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Subring subgroups in Chevalley groups with doubly laced root systems[☆]

Alexei Stepanov^{a,b,*}^a Department of Mathematics and Mechanics, Saint Petersburg State University, Russian Federation^b Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan

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ABSTRACT

We study the lattice \mathfrak{L} of subgroups of a Chevalley group $G(\Phi, A)$ over a ring A , containing its elementary subgroup $E(\Phi, S)$ over a subring $S \subseteq A$. The standard description asserts that for any $H \in \mathfrak{L}$ there exists a unique subring $R \supseteq S$ of A such that H contains $E(\Phi, R)$ as a normal subgroup.

The standard description was obtained by Ya. Nuzhin for algebraic field extensions. On the other hand, it was expected that the standard description does not hold for transcendental extensions. This was recently proved by the author for simply laced root systems.

Now, suppose that Φ is doubly laced, i.e. $\Phi = B_l, C_l$ or F_4 , and that 2 is invertible in S . In the article we prove that in these settings the standard description of \mathfrak{L} holds for an arbitrary pair of rings $S \subseteq A$.

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Introduction

Let $G = G_P(\Phi, _)$ denote a Chevalley–Demazure group scheme with root system Φ and weight lattice P . All rings throughout the paper are commutative with the unit element and all ring homomorphisms preserve the units. For a ring A denote by $E(A) = E_P(\Phi, A)$ the elementary subgroup

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* Correspondence to: Department of Mathematics and Mechanics, Saint Petersburg State University, Russian Federation.

E-mail address: stepanov239@gmail.com.

of $G(A)$, i.e. the subgroup generated by all elementary root unipotent elements $x_{\alpha}(t)$, $\alpha \in \Phi$, $t \in A$. Let S be a subring of A . We study the lattice $\mathcal{L} = L(E(S), G(A))$ of subgroups of $G(A)$, containing $E(S)$.

The standard answer to this problem is called *sandwich classification theorem*. It asserts that for each $H \in \mathcal{L}$ there exists a subring R between S and A such that H normalizes $E(R)$. In other words, \mathcal{L} is partitioned into disjoint union of sandwiches $L(E(R), N_A(R))$, where $N_A(R)$ is the normalizer of $E(R)$ in $G(A)$.

In this article we prove the standard description of the lattice \mathcal{L} for a doubly laced root system and an *arbitrary pair of rings* $S \subseteq A$, provided that 2 is invertible in S (for $\Phi = B_{2k+1}$ we assume in addition that -1 is a square in S).

Theorem. *Let $\Phi = B_l, C_l$ or F_4 and $l \geq 2$. Let $S \subseteq A$ be a pair of rings such that 2 is invertible in S and if $\Phi = B_{2k+1}$, then -1 is a square in S . Then, given a subgroup $H \leq G(A)$ containing $E(S)$, there exists a unique subring $R \subseteq A$ containing S such that*

$$E(R) \leq H \leq N_A(R).$$

This result is very surprising. Indeed, for simply laced root systems the situation is dramatically different. Namely, the author has recently proved in [24] and [25] that if A contains a transcendental element over S and Φ is simply laced, then the lattice \mathcal{L} is far from being standard. More precisely, in the adjoint group $G(A) = G_{\text{ad}}(\Phi, A)$ there exists a subgroup $H \in \mathcal{L}$ isomorphic to the free product of $E(S)$ and a cyclic subgroup C . Clearly, there is no hope to describe the lattice $L(E(S), E(S) \star C) \subset \mathcal{L}$ in terms of sandwich classification.

Now, using the result by Nuzhin [17] we obtain an almost complete answer to the question: “When the lattice \mathcal{L} is standard?” for fields $S \subseteq A$ of characteristic $\neq 2$ and $\Phi \neq A_1, G_2$.

Corollary. *Let A/S be a field extension. Suppose that $\text{char } S \neq 2$, $\Phi \neq A_1, G_2$ and if $\Phi = B_{2k+1}$, then -1 is a square in S . The lattice $\mathcal{L} = L(E(S), G(A))$ is standard if and only if A is algebraic over S or $\Phi = B_l, C_l, F_4$.*

Possible generalizations of these two results will be discussed below.

Background and history. The study of subgroup structure of linear groups over fields formed an important part of the group theory. In the middle of the 20-th century, when the foundations of the theory of algebraic groups were being developed, many outstanding mathematicians including Bruhat, Borel, and Tits investigated various problems concerning lattices of Zariski closed subgroups of reductive groups.

The subject got a new life in the 1970-ies from activity around the classification of finite simple groups. In this context, subgroups of classical and exceptional Chevalley groups over fields were extensively studied by Aschbacher, Bashkirov, Cooperstein, Dye, King, Kantor, Liebeck, Li Shangzhi, Nuzhin, Timmesfeld, Wagner, Zaleskii, Seitz, and many others. In 1984, in the framework of the *Maximal Subgroup Classification Project M*, Aschbacher defined classes C_1, \dots, C_8 of subgroups which were expected to be maximal subgroups of the finite classical groups.

At the same time Borevich and Vavilov [9] initiated the study of subgroup structure of classical groups over rings which extends later to a sort of *Large Subgroup Classification Project* in linear groups over rings. It turns out that for a sufficiently large subgroup D of a Chevalley group $G(A) = G(\Phi, A)$ the lattice $\mathcal{L} = L(D, G(A))$ breaks into disjoint union of “sandwiches” $L(E_i, N_i)$, where N_i is the normalizer of E_i in G , and each E_i is generated by unipotent elements. If such a description holds, we say that the lattice \mathcal{L} is standard. Vavilov [33] picked out 5 classes of subgroups corresponding to Aschbacher classes:

- $C_1 + C_2$ (subsystem subgroups);
- C_3 (ring extension subgroups);
- $C_4 + C_7$ (tensor product subgroups);

- C_5 (subring subgroups);
- C_8 (classical groups in natural representations).

For subgroups from these classes he conjectured that the lattice \mathcal{L} is standard under certain natural assumptions. If this holds, then $\mathcal{L} = \bigsqcup L(E_i, N_i)$, and to describe all subgroups from \mathcal{L} one studies the properties of the quotients N_i/E_i . Usually, it turns out that these groups look like some nonstable K_1 -groups and one can apply K-theoretic ideas to investigate them.

In survey [33] Vavilov described results on class $C_1 + C_2$ for classical groups over fields and rings and on class C_3 for GL_n over fields. The results on class C_8 for overgroups of classical groups in the natural representations were obtained in [37] by Hong You and in a series of papers [30–32,19] by Petrov and Vavilov. Quite recently substantial progress for the class $C_4 + C_7$ was obtained by Vavilov and his students Ananievsii and Sinchuk [4]. A set of open questions in this subject can be found in [35].

In the present paper we consider the lattice of overgroups of subring subgroups which constitute the class C_5 . This lattice was known to be standard in the following situations.

0. $A = S$, $\Phi \neq A_1$. In this case the standard description follows from normality of $E(A)$ in $G(A)$, i.e. the whole lattice is one big sandwich (Taddei [26], 1986).
1. $\Phi = A_n$, $n \geq 2$, A is the field of fractions of a Dedekind domain S (Shmidt [21], 1979).
2. $\Phi \neq A_1$, A is the field of fractions of a principal ideal domain S (Nuzhin and Yakushevich [18], 2000).
3. $\Phi \neq A_1$, a field A is an algebraic extension of a field S , provided that all structure constants $N_{\alpha\beta ij}$ are invertible in S or S is a perfect field, and that S is not too small for certain root systems (Nuzhin [17], 1983). A weaker result was obtained 15 years later by Wang Dengyin and Li Shangzhi in [36].

Our main result is a wide generalization of items (2) and (3) above for doubly laced root systems.

Techniques. The main technical tools in Sections 1–5 will be developed for all root systems of rank greater than 1 under assumption that all structure constants $N_{\alpha\beta ij}$ of the corresponding complex Lie algebra are invertible in S , i.e. 2 is invertible if $\Phi = C_l, B_l, F_4$ and 6 is invertible if $\Phi = G_2$. In subsequent papers, we plan to use these techniques to obtain the standard description of the lattice \mathcal{L} for all root systems under certain additional assumptions on the pair $S \subseteq A$.

For the proof of our main result we shall use the following strategy. Let H be a subgroup from \mathcal{L} . Denote by E_H the group generated by all elementary root unipotent elements of H and let N_H be its normalizer in $G(A)$. Let g be an arbitrary element of H . It suffices to prove that g belongs to N_H . Assume that we can find a generating set $X = X(g)$ of the group $E(S)$ such that for each $x \in X$ the conjugate x^g lies in a standard parabolic subgroup (this trick is called “reduction to a proper parabolic”). Then to finish the proof that the description of the lattice \mathcal{L} is standard, it suffices to prove the following statements.

Lemma 1. *If $E(S)^g \leq N_H$, then $g \in N_H$.*

Lemma 2. *If $y \in H \cap P$ for a standard parabolic subgroup P , then $y \in N_H$.*

Similar strategy was successfully used several times for investigation of certain classes of subgroups in Chevalley groups, see [33] and [35] for more details and references. For the reduction to a proper parabolic subgroup we use the identity with constants (Lemma 6.2) obtained by Gordeev in [11] (the existence of an identity with constants in classical groups was studied by Golubchik and Mikhalev in [10]). The appropriate “short” identity for $\Phi = F_4$ was recently obtained by Nesterov and the author in [16]. For other root systems the short identity does not hold. In fact, using an approach suggested by Tomanov in [28], Gordeev in [11] proved that there are no identities with constants for a simply laced root system Φ , and Nesterov with the author [16] have shown that short identity does not hold for $\Phi = G_2$.

