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Oscillation Criteria for Certain *n*th Order Differential Equations with Deviating Arguments

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Oscillation criteria for nth order differential equations with deviating arguments of the form

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t)\right)' + F(t,\ x[g(t)]) = 0, \qquad n \text{ even}$$

are established, where $g \in C([t_0,\ \infty), \mathbb{R}), F \in C([t_0,\ \infty) \times \mathbb{R}, \mathbb{R})$, and $\alpha > 0$ is a constant. © 2001 Academic Press

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1. INTRODUCTION

In this paper we shall study the oscillatory behavior of the functional differential equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t)\right)' + F(t, x[g(t)]) = 0, \qquad n \text{ even} \tag{1.1}$$

where α is a positive constant, $g(t) \in C([t_0, \infty), \mathbb{R})$, $\lim_{t\to\infty} g(t) = \infty$, and $F(t, x) \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, sgn $F(t, x) = \operatorname{sgn} x$, $t \geq t_0$.

We shall assume that there exist a constant $\beta > 0$ and a function $q(t) \in C([t_0, \infty), \mathbb{R}^+)$ such that

$$F(t, x)\operatorname{sgn} x \geq q(t)|x|^{\beta} \qquad \text{for } x \neq 0 \quad \text{and} \quad t \geq t_0. \tag{1.2}$$

By a solution of Eq. (1.1) we mean a function $x(t) \in C^{n-1}([T_x, \infty), \mathbb{R})$ for some $T_x \geq t_0$ which has the property that $\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t) \in C^1([T_x, \infty), \mathbb{R})$ and satisfies equation (1.1) on $[T_x, \infty)$. A nontrivial solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1.1) is oscillatory if all of its solutions are oscillatory.

The equation (1.1) with $n = 2$, namely, the equation

$$\left(|x'(t)|^{\alpha-1}x'(t)\right)' + F(t, x[g(t)]) = 0,$$

and/or related equations have been the subject of intensive studies in recent years because these equations are natural generalizations of the equation

$$x''(t) + F(t, x[g(t)]) = 0.$$

For recent contributions we refer the reader to [2–5, 15, 19, 20] and references therein. As far as we know the equation (1.1) has never been the subject of systematic investigations.

In Section 2, we shall present some oscillation criteria for Eq. (1.1) which extend several known results established in [2–10, 16, 18–20]. Section 3 contains extensions of some of the results presented in Section 2 to a special case of (1.1), namely, the equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t)\right)' + q(t)f(x[g(t)]) = 0, \tag{1.3}$$

where $\alpha > 0$ is a constant, $q(t) \in C([t_0, \infty), \mathbb{R}^+)$, $g(t) \in C([t_0, \infty), \mathbb{R})$, $f(x) \in C(\mathbb{R}, \mathbb{R})$, $\lim_{t\to\infty} g(t) = \infty$, and $xf(x) > 0$ for $x \neq 0$. The function f in equation (1.3) need not be a monotonic function. Here, we shall also consider equations of neutral type of the form

$$\frac{d}{dt}\left(\left|[x(t) + p(t)x[\tau(t)]]^{(n-1)}\right|^{\alpha-1}(x(t) + p(t)x[\tau(t)])^{(n-1)}\right) + F(t, x[g(t)]) = 0, \tag{1.4}$$

where α, F, and g are as in Eq. (1.1), $p(t) \in C([t_0, \infty), \mathbb{R}_0), \mathbb{R}_0 = [0, \infty)$, $\tau(t) \in C^1([t_0, \infty), \mathbb{R})$, and $\lim_{t\to\infty} \tau(t) = \infty$. The obtained results extend those presented in [10, 12, 16]. In Section 4, we shall consider the more general equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + F\left(t, x[g(t)], \frac{d}{dt} x[h(t)]\right) = 0, \tag{1.5}$$

where α is a positive constant, $g(t), h(t) \in C([t_0, \infty), \mathbb{R})$, $h(t) \leq t$, $h'(t) > 0$ for $t \geq t_0$, $\lim_{t\to\infty} g(t) = \infty = \lim_{t\to\infty} h(t)$, and $F \in C([t_0, \infty) \times \mathbb{R}^2, \mathbb{R})$.

We shall assume that there exist a function $q(t) \in C([t_0, \infty), \mathbb{R}^+)$ and positive constants β and μ such that

$$F(t, x, y) \operatorname{sgn} x \geq q(t)|x|^{\beta}|y|^{\mu} \qquad \text{for } xy \neq 0 \quad \text{and} \quad t \geq t_0. \tag{1.6}$$

The results presented in this section extend some of our earlier work in [1, 2, 6].

2. MAIN RESULTS

We shall need the following:

LEMMA 2.1 [18]. *Let $x(t) \in C^n([t_0, \infty), \mathbb{R}^+)$. If $x^{(n)}(t)$ is eventually of one sign for all large t, say, $t_1 \geq t_0$, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$, with $n + l$ even for $x^{(n)}(t) \geq 0$, or $n + l$ odd for $x^{(n)}(t) \leq 0$ such that*

$$l > 0 \quad \text{implies that } x^{(k)}(t) > 0 \qquad \text{for } t \geq t_x, \quad k = 0, 1, \ldots, l-1$$

and

$$l \leq n-1 \quad \text{implies that } (-1)^{l+k} x^{(k)}(t) > 0$$
$$\text{for } t \geq t_x,\ k = l, l+1, \ldots, n-1.$$

LEMMA 2.2 [18]. *If the function $x(t)$ is as in Lemma 2.1 and $x^{(n-1)}(t)x^{(n)}(t) \leq 0$ for $t \geq t_x$, then there exists a constant θ, $0 < \theta < 1$, such that*

$$x(t) \geq \frac{\theta}{(n-1)!} t^{n-1} x^{(n-1)}(t) \qquad \text{for all large } t$$

and

$$x'[t/2] \geq \frac{\theta}{(n-2)!} t^{n-2} x^{(n-1)}(t) \qquad \text{for all large } t.$$

LEMMA 2.3. [11]. *If X and Y are nonnegative numbers, then*

$$X^\lambda - \lambda X Y^{\lambda-1} + (\lambda - 1)Y^\lambda \geq 0, \qquad \lambda > 1$$

and

$$X^\lambda - \lambda X Y^{\lambda-1} - (1 - \lambda)Y^\lambda \leq 0, \qquad 0 < \lambda < 1.$$

In the above inequalities the equality holds if and only if $X = Y$.

THEOREM 2.1. *Let condition* (1.2) *hold with* $\alpha = \beta$. *If there exist* $\sigma(t), \rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$, *and a constant* $\theta > 1$ *such that*

$$\sigma(t) \leq \inf\{t, g(t)\}, \qquad \lim_{t\to\infty} \sigma(t) = \infty \qquad \text{and} \qquad \sigma'(t) > 0 \qquad \text{for } t \geq t_0 \tag{2.1}$$

and for $T \geq t_0$,

$$\limsup_{t\to\infty} \int_T^\infty \left[\rho(s)q(s) - \lambda\theta \frac{(\rho'(s))^{\alpha+1}}{(\rho(s)\sigma^{n-2}(s)\sigma'(s))^\alpha}\right] ds = \infty, \tag{2.2}$$

where $\lambda = (1/(\alpha+1))^{\alpha+1}(2(n-1)!)^\alpha$, *then Eq.* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that Eq. (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 \geq t_0 \geq 0$. Since

