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Note

A note on the relationship between the graphical traveling salesman polyhedron, the Symmetric Traveling Salesman Polytope, and the metric cone

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ABSTRACT

In this short communication, we observe that the Graphical Traveling Salesman Polyhedron is the intersection of the positive orthant with the Minkowski sum of the Symmetric Traveling Salesman Polytope and the polar of the metric cone. This follows almost trivially from known facts. There are nonetheless two reasons why we find this observation worth communicating: It is very surprising; it helps us understand the relationship between these two important families of polyhedra.

1. Introduction

The *Symmetric Traveling Salesman Polytope* is the convex hull of all characteristic vectors of edge sets of cycles (i.e., circuits) on the vertex set $V_n := \{1, \dots, n\}$ (in other words, Hamiltonian cycles in the complete graph with vertex set V_n). For the formal definition, denote by E_n the set of all two-element subsets of V_n. This is the set of all possible edges of a graph with vertex set V_n. The Symmetric Traveling Salesman Polytope is then the following set:

$$S_n := \operatorname{conv}\left\{\chi^C \mid C \text{ is the edge set of a Hamiltonian cycle with vertex set } V_n\right\} \subset \mathbb{R}^{E_n}.$$

Here, for an edge set F, χ^F is the characteristic vector in $\mathbb{R}^{E_n}$ with $\chi^F_e = 1$ if $e \in F$, and zero otherwise. The importance of the Symmetric Traveling Salesman Polytope comes mainly, but not exclusively, from its use in the solution of the so-called Symmetric Traveling Salesman Problem, which consists of finding a Hamiltonian cycle of minimum cost.

The *Graphical Traveling Salesman Polyhedron* is the convex hull of all characteristic vectors of edge multi-sets of connected Eulerian multi-graphs on the vertex set V_n. A multi-graph with vertex set V_n has as its edge set a sub-multi-set of E_n, which is to say that our multi-graphs can have parallel edges but no loops. By defining, for any multi-set F of edges of K_n, its characteristic vector $\chi^F \in \mathbb{R}^{E_n}$ in such a way that χ^F_e counts the number of occurrences of e in F, the Graphical Traveling Salesman Polyhedron is formally defined as

$$P_n := \operatorname{conv}\left\{\chi^F \mid F \text{ is the edge multi-set of a connected Eulerian multi-graph with vertex set } V_n\right\} \subset \mathbb{R}^{E_n}.$$

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Ever since the seminal work of Naddef & Rinaldi [4,5] on the two polyhedra, P_n is considered to be an important tool for investigating the facets of S_n. Moreover, in works of Carr [2] and Applegate, Bixby, Chvàtal & Cook [1], P_n has been used algorithmically in contributing to solution schemes for the Symmetric Traveling Salesman Problem.

Numerous authors have expressed how close the connection between Graphical and Symmetric Traveling Salesman Polyhedra is. The most basic justification for this opinion is the fact that S_n is a face of P_n — consisting of all points x whose "degree" is two at every vertex: $\sum_{v\neq u} x_{uv} = 2$ for all $u \in V_n$. However, the connections are far deeper (see [3] or [6] and the references therein). In this short communication, we contribute the following surprising geometric observation to the issue of the relationship between these two polyhedra:

Theorem. *P_n is the intersection of the positive orthant with the Minkowski sum of S_n and the polar $C_n^{\triangle}$ of the metric cone C_n:*

$$P_n = (S_n + C_n^{\triangle}) \cap \mathbb{R}_+^{E_n}. \tag{1}$$

The metric cone consists of all $a \in \mathbb{R}^{E_n}$ which satisfy the triangle inequality:

$$a_{uv} \leq a_{uw} + a_{wv} \tag{2}$$

for all pairwise distinct vertices $u, v, w \in V_n$. Consequently, its polar is generated as a cone by the vectors (we abbreviate $\chi^{\{e\}}$ to χ^e)

$$\chi^{uw} + \chi^{wv} - \chi^{uv}. \tag{3}$$

The proof of this theorem is an application of three or four known facts or techniques in the area of Symmetric and Graphical Traveling Salesman polyhedra.

2. Proof

We start with showing that $P_n \subset (S_n + C_n^{\triangle}) \cap \mathbb{R}_+^{E_n}$. While $P_n \subset \mathbb{R}_+^{E_n}$ holds trivially, $P_n \subset S_n + C_n^{\triangle}$ follows from an argument of [5], which we reproduce here for the sake of completeness.