Gordeev’s result was used by the author in [25] to show that the lattice \mathcal{L} is not standard if $A = S[t]$ is a polynomial ring and Φ is simply laced. Thus, for extensions of transcendental type, existence of the identity with constants is crucial.

Further problems. The current state of the problem for simply laced root systems is described in [25]. Here we discuss further problems for multiply laced root systems.

First, suppose $\Phi = B_{2k+1}$, 2 is invertible in S , and -1 is not a square in S . Then there are no small semisimple elements in the group $E(S)$. More precisely, in the simply connected case there are no small semisimple elements in $G_{sc}(B_l, S)$, in the adjoint case small semisimple elements exist but does not belong to the elementary group. It seems that in both cases one can obtain a free product subgroup (with amalgamated center) between $E(S)$ and $G(S[t])$ with the same method as in [25]. On the other hand, for the adjoint group SO_{4k+3} sandwich classification should hold for the lattice of subgroups of $SO_{4k+3}(A)$ containing $EO_{4k+3}(S)$ and a small semisimple element.

If 2 is not invertible in S , then the answer should be modified due to the existence of subgroups, corresponding to form rings (see [6]). The problem is the same as for the normal structure of a Chevalley group (see [1] and [29]). Namely, the bottom layer of a sandwich is generated by subgroups $x_\alpha(q_\alpha)$, where q_α are additive subgroups of A which are equal to the same ideal of A if 2 is invertible; otherwise they can be different for roots of different length. In our setting even the properties of q_α strongly depend on the root system and the proof is expected to be much more subtle. Nevertheless, if Φ is doubly laced and $2 = 0$, then the key place of the proof, reduction to a proper parabolic, still works. Therefore, we believe that the description in this case is standard and expect only technical difficulties.

If 2 is not invertible and $2 \neq 0$, one will have substantial problems in this place of the proof. In this case the identity with constants containing small semisimple element ensures only that $E(2S)^g \leq N_H$ instead of $E(S)^g \leq N_H$. On the other hand, one can try to play with identity with constants containing a small unipotent element over the quotient ring modulo 2 but result of this game is not quite clear now.

If $\Phi = G_2$, then there is a “long” identity with constants which guarantees that there are no free product subgroups inside \mathcal{L} . On the other hand, without short identity the proof of the present paper does not work. Using the long identity one can still extract some unipotent elements. Then one can look to a representation of $G(G_2, A)$ to prove that there are enough unipotents or to discover an obstruction.

The rest of the article is organized as follows. In the first section we investigate the structure of the group E_H . In Section 2 we develop some tools from representation theory to show that the group $N_A(R)/G(R)$ is solvable. This is done at the beginning of Section 3. The rest of Section 3 is to prove Lemma 1. The proof of this lemma bases on the result of A. Bak [7], R. Hazrat and N. Vavilov [12] on nilpotent structure of the group $K_1(\Phi, R)$ over a Noetherian commutative ring R .

In Section 4 we show that the representation of $G_{ad}(\Phi, A)$ in an internal Chevalley module is faithful. This result is used in Section 5 to prove Lemma 2. In Section 6 we prove the main result of the article. In the last section we use group theoretic arguments to extend the description to subgroups normalized by $E(S)$ and to investigate the top layers of sandwiches, arising in this description.

Notation. Let H be a group. For two elements $x, y \in H$ we write $[x, y] = x^{-1}y^{-1}xy$ for their commutator and $x^y = y^{-1}xy$ for the element, conjugate to x by y . For subgroups $X, Y \leq H$ we let X^Y denote the normal closure of X in the subgroup generated by X and Y while $[X, Y]$ stands for their mutual commutator subgroup. The commutator subgroup $[X, X]$ of X will be denoted also by $D(X)$ and we set $D^k(X) = [D^{k-1}(X), D^{k-1}(X)]$. Recall that a group X is called *perfect* if $D(X) = X$.

Let Φ be a reduced irreducible root system and let P be a lattice between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. We denote by $G = G_P(\Phi, _)$ the Chevalley–Demazure group scheme of type (Φ, P) . For groups of adjoint types we write G_{ad} instead of $G_{Q(\Phi)}$. By $L(\Phi, R)$ we denote the Lie algebra of the group $G(R)$.

All subschemes of G are assumed to be defined over \mathbb{Z} . Fix a split maximal torus T of G . For a root $\alpha \in \Phi$ and an element $r \in R$ we denote by $x_\alpha(r)$ the corresponding elementary root unipotent

element and by $X_\alpha = \{x_\alpha(r) \mid r \in R\}$ the root subgroup. Fix a set Π of fundamental roots. The set of positive roots is denoted by Φ^+ .

Let \mathfrak{q} be an ideal of a ring R . As usual, $G(R, \mathfrak{q})$ denotes the principal congruence subgroup of $G(R)$ of level \mathfrak{q} , i.e. the kernel of the reduction homomorphism $G(R) \rightarrow G(R/\mathfrak{q})$. Denote by $E(\mathfrak{q})$ the subgroup of $E(R)$ generated by elementary root unipotents $x_\alpha(t)$ for all $\alpha \in \Phi$ and $t \in \mathfrak{q}$, and by $E(R, \mathfrak{q}) = E(\mathfrak{q})^{E(R)}$ the relative elementary subgroup.

Let $\varphi : G \rightarrow GL_n$ be a faithful representation of G . In the proofs we identify the elements of $G(R)$ and its Lie algebra $L(\Phi, R)$ with their images in the matrix ring $M_n(R)$ under homomorphisms induced by φ . For a matrix $g \in M_n(R)$ we write g_{ij} to denote its entry in position (i, j) . For an invertible g the entries of g^{-1} will be denoted by g'_{ij} .

Let X be a subset of a ring R . By $\langle X \rangle$ we denote the smallest multiplicative subset, containing X , and by $\langle X \rangle^{-1}R$ the localization of R in this multiplicative subset.

1. Subgroups generated by elementary root unipotents

Let S be a subring of a ring A . Let H be a subgroup of $G(A)$, normalized by $E(S)$. The following lemma shows that if the right hand side of a Chevalley commutator formula belongs to H , then each factor also lies in H . It was obtained in [1, Lemma 2.2] for the case $S = A$. The proof is essentially the same.

Lemma 1.1. *Let $\alpha, \beta \in \Phi$ be a set of fundamental roots for a subsystem Φ' of type C_2 or G_2 with α short. Let*

$$g = x_{\alpha+\beta}(s)x_{2\alpha+\beta}(t), \quad \text{when } \Phi' = C_2;$$

$$g = x_{\alpha+\beta}(s)x_{2\alpha+\beta}(t)x_{3\alpha+\beta}(u)x_{3\alpha+2\beta}(v), \quad \text{when } \Phi' = G_2.$$

If $g \in H$, then each factor of g belongs to H .

Put $\mathfrak{q}_\alpha = \mathfrak{q}_\alpha(H) = \{t \in A \mid x_\alpha(t) \in H\}$. Since the Weyl group acts transitively on the set of roots of the same length, it is easy to see that $\mathfrak{q}_\alpha = \mathfrak{q}_\beta$ if $|\alpha| = |\beta|$, in fact this has been already used in the proof of the previous lemma.

Lemma 1.2. *Suppose that $\Phi \neq A_1, 2$ is invertible in S if $\Phi = B_1, C_1, F_4$, and 3 is invertible in S if $\Phi = G_2$. For any $\alpha, \beta \in \Phi$ we have $\mathfrak{q}_\alpha = \mathfrak{q}_\beta$. Denote the set \mathfrak{q}_α ($\alpha \in \Phi$) by $\mathfrak{q}(H)$ or simply by \mathfrak{q} . Then \mathfrak{q} is an S -module closed under multiplication.*

Proof. Clearly, each \mathfrak{q}_α is an additive subgroup of A . The first assertion is obvious if $|\alpha| = |\beta|$. Let α be a short root. We may assume that β is a long root such that $\alpha + \beta \in \Phi$. If $\Phi \neq G_2$, then $\alpha + \beta$ is short and $2\alpha + \beta$ is long. For $t \in \mathfrak{q}_\beta$ put

$$a = [x_\alpha(1), x_\beta(t)] = x_{\alpha+\beta}(\pm t)x_{2\alpha+\beta}(\pm t).$$

Since $a \in H$, by Lemma 1.1, $x_{\alpha+\beta}(\pm t) \in H$. Therefore, $\mathfrak{q}_\beta \subseteq \mathfrak{q}_{\alpha+\beta} = \mathfrak{q}_\alpha$. Conversely, for $v \in S$ we have $[x_\alpha(u), x_{\alpha+\beta}(v)] = x_{2\alpha+\beta}(2uv) \in H$, hence $2S\mathfrak{q}_\alpha \subseteq \mathfrak{q}_{2\alpha+\beta} = \mathfrak{q}_\beta$. Since $\frac{1}{2} \in S$, we obtain $\mathfrak{q}_\alpha = \mathfrak{q}_\beta$.

If $\Phi = G_2$, then we assume that α and β is a base of Φ with α short. Take $t \in \mathfrak{q}_\beta$. Then $[x_\alpha(1), x_\beta(t)] = x_{\alpha+\beta}(\pm t)x_{2\alpha+\beta}(\pm t)x_{3\alpha+\beta}(\pm t)x_{3\alpha+2\beta}(\pm t^2) \in H$ and by Lemma 1.1 $t \in \mathfrak{q}_{\alpha+\beta} = \mathfrak{q}_\alpha$, hence $\mathfrak{q}_\beta \subseteq \mathfrak{q}_\alpha$. Conversely, for $s \in \mathfrak{q}_\alpha$ we have $[x_{2\alpha+\beta}(\frac{1}{3}), x_\alpha(s)] = x_{3\alpha+\beta}(\pm s) \in H$, hence $\mathfrak{q}_\alpha \subseteq \mathfrak{q}_{3\alpha+\beta} = \mathfrak{q}_\beta$.