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{n-1}(t)\right)' = -F(t, x[g(t)]) \leq 0$$

it follows that the function $|x^{(n-1)}(t)|^{\alpha-1} x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. If $x^{(n-1)}(t) < 0$ eventually, then since

$$0 \geq \left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' = \alpha\left(-x^{(n-1)}(t)\right)^{\alpha-1} x^{(n)}(t),$$

we find that $x^{(n)}(t) \leq 0$ eventually. But then Lemma 2.1 implies that $x^{(n-1)}(t) > 0$ eventually. Further, when $x^{(n-1)}(t) > 0$ eventually then again from Lemma 2.1 (note n is even) we have $x'(t) > 0$ eventually. Thus there exists a $t_2 \geq t_1$ such that

$$x'(t) > 0 \qquad \text{and} \qquad x^{(n-1)}(t) > 0 \qquad \text{for } t \geq t_2. \tag{2.3}$$

Define

$$w(t) = \rho(t) \frac{\left(x^{(n-1)}(t)\right)^\alpha}{x^\beta[\sigma(t)/2]}, \qquad t \geq t_2.$$

Then, for $t \geq t_2$, in view of (1.2) we have

$$w'(t) \leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\beta\sigma'(t)}{2} \rho(t) \frac{\left(x^{(n-1)}(t)\right)^\alpha x'[\sigma(t)/2]}{x^{\beta+1}[\sigma(t)/2]}. \tag{2.4}$$

By Lemma 2.2 (notice since $x^{(n-1)}(t) > 0$ for $t \geq t_2$, we have $[(x^{(n-1)}(t))^\alpha]' \leq 0$ for $t \geq t_2$, which in turn implies $x^{(n)}(t) \leq 0$ for $t \geq t_2$), there exists a $t_3 \geq t_2$ and a constant $\theta_1, 0 < \theta_1 < 1$ such that

$$x'[\sigma(t)/2] \geq \frac{\theta_1}{(n-2)!}\sigma^{n-2}(t)x^{(n-1)}(t) \qquad \text{for } t \geq t_3, \tag{2.5}$$

since $x^{(n-1)}(\sigma(t)) \geq x^{(n-1)}(t)$ for $t \geq t_3$.

Using (2.5) in (2.4) with $\alpha = \beta$, we find

$$\begin{aligned} w'(t) \leq{}& -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) \\ & - \frac{\alpha\theta_1}{2(n-2)!}\sigma^{(n-2)}(t)\sigma'(t)\rho^{-1/\alpha}(t)w^{(\alpha+1)/\alpha}(t). \end{aligned}$$

Fix $t \geq t_3$, and set

$$X = \left(\frac{\alpha\theta_1}{2(n-2)!}\sigma^{n-2}(t)\sigma'(t)\right)^{\alpha/(\alpha+1)} \frac{w(t)}{\rho^{1/(\alpha+1)}(t)}, \qquad \lambda = (\alpha+1)/\alpha > 1$$

and

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^\alpha \left[\frac{\rho'(t)}{\rho(t)}\rho^{1/(\alpha+1)}(t)\left(\frac{\alpha\theta_1}{2(n-2)!}\sigma^{n-2}(t)\sigma'(t)\right)^{-\alpha/(\alpha+1)}\right]^\alpha.$$

Then, by Lemma 2.3, we obtain

$$\begin{aligned} &\frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\theta_1}{2(n-2)!}\sigma^{n-2}(t)\sigma'(t)\rho^{-1/\alpha}(t)w^{(\alpha+1)/\alpha}(t) \\ &\leq \left(\frac{1}{\alpha+1}\right)^{\alpha+1}\left[\rho(t)\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}\left(\frac{\theta_1}{2(n-2)!}\sigma^{n-2}(t)\sigma'(t)\right)^{-\alpha}\right], \qquad t \geq t_3. \end{aligned}$$

Now, inequality (2.4) reduces to

$$w'(t) \leq -\rho(t)\left[q(t) - \frac{\lambda\rho'(t)}{\rho(t)}\left(\frac{\rho'(t)}{\theta_1\rho(t)\sigma^{n-2}(t)\sigma'(t)}\right)^\alpha\right] \qquad \text{for } t \geq t_3.$$

Integrating the above inequality from t_3 to t, we get

$$\begin{aligned} 0 &< w(t) \\ &\leq w(t_3) - \int_{t_3}^{t}\left[\rho(s)q(s) - \lambda\rho'(s)\left(\frac{\rho'(s)}{\theta_1\rho(s)\sigma^{n-2}(s)\sigma'(s)}\right)^\alpha\right]ds. \end{aligned} \tag{2.6}$$

Taking lim sup on both sides of (2.6) as $t \to \infty$, we obtain a contradiction to condition (2.2). This completes the proof. ∎

We can apply Theorem 2.1 to the second order half-linear equation

$$\left(|x'(t)|^{\alpha-1}x'(t)\right)' + q(t)|x[g(t)]|^{\alpha-1}x[g(t)] = 0, \tag{2.7}$$

where $\alpha > 0$ is a constant, $q(t) \in C([t_0, \infty), \mathbb{R}^+)$, $g(t) \in C([t_0, \infty), \mathbb{R})$, and $\lim_{t\to\infty} g(t) = \infty$. In fact, we get the following new result.

COROLLARY 2.1. *If there exist two functions* $\rho(t), \sigma(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that condition* (2.1) *holds, and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(s))^{\alpha+1}}{(\rho(s)\sigma'(s))^{\alpha}}\right] ds = \infty, \tag{2.8}$$

then Eq. (2.7) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2.7), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$. It is easy to check that $x'(t) > 0$ and $x'[\sigma(t)] \geq x'(t)$ for $t \geq t_2 \geq t_1$. Next, we define

$$w(t) = \rho(t)\left(\frac{x'(t)}{x[\sigma(t)]}\right)^{\alpha}, \qquad t \geq t_2.$$

Then,

$$w'(t) \leq \rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} - \alpha\rho^{-1/\alpha}(t)w^{(\alpha+1)/\alpha}(t) \qquad \text{for } t \geq t_2.$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. ∎

The following example illustrates our theory.

EXAMPLE 2.1. Consider the second order half-linear differential equation

$$\left(|x'(t)|^{\alpha-1}x'(t)\right)' + \frac{1}{t^{\alpha+1}}|x(t)|^{\alpha-1}x(t) = 0, \qquad t > 0, \tag{2.9}$$

where $\alpha > 0$ is a constant. Here, we take $\rho(t) = t^{\alpha}$. Then,

$$\begin{aligned} &\int_{T}^{t} \left[\rho(s)q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\right] ds \\ &\quad = \int_{T}^{t} \left[1 - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\right] \frac{1}{s}\, ds \\ &\quad = \left[1 - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\right] \ln\frac{t}{T} \to \infty \qquad \text{as } t \to \infty. \end{aligned}$$

All conditions of Corollary 2.1 are satisfied and hence Eq. (2.9) is oscillatory. We note that the above conclusion do not appear to follow from the known oscillation criteria in the literature.