Let $x \in \mathbb{Z}_+^{E_n}$ be a the characteristic vector of the edge multi-set of a connected Eulerian multi-graph G with vertex set V_n. We prove by induction on the number m of edges of G, that x can be written as a sum of a cycle and a number of vectors (3). If $m = n$, then there is nothing to prove. Let $m \geq n + 1$. There exists a vertex w of degree at least four in G. We distinguish two cases. The easy case occurs when $G \setminus w$ is still connected. Here, we let u and v be two arbitrary (possibly identical) neighbors of w. By either replacing the edges uw and wv of G with the new edge uv, if $u \neq v$, or deleting uw and wv, if $u = v$, one obtains a connected Eulerian multi-graph G' with fewer edges than G. The change in the vector x amounts to subtracting the expression (3): $x' = x - (\chi^{uw} + \chi^{wv} - \chi^{uv})$, if $u \neq v$, and $x' = x - (\chi^{uw} + \chi^{wv})$, if $u = v$. In the slightly more difficult case when the graph $G \setminus w$ has at least two connected components, we can let u and v be two neighbors of w in distinct components of $G \setminus w$. This makes sure that the graph G' is still connected. We conclude by induction that x', and hence x, can be written as a sum of a cycle and a number of vectors (3).

We now prove $P_n \supset (S_n + C_n^{\triangle}) \cap \mathbb{R}_+^{E_n}$. For this, we show that any inequality which is facet-defining for P_n is valid for $(S_n + C_n^{\triangle}) \cap \mathbb{R}_+^{E_n}$.

We again invoke an argument from [5]: Naddef & Rinaldi have shown[1] that the inequalities defining facets of P_n fall into one of two categories: the non-negativity inequalities $x_e \geq 0$, with $e \in E_n$ (or positive scalar multiples thereof), or inequalities whose coefficient vectors satisfy the triangle inequality (2). We reproduce the proof of this statement.

First recall that an inequality $a \cdot x \geq \alpha$ is said to be *dominated* by another inequality $b \cdot x \geq \beta$, if the face defined by the first inequality is contained in the face defined by the second inequality.

Suppose that $a \cdot x \geq \alpha$ is not dominated by a non-negativity inequality (it need not be define a facet, though), and let u, v, w be three distinct vertices in V_n. Then there exists an $x \in \mathbb{Z}_+^{E_n}$ defining the edge multi-set of a connected Eulerian multi-graph G which has an edge between u and v, such that $a \cdot x = \alpha$. If we replace the edge uv of G by the two edges uw and wv, then we obtain a connected Eulerian multi-graph, whose edge multi-set is given, in terms of its characteristic vector, by $x' := x + \chi^{uw} + \chi^{wv} - \chi^{uv}$. Now $a \cdot x' \geq \alpha$, implies $a_{uw} + a_{wv} - a_{uv} \geq 0$, i.e., the triangle inequality.

We now conclude the proof of the inclusion $P_n \supset (S_n + C_n^{\triangle}) \cap \mathbb{R}_+^{E_n}$. Let $a \cdot x \geq \alpha$ be an inequality which is facet-defining for P_n. First note that the non-negativity inequalities are clearly satisfied by the right hand side of (1). Hence, using what we have just discussed, let us assume that a satisfies the triangle inequality. This means that a is a member of the metric cone C_n. Consequently, the inequality $a \cdot x \geq 0$ is valid for $C_n^{\triangle}$. Further, since $S_n \subset P_n$, the inequality $a \cdot x \geq \alpha$ is clearly valid for S_n. Hence the inequality is valid for $S_n + C_n^{\triangle}$.

This concludes the proof of the theorem. □

Note that, in passing, we have proved the following. If we define P_n' to be the set of all $y \in \mathbb{R}^{E_n}$ which satisfy $a \cdot y \geq \alpha$ for every inequality $a \cdot x \geq \alpha$ defining a facet of P_n but not being a scalar multiple of a non-negativity inequality, then we have $S_n + C_n^{\triangle} \subset P_n'$.

[1] In fact, Proposition 2.2 of [5] states that the facet-defining inequalities for P_n fall into three classes — one of which is the class of non-negativity inequalities and the other two satisfy the triangle inequality.

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