-1} \circ M$</td>
</tr>
<tr>
<td>$\bar{D}_a$</td>
<td>$0$</td>
<td>$K^{-1}$</td>
<td>$K^{-1}$</td>
<td>$M^{-1} \circ K \circ M^{-1}$</td>
</tr>
</tbody>
</table>
We conclude that \( F(x,y) = N(F(M^{-1}(y), M(x))) \) is satisfied for those couples \((x,y) \in [0,1]^2\) s.t. \( F(x,y) < \beta \). In case \( \beta < F(x,y) \) it follows from Eq. (6) that \( F(M^{-1}(y), M(x)) < \beta \). Denoting \( M^{-1}(y) \) and \( M(x) \) by, resp., \( u \) and \( v \), we can repeat the above procedure and conclude that
\[
F(M^{-1}(y), M(x)) = F(u, v) = N(F(M^{-1}(v), M(u))) = N(F(x,y)).
\]
From the continuity of \( F \), \( M \) and \( N \) it follows that Eq. (4) also holds whenever \( x \in (0,1) \) or \( y \in (0,1) \). □

To invert Theorem 13 we need to impose some additional boundary conditions on \( F \). Suppose for example that \( F(x,y) = 1 \), for every \((x,y) \in [0,1]^2\). Then \( F \) is trivially continuous, \( C_a = C_b = 0 \) whenever \( a \in [0,1], C_1 = C_2 = 1 \) and \( D_a = D_b = 0 \) for every \( a \in [0,1] \). The assertions of Theorem 13 hold but \( F(0,0) = 1 > 0 = F(1,1)^\beta \). Note that these assertions do not force \( F \) to satisfy \( F(x,x^\beta) = \beta \), which is necessary for Eq. (4) to hold. Simply, requiring that \( F(0,1) = F(1,0) = \beta \) counter this deficiency.

**Theorem 15.** Consider a strict negator \( M \), an involutive negator \( N \) with unique fixpoint \( \beta \) and an increasing \([0,1]^2 \to [0,1] \) function \( F \). If \( F \) is continuous, then the following assertions are equivalent:

1. \( F \) satisfies Eq. (4).
2. \( C_{N(a)} \in Q(D_{a} M) \), for every \( a \in [0,\beta] \), and \( F(0,1) = F(1,0) = \beta \).
3. \( D_{N(a)} \in Q(C_{a} M) \), for every \( a \in [0,\beta] \), and \( F(0,1) = F(1,0) = \beta \).
4. \( C_{N(a)} \in Q(D_{a} M^{-1}) \), for every \( a \in [0,\beta] \), and \( F(0,1) = F(1,0) = \beta \).
5. \( D_{N(a)} \in Q(C_{a} M^{-1}) \), for every \( a \in [0,\beta] \), and \( F(0,1) = F(1,0) = \beta \).

**Proof.** For assertion 1 to hold it is always necessary that assertions 2, 3, 4 are satisfied (Theorem 13) and \( F(x,x^\beta) = \beta \), for every \( x \in [0,1] \). Invoking Theorems 4, 5 and the observation that \( C_1 = 1 \in Q(0,M) = Q(D_0 M) \), it follows that assertion 3 implies assertion 2. Similarly, Theorems 4 and 5 together with \( C_1 = 1 \in Q(0,M) = Q(D_0 M) \) yields that assertion 5 implies assertion 4. This leaves us to prove that each of the assertions 2 and 4 imply assertion 1. We prove the first implication, the proof of the second one being analogous.