For each $t \geq t_0$, we let $g(t) \leq t$ and define $\gamma(t) = \sup\{s \geq t_0 : g(s) \leq t\}$. Clearly, $\gamma(t) \geq t$ and $g \circ \gamma(t) = t$. Our next result is embodied in the following:

THEOREM 2.2. *Let condition* (1.2) *hold with* $\alpha = \beta$. *If*

$$\limsup_{t\to\infty} t^{\alpha(n-1)} \int_{\gamma(t)}^{\infty} q(s)\, ds > ((n-1)!)^{\alpha}, \tag{2.10}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1.1), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in the proof of Theorem 2.1, we obtain (2.3) for $t \geq t_2$. Now integrating Eq. (1.1) from $t \geq t_2$ to u and letting $u \to \infty$, we get

$$\left(x^{(n-1)}(t)\right)^{\alpha} \geq \int_t^{\infty} q(s) x^{\alpha}[g(s)]\, ds.$$

By Lemma 2.2 there exist a constant θ, $0 < \theta < 1$ and $t_3 \geq t_2$ such that

$$x(t) \geq \frac{\theta}{(n-1)!} t^{n-1} x^{(n-1)}(t) \qquad \text{for } t \geq t_3. \tag{2.11}$$

Thus,

$$\begin{aligned} x^{\alpha}(t) &\geq \left(\frac{\theta}{(n-1)!} t^{n-1}\right)^{\alpha} (x^{(n-1)}(t))^{\alpha} \\ &\geq \left(\frac{\theta}{(n-1)!} t^{n-1}\right)^{\alpha} \int_t^{\infty} q(s) x^{\alpha}[g(s)]\, ds \qquad \text{for } t \geq t_3. \end{aligned}$$

Now by $\gamma(t) \geq t$ and the fact that $x'(t) > 0$ and $g(s) \geq t$ for $s \geq \gamma(t)$, it follows that

$$\begin{aligned} x^{\alpha}(t) &\geq \left(\frac{\theta}{(n-1)!} t^{n-1}\right)^{\alpha} \int_{\gamma(t)}^{\infty} q(s) x^{\alpha}[g(s)]\, ds \\ &\geq \left(\frac{\theta}{(n-1)!} t^{n-1}\right)^{\alpha} x^{\alpha}(t) \int_{\gamma(t)}^{\infty} q(s)\, ds. \end{aligned}$$

Dividing both sides of the above inequality by $x^{\alpha}(t)$, we get

$$\left(\frac{\theta}{(n-1)!} t^{n-1}\right)^{\alpha} \int_{\gamma(t)}^{\infty} q(s)\, ds \leq 1 \qquad \text{for } t \geq t_3. \tag{2.12}$$

Thus,

$$\limsup_{t\to\infty} \left(\frac{t^{n-1}}{(n-1)!}\right)^{\alpha} \int_{\gamma(t)}^{t} q(s)\, ds = c < \infty.$$

Suppose (2.10) holds. Then there exists a sequence $\{T_m\}_{m=1}^{\infty}$, with $T_m \to \infty$ as $m \to \infty$ such that

$$\lim_{m\to\infty} \left(\frac{T_m}{(n-1)!}\right)^{\alpha} \int_{\gamma(T_m)}^{\infty} q(s)\,ds = c > 1.$$

Thus, for $\epsilon = (c-1)/2 > 0$, there exists $N > 0$ such that

$$\frac{c+1}{2} = c - \epsilon < \left(\frac{T_m}{(n-1)!}\right)^{\alpha} \int_{\gamma(T_m)}^{\infty} q(s)\,ds \qquad \text{for } m > N. \tag{2.13}$$

Choose $K \in (2/(c+1)^{1/\alpha}, 1)$. From (2.12) and (2.13), we get

$$1 \geq K^{\alpha}\left(\frac{T_\lambda}{(n-1)!}\right)^{\alpha} \int_{\gamma(T_\lambda)}^{\infty} q(s)\,ds > \frac{2}{c+1}\frac{c+1}{2} = 1$$

for T_λ sufficiently large. This contradiction proves that condition (2.10) is not satisfied. This completes the proof. ∎

In Theorem 2.2 if $g(t) \geq t$, i.e., $g(t)$ is an advanced argument, and $g'(t) \geq 0$ for $t \geq t_0$, we find that Theorem 2.2 takes the following form.

THEOREM 2.3. *Let condition* (1.2) *hold with* $\alpha = \beta, g(t) \geq t$, *and* $g'(t) \geq 0$ *for* $t \geq t_0$. *If.*

$$\limsup_{t\to\infty} t^{\alpha(n-1)} \int_t^{\infty} q(s)\,ds > ((n-1)!)^{\alpha}, \tag{2.14}$$

then Eq. (1.1) *is oscillatory.*

EXAMPLE 2.2. Consider the half-linear differential equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + ct^{-\alpha(n-1)-1}|x[g(t)]|^{\alpha-1}x[g(t)] \tag{2.15}$$
$$= 0, \qquad t > 0,$$

where α and c are positive constants, $g(t) \in C([t_0, \infty), \mathbb{R})$, and $\lim_{t\to\infty} g(t) = \infty$. We conclude the following:

(i) If $g(t) = t/2$, then $\gamma(t) = 2t$, and hence Eq. (2.15) is oscillatory by Theorem 2.2 provided that

$$c > 2^{-\alpha(n-1)}[\alpha(n-1)((n-1)!)^{\alpha}].$$

(ii) If $g(t) \geq t$ and $g'(t) \geq 0$, then Eq. (2.15) is oscillatory by Theorem 2.3 provided that

$$c > \alpha(n-1)((n-1)!)^{\alpha}.$$

Next, we have the following comparison result.

THEOREM 2.4. *Let condition* (1.2) *hold and assume that there exist a function* $\sigma(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ *and a constant* $\theta, 0 < \theta < 1$ *such that*

$$\sigma(t) \leq \inf\{t, g(t)\}, \qquad \lim_{t\to\infty} \sigma(t) = \infty \qquad \text{and} \tag{2.16}$$
$$\sigma'(t) \geq 0 \qquad \text{for } t \geq t_0.$$

If every solution of the delay equation

$$y'(t) + \left(\frac{\theta}{(n-1)!}\right)^{\alpha} \sigma^{\alpha(n-1)}(t)|y[\sigma(t)]|^{\beta/\alpha}\operatorname{sgn} y[\sigma(t)] = 0 \tag{2.17}$$

is oscillatory, then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in the proof of Theorem 2.1, we see that $x^{(n-1)}(t) > 0$ for $t \geq t_2 \geq t_1$. By Lemma 2.2 there exist a constant θ, $0 < \theta < 1$ and $t_3 \geq t_2$ such that

$$x[\sigma(t)] \geq \frac{\theta}{(n-1)!}\sigma^{n-1}(t)x^{(n-1)}[\sigma(t)] \qquad \text{for } t \geq t_3. \tag{2.18}$$

Using (2.18) in Eq. (1.1), for $t \geq t_3$ we obtain

$$\begin{aligned}\left(\left(x^{(n-1)}(t)\right)^{\alpha}\right)' &+ \left(\frac{\theta}{(n-1)!}\sigma^{n-1}(t)\right)^{\beta} q(t)\left(x^{(n-1)}[\sigma(t)]\right)^{\beta} \\ &\leq \left(\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\beta}[\sigma(t)] \leq 0.\end{aligned} \tag{2.19}$$

Let $y(t) = \left(x^{(n-1)}(t)\right)^{\alpha}$, $t \geq t_3$, to get

$$y'(t) + \left(\frac{\theta}{(n-1)!}\sigma^{n-1}(t)\right)^{\beta} q(t)\left(y^{\beta/\alpha}[\sigma(t)]\right) \leq 0 \qquad \text{for } t \geq t_3. \tag{2.20}$$

Integrating inequality (2.20) from $t \geq t_3$ to u and letting $u \to \infty$, we find

$$y(t) \geq \int_t^{\infty} \left(\frac{\theta}{(n-1)!}\sigma^{n-1}(s)\right)^{\beta} q(s)y^{\beta/\alpha}[\sigma(s)]\,ds \qquad \text{for } t \geq t_3.$$

The function $y(t)$ is obviously decreasing on $[t_3, \infty)$. Hence, by Theorem 1 in [17], we conclude that there exists a positive solution $y(t)$ of Eq. (2.17) with $\lim_{t\to\infty} y(t) = 0$, which contradicts the fact that Eq. (2.17) is oscillatory. This completes the proof. ∎

We can apply the results established in [14] to obtain the following corollary.