If $\Phi \neq C_2$, then it contains a subsystem of type A_2 , where the second assertion of the lemma is easy to prove. If $\Phi = C_2$, then the formula for a together with Lemma 1.1 show that $st \in \mathfrak{q}$ whenever $s \in \mathfrak{q}$ and $t \in S \cup \mathfrak{q}$. \square

Note that if H is a subgroup of $G(A)$ containing $E(S)$, then $q(H)$ is a S -algebra. In other words, $q(H)$ is the largest subring of A such that $E(q(H)) \leq H$. The set $q(H)$ will be called a subring associated with H .

The following lemma will be used for the proof of the uniqueness statement of the main theorem. It is well known for the case $S = A$.

Lemma 1.3. *Suppose that $\Phi \neq A_1$, 2 is invertible in A if $\Phi = B_l, C_l, F_4$, and 3 is invertible in A if $\Phi = G_2$. Then $[E(S), E(A)] = E(A)$.*

Proof. The subgroup $H = [E(S), E(A)]$ is normal in $E(A)$. We have to show that $q(H) = A$. By Lemma 1.2 it suffices to show that H contains $x_\gamma(1)$ for some $\gamma \in \Phi$. This is clear if Φ contains a subsystem of type A_2 . Otherwise $\Phi = C_2$ or $\Phi = G_2$. Take two short roots α and β and put $\gamma = \alpha + \beta$. Then $x_\gamma(1) = [x_\alpha(1), x_\beta(\pm\varepsilon)] \in H$, where $\varepsilon = \frac{1}{2}$ for $\Phi = C_2$ and $\varepsilon = \frac{1}{3}$ for $\Phi = G_2$. \square

2. Irreducibility of rational representations

Let R be a subring of a ring A . To describe the normalizer $N_A(R)$ of the group $E(R)$ in $G(A)$ we use representation theory. A rational representation φ of an affine group scheme G over \mathbb{Z} defines a Hopf algebra homomorphism $\varphi^* : \mathbb{Z}[GL_n] \rightarrow \mathbb{Z}[G]$. Let $i : R \hookrightarrow A$ be an embedding of rings. In general, the square

$$\begin{array}{ccc}
 G(R) & \xrightarrow{\varphi_R} & GL_n(R) \\
 G(i) \downarrow & & \downarrow GL_n(i) \\
 G(A) & \xrightarrow{\varphi_A} & GL_n(A)
 \end{array} \tag{*}$$

is not a pullback. But for a faithful representation it is.

Lemma 2.1. *Let φ be a faithful representation of a group scheme G . Then the square (*) is a pullback square. In other words, if we identify elements of $G(R)$, $G(A)$, and $GL_n(R)$ with their images in $GL_n(A)$, then*

$$G(R) = G(A) \cap GL_n(R).$$

Proof. Recall that a representation φ is called faithful if it is a monomorphism in the category of group schemes. In this case φ^* is surjective. Take $x \in GL_n(R) = \text{Hom}_{\text{Rings}}(\mathbb{Z}[GL_n], R)$ and $y \in G(A) = \text{Hom}_{\text{Rings}}(\mathbb{Z}[G], A)$ such that $GL_n(i)(x) = \varphi_A(y)$. The latter condition can be expressed in terms of a commutative diagram of rings and ring homomorphisms

$$\begin{array}{ccc}
 \mathbb{Z}[GL_n] & \longrightarrow & \mathbb{Z}[G] \\
 \downarrow x & \nearrow z & \downarrow y \\
 R & \longrightarrow & A
 \end{array}$$

To prove the lemma it suffices to find an element $z \in G(R) = \text{Hom}_{\text{Rings}}(\mathbb{Z}[G], R)$ making the diagram commute. Given $a \in \mathbb{Z}[G]$ there exists its preimage $b \in \mathbb{Z}[GL_n]$, and we set $z(a) = x(b)$. We have $i(z(a)) = i(x(b)) = y(a)$. Since i is injective, $z(a)$ does not depend on the choice of b . The fact that z is a ring homomorphism is trivial. \square

Next, we consider a notion of absolute irreducibility of a representation of a group scheme. We call a representation $\varphi : G \rightarrow GL_n$ absolutely irreducible over a ring S if the image of φ_S generates the full matrix ring $M_n(S)$ as an S -module. Since for any S -algebra R the matrix ring $M_n(R)$ is generated by the image of $M_n(S)$ as an R -module, absolute irreducibility over S implies absolute irreducibility over any S -algebra.

For an algebraically closed field our definition coincides with the usual one by the Burnside theorem. We shall show that for a Chevalley group scheme a representation is absolutely irreducible over S iff it is irreducible over any S -algebra, which is an algebraically closed field, provided that small primes are invertible in S .

Let $G = G(\Phi, _)$ be a Chevalley group scheme, $\varphi : G \rightarrow GL_n$ its representation. Denote by $L(\Phi, R)$ the Chevalley algebra of type Φ over a ring R , i.e. the Lie algebra over R obtained from the simple complex Lie algebra of type Φ . Denote by e_α the root element of $L(\Phi, \mathbb{Z})$, corresponding to a root $\alpha \in \Phi$. For a ring R we denote by the same symbol the image of e_α in $L(\Phi, R) = L(\Phi, \mathbb{Z}) \otimes R$. The elements of the group $G(R)$ and its Lie algebra $L(\Phi, R)$ are identified with their images in $M_n(R)$ under the homomorphisms induced by φ . Denote by $\mathbf{m}(\varphi)$ the largest integer such that $e_\alpha^{\mathbf{m}(\varphi)} \neq 0$ over \mathbb{Z} for some $\alpha \in \Phi$.

Lemma 2.2. *Let φ be a representation of a Chevalley group scheme G . Let \mathcal{P} be a set of primes, containing all primes $p \leq \mathbf{m}(\varphi)$, and $S = \langle \mathcal{P} \rangle^{-1}\mathbb{Z}$. Representation φ is absolutely irreducible over S if and only if it is irreducible over any S -algebra F which is an algebraically closed field. Moreover, in this case for any S -algebra R the R -module generated by the image of the elementary subgroup $E(R)$ already coincides with $M_n(R)$.*

Proof. Let K be a prime field with $\text{char } K \notin \mathcal{P}$ and let \bar{K} be its algebraic closure. Since $K \cong S / (\text{char } K)S$ or $K \cong \mathbb{Q}$, it is an S -algebra. By the Burnside lemma $G(\bar{K})$ spans $M_n(\bar{K})$ as a \bar{K} -vector space. For an algebraically closed field $G(\bar{K}) = E(\bar{K})$. Since this group is generated by the root unipotent elements $x_\alpha(r)$, the set $\{x_\alpha(r) \mid \alpha \in \Phi, r \in \bar{K}\}$ generates $M_n(\bar{K})$ as a \bar{K} -algebra. We want to show that the image of $E(\mathbb{Z})$ in $M_n(K)$ generates this matrix ring as a K -module. With this end we express $x_\alpha(r)$ as a linear combination of $x_\alpha(j)$, where $j = 0, \dots, \mathbf{m}(\varphi)$. We have

$$x_\alpha(r) = \sum_{k=0}^m \frac{1}{k!} r^k e_\alpha^k,$$

where $m = \mathbf{m}(\varphi)$. By the condition on \mathcal{P} we have $\text{char } K > m$, therefore the images of integers $0, \dots, m$ in K are distinct. Since the Vandermonde determinant is nonzero, the system of linear equations $\sum_{j=0}^m u_j j^k = r^k, k = 0, \dots, m$, has a unique solution (here $0^0 = 1$). Therefore, given $r \in \bar{K}$ and $\alpha \in \Phi$ there are $u_0, \dots, u_m \in \bar{K}$ such that

$$x_\alpha(r) = \sum_{k=0}^m \frac{1}{k!} \left(\sum_{j=0}^m u_j j^k \right) e_\alpha^k = \sum_{j=0}^m u_j x_\alpha(j),$$

i.e. $x_\alpha(r)$ belongs to the \bar{K} -subalgebra of $M_n(\bar{K})$ generated by $x_\alpha(1)$. Hence, the \bar{K} -subalgebra generated by $\{x_\alpha(1) \mid \alpha \in \Phi\}$, which is equal to the \bar{K} -subspace generated by the image of $E(\mathbb{Z})$, coincides with $M_n(\bar{K})$. It follows that there exist n^2 linearly independent over \bar{K} elements of the image of $E(\mathbb{Z})$. Thus, the image of $E(\mathbb{Z})$ generates an n^2 -dimensional K -subspace of $M_n(K)$ which must coincide with $M_n(K)$.

Put $K = \mathbb{Q}$. It follows that each matrix unit e_{hj} can be expressed as a rational linear combination of some elements of $E(\mathbb{Z})$. Denote by \mathcal{P}' the set of all primes, dividing the denominators of the coefficients of these linear combinations. Let s be the product of all primes from $\mathcal{P}' \setminus \mathcal{P}$. Then for some $d \in \mathbb{N}$ all matrices $s^d e_{hj}$ belong to the S -module, generated by $E(\mathbb{Z})$.