Assume that \( C_{N(a)} \in Q(D_{a} M) \), for every \( a \in [0,\beta] \), and \( F(0,1) = F(1,0) = \beta \). Then, equivalently, \( M^{-1} \circ C_{N(a)} \circ M^{-1} \in Q(D_{a} \text{id}) \), for every \( a \in [0,\beta] \) (Theorem 6). The left continuity of \( F \) ensures that every \( C_{N(a)} \circ M^{-1} \) must be right-continuous. As \( F(0,1) = F(1,0) = \beta \), it holds that \( M^{-1} \circ C_{N(a)} \circ M^{-1}(1) = 0 \) and \( D_{a}(1) = 0 \), for every \( a \in [0,\beta] \). Invoking Theorem 7 these considerations lead to \( M^{-1} \circ C_{N(a)} \circ M^{-1} = D_{a} \text{id} \). From the decreasingness of \( D_{a} \) and the continuity of \( F \), we obtain the following chain of equalities:
\[
M^{-1}(C_{N(a)}(M^{-1}(x))) = \sup \{ t \in [0,1] | D_{a}(t) > x \} = \inf \{ t \in [0,1] | D_{a}(t) < x \} = \inf \{ t \in [0,1] | F(t,x) \geq a \} = D_{a}(x),
\]
for every \((x,a) \in [0,1] \times [0,\beta] \). Moreover, \( F(0,1) = F(1,0) = \beta \) implies that \( M^{-1}(C_{N(a)}(M^{-1}(1))) = 0 = D_{a}(1) \) is fulfilled for every \( a \in [0,\beta] \). We conclude that \( M^{-1} \circ C_{N(a)} \circ M^{-1} = D_{a} \) whenever \( a \in [0,\beta] \) and thus \( F \) satisfies Eq. (4) (Theorem 14). □

**7. Concluding remarks**

Dealing with increasing \([0,1]^2 \to [0,1] \) functions we have discussed a generalized version of the commutativity property which captures also properties linked to self-duality. In particular, the solutions of this functional equation are either transformations of a commutative or of a self-dual function. Geometrically, the property bounds up the value of an increasing \([0,1]^2 \to [0,1] \) function \( F \) in a point \((x,y) \in [0,1]^2 \) to its value in the \( \Phi \)-inverse point \( (\Phi^{-1}(y),\Phi(x)) \), with \( \Phi \) a given monotone \([0,1] \to [0,1] \) bijection. Describing the solutions in terms of contour lines converts the three-dimensional ‘symmetry’ comprised in the functional equation into a two-dimensional symmetry for decreasing \([0,1] \to [0,1] \) functions. For increasing \([0,1] \to [0,1] \) bijections \( \Phi \) we obtain the \( \Phi^{-1} \)-symmetry of the contour lines. In case \( \Phi \) is decreasing the four different types of contour line get paired. Each type can then be interpreted as the \( \Phi^{-1} \)-or \( \Phi^{-1} \)-inverse of another type.

The results of this paper can be easily modified to admit other types of monotone \([0,1]^2 \to [0,1] \) functions \( F \). Let \( N \) be an involutive negator. Consider the order 4 transformation \( \sigma_N : [0,1]^2 \to [0,1]^2 : (x,y) \mapsto (N(y),x) \). In case \( N \) equals the standard negator \( \mathcal{N} \), the transformation \( \sigma_N \) determines the following rotation:
\[
\sigma^x \circ \sigma^y \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} 1 - y \\ x \end{array} \right) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \circ \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right),
\]
The rotation matrix used in Eq. (7) has order 4 and determines a rotation of 90 degrees around the origin. In case \( N \) is not linear, we can rewrite \( \sigma_N \) as \( \sigma_N = \sigma \circ \sigma^y \circ \sigma^y \circ \sigma^x \). Since \( \sigma^y \circ \sigma^x \) determines an increasing \([0,1]^2 \to [0,1]^2 \) bijection, \( \sigma_N \) can be understood as some distorted rotation around the point \((\beta,\beta)\), with \( \beta \) the unique fixpoint of \( N \).

Rotating the arguments of an increasing \([0,1]^2 \to [0,1] \) function \( F \) by means of a transformation \( \gamma_N \in \{ \sigma_N, \sigma_N^2, \sigma_N^3 \} \) yields a \([0,1]^2 \to [0,1] \) function \( G = F \circ \gamma_N \) that has monotone partial functions \( G(x,\bullet) \) and \( G(\bullet,x) \). If \( F \) satisfies Eq. (2), then \( G \) satisfies a structurally identical functional equation:
Furthermore, when defining the contour lines of \( G \), \( \inf \) and \( \sup \) have to be interchanged whenever the associated partial function is decreasing. For example, if \( \gamma_N = \sigma_N \), then \( \mathcal{C}_\sigma^G(x) = \inf \{ t \in [0, 1] | G(x, t) \leq a \} \) instead of \( \mathcal{C}_\phi^F(x) = \sup \{ t \in [0, 1] | F(x, t) \leq a \} \). In case \( \gamma_N = \sigma_N \) or \( \gamma_N = \sigma_N^3 \), the contour lines of \( G \) will be increasing; if \( \gamma_N = \sigma_N^3 \), they will be decreasing.

References