COROLLARY 2.2. *Let conditions* (1.2) *and* (2.16) *hold. If*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\sigma^{\alpha(n-1)}(s)q(s)\,ds > \frac{((n-1)!)^{\alpha}}{e} \qquad \text{when } \alpha=\beta \quad (2.21)$$

or

$$\int^{\infty}\sigma^{\beta(n-1)}(s)q(s)\,ds=\infty \qquad \text{when } 0<\beta/\alpha<1, \qquad (2.22)$$

then equation (1.1) *is oscillatory.*

THEOREM 2.5. *Let condition* (1.2) *hold with* $\alpha>1$ *and* $\beta>1$, *and assume that there exist two functions* $\sigma(t),\rho(t)\in C^1([t_0,\infty),\mathbb{R}^+)$ *such that condition* (2.1) *is satisfied, and*

$$\rho'(t)\ge 0 \qquad \text{and} \qquad \left(\frac{\rho'(t)}{\sigma^{n-2}(t)\sigma'(t)}\right)'\le 0 \qquad \text{for } t\ge t_0. \qquad (2.23)$$

If

$$\int^{\infty}\rho(s)q(s)\,ds=\infty, \qquad (2.24)$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say, $x(t)>0$ for $t\ge t_1\ge t_0$. As in the proof of Theorem 2.1 we obtain (2.3) for $t\ge t_2$. Next, we define $w(t)$ as in the proof of Theorem 2.1 to obtain (2.4) which takes the form

$$w'(t)\le -\rho(t)q(t)+\rho'(t)\frac{\left(x^{(n-1)}(t)\right)^{\alpha}}{x^{\beta}[\sigma(t)/2]} \qquad \text{for } t\ge t_2. \qquad (2.25)$$

Since $x^{(n-1)}(t)$ is nonincreasing on $[t_2,\infty)$, there exist a $t_3\ge t_2$ and positive constants b and θ_1, $0<\theta_1<1$ such that $(x^{(n-1)}(t))^{\alpha-1}\le b$ for $t\ge t_3$, and (2.5) holds for $t\ge t_3$. Now (2.25) takes the form

$$w'(t)\le -\rho(t)q(t)+b\frac{(n-2)!}{\theta_1}\frac{\rho'(t)}{\sigma^{n-2}(t)}\frac{x'[\sigma(t)/2]}{x^{\beta}[\sigma(t)/2]}, \qquad t\ge t_3. \quad (2.26)$$

But, by the Bonnet theorem for a fixed $t\ge t_3$ and for some $\xi\in[t_3,t]$, we have

$$\begin{aligned}\int_{t_3}^{t}&\frac{\rho'(s)}{\sigma^{n-2}(s)\sigma'(s)}\frac{x'[\sigma(s)/2]\sigma'(s)/2}{x^{\beta}[\sigma(s)/2]}\,ds\\&=\left(\frac{\rho'(t_3)}{\sigma^{n-2}(t_3)\sigma'(t_3)}\right)\int_{t_3}^{\xi}\frac{x'[\sigma(s)/2]\sigma'(s)/2}{x^{\beta}[\sigma(s)/2]}\,ds\\&=\left(\frac{\rho'(t_3)}{\sigma^{n-2}(t_3)\sigma'(t_3)}\right)\int_{x[\sigma(t_3)/2]}^{x[\sigma(\xi)/2]}w^{-\beta}\,dw\end{aligned}$$

and hence, since $\rho'(t_3) \geq 0$ and

$$\int_{x[\sigma(t_3)/2]}^{x[\sigma(\xi)/2]} \frac{dw}{w^\beta} = \frac{1}{\beta-1}\left(x^{1-\beta}[\sigma(t_3)/2] - x^{1-\beta}[\sigma(\xi)/2]\right) < \frac{1}{\beta-1} x^{1-\beta}[\sigma(t_3)/2],$$

we find

$$\int_{t_3}^{t} \frac{\rho'(s)}{\sigma^{n-2}(s)\sigma'(s)} \frac{x'[\sigma(s)/2]\sigma'(s)/2}{x^\beta[\sigma(s)/2]}\, ds \leq K \qquad \text{for } t \geq t_3, \tag{2.27}$$

where

$$K = \frac{\rho'(t_3)}{\sigma^{n-2}(t_3)\sigma'(t_3)} \frac{1}{\beta-1} x^{1-\beta}[\sigma(t_3)/2].$$

Now in view of (2.27) it follows that

$$\int_{t_3}^{t} \rho(s)q(s)\, ds \leq -w(t) + w(t_3) + K < \infty.$$

This contradicts (2.24) and so the proof is complete. ∎

THEOREM 2.6. *Let condition* (2.23) *in Theorem* 2.5 *be replaced by*

$$\rho'(t) \geq 0 \qquad \textit{and} \qquad \int_{t_0}^{\infty} \left| \left(\frac{\rho'(s)}{\sigma^{n-2}(s)\sigma'(s)} \right)' \right| ds < \infty;$$

then the conclusion of Theorem 2.5 *holds*.

Proof. The proof is similar to that of Theorem 2.5 and hence is omitted. ∎

THEOREM 2.7. *Let condition* (1.2) *hold with* $\beta > \alpha$ *and assume that there exists* $\sigma(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that condition* (2.1) *is satisfied. If*

$$\int^{\infty} \sigma^{n-2}(s)\sigma'(s)\left(\int_s^{\infty} q(u)du\right)^{1/\alpha} ds = \infty, \tag{2.28}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in the proof of Theorem 2.5 we take $\rho(t) = 1$ and obtain

$$\int_{t_2}^{\infty} q(s)\, ds \leq \frac{\left(x^{(n-1)}(t_2)\right)^\alpha}{x^\beta[\sigma(t_2)/2]} < \infty,$$

and therefore for $t \geq t_2$,

$$\int_t^\infty q(s)\,ds \leq \frac{\left(x^{(n-1)}(t)\right)^\alpha}{x^\beta[\sigma(t)/2]} \qquad \text{or}$$

$$\left(\int_t^\infty q(s)\,ds\right)^{1/\alpha} \leq \frac{x^{(n-1)}(t)}{x^{\beta/\alpha}[\sigma(t)/2]}.$$

Now by Lemma 2.2 there exist a $t_3 \geq t_2$ and a constant θ_1, $0 < \theta_1 < 1$ such that (2.5) holds for $t \geq t_3$. Thus, for $t \geq t_3$,

$$\begin{aligned}
&\left(\frac{\theta_1}{2(n-2)!}\sigma^{n-2}(t)\sigma'(t)\right)\left(\int_t^\infty q(s)\,ds\right)^{1/\alpha} \\
&\quad \leq \frac{\theta_1}{2(n-2)!}\sigma^{n-2}(t)\sigma'(t)\frac{x^{(n-1)}(t)}{x^{\beta/\alpha}[\sigma(t)/2]} \\
&\quad \leq \frac{x'[\sigma(t)/2]\sigma'(t)/2}{x^{\beta/\alpha}[\sigma(t)/2]}.
\end{aligned}$$

Integrating the above inequality from t_3 to t, we get

$$\begin{aligned}
\frac{\theta_1}{2(n-2)!}\int_{t_3}^t \sigma^{n-2}(s)\sigma'(s)\left(\int_s^\infty q(u)du\right)^{1/\alpha} ds &\leq \int_{t_3}^t \frac{x'[\sigma(s)/2]\sigma'(s)/2}{x^{\beta/\alpha}[\sigma(s)/2]}\,ds \\
&= \int_{x[\sigma(t_3)/2]}^{x[\sigma(t)/2]} w^{-\beta/\alpha}\,dw \\
&\leq \frac{\alpha}{\beta-\alpha}x^{(\alpha-\beta)/\alpha}[\sigma(t_3)/2] < \infty,
\end{aligned}$$

which contradicts condition (2.28). This completes the proof. ∎

EXAMPLE 2.3. The equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t)\right)' + t^{-(n-1)\alpha-1}|x[\gamma t]|^\beta \operatorname{sgn} x[g(t)] = 0, \qquad t \geq t_0 > 0,$$

which α, β, and γ are positive constants, $\beta > \alpha$, and $\gamma \leq 1$, is oscillatory by Theorem 2.7.