The ring S/S is isomorphic to the direct sum of fields \mathbb{F}_p over all $p \in \mathcal{P}' \setminus \mathcal{P}$. The first paragraph of the proof with $K = \mathbb{F}_p$ shows that $E(\mathbb{F}_p)$ generates $M_n(\mathbb{F}_p)$ as \mathbb{F}_p -vector space. Hence, the S/S -submodule generated by $E(S/S)$ coincides with $M_n(S/S)$. Taking inverse images, one concludes that the elements $(1 + sv_{hj})e_{hj}$ belong to the S -module spanned by $E(S)$ for all h, j and for some $v_{hj} \in S$. Clearly, $1 + sv_{hj}$ and s^d are relatively prime in S , hence all matrix units belong to the S -module spanned by $E(S)$. Thus, this S -module coincides with $M_n(S)$.

The converse implication is trivial. \square

Corollary 2.3. *Let*

- $\mathcal{P} = \{2, 3\}$ if $\Phi = G_2$ or $\Phi = E_6$;
- $\mathcal{P} = \{2\} \cup \{\text{prime divisors of } l + 1\}$ if $\Phi = A_l$;
- $\mathcal{P} = \{2\}$ otherwise.

Then the adjoint representation of the group scheme $G(\Phi, _)$ is absolutely irreducible over $S = \langle \mathcal{P} \rangle^{-1}\mathbb{Z}$.

Proof. The center of the Chevalley algebra $L(\Phi, R)$ is trivial if and only if the kernel of the Cartan matrix over R is trivial (see [14]). The determinant of the Cartan matrix is a power of 2 except for $\Phi = E_6$ where it equals 3 and A_l where it is $l + 1$. Thus, under the conditions of the Corollary the center of $L(\Phi, S)$ is trivial.

For a field F of characteristic not 2 (and 3 if $\Phi = G_2$) the representation of $G(\Phi, F)$ on $L(\Phi, F)$ /Center is absolutely irreducible by [13].

Using description of a simple complex Lie algebra by generators and relations one shows that $m(\text{adj}) = 2$. This can be also obtained using the weight diagrams of adjoint representations from [20]. Indeed, there are at most 2 adjacent edges with the same label in these diagrams which means that the cube of each root element e_α of $L(\Phi, \mathbb{Z})$ vanishes on each weight subspace.

Now, by Lemma 2.2 we conclude that the adjoint representation is absolutely irreducible over S . \square

3. The normalizer

Denote by $N_A(R)$ the normalizer of $E(R)$ in $G(A)$. By Taddei’s theorem [26] we know that $E(R)$ is normal in $G(R)$, i.e. $G(R) \leq N_A(R)$. Moreover, the main theorem of Hazrat and Vavilov [12] asserts that $G(R)/E(R)$ is an extension of a nilpotent group by an abelian group provided R is Noetherian. The following lemma gives a criterion for $g \in G(A)$ to belong to $N_A(R)$ in terms of its matrix entries in a faithful representation φ . As usual, g is identified with $\varphi(g)$ and its matrix entries are denoted by g_{ij} . The entries of g^{-1} are denoted by g'_{ij} .

Lemma 3.1. *Let $\varphi : G \rightarrow GL_n$ be a faithful representation which is absolutely irreducible over S and let $R \subseteq A$ be S -algebras. For an element $g \in G(A)$ we have*

$$E(R)^g \leq G(R) \iff g'_{ij}g_{kl} \in R \text{ for all } i, j, k, l \in \{1, \dots, n\} \iff G(R)^g \leq G(R).$$

Proof. Since φ is absolutely irreducible over S , by Lemma 2.2 any matrix unit e_{jk} can be written in the form $\sum s_m a^{(m)}$ for some $s_m \in R$ and $a^{(m)} \in E(R)$. If $E(R)^g \leq G(R)$, then $g^{-1}a^{(m)}g \in G(R)$. Therefore, $g^{-1}e_{jk}g = \sum s_m g^{-1}a^{(m)}g \in M_n(R)$, hence $g'_{ij}g_{kl} \in R$.

Suppose that $g'_{ij}g_{kl} \in R$ for all $i, j, k, l \in \{1, \dots, n\}$ and take $a \in G(R)$. Then $g^{-1}ag, g^{-1}a^{-1}g \in M_n(R)$. Hence $g^{-1}ag \in GL_n(R)$, and by Lemma 2.1, $g^{-1}ag \in G(R)$. The remaining implication is trivial. \square

By Lemma 2.1 we have $G(A) \cap GL_n(R) = G(R)$. Therefore, to prove that $g \in G(A)$ is defined over R , i.e. belongs to $G(R)$, it suffices to show that $g \in GL(n, R)$.

Corollary 3.2. *Under conditions of the previous lemma one has $[N_A(R), N_A(R)] \leq G(R)$.*

Proof. If $g, h \in N_A(R)$ and $a = [g, h]$, then for all indexes p, q we have

$$a_{pq} = \sum_{i,j,k=1}^n g'_{pi} h'_{ij} g_{jk} h_{kq} = \sum_{i,j,k=1}^n (g'_{pi} g_{jk})(h'_{ij} h_{kq}) \in R. \quad \square$$

Not for all weight lattices P the group scheme $G_P(\Phi, -)$ has a faithful representation which is absolutely irreducible over \mathbb{Z} or $\mathbb{Z}[\frac{1}{2}]$. But for each root system Φ there exists at least one weight lattice P such that such a representation exists. Lemma 3.5 shows that in this case the group $N_A(R)/G(R)$ is solvable. To prove this we need one more statement which is well known for specialists (although I could not find an exact reference).

Lemma 3.3. *Let $Q \subseteq P$ be weight lattices and $\rho : G_P(\Phi, -) \rightarrow G_Q(\Phi, -)$ the corresponding morphism of group schemes. Then for any commutative ring R the cokernel of ρ_R is an abelian group, i.e. $[G_Q(\Phi, R), G_Q(\Phi, R)] \leq \rho_R(G_P(\Phi, R))$.*

Proof. The group functor $K(R) = \text{Ker } \rho_R$ is an affine group scheme. Since the kernel of ρ_R is central, K is abelian. The exact sequence of group schemes $1 \rightarrow K \rightarrow H \rightarrow G_Q(\Phi, -) \rightarrow 1$ gives rise to an exact sequence (see [15, Lemma 2.6.1])

$$1 \rightarrow K(R) \rightarrow G_P(\Phi, R) \rightarrow G_Q(\Phi, R) \rightarrow \check{H}_{\text{et}}^1(R, K)$$

where $\check{H}_{\text{et}}^1(R, K)$ is the Čech cohomology pointed set. But, since K is abelian, $\check{H}_{\text{et}}^1(R, K)$ is an abelian group which implies that the cokernel of ρ_R is abelian. \square

In the next corollary we need to include weight lattice P in the notation of the normalizer: $N_A(R) = N_A^P(R)$.

Corollary 3.4. *Let $\psi_A : N_A^P(R) \rightarrow N_A^Q(R)$ be the homomorphism induced by ρ_A . Suppose that $\Phi \neq A_1$ and if $\Phi = C_2$ then 2 is invertible in R . Then the kernel and cokernel of ψ_A are abelian.*

Proof. Since the restriction of $\rho_A : E_P(\Phi, R) \rightarrow E_Q(\Phi, R)$ is surjective, we have $\rho_A(N_A^P(R)) \leq N_A^Q(R)$, so the map ψ_A is correctly defined. Obviously, its kernel is abelian. By Lemma 3.3 for $a, b \in N_A^Q(R)$ the commutator $[a, b]$ lies in $\rho_A(G_P(\Phi, A))$. Let c be its preimage in $G_P(\Phi, A)$. Since $[a, b]$ normalizes $E_Q(\Phi, R)$, the image of the group $E_P(\Phi, R)^c$ lies in $E_Q(\Phi, R)$. Therefore, $E_P(\Phi, R)^c \leq E_P(\Phi, R) \cdot K(A)$, where $K(A) = \text{Ker } \rho_A$ is central. The conditions of the corollary imply that $E_P(\Phi, R)$ is a perfect group (see [22]). Thus,

$$E_P(\Phi, R)^c = [E_P(\Phi, R)^c, E_P(\Phi, R)^c] \leq [E_P(\Phi, R) \cdot K(A), E_P(\Phi, R) \cdot K(A)] \leq E_P(\Phi, R).$$

Therefore, $c \in N_A^P(R)$ which means that $[a, b]$ belongs to the image of ψ_A . \square

Lemma 3.5. *Suppose that $\Phi \neq A_1$, 2 is invertible in R if $\Phi = C_2, F_4$ or E_8 and 6 is invertible in R if $\Phi = G_2$. Then the group $N_A(R)/G(R)$ is solvable.*

Proof. For $\Phi = G_2, F_4, E_8$ put $Q = Q(\Phi)$ so that $G_Q(\Phi, -) = G_{\text{ad}}(\Phi, -)$. By Corollary 2.3 this group scheme has a faithful representation, absolutely irreducible over R .

Otherwise, $L(\Phi, \mathbb{C})$ has a minuscule representation φ . Let Q be the weight lattice of φ (here $Q \neq Q(\Phi)$). Then φ induces a faithful representation of $G_Q(\Phi, -)$. This representation, which we

denote also by φ , is known to be irreducible over all closed fields and has $\mathbf{m}(\varphi) = 1$ (see, for instance, [20]). By Lemma 2.2 φ is absolutely irreducible over \mathbb{Z} .

In both cases, by Corollary 3.2, we have $[N_A^Q(R), N_A^Q(R)] \leq G_Q(\Phi, R)$. Recall that for a group H we denote by $D^k(H)$ its k -th commutator subgroup. With this notation we have $D^1(N_A^Q(R)) \leq G_Q(\Phi, R)$.

By Corollary 3.4 $D^1(N_A^{\text{ad}}(R))$ is a subgroup of the image of $N_A^Q(R)$ in $G_{\text{ad}}(\Phi, A)$. Therefore, its commutator subgroup $D^2(N_A^{\text{ad}}(R))$ lies in the image of $G_Q(\Phi, R)$ in $G_{\text{ad}}(\Phi, A)$. Thus, for the adjoint group we have

$$D^2(N_A^{\text{ad}}(R)) \leq G_{\text{ad}}(R).$$

In general, let P be an arbitrary weight lattice. Denote by M the image of $N_A^P(R)$ in $G_{\text{ad}}(A)$. Since $M \leq N_A^{\text{ad}}(R)$, we have $D^2(M) \leq G_{\text{ad}}(R)$. Then, $D^3(M)$ lies in the commutator subgroup of $G_{\text{ad}}(R)$ which, by Lemma 3.3, lies in the image of $G^P(\Phi, R)$ in $G_{\text{ad}}(A)$. Taking preimages under the homomorphism $\rho_A : G_P(\Phi, A) \rightarrow G_{\text{ad}}(\Phi, A)$, we obtain the inclusion $D^3(N_A^P(R)) \leq G^P(\Phi, R) \text{Ker } \rho_A$. Since the kernel of ρ_A is central, $D^4(N_A^P(R)) \leq G^P(\Phi, R)$, i.e. the group $N_A^P(R)/G_P(\Phi, R)$ is solvable. \square

Remark. It seems that in fact the group $N_A(R)/G(R)$ is abelian and the condition that 2 is invertible is extra at least for $\Phi = E_8$. Therefore, in the actual proof I did not take care on the solvable degree of this group.

The following statement is crucial for the proof of our main theorem. For Noetherian rings it is an almost immediate consequence of the previous proposition.

Lemma 3.6. *Suppose that $\Phi \neq A_1$, 2 is invertible in R if $\Phi = F_4, C_2$ or $\Phi = E_8$, and 6 is invertible in R if $\Phi = G_2$. For an element $g \in G(A)$: if $E(R)^g \leq N_A(R)$, then $g \in N_A(R)$. Moreover, $E(R)$ is a characteristic subgroup of $N_A(R)$.*

Proof. Let θ be an automorphism of $N_A(R)$ or an automorphism of $G(A)$ such that $E(R)^\theta \leq N_A(R)$ (we denote by b^θ the image of an element $b \in N_A(R)$ under the action of θ).

Since $E(R)$ is perfect (see [22]) and $D^k(N_A(R)) \leq G(R)$ for sufficiently large integer k (Lemma 3.5), we have

$$E(R)^\theta = D^k(E(R)^\theta) \leq D^k(N_A(R)) \leq G(R).$$

Fix $r \in R$ and $\alpha \in \Phi$. We prove that $x_\alpha(r)^\theta \in E(R)$.

Put $Z = \mathbb{Z}[1/2]$ if $\Phi = F_4, C_2$ or $\Phi = E_8$, $Z = \mathbb{Z}[1/6]$ if $\Phi = G_2$, and $Z = \mathbb{Z}$ otherwise. Let S be the image of Z in R . The ring $R' = S[r]$ is a finitely generated \mathbb{Z} -algebra. Therefore, the group $E(R')$ is finitely generated by the elements $x_\alpha(\varepsilon)$ and $x_\alpha(r)$ for all $\alpha \in \Phi$ (here $\varepsilon = 1/2$ for $\Phi = C_2, F_4, E_8$, $\varepsilon = 1/6$ for $\Phi = G_2$, and $\varepsilon = 1$ otherwise). Clearly, there exists a finitely generated R' -algebra R'' such that the finite set $\{x_\alpha(\varepsilon)^\theta, x_\alpha(r)^\theta \mid \alpha \in \Phi\}$ is contained in $G(R'')$. Inclusion $E(R)^\theta \leq G(R)$ shows that $R'' \subseteq R$. Since R'' is Noetherian, by the main theorem of [12] the group $G(R'')/E(R'')$ is solvable, hence $E(R'')$ is the largest perfect subgroup of $G(R'')$. Since $E(R')^\theta$ is perfect and (by the choice of R'') lies in $G(R'')$, we have $E(R')^\theta \leq E(R'')$. In particular, $x_\alpha(r)^\theta \in E(R)$. Thus, $E(R)^\theta \leq E(R)$.

When θ is an automorphism of $N_A(R)$, this means that $E(R)$ is a characteristic subgroup of $N_A(R)$. When θ is an inner automorphism defined by $g \in G(A)$, this proves the first assertion of the lemma. \square

The following straightforward corollary shows that the normalizers of all subgroups of the sandwich $L(E(R), N_A(R))$ lie in the same sandwich.

Corollary 3.7. *Under the conditions of Lemma 3.6 for any $H \leq N_A(R)$ containing $E(R)$ its normalizer is contained in $N_A(R)$. In particular, the group $N_A(R)$ is self normalizing.*

4. Internal Chevalley module

Our next goal is to prove Lemma 2 from the introduction (= Lemma 5.2). Denote by B the standard Borel subgroup of $G(\Phi, _)$, i.e. the subgroup containing T with the root system Φ^+ (recall that a torus T and a set of positive roots Φ^+ were fixed from the beginning). Let U be the unipotent radical of B . A standard parabolic subgroup is a parabolic subgroup containing B . Let $\Gamma \subseteq \Pi$. We say that a standard parabolic subgroup P has type Γ if its root system is a union of Φ^+ with a closed subset of roots generated by $-\Gamma$.

Let P be a standard maximal parabolic subgroup, i.e. a standard parabolic subgroup of type $\Pi \setminus \{\alpha\}$ for some $\alpha \in \Pi$. To prove the lemma we need to show that the representation of L_P in the internal Chevalley module $U_P/[U_P, U_P]$ is faithful. This will be done in this section.

For a root $\beta \in \Phi^+$ denote by $m_\alpha(\beta)$ the coefficient at α in decomposition of β to a linear combination of simple roots. Put $Y_\alpha(k) = \{\beta \in \Phi \mid m_\alpha(\beta) = k\}$.

Lemma 4.1. *For a root $\beta \in \pm Y_\alpha(0)$ there exists a root $\gamma \in Y_\alpha(1)$ such that $\beta + \gamma \in \Phi$ (in this case $\beta + \gamma$ automatically belongs to $Y_\alpha(1)$).*

Proof. Let $\beta \in Y_\alpha(0)$ and let $\beta = \sum_{i=1}^k m_k \beta_k$ be its presentation as a linear combination of simple roots. On the Dynkin diagram take a chain $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_n = \beta_j$ connecting α with a nearest simple root β_j , so that $\alpha_h \neq \beta_i$ for all $h = 0, \dots, n-1$ and $i = 1, \dots, k$. Then $\gamma = \alpha_0 + \alpha_1 + \dots + \alpha_{n-1} \in Y_\alpha(1)$ and the inner product $\langle \gamma, \beta \rangle = \langle \alpha_{n-1}, m_j \beta_j \rangle$ is negative. Therefore, $\beta + \gamma \in \Phi$ and, moreover, $\beta + \gamma \in Y_\alpha(1)$.

If $\beta \in -Y_\alpha(0)$, then by the first paragraph of the proof there exists $\gamma \in Y_\alpha(1)$ such that $\gamma - \beta \in Y_\alpha(1)$ and $\beta + (\gamma - \beta)$ is a root. \square

Lemma 4.2. *The set $Y_\alpha(1)$ generates the linear span of Φ .*

Proof. By the previous lemma for any simple root $\beta \neq \alpha$ there is a root $\gamma \in Y_\alpha(1)$ such that $\gamma + \beta = \delta \in Y_\alpha(1)$. Hence all simple roots lie in the span of $Y_\alpha(1)$ which implies the result. \square

Throughout this section we assume that all structure constants $N_{\alpha, \beta, i, j}$ in the Chevalley commutator formula for $G(\Phi, _)$ are invertible in S , i.e. 2 is invertible if $\Phi = C_l, B_l, F_4$ and 6 is invertible if $\Phi = G_2$.

Let $\alpha \in \Pi$ and let P be the maximal standard parabolic subgroup of type $\Pi \setminus \{\alpha\}$. Then the Levi subgroup L_P of P (i.e. the reductive part of P) has root system $\Delta = \pm Y_\alpha(0)$. The root system of the unipotent radical U_P of P is equal to $\bigcup_{i>0} Y_\alpha(i)$. It follows that the group of points $U_P(R)$ of U_P over a ring R is generated by $x_\alpha(r)$ for all $\alpha \in \bigcup_{i>0} Y_\alpha(i)$ and $r \in R$.

By [5, Lemma 4] the group $[U_P(A), U_P(A)]$ is generated by $X_\beta(A)$ for all $\beta \in \bigcup_{i>0} Y_\alpha(i)$ and $U_P(A)/[U_P(A), U_P(A)] \cong \prod_{\beta \in Y_\alpha(1)} X_\beta(A)$ has a natural structure of a free A -module (see [5, Theorem 2]).¹ The action of L_P on U_P by conjugation gives rise to the action of L_P on this free module. This action is called the representation of $L_P(A)$ in the internal Chevalley module, it is functorial with respect to A .

Choose a basis of $U_P(A)/[U_P(A), U_P(A)]$ consisting of $x_\gamma(1)$ for all $\gamma \in Y_\alpha(1)$. Thus, we get a morphism of group schemes $\pi : L_P \rightarrow GL_m$. Denote by K the kernel of π . First, we show that $K \cap TX_\delta$ is central for all $\delta \in \Delta$.

Lemma 4.3. *For any S -algebra A the intersection $K(A) \cap T(A)X_\alpha(A)$ is central.*

¹ The fact that $A = K$ is a field is not used in the proof of these statements.