3. SOME EXTENSIONS

Here we shall extend our results of Section 2 to Eqs. (1.3) and (1.4). For Eq. (1.3) when the function f need not be monotonic we need the following notations and a lemma due to Mahfoud [16],

$$\mathbb{R}_{t_0} = \begin{cases} (-\infty, -t_0] \cup [t_0, \infty) & \text{if } t_0 > 0 \\ (-\infty, 0) \cup (0, \infty) & \text{if } t_0 = 0, \end{cases}$$

$$C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ is continuous and } xf(x) > 0 \text{ for } x \neq 0\}$$

and

$$C_B(\mathbb{R}_{t_0}) = \big\{f \in C(\mathbb{R}) : f \text{ is of bounded variation on any interval } [a, b] \subset \mathbb{R}_{t_0}\big\}.$$

LEMMA 3.1. *Suppose* $t_0 > 0$ *and* $f \in C(\mathbb{R})$. *Then,* $f \in C_B(\mathbb{R}_{t_0})$ *if and only if* $f(x) = H(x)G(x)$ *for all* $x \in \mathbb{R}_{t_0}$, *where* G: $\mathbb{R}_{t_0} \to \mathbb{R}^+ = (0, \infty)$ *is nondecreasing on* $(-\infty, -t_0)$ *and nonincreasing on* (t_0, ∞), *and* H: $\mathbb{R}_{t_0} \to \mathbb{R}$ *is nondecreasing on* $\mathbb{R}_{t_0}$.

To obtain an extension, we assume that $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, and let G and H be a pair of continuous components of f with H being the nondecreasing one. Also, we assume that

$$H(x) \operatorname{sgn} x \geq |x|^\beta \qquad \text{for } x \neq 0 \qquad \text{and} \qquad \beta > 0 \text{ is a constant.} \tag{3.1}$$

As in Section 2, if $x(t)$ is a nonoscillatory solution of Eq. (1.3), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$, then there exists a $t_2 \geq t_1$ such that (2.3) holds for all $t \geq t_2$. Next, there exist a $t_3 \geq t_2$ and a constant $b > 0$ such that

$$x^{(n-1)}(t) \leq b \qquad \text{for } t \geq t_3. \tag{3.2}$$

Integrating (3.2) $(n-1)$ times, there exist a $t_4 \geq t_3$ and a positive constant $K > 0$ such that

$$x[g(t)] \leq Kg^{n-1}(t) \qquad \text{for } t \geq t_4. \tag{3.3}$$

Now it follows from Eq. (1.3) that

$$\begin{aligned} 0 &= \frac{d}{dt}\big(x^{(n-1)}(t)\big)^\alpha + q(t)G(x[g(t)])H(x[g(t)]) \\ &\geq \frac{d}{dt}\big(\big(x^{(n-1)}(t)\big)^\alpha\big) + q(t)G(x[g(t)])x^\beta[g(t)] \\ &\geq \frac{d}{dt}\big(\big(x^{(n-1)}(t)\big)^\alpha\big) + q(t)G(Kg^{n-1}(t))x^\beta[\sigma(t)] \qquad \text{for } t \geq t_4, \end{aligned} \tag{3.4}$$

where

$$\sigma \in C^1([t_0, \infty), \mathbb{R}^+), \qquad \sigma(t) \leq \inf\{t, g(t)\} \to \infty \quad \text{as } t \to \infty, \qquad \sigma'(t) \geq 0 \qquad \text{for } t \geq t_0. \tag{3.5}$$

Integrating the above inequality from t to u ($t_4 \leq t \leq u$) and letting $u \to \infty$, we obtain

$$x^{(n-1)}(t) \geq \left(\int_t^\infty q(s)G(K[g^{n-1}(s)])x^\beta[\sigma(s)]\,ds\right)^{1/\alpha}.$$

Following similar steps as in the proof of Lemma 2.1 in [13], we find that if inequality (3.4) has an eventually positive solution, then so does the equation

$$\frac{d}{dt}\left(y^{(n-1)}(t)\right)^{\alpha} + q(t)G(Kg^{n-1}(t))y^{\beta}[\sigma(t)] = 0. \tag{3.6}$$

Thus, to extend the results of Section 2, we shall need to apply the following theorem.

THEOREM 3.1. *Assume that* $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one. Moreover, assume that conditions* (3.1) *and* (3.5) *hold. If, for every* $K > 0$, *the equation*

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + q(t)G(Kg^{n-1}(t))|x[\sigma(t)]|^{\beta-1}x[\sigma(t)] = 0$$

is oscillatory, then Eq. (1.3) *is also oscillatory.*

We note that Theorem 3.1 together with the results of Section 2 can be applied to equations of type (1.3) with f being any of the following functions:

(i) $f(x) = |x|^{\beta-1}x/(1+|x|^{\gamma})$, β, γ are positive constants,
(ii) $f(x) = |x|^{\beta-1}x \exp(-|x|^{\gamma})$, β, γ are positive constants,
(iii) $f(x) = |x|^{\beta-1}x \operatorname{sech} x$, β is a positive constant.

However, the results of Section 2 are not applicable to Eq. (1.3) with any one of the above choices of f.

Next, we shall extend the results of Section 2 to neutral equations of type (1.4). In fact, if we define $z(t) = x(t) + p(t)x[\tau(t)]$, then Eq. (1.4) becomes

$$\left(|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\right)' + F(t, x[g(t)]) = 0. \tag{3.7}$$

Now if $x(t)$ is a nonoscillatory solution of Eq. (1.4), say, $x(t) > 0$ and $x[\tau(t)] > 0$ for $t \geq t_1 \geq t_0$. Then, $z(t) > 0$ for $t \geq t_1$ and there exists a $t_2 \geq t_1$ such that $z^{(n-1)}(t) > 0$ and $z'(t) > 0$ for $t \geq t_2$. In what follows we shall examine the following two cases for $\tau(t)$ and $p(t)$:

(i) $\{0 \leq p(t) \leq 1, \tau(t) < t\}$ and
(ii) $\{p(t) \geq 1, \tau(t) > t\}$.

For case (i), we assume that

$$\begin{aligned} &0 \leq p(t) \leq 1,\ \tau(t) < t \text{ and } \tau(t) \text{ is strictly} \\ &\text{increasing for } t \geq t_0 \text{ and } p(t) \not\equiv 1 \text{ eventually.} \end{aligned} \tag{3.8}$$

Now,

$$\begin{aligned} x(t) &= z(t) - p(t)x[\tau(t)] \\ &= z(t) - p(t)[z[\tau(t)] - p[\tau(t)]x[\tau\circ\tau(t)] \\ &\geq z(t) - p(t)z[\tau(t)]] \geq (1-p(t))z(t) \qquad \text{for } t \geq t_2. \end{aligned} \tag{3.9}$$

Using conditions (1.2) and (3.9) in Eq. (3.7), we get

$$\frac{d}{dt}\left(z^{(n-1)}(t)\right)^{\alpha} + q(t)(1-p[g(t)])^{\beta} z^{\beta}[g(t)] \leq 0 \qquad \text{for } t \geq t_3 \geq t_2.$$

Now if (3.5) holds, then

$$\frac{d}{dt}(z^{(n-1)}(t))^{\alpha} + q(t)(1-p[g(t)])^{\beta} z^{\beta}[\sigma(t)] \leq 0 \qquad \text{for } t \geq t_3. \tag{3.10}$$