Proof. Let $tx_\beta(r) \in L_P(A)$, for some $\beta \in \Delta$, $r \in A$ and $t \in T$. By Lemma 4.1 there exists $\gamma \in Y_\alpha(1)$ such that $\beta + \gamma \in Y_\alpha(1)$. Calculation shows that $x_\gamma(1)^{tx_\beta(r)} = x_\gamma(\gamma(t))x_{\beta+\gamma}(N_{\gamma,\beta,1,1}\gamma(t)r) \cdots$, where dots denote a product of certain elements from other elementary root subgroups. If $tx_\beta(r) \in K(A)$, then $N_{\gamma,\beta,1,1}\gamma(t)r = 0$ and, since $N_{\gamma,\beta,1,1}$ is invertible in A , we get $r = 0$. It follows that t centralizes $U_P(A)/[U_P(A), U_P(A)]$, therefore $x_\gamma(1)^t = x_\gamma(\gamma(t)) = x_\gamma(1)$, i.e. $\gamma(t) = 1$ for all $\gamma \in Y_\alpha(1)$. By Lemma 4.2 we conclude that $\gamma(t) = 1$ for all $\gamma \in \Phi$, hence t lies in the center of $G(A)$. \square

Lemma 4.4. *Let F be a field and an S -algebra. Then $K(F)$ is central.*

Proof. Denote by \bar{F} the algebraic closure of F . Then $L_P(\bar{F}) = T(\bar{F})(H_1 \times H_2)$, where each H_i is either trivial or a Chevalley group over \bar{F} , and is normalized by the torus. By the main theorem of [27] the quotient of H_i by the center $\text{Center}(H_i)$ is a simple group. Suppose that there exists an element $th_1h_2 \in L_P(\bar{F}) \cap K(\bar{F})$ such that h_1 is not central in H_1 . By standard group theoretical arguments there exists an element $g \in H_1$ such that $[g, h_1] \notin \text{Center}(H_1)$. It follows that $[K(\bar{F}), H_1]$ is a noncentral normal subgroup of H_1 . It follows that it contains the product of a central element with an elementary root unipotent element which is impossible by Lemma 4.3. Hence h_1 is central. By the same arguments h_2 is central in H_2 . Since the center of H_1 and H_2 are contained in the torus, $K(\bar{F}) \leq T(\bar{F})$. Again by Lemma 4.3 we deduce that $K(\bar{F})$ lies in the center of $G(\bar{F})$.

Since the natural map $G(F) \rightarrow G(\bar{F})$ is injective, $K(F)$ lies in the center of $G(F)$. \square

The following lemma is proved in [34] for the unipotent radical of a Borel subgroup of a Chevalley group. The proof for a Borel subgroup of a split reductive algebraic group is essentially the same.

Lemma 4.5. *Let L be a split reductive algebraic group with a split maximal torus T . Suppose that the root system Ψ of L is irreducible. Denote by B a Borel subgroup of L containing T . Let H be a normal subgroup of L and R a commutative ring. If $H(R) \cap B(R)$ contains a noncentral element, then H contains a nontrivial elementary root unipotent element.*

Proposition 4.6. *Let $S = \mathbb{Z}[\frac{1}{2}]$ if $\Phi = B_1, C_1, F_4$, $S = \mathbb{Z}[\frac{1}{6}]$ if $\Phi = G_2$, and $S = \mathbb{Z}$ otherwise. Let P be a standard maximal parabolic subgroup of a Chevalley group G of adjoint type over S . Then the representation of L_P in the internal Chevalley module $U_P/[U_P, U_P]$ is faithful.*

Proof. We have to prove that $K(A)$ is trivial for any S -algebra A . Let $h \in K(A)$. Since $K(A)$ is normal in $L_P(A)$, all commutators $[h, x_\delta(r)]$ lie in $K(A)$. If h is central, then it belongs to $T(A)$ and by Lemma 4.3 equals to 1. Otherwise, $g = [h, x_\delta(r)] \in K(A)$ is not central in $L(A, J)$ for some $\delta \in \Delta$ and $r \in A$ (see [2]). Note that g belongs to an irreducible component of $L_P(A)$. Denote this component by L and its root system by Ψ .

By Lemma 4.4 K vanishes modulo all maximal ideals. Therefore g lies in the principle congruence subgroup $L(A, J)$, where J denotes the Jacobson radical of A . Put $\Psi^+ = \Phi^+ \cap \Psi$. By the Gauss decomposition (see [3, Proposition 2.3]) g can be written in the form $g = ub$, where b lies in the standard Borel subgroup of L and u in the unipotent radical of the opposite Borel subgroup. Denote by V the free A -module $U_P(A)/[U_P(A), U_P(A)]$. Let γ be a root of minimal height in the set $\{\delta \in Y_\alpha(1) \mid x_\delta(1)^b \neq x_\delta(1)\}$. Put $Z_\gamma = \{\delta \in Y_\alpha(1) \mid \text{ht } \delta < \text{ht } \gamma\}$. Since u is a product of $x_\beta(r_\beta)$, as β ranges over Ψ^- and $r_\beta \in A$, we have

$$x_\gamma(1)^u \equiv x_\gamma(1) \prod_{\delta \in Z_\gamma} x_\delta(r_\delta) \pmod{[U_P(A), U_P(A)]}$$

for some $r_\delta \in A$. In the module V we have

$$x_\gamma(1) = x_\gamma(1)^g = \left(x_\gamma(1) \prod_{\delta \in Z_\gamma} x_\delta(r_\delta) \right)^b.$$

By minimality of γ the element b stabilizes all $x_\delta(r_\delta)$. Therefore, $x_\gamma(1) = x_\gamma(1)^b$ which contradicts the choice of γ . The contradiction shows that b acts trivially on V . It follows that both b and u belong to $K(A) \cap L(A)$ which is a normal subgroup of a split reductive group $L(A)$ with an irreducible root system.

By Lemma 4.3 $K(A)$ contains no elementary root unipotent elements. Now, Lemma 4.5 shows that b and u lies in the center of $L(A)$ which is contained in the torus $T(A)$. Since $K(A) \cap T(A)$ is trivial, $g = 1$. \square

5. Inside a parabolic subgroup

In this section we prove Lemma 2 from the introduction (= Lemma 5.2). For this we still need the assumption that all structure constants $N_{\alpha\beta ij}$ are invertible in the ground ring S . Let H be an overgroup of $E(S)$ with associated subring $R = q(H)$. Let P be a standard maximal parabolic subgroup, and $g \in P \cap H$. We have to prove that $g \in N_A(R)$.

Write $g = ab$, where a belongs to the Levi subgroup L_P of P and b is in the unipotent radical U_P of P . The action of g on the internal Chevalley module $U_P/[U_P, U_P]$ coincides with the action of a . If $x_\gamma(1) \in U_P$, then $x_\gamma(1)^a \in H \cap U_P$. Lemma 5.1 shows that in this case $x_\gamma(1)^a \in G(R)$. If G is of adjoint type, then Proposition 4.6 shows that the representation of L_P in the internal Chevalley module $U_P/[U_P, U_P]$ is faithful and by Lemma 2.1 $a \in G(R)$. In general we use the fact that the kernel of the canonical morphism $G \rightarrow G_{ad}(\Phi, _)$ is central to show that $a \in N_A(R)$. Finally we apply Lemma 5.1 again to show that $b \in N_A(R)$.

Let U denote the unipotent radical of the standard Borel subgroup, i.e. $U(A)$ is generated by all $x_\alpha(s)$, where $\alpha \in \Phi^+$ and $s \in A$. Let H be a subgroup of $G(A)$, normalized by $E(S)$.

Lemma 5.1. *Let $R = q(H)$. Then $U(A) \cap H \leq E(R)$. Moreover, if $g \in U$ and $E(R)^g \leq H$, then $g \in E(R)$.*

Proof. We prove the “moreover” statement. Let $g = \prod_{\alpha \in \Phi^+} x_\alpha(s_\alpha) \in U$ and $E(R)^g \in H$. Here we assume that the ordering of factors agrees with the height of roots. We have to prove that $s_\alpha \in R$ for any $\alpha \in \Phi^+$. Denote by $U^{(h)}$ the subgroup of $G(A)$ generated by all $x_\alpha(s)$ with $s \in A$ and $\alpha \in \Phi^+$ of height $\geq h$. Let n be the largest integer such that $g \in U^{(n)}$. We proceed by induction on $m - n$ where m denotes the height of the maximal root. If $m - n = 0$, then $g = x_\gamma(s)$, where γ is the maximal root which is long. There exists a root δ such that $\gamma + \delta \in \Phi$. Then $[g, x_\delta(1)] = x_{\gamma+\delta}(\pm s) \prod_{i>1} x_{\gamma+i\delta}(\ast) \in H$ and Lemma 1.1 shows that $x_{\gamma+\delta}(\pm s) \in H$, i.e. $s \in R$.