As in the above discussion, we conclude that if inequality (3.10) has an eventually positive solution, then so does the equation

$$\frac{d}{dt}(y^{(n-1)}(t))^{\alpha} + q(t)(1-p[g(t)])^{\beta} y^{\beta}[\sigma(t)] = 0. \tag{3.11}$$

Thus, we have the following result:

THEOREM 3.2. *Let conditions* (1.2), (3.5), *and* (3.8) *hold. If the equation*

$$\left(|y^{(n-1)}(t)|^{\alpha-1}y^{(n-1)}(t)\right)' + q(t)(1-p[g(t)])^{\beta}|y[\sigma(t)]|^{\beta-1}y[\sigma(t)] = 0$$

is oscillatory, then Eq. (1.4) *is also oscillatory.*

For case (ii), we assume that

$$p(t) \geq 1,\ p(t) \not\equiv 1 \text{ eventually}, \qquad \tau(t) > t \text{ and } \tau(t) \text{ is strictly increasing for } t \geq t_0, \tag{3.12}$$

and there exists $\sigma^*(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ such that

$$\sigma^*(t) \leq \inf\{t, \tau^{-1} o g(t)\} \to \infty \qquad \text{as } t\to\infty \qquad \text{and} \qquad (\sigma^*(t))' \geq 0 \qquad \text{for } t \geq t_0, \tag{3.13}$$

where τ^{-1} is the inverse function of τ. We also let

$$P^*(t) = \frac{1}{p[\tau^{-1}(t)]}\left(1 - \frac{1}{p[\tau^{-1}\circ\tau^{-1}(t)]}\right) \qquad \text{for all large } t.$$

Now, since $z'(t) > 0$ for $t \geq t_2$, we obtain

$$\begin{aligned} x(t) &= \frac{1}{p[\tau^{-1}(t)]}\left(z[\tau^{-1}(t)] - x[\tau^{-1}(t)]\right) \\ &= \frac{z[\tau^{-1}(t)]}{p[\tau^{-1}(t)]} - \frac{1}{p[\tau^{-1}(t)]}\left(\frac{\tau[\tau^{-1}\circ\tau^{-1}(t)]}{p[\tau^{-1}\circ\tau^{-1}(t)]} - \frac{x[\tau^{-1}\circ\tau^{-1}(t)]}{p[\tau^{-1}\circ\tau^{-1}(t)]}\right) \\ &\geq \frac{z[\tau^{-1}(t)]}{p[\tau^{-1}(t)]} - \frac{z[\tau^{-1}\circ\tau^{-1}(t)]}{p[\tau^{-1}(t)]p[\tau^{-1}\circ\tau^{-1}(t)]} \\ &\geq \frac{1}{p[\tau^{-1}(t)]}\left[1 - \frac{1}{p[\tau^{-1}\circ\tau^{-1}(t)]}\right]z[\tau^{-1}(t)] \\ &= P^*(t)z[\tau^{-1}(t)] \qquad \text{for } t \geq t_2. \end{aligned} \tag{3.14}$$

Using (1.2), (3.13), and (3.14) in Eq. (3.7), we have

$$\begin{aligned} 0 &\geq \frac{d}{dt}\left(z^{(n-1)}(t)\right)^{\alpha} + q(t)(P^*[g(t)])^{\beta} z^{\beta}[\tau^{-1}\circ g(t)] \\ &\geq \frac{d}{dt}\left(z^{(n-1)}(t)\right)^{\alpha} + q(t)(P^*[g(t)])^{\beta} z^{\beta}[\sigma^*(t)] \qquad \text{for } t \geq t_3 \geq t_2. \end{aligned}$$

Thus, similar to Theorem 3.2 we have the following result:

THEOREM 3.3. *Let conditions* (1.2), (3.12), *and* (3.13) *hold. If the equation*

$$\left(\left|y^{(n-1)}(t)\right|^{\alpha-1} y^{(n-1)}(t)\right)' + q(t)(P^*[g(t)])^{\beta}|y[\sigma^*(t)]|^{\beta-1}y[\sigma^*(t)] = 0$$

is oscillatory, then Eq. (1.4) *is also oscillatory.*

Remark 3.1. Further extensions to equations of the form

$$\begin{aligned} &\left(\left|[x(t) + p(t)x[\tau(t)]]^{(n-1)}\right|^{\alpha-1}(x(t) + p(t)x[\tau(t)_0])^{(n-1)}\right)' + q(t)f(x[g(t)]) \\ &\quad = 0, \end{aligned}$$

where f need not be monotonic, can be obtained easily by Theorems 3.1–3.3.

EXAMPLE 3.1. For the equation

$$\begin{aligned} &\frac{d}{dt}\left(\left|[x(t) + px[\gamma t]]^{(n-1)}\right|^{\alpha-1}[x(t) + px[\gamma t]]\right) + t^{-(n-1)\alpha-1}|x[\lambda t]|^{\beta} \operatorname{sgn} x[\lambda t] \\ &\quad = 0 \qquad \text{for } t \geq t_0 > 0, \end{aligned} \tag{3.15}$$

where p, α, β, γ, and λ are positive constants, and $\beta > \alpha$, we conclude the following:

(i) If $p < 1, \gamma < 1$, and $\lambda \leq 1$, then Eq. (3.15) is oscillatory by Theorems 3.1 and 2.7.

(ii) If $p > 1, \gamma > 1$, and $\lambda \leq \gamma$, then Eq. (3.15) is oscillatory by Theorems 3.2 and 2.7.

4. FURTHER OSCILLATION CRITERIA

Our first oscillatory criterion for the equation (1.5) is embodied in the following theorem.

THEOREM 4.1. *Let condition* (1.6) *hold, and* $h(t) \le g(t) \le t$ *for* $t \ge t_0$. *If for every* $\theta_i, 0 < \theta_i < 1, i = 1, 2$ *the equations*

$$y'(t) + \left[\frac{\theta_1}{(n-1)^{\beta}((n-2)!)^{\lambda}}\right](h(t))^{(n-2)\lambda} H(t)q(t)|y[h(t)]|^{\lambda/\alpha} \operatorname{sgn} y[h(t)] = 0 \tag{4.1}$$

and

$$z'(t) + \left[\frac{\theta_2}{(2^{n-2}(n-2)!)^{\lambda}}\right](t-h(t))^{(n-2)\lambda} H(t)q(t)\left|z\left[\frac{t+h(t)}{2}\right]\right|^{\lambda/\alpha} \times \operatorname{sgn} z\left[\frac{t+h(t)}{2}\right] = 0, \tag{4.2}$$

where $\lambda = \beta + \mu \le \alpha$ *and* $H(t) = (h(t))^{\beta}(h'(t))^{\lambda}$ *are oscillatory, then Eq.* (1.5) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.5), say, $x(t) > 0$ for $t \ge t_1 \ge t_0$. It is easy to check that there exists a $t_2 \ge t_1$ such that $x^{(n-1)}(t) > 0$ and $x'(t) > 0$ for $t \ge t_2$. We distinguish the following two cases:

(I) $x^{(n-1)}(t) > 0, \ldots, x''(t) > 0$ and $x'(t) > 0$ for $t \ge t_2$, and

(II) $x^{(n-1)}(t) > 0, \ldots, x''(t) < 0$ and $x'(t) > 0$ for $t \ge t_2$.