Now, let $m - n > 0$. Take a factor $x_\beta(s_\beta)$ of g with β of height n . There exists a simple root $\delta \in \Pi$ such that $\beta + \delta$ is a root. Consider the element $g' = [g, x_\delta(1)] \in H \cap U^{(n+1)}$. Modulo $U^{(n+2)}$ this element is a product of commutators $[x_\alpha(s_\alpha), x_\delta(1)] \equiv x_{\alpha+\delta}(\pm N_{\alpha\delta 11} s_\alpha) \pmod{U^{(n+2)}}$ over all α of height n such that $\alpha + \delta \in \Phi$. Therefore, $x_{\beta+\delta}(\pm N_{\beta\delta 11} s_\beta)$ is a factor of g' and, by induction arguments, $\pm N_{\beta\delta 11} s_\beta \in R$. Since $N_{\beta\delta 11}$ is invertible in R , we get $s_\beta \in R$ for any root β of height n . Multiplying g by $x_\beta(-s_\beta)$ for all β of height n we get an element $g'' \in U^{(n+1)}$ such that $E(R)^{g''} \in H$. By induction hypothesis $g'' \in E(R)$. Thus, $g \in E(R)$. \square

Lemma 5.2. *Let H be a subgroup of $G(A)$, containing $E(S)$, and let $R = q(H)$ be an S -algebra, associated with H . Suppose that $g \in H$ belongs to a standard maximal parabolic subgroup P of $G(A)$. Then $g \in N_A(R)$.*

Proof. Let U_P be the unipotent radical of P and L_P its Levi subgroup. Then $g = ab$ for some $b \in U_P(A)$ and $a \in L_P(A)$. For any $d \in U_P(R)$ the element d^g belongs to $H \cap U_P(A)$. By Lemma 5.1, $d^g \in E(R)$.

First, suppose that $G = G_{ad}(\Phi, _)$. Consider a representation π of the group P in the internal Chevalley module $U_P/[U_P, U_P]$. Clearly, U_P lies in $\text{Ker } \pi$. By Proposition 4.6 the restriction of π on L_P is faithful. We have $\pi(a) = \pi(g) \in \text{GL}_m(R)$. By Lemma 2.1 $a \in G(R)$. In particular, a normalizes $E(R)$, and therefore, $E(R)^b = E(R)^g \leq H$. Now, Lemma 5.1 shows that $b \in E(R)$, and thus, $g \in G(R)$.

Now, let G have an arbitrary weight lattice and let $\rho : G \rightarrow G_{ad}(\Phi, _)$ be the canonical morphism of schemes. Then $C(A) = \text{Ker } \rho_A$ is central for any ring A . Let $\tilde{H} = \rho_A(H)$. Note that $q(\tilde{H}) = R$. Indeed,

if $x_\alpha(r) \in \bar{H}$, then $cx_\alpha(r) \in H$ for some central element c . Take $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$. Then $[cx_\alpha(r), x_\beta(1)] = [x_\alpha(r), x_\beta(1)]$ and Lemma 1.1 shows that $x_{\alpha+\beta}(r) \in H$, i.e. $r \in R$.

We have already proved that $\rho_A(g)$ belongs to $G_{\text{ad}}(\Phi, q(\bar{H})) = G_{\text{ad}}(\Phi, R)$. In particular, $E_{\text{ad}}(\Phi, R)^g = E_{\text{ad}}(\Phi, R)$. Since the image of $E(R)$ under ρ_A is equal to $E_{\text{ad}}(\Phi, R)$, we have $E(R)^g \leq E(R)C(A)$. It follows that $E(R)^g = [E(R)^g, E(R)^g] \leq [E(R)C(A), E(R)C(A)] = E(R)$, i.e. $g \in N_A(R)$. \square

6. Sandwich classification theorem

Recall that a noncentral semisimple element h of $G(A)$ is called *small* if it is annihilated by all long roots, see [11]. In other words, h is conjugate to an element $h' \in T$ such that h' commutes with all long root subgroups. Clearly, if all roots in Φ have the same length, then there are no small semisimple elements.

Lemma 6.1. *Let R be a ring such that $2 \neq 0$ in R . Let $\Phi = B_n, C_n, F_4$. Suppose that if $\Phi = B_{2k+1}$ ($k \geq 1$), then -1 is a square in R . Then, there exists a small semisimple element $h \in T(R) \cap E(R)$ defined over \mathbb{Z} or $\mathbb{Z}[\sqrt{-1}]$. For any root $\alpha \in \Phi$ we have $\alpha(h) = \pm 1$.*

Proof. Any root semisimple element $h_\alpha(-1) \in T(R)$ by definition belongs to $E(R)$ and is noncentral if $\Phi \neq A_1$. Note that $h_\alpha(-1)$ commutes with X_α . If $\Phi = C_n$ ($n \geq 2$), then $h_\alpha(-1)$ is small semisimple for any long root α . Indeed, all long roots in C_n are orthogonal, therefore $h_\alpha(-1)$ commutes with all long root subgroups.

In the case $\Phi = B_{2k}$ consider the element $h = \prod_{i=1}^k h_{\alpha_{2i-1}}(-1)$, where $\alpha_1, \dots, \alpha_{2k-1} \in \Pi$ are long fundamental roots. Since $\alpha_{2i-1} \perp \alpha_{2j-1}$ for all $i \neq j$, the element h commutes with root subgroups $X_{\alpha_{2i-1}}$. On the other hand,

$$\alpha_{2i}(h) = \alpha_{2i}(h_{\alpha_{2i-1}}(-1)h_{\alpha_{2i+1}}(-1)) = (-1)(-1) = 1.$$

Thus, h is small semisimple.

The set of all long roots in $\Phi = F_4$ is contained in a root system of type B_4 . Clearly, the image in $E(F_4, R)$ of the small semisimple element of $E(B_4, R)$ constructed above is small semisimple.

Finally, if $\Phi = B_{2k+1}$ and -1 is a square in R , then a small semisimple element in $T(R) \cap E(R)$ is shown in Section 2 of Gordeev.

Since h is defined over \mathbb{Z} or over $\mathbb{Z}[\sqrt{-1}]$, to prove the last assertion of the lemma it suffices to show that $\beta(h)^2 = 1$ for all short roots $\beta \in \Phi$. Indeed, as our root system is doubly laced, there exists a long root $\alpha \in \Phi$ such that $\alpha + 2\beta \in \Phi$ is a long root. Then $\beta(h)^2 = (\alpha + 2\beta)(h) = 1$. \square

The following lemma is a key technical step for the proof of the main theorem. It shows why we can prove the theorem only for doubly laced root systems.

Lemma 6.2. *(See Gordeev [11], Nesterov and Stepanov [16].) Let $\Phi = B_n, C_n$ or F_4 . Let T be a split maximal torus, α a long root, and $g \in G(A)$. If $h \in T(R)$ is a small semisimple element, then $X_\alpha^{h^g}$ commutes with X_α . Hence $X_\alpha^{h^g}$ is contained in a standard proper parabolic subgroup of $G(A)$.*

Remark. The lemma was proved in [11,16] over an algebraically closed field. But it is easy to see that an identity with constants is inherited by subrings and quotient rings, and any ring is a quotient of a polynomial ring which is a subring of a closed field.

The following result is a consequence of the normal structure of a Chevalley group, see [1] or [29]. However, it is easier to give a direct proof than to deduce the lemma from the normal structure theorem.

Lemma 6.3. Let G be a Chevalley–Demazure group scheme with root system $\Phi = B_l, C_l, F_4$ and E its elementary subgroup functor. Let R be a ring, T a split maximal torus, and $h \in T(R)$ a small semisimple element defined over \mathbb{Z} or $\mathbb{Z}[\sqrt{-1}]$. Suppose that 2 is invertible in R . If H is a subgroup of $G(R)$ normalized by $E(R)$ and $h \in H$, then $H \geq E(R)$.

Proof. Since h is noncentral, by Lemma 6.1 there exists $\beta \in \Phi$ such that $\beta(h) = -1$. Then $[x_\beta(\frac{1}{2}), h] = x_\beta(-1) \in H$. By Lemma 1.2 with $S = R$ we have $H \geq E(R)$. \square

Now we are prepared to prove the main result of the article.

Theorem 6.4. Let G be a Chevalley–Demazure group scheme with root system Φ , where $\Phi = B_l, C_l$ or F_4 and $l \geq 2$. Let $S \subseteq A$ be a pair of rings such that 2 is invertible in S . In the case $\Phi = B_{2k+1}$ ($k \geq 1$) suppose in addition that -1 is a square in S . Then, given a subgroup $H \leq G(A)$ containing $E(S)$ there exists a unique subring $R \subseteq A$ containing S such that

$$E(R) \leq H \leq N_A(R).$$

Proof. Let $R = q(H)$ be the subring associated with H , so that $E(R) \leq H$. For an element $g \in H$ we show that $g \in N_A(R)$.

Let $h \in G(R)$ be a small semisimple element from a chosen split maximal torus T . Let α be a long root. Take two arbitrary elements a, b from $E(R)$ and consider the element $c = x_\alpha(1)^{h^{agb}} \in H$. By Lemma 6.2 this element belongs to a standard proper parabolic subgroup and by Lemma 5.2 $c \in N_A(R)$. Rewrite c in the form

$$c = b^{-1}(g^{-1}(a^{-1}h^{-1}a)g(bx_\alpha(1)b^{-1})g^{-1}(a^{-1}ha)g)b \in N_A(R).$$

Since $b \in E(R)$, we have

$$bcb^{-1} = (bx_\alpha(1)b^{-1})^{g^{-1}(a^{-1}ha)g} \in N_A(R).$$

Fix a and let b vary. The subgroup generated by $\{bx_\alpha(1)b^{-1} \mid b \in E(R)\}$ is normal in $E(R)$ and contains $x_\alpha(1)$. By Lemma 1.2 with $S = R$ it must coincide with $E(R)$. Thus, $E(R)^{g^{-1}(a^{-1}ha)g} \leq N_A(R)$, and by Lemma 3.6 $(a^{-1}ha)^g \in N_A(R)$.

Again, elements of the form $a^{-1}ha$ generate a subgroup normalized by $E(R)$ as a ranges over $E(R)$. Since this subgroup contains h , by Lemma 6.3 it contains $E(R)$. Therefore, $E(R)^g \in N_A(R)$. By Lemma 3.6 one has $g \in N_A(R)$. Thus, an arbitrary element from H is contained in $N_A(R)$, i.e. $H \leq N_A(R)$.

It remains to show that each subgroup H belongs to a unique sandwich $L(E(R), N_A(R))$. By Lemma 1.3 the normal closure of $E(S)$ in $E(R)$ equals to $E(R)$. Therefore,

$$E(R) = E(S)^{E(R)} \leq E(S)^H \leq E(R)^{N_A(R)} = E(R),$$

i.e. $E(S)^H = E(R)$. Since $E(R) = E(R')$ implies $R = R'$, two sandwiches $L(E(R), N_A(R))$ and $L(E(R'), N_A(R'))$ have empty intersection for different subrings R and R' . \square

7. Subgroups normalized by $E(S)$

Here we use group theoretic arguments developed in [23] to extend the description of subgroups, containing $E(S)$, to subgroups, normalized by $E(S)$.

Recall some definitions introduced by Z.I. Borevich in [8]. Let D be a subgroup of G . A subgroup F of G is called D -full if $D^F = F$. A subgroup D is called *polynormal* in G if for each subgroup H

between D and G the subgroup D^H is D -full, i.e. $D^{D^H} = D^H$. In terms of distribution of intermediate subgroups the definition of polynormality can be reformulated as follows. A subgroup D is polynormal in G if given a subgroup H such that $D \leq H \leq G$ there exists a unique D -full subgroup F such that $F \leq H \leq N_G(F)$. One notice that Theorem 6.4 asserts that $E(S)$ is polynormal in $E(A)$ and the subgroups $E(R)$ for all subrings $R: A \subseteq R \subseteq S$ exhaust all $E(S)$ -full subgroups.

The notion of polynormality was extended in [23] to take into account all subgroups normalized by D . Namely, a subgroup F of G is called D -perfect if $[F, D] = F$. A subgroup D is called *strongly polynormal* in G if for any subgroup $H \leq G$, normalized by D , the group $[H, D]$ is D -perfect. In other words, given $H \leq G$, normalized by D , there exists a unique D -perfect subgroup F such that $F \leq H \leq C_{G,D}(F)$, where $C = C_{G,D}(F)$ is the largest subgroup normalizing F and satisfying the property $[C, D] \leq F$.

It is an exercise to show, using the Hall–Witt commutator identity, that a polynormal perfect subgroup is strongly polynormal, see Theorem 1 of [23]. In the following proposition we observe that in this case D -perfect subgroups are known as soon as the normal structure of each D -full subgroup is established.

Proposition 7.1. *Let D be a perfect polynormal subgroup of G with the set $\{F_i \mid i \in I\}$ of D -full subgroups. Then for any subgroup $H \leq G$, normalized by D , there exist $i \in I$ and an F_i -perfect subgroup $F \leq F_i$ such that $F \leq H \leq C_{G,D}(F)$. Moreover, $C_{G,D}(F)$ is the largest subgroup of G such that $[C_{G,D}(F), F_i] = F$.*

Proof. By Theorem 1 of [23] D is strongly polynormal in G . Thus, it suffices to show that a D -perfect subgroup F is an F_i -perfect subgroup of F_i for some index i . Indeed, FD is a D -full subgroup, and therefore, coincides with F_i for some index i . Thus, $[F_i, F] = [D, F] \cdot [F, F] = F$.

Since $F_i = FD$, the characterization of $C_{G,D}(F)$ in the proposition is equivalent to its definition. \square

Now we return to the study of subgroups in a Chevalley group and formulate the corollary of the main theorem for subgroups normalized by $E(S)$. Recall that $E(R, \mathfrak{q})$ denotes the relative elementary subgroup of $E(R)$ corresponding to an ideal \mathfrak{q} .

Theorem 7.2. *Let G be a Chevalley–Demazure group scheme with root system Φ , where $\Phi = B_l, C_l$ or F_4 and $l \geq 2$. Let $S \subseteq A$ be a pair of rings such that 2 is invertible in S . In the case $\Phi = B_{2k+1}$ ($k \geq 1$) suppose in addition that -1 is a square in S . Given a subgroup $H \leq G(A)$ normalized by $E(S)$ there exists a unique subring $R \subseteq A$ containing S and an ideal \mathfrak{q} of R such that*

$$E(R, \mathfrak{q}) \leq H \leq C_A(R, \mathfrak{q}),$$

where $C_A(R, \mathfrak{q})$ is the largest subgroup of $G(A)$ satisfying condition $[C_A(R, \mathfrak{q}), E(R)] = E(R, \mathfrak{q})$.

Proof. By Theorem 6.4 the group $E(S)$ is polynormal in $G(A)$ and $\{E(R) \mid S \subseteq R \subseteq A\}$ is the set of all $E(S)$ -full subgroups. By the normal structure of $E(R)$ (see [29] or [1]) we know that only relative elementary subgroups are $E(R)$ -perfect. Now the result follows from Proposition 7.1. \square

Finally, we characterize elements of the group $C_A(R, \mathfrak{q})$ in a faithful absolutely irreducible representation by congruences. As usual, δ_{ij} denotes the Kronecker symbol.

Proposition 7.3. *Let $\varphi : G \rightarrow GL_n$ be a faithful representation, absolutely irreducible over S . Under conditions of the corollary the group $C_A(R, \mathfrak{q})$ is an abelian extension of the principal congruence subgroup $G(R, \mathfrak{q})$ and its elements can be characterized as follows. An element $g \in G(A)$ belongs to $C_A(R, \mathfrak{q})$ if and only if for all indexes $h, i, j, k \in \{1, \dots, n\}$*

$$g'_{hi} g_{jk} \equiv \delta_{hi} \delta_{jk} \pmod{\mathfrak{q}}.$$

Moreover, both $E(R, \mathfrak{q})$ and $G(R, \mathfrak{q})$ are normal in $N_A(R)$ and $C_A(R, \mathfrak{q})$ is the full preimage of the center under the canonical homomorphism $N_A(R) \rightarrow N_A(R)/G(R, \mathfrak{q})$.

Proof. Since φ is absolutely irreducible, $E(R)$ generates $M_n(R)$ as an R -module. In particular, given $i, j \in \{1, \dots, n\}$ a matrix unit is a linear combination $e_{ij} = \sum_{p=1}^m s_p a_p$ for some $s_1, \dots, s_m \in R$ and $a_1, \dots, a_m \in E(R)$. For $g \in C_A(R, \mathfrak{q})$ we have $g^{-1} a_p g \in a_p E(R, \mathfrak{q})$. It follows that $g^{-1} a_p g \in a_p + M_n(\mathfrak{q})$. Conjugating e_{ij} by g we get

$$g^{-1} e_{ij} g = \sum_{p=1}^m s_p g^{-1} a_p g \equiv \sum_{p=1}^m s_p a_p \equiv e_{ij} \pmod{M(\mathfrak{q})}.$$

Hence $g'_{hi} g_{jk} \equiv \delta_{hi} \delta_{jk} \pmod{\mathfrak{q}}$.

Now, let C be the subset of $G(A)$ consisting of matrices satisfying the above congruences. It is easy to see that C is a subgroup and we have already proved that $C_A(R, \mathfrak{q}) \leq C$. Let $g \in C$ and $a \in N_A(R)$. Put $b = [g, a]$. Then $b_{hm} = \sum_{i,j,k=1}^n g'_{hi} a'_{ij} g_{jk} a_{km}$. If $h \neq i$ or $j \neq k$, then $g'_{hi} g_{jk} \in \mathfrak{q}$, and by Lemma 3.6 $a'_{ij} a_{km} \in R$. Therefore,

$$b_{hm} \equiv \sum_{k=1}^n g'_{hh} a'_{hk} g_{kk} a_{km} \equiv \sum_{k=1}^n a'_{hk} a_{km} \equiv \delta_{hm} \pmod{\mathfrak{q}},$$

which means that $b \in G(R, \mathfrak{q})$. Hence $[C, N_A(R)] \leq G(R, \mathfrak{q})$. In particular, C and $G(R, \mathfrak{q})$ are normal in $N_A(R)$, C is an abelian extension of $G(R, \mathfrak{q})$, and $[C, E(R)] \leq G(R, \mathfrak{q})$. By the standard commutator formula (see [29])

$$[[C, E(R)], E(R)] \leq [G(R, \mathfrak{q}), E(R)] = E(R, \mathfrak{q})$$

and the converse inclusion is obvious. Being perfect normal subgroup, $E(R)$ is strongly polynormal in $N_A(R)$. Therefore,

$$[C, E(R)] = [[C, E(R)], E(R)] = E(R, \mathfrak{q}).$$

It follows from the maximality of $C_A(R, \mathfrak{q})$ that $C = C_A(R, \mathfrak{q})$.

Now, let C' be the full preimage of the center of $N_A(R)/G(R, \mathfrak{q})$. The inclusion $[C, N_A(R)] \leq G(R, \mathfrak{q})$ shows that $C \leq C'$. Conversely, since $[C', N_A(R)] \leq G(R, \mathfrak{q})$ we have

$$[C', E(R)] = [[C', E(R)], E(R)] \leq [G(R, \mathfrak{q}), E(R)] = E(R, \mathfrak{q}).$$

Since $C = C_A(R, \mathfrak{q})$ is the largest subgroup with this property, we have $C' = C$.

Finally, since $G(R, \mathfrak{q})$ and $E(R)$ are normal in $N_A(R)$, we conclude that $E(R, \mathfrak{q}) = [G(R, \mathfrak{q}), E(R)]$ is also normal. \square

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