Assume (I) holds. By Lemma 2.2 there exist a $t_3 \ge t_2$ and $b_i > 0, 0 < b_i < 1, i = 1, 2$ such that, for $t \ge t_3$,

$$x[g(t)] \ge x[h(t)] \ge \frac{b_1}{(n-1)!} h^{n-1}(t) x^{(n-1)}[h(t)] \tag{4.3}$$

and

$$\frac{d}{dt} x[h(t)] = x'[h(t)]h'(t) \ge \frac{b_2}{(n-2)!} h^{n-2}(t)h'(t)x^{(n-1)}[h(t)]. \tag{4.4}$$

Using conditions (1.6), (4.3), and (4.4) in Eq. (1.5), we get

$$\frac{d}{dt}\left(x^{(n-1)}(t)\right)^{\alpha} + \left(\frac{b_1}{(n-1)!}\right)^{\beta}\left(\frac{b_2}{(n-2)!}\right)^{\mu} h^{(n-2)\lambda}(t)H(t)q(t) \times \left(x^{(n-1)}[h(t)]\right)^{\lambda} \le 0 \qquad \text{for } t \ge t_3.$$

Setting $w(t) = (x^{(n-1)}(t))^{\alpha}$, $t \geq t_3$ we have

$$w'(t) + \frac{\theta_1^{\beta}\theta_2^{\mu}}{(n-1)^{\beta}((n-2)!)^{\lambda}} h^{(n-2)\lambda}(t)H(t)q(t)w^{\lambda/\alpha}[h(t)] \leq 0 \qquad \text{for } t \geq t_3. \tag{4.5}$$

Integrating (4.5) from $t \geq t_3$ to u and letting $u \to \infty$, we find

$$w(t) \geq \left[\frac{\theta_1^{\beta}\theta_2^{\mu}}{(n-1)^{\beta}((n-2)!)^{\lambda}}\right] \int_t^{\infty} h^{(n-2)\lambda}(s)H(s)q(s)w^{\lambda/\alpha}[h(s)]\,ds.$$

The function $w(t) = (x^{(n-1)}(t))^{\alpha}$ is clearly strictly decreasing for $t \geq t_3$. Hence, by Theorem 1 in [17], there exists a positive solution $y(t)$ of Eq. (1.5) with $y(t) \to 0$ as $t \to \infty$. But this contradicts the assumption that Eq. (4.1) is oscillatory.

Assume (II) holds. By Lemma 2.2 there exists a $T_1 \geq t_1$ and a constant a, $0 < a < 1$ such that

$$x[g(t)] \geq x[h(t)] \geq ah(t)x'[h(t)] \qquad \text{for } t \geq T_1. \tag{4.6}$$

Using conditions (1.6) and (4.6) in Eq. (1.5) and setting $v(t) = x'(t)$ for $t \geq T_1$, we obtain

$$\frac{d}{dt}\left(v^{(n-2)}(t)\right)^{\alpha} + a^{\beta}H(t)q(t)v^{\lambda}[h(t)] \leq 0 \qquad \text{for } t \geq T_1. \tag{4.7}$$

It is clear that function $v(t)$ satisfies

$$(-1)^i v^{(i)}(t) > 0, \qquad i = 0, 1, \ldots, n-2 \qquad \text{and} \qquad t \geq T_1. \tag{4.8}$$

Now by Lemma 2.2.4 in [2], there exists a $T \geq T_1$ such that

$$v[h(t)] \geq \left[\frac{(t-h(t))^{n-2}}{2^{n-2}(n-2)!}\right] v^{(n-2)}\left[\frac{t+h(t)}{2}\right] \qquad \text{for } T \leq h(t) \leq \frac{t+h(t)}{2}. \tag{4.9}$$

Thus, (4.7) takes the form

$$w'(t) + \frac{a^{\beta}}{[2^{n-2}(n-2)!]^{\lambda}}[t-h(t)]^{(n-2)\lambda}H(t)q(t)w^{\lambda/\alpha}[h(t)] \leq 0, \qquad t \geq T, \tag{4.10}$$

where $w(t) = \left(v^{(n-2)}(t)\right)^{\alpha}$, $t \geq T$. The rest of the proof is similar to that of case (I) and hence is omitted. ∎

Now applying the results established in [14] to Theorem 4.1, we obtain

COROLLARY 4.1. *Let condition* (1.6) *hold, and let* $h(t) \leq g(t) \leq t$ *for* $t \geq t_0$. *If*

(I_1)

$$\liminf_{t\to\infty} \int_{h(t)}^{t} h^{(n-2)\alpha}(s)H(s)q(s)\,ds > \frac{(n-1)^{\beta}((n-2)!)^{\alpha}}{e}$$

and

$$\liminf_{t\to\infty} \int_{(t+h(t))/2}^{t} [s-h(s)]^{(n-2)\alpha}H(s)q(s)\,ds > \frac{(2^{n-2}(n-2)!)^{\alpha}}{e}$$

are satisfied when $\lambda = \beta + \mu = \alpha$,

(I_2)

$$\int^{\infty} h^{(n-2)\lambda}(s)H(s)q(s)\,ds = \infty$$

and

$$\int^{\infty} [s-h(s)]^{(n-2)\lambda}H(s)q(s)\,ds = \infty$$

hold when $\lambda < \alpha$, *then Eq.* (1.5) *is oscillatory.*

Next we shall provide sufficient conditions for the oscillation of Eq. (1.5) when $\beta \leq \alpha$ and $\mu \leq \alpha$.

THEOREM 4.2. *Let condition* (1.6) *hold, and let* $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$. *If for every positive constant* θ_i, $i = 1, 2$ *the equations*

$$y'(t) + \left[\frac{\theta_1}{((n-1)!)^{\beta}}\right](h'(t))^{\mu} g^{(n-1)\beta}(t)q(t)|y[g(t)]|^{\beta/\alpha} \times \operatorname{sgn} y[g(t)] = 0 \tag{4.11}$$

and

$$z'(t) + \frac{\theta_2}{(2^{n-2}(n-2)!)^{\mu}}(t-h(t))^{(n-2)\mu}(h'(t))^{\mu}q(t)\left|z\left[\frac{t+h(t)}{2}\right]\right|^{\mu/\alpha} \times \operatorname{sgn} z\left[\frac{t+h(t)}{2}\right] = 0 \tag{4.12}$$

are oscillatory, then Eq. (1.5) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.5), say $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in Theorem 4.1, we have cases (I) and (II) for $t \geq t_2$.

Assume (I) holds. Then there exist a $t_3 \geq t_2$ and positive constants a and b such that

$$\frac{d}{dt}x[h(t)] \geq ah'(t) \qquad \text{for } t \geq t_3 \tag{4.13}$$

and

$$x[g(t)] \geq \frac{b}{(n-1)!}g^{n-1}(t)x^{(n-1)}[g(t)] \qquad \text{for } t \geq t_3. \tag{4.14}$$

Using conditions (1.6), (4.13), and (4.14) in Eq. (1.5), we obtain

$$w'(t) + \left[\frac{a^{\mu}b^{\beta}}{((n-1)!)^{\beta}}\right](h'(t))^{\mu}g^{(n-1)\beta}(t)q(t)w^{\beta/\alpha}[g(t)] \leq 0 \qquad \text{for } t \geq t_3,$$

where $w(t) = (x^{(n-1)}(t))^{\alpha}$, $t \geq t_3$. Now proceeding as in the proof of Theorem 4.1(I), we arrive at the desired contradiction.

Assume (II) holds. Then there exist a $T \geq t_1$ and a positive constant a_1 such that (4.9) holds, and

$$x[g(t)] \geq a_1 \qquad \text{for } t \geq T. \tag{4.15}$$

Thus (4.10) takes the form

$$w'(t) + \left[\frac{a_1^{\beta}}{(2^{n-2}(n-2)!)^{\mu}}\right](t-h(t))^{(n-2)\mu}(h'(t))^{\mu}q(t)w^{\mu/\alpha}\left[\frac{t+h(t)}{2}\right] \leq 0 \qquad \text{for } t \geq T.$$

The rest of the proof is similar to that of Theorem 4.1(II) and hence is omitted. ∎

The following result provides sufficient conditions for the oscillation of Eq. (1.5) when β and μ are arbitrary positive constants.

THEOREM 4.3. *Let condition* (1.6) *hold, and let* $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$. *If for every positive constant* θ_1, θ_2 *the equation*

$$\left(\left|y^{(n-1)}(t)\right|^{\alpha-1}y^{(n-1)}(t)\right)' + \theta_1(h'(t))^{\mu}q(t)|y[g(t)]|^{\beta}\operatorname{sgn} y[g(t)] = 0 \tag{4.16}$$

is oscillatory, and every bounded solution of the equation

$$\left(\left|z^{(n-2)}(t)\right|^{\alpha-1}z^{(n-2)}(t)\right)' + \theta_2(h'(t))^{\mu}q(t)|z[h(t)]|^{\mu}\operatorname{sgn} z[h(t)] = 0 \tag{4.17}$$

is oscillatory, then Eq. (1.5) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.5), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in the proof of Theorem 4.1 we consider cases (I) and (II).

In case (I) inequality (4.13) holds for $t \geq t_3$. Thus, Eq. (1.5) leads to

$$\frac{d}{dt}\left(x^{(n-1)}(t)\right)^{\alpha} + a^{\mu}(h'(t))^{\mu} q(t) x^{\beta}[g(t)] \leq 0 \qquad \text{for } t \geq t_3.$$

Using an argument presented in Section 3, we find that the equation

$$\frac{d}{dt}\left(x^{(n-1)}(t)\right)^{\alpha} + a^{\mu}(h'(t))^{\mu} q(t) x^{\beta}[g(t)] = 0$$

has a positive solution, which is a contradiction.

If (II) holds, then (4.15) is satisfied for $t \geq T \geq t_1$, and hence we have

$$\frac{d}{dt}\left(v^{(n-2)}(t)\right)^{\alpha} + a_1^{\beta}(h'(t))^{\mu} q(t) v^{\beta}[h(t)] \leq 0 \qquad \text{for } t \geq T, \tag{4.18}$$

where $v(t) = x'(t)$ and (4.8) holds for $t \geq T$.

Integrating inequality (4.18) $(n-1)$ times from $t \geq T$ to u, using (4.8), and letting $u \to \infty$, we find

$$v(t) \geq a_1^{\beta} \int_t^{\infty} \frac{(s-t)^{n-3}}{(n-3)!} \left(\int_s^{\infty} (h'(\tau))^{\mu} q(\tau) v^{\beta}[h(\tau)] d\tau \right)^{1/\alpha} ds.$$

Now following similar steps of the proof of Theorem 1 in [17], we conclude that Eq. (4.17) has a solution $z(t)$ with $\lim_{t\to\infty} z(t) = 0$, which is a contradiction. This completes the proof. ∎

EXAMPLE 4.1. Consider the equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + q(t)\left|x\left[\frac{t}{2}\right]\right|^{\beta}\left|\frac{d}{dt}x\left[\frac{t}{2}\right]\right|^{\mu} \operatorname{sgn} x\left[\frac{t}{2}\right] = 0, \tag{4.19}$$

where $q(t) \in C([t_0, \infty), \mathbb{R}^+)$, α, β, and μ are positive constants. Let $\beta \leq \alpha$ and $\mu \leq \alpha$. Then, by Theorem 4.2, Eq. (4.19) is oscillatory if for every positive constant θ_1, θ_2 the equations

$$y'(t) + \left[\frac{\theta_1}{2^{\mu}(2^{n-1}(n-1)!)^{\beta}}\right] t^{(n-1)\beta} q(t) \left|y\left[\frac{t}{2}\right]\right|^{\beta/\alpha} \operatorname{sgn} y\left[\frac{t}{2}\right] = 0$$

and

$$z'(t) + \left[\frac{\theta_2}{(2^{2n-3}(n-2)!)^{\mu}}\right] t^{(n-2)\mu} q(t) \left|z\left[\frac{3t}{4}\right]\right|^{\mu/\alpha} \operatorname{sgn} z\left[\frac{t}{2}\right] = 0$$

are oscillatory. We also note that Eq. (4.19) is oscillatory if we take $q(t) = t^{n-2-k}$, $0 < k < 1$ when $\alpha = \beta = \mu$, and

$$q(t) = \frac{1}{t} \min\{t^{(n-1)\beta}, t^{(n-2)\mu}\}, \qquad t > 1$$

when $\beta < \alpha$ and $\mu < \alpha$.

REFERENCES

1. R. P. Agarwal and S. R. Grace, Oscillation of certain functional differential equations, *Comput. Math. Appl*. **38** (1999), 143–153.
2. R. P. Agarwal, S. R. Grace, and D. O'Regan, "Oscillation Theory for Difference and Functional Differential Equations," Kluwer, Dordrecht, 2000.
3. R. P. Agarwal, S.-H. Shieh, and C. C. Yeh, Oscillation criteria for second-order retarded differential equations, *Math. Comput. Model*. **26** (1997), 1–11.
4. A. Elbert, A half-linear second order differential equation, *in* "Proceedings of the Colloquia Math. Soc. János Bolyai 30: Qualitative Theory of Differential Equations," Szeged, 1979, pp. 153–180.
5. A. Elbert and T. Kusano, Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations, *Acta Math. Hungar*. **56** (1990), 325–336.
6. S. R. Grace, Oscillatory and asymptotic behavior of delay differential equations with a nonlinear damping term, *J. Math. Anal. Appl*. **168** (1992), 306–318.
7. S. R. Grace, Oscillation theorems for damped functional differential equations, *Funkcial. Ekvac*. **35** (1992), 261–278.
8. S. R. Grace, Oscillation theorems for certain functional differential equations, *J. Math. Anal. Appl*. **184** (1994), 100–111.
9. S. R. Grace, Oscillation criteria of comparison type for nonlinear functional differential equations, *Math. Nachr*. **173** (1995), 177–192.
10. S. R. Grace and B. S. Lalli, Oscillation theorems for *n*th order delay differential equations, *J. Math. Anal. Appl*. **91** (1983), 352–366.
11. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1988.
12. J. Jeroš and T. Kusano, Oscillation properties of first order nonlinear functional differential equations of neutral type, *Differential Integral Equations* **4** (1991), 425–436.
13. A. G. Kartsatos, On *n*th order differential inequalities, *J. Math. Anal. Appl*. **52** (1975), 1–9.
14. R. G. Koplatadze and T. A. Chanturia, On oscillatory and monotone solutions of first order differential equations with deviating arguments, *Differencial'nye Uravnenija* **18** (1982), 1463–1465.
15. T. Kusano and B. S. Lalli, On oscillation of half-linear functional differential equations with deviating arguments, *Hiroshima Math. J*. **24** (1994), 549–563.
16. W. E. Mahfoud, Remarks on some oscillation theorems for *n*th order differential equations with a retarded argument, *J. Math. Anal. Appl*. **62** (1978), 68–80.
17. Ch. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math*. **36** (1981), 168–178.
18. Ch. G. Philos, A new criterion for the oscillatory and asymptotic behavior of delay differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Mat*. **39** (1981), 61–64.
19. P. J. Y. Wong and R. P. Agarwal, Oscillation theorems and existence criteria of asymptotically monotone solutions for second order differential equations, *Dynam. Systems Appl*. **4** (1995), 477–496.
20. P. J. Y. Wong and R. P. Agarwal, Oscillatory behaviour of solutions of certain second order nonlinear differential equations, *J. Math. Anal. Appl*. **198** (1996), 337–354.