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Some New Results on Ordered Fields

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INTRODUCTION

This paper is organized as follows. In the first section we construct an ordered non-archimedean field whose unique automorphism is the identity. Obviously, archimedean ordered fields have a unique automorphism (order preserving). The converse was first studied in [1], and we show here that it not true.

In the second section we are concerned with real closed fields. We prove that Bolzano's theorem and the maximum principle for poynomials characterize real closed fields.

In the last section we continue the study of fields with the extension property (EP) initiated in [7]. A field F has the extension property if every automorphism of $F(x)$, where x is transcendent over F, is the extension of an automorphism of F. These fields play a crucial role in the study of "homogeneity" conditions in the space of orderings of a field (see [7]), and we prove here some new results about them. In particular, we show that pythagorean fields and n-maximal fields (see [6]) have the extension property. We end this section by proposing two questions about EP fields.

1. A NON-ARCHIMEDEAN ORDERED FIELD WITH A UNIQUE AUTOMORPHISM

Archimedean real closed fields have, of course, a unique automorphism. However, in the non-archimedean case one does not know that an automorphism has to fix a subfield over which the real closed field is algebraic. As far as I know, the following remains open:

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QUESTION. *Does there exist a non-archimedean real closed field with a unique automorphism?*

In the paper an *ordered field* is a couple (K, P), where K is a field and P is a subset of K, verifying

$$P+P\subset P; \qquad P\cdot P\subset P; \qquad P\cap(-P)=\{0\}; \qquad P\cup(-P)=K.$$

We shall use $a\leqslant_P b$ (resp. $a<_P b$) instead of $b-a\in P$ (resp. $b-a\in P\backslash\{0\}$).

We devote this section to the construction of a non-archimedean ordered field (K, P) with a unique automorphism.

We take $K=\mathbb{R}(x, y)$, where $\mathbb{R}$ is the field of real numbers and x, y are algebraically independent elements over $\mathbb{R}$.

Given an ordering P in K and $f\in K$, we set the following [5]:

$$D(f)=\{a\in\mathbb{R}: a<_P f\}.$$

Also we set

$$A=\{f\in K: \varnothing\neq D(f)\neq\mathbb{R}\}$$

and $\mathfrak{m}=\{f\in A: D(f)$ equals $(\leftarrow, 0)$ or $(\leftarrow, 0]\}$. A is a valuation ring with maximal ideal $\mathfrak{m}$.

Clearly, $P^*=\{f+\mathfrak{m},\ f\in P\cap A\}$ is an ordering in the residual field $A/\mathfrak{m}$ and $(A/\mathfrak{m}, P^*)$ is archimedean over $\mathbb{R}$ via $\mathbb{R}\to A/\mathfrak{m}$: $r\to r+\mathfrak{m}$. So, we identify $\mathbb{R}$ with $A/\mathfrak{m}$ and consider the signed place $p: K\to(A/\mathfrak{m}=\mathbb{R})\cup\{+\infty, -\infty\}$ associated with P, i.e.,

$$p(f)=\begin{cases} f+\mathfrak{m} & f\in A,\\ +\infty & f\in K\backslash A, f>_P 0,\\ -\infty & f\in K\backslash A, f<_P 0.\end{cases}$$

If P is centered at the origin, i.e., $p(x)=p(y)=0$, the place p admits a unique extension, which will also be called p, to $F(y)$, F being the real closure of $(\mathbb{R}(x), P\cap\mathbb{R}(x))$,

Let q be a large enough natural number. We have proved in [2] the existence of a unique ordering P in K such that

(1) $p(x)=p(y)=0, \qquad x, y\in P;$

(2) $\sup\left\{\dfrac{m}{n}\in\mathbb{Q}^+: \dfrac{y^m}{x^n}\notin A\right\}=\dfrac{2}{5}, \qquad p\left(\dfrac{y^2}{x^5}\right)=1;$

$$(3)\quad y^2-x^5\in P,\qquad \sup\left\{\frac{m}{n}\in\mathbb{Q}^+ : \frac{(y^2-x^5)^m}{x^n}\notin A\right\}=\frac{2}{q},\qquad [*]$$

$$p\left(\frac{(y^2-x^5)^2}{x^q}\right)=1;$$

$$(4)\quad (y^2-x^5)^2-x^q\in P,\qquad \left\{\frac{m}{n}\in\mathbb{Q}^+ : \frac{[(y^2-x^5)^2-x^q]^m}{x^n}\notin A\right\}=\varnothing.$$

Obviously, (K, P) is non-archimedean and we will show that (K, P) has a unique automorphism. Let σ be an automorphism of (K, P). We will see that $\sigma(\mathbb{R})\subset\mathbb{R}$. Otherwise $M=\{a\in\mathbb{R}^+ : \sigma(a)=f_a/g_a\in K\backslash R\}\neq\varnothing$ and we choose $a\in M$ such that [total degree of f_a + total degree of g_a] $=s\neq 0$ is minimum in M. Then, if b is the positive square root of a, we have $b\in M$, $f_a/g_a=\sigma(a)=\sigma(b)^2=f_b^2/g_b^2$, and $s=2s'$, where $s'=$ (total degree of f_b + total degree of g_b), which contradicts the choice of a. Moreover, since $\sigma(\mathbb{R})\subset\mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, we conclude that $\sigma/\mathbb{R}=\mathrm{id}$.

Now, applying Theorem 5.2 of [10], note that σ can be extended to an automorphism of $K(\sqrt{-1})=\mathbb{C}(x, y)$, and we know that

$$\sigma(x)=\frac{A_1+B_1x+C_1y+D_1x^2+E_1y^2+F_1xy}{A_3+B_3x+C_3y+D_3x^2+E_3y^2+F_3xy}$$

$$\sigma(y)=\frac{A_2+B_2x+C_2y+D_2x^2+E_2y^2+F_2xy}{A_4+B_4x+C_4y+D_4x^2+E_4y^2+F_4xy},$$

where all the coefficients are real numbers.

Clearly, if we write $t=\sigma(x)$ and $u=\sigma(y)$, conditions [*] must hold, changing x and y by t and u, respectively. Thus $p(t)=p(u)=0$ and so $A_1=A_2=0$.

We will prove that A_3 and A_4 are not zero. For instance, suppose $A_3=0$. Then $B_1=0$ because $p(t)=0$. Since σ is an automorphism, B_1 and B_2 cannot vanish simultaneously, and so $B_2\neq 0$. But $0=p(u/t^2)=B_2\cdot B_3^2$. It follows that $B_3=0$.

Now, $p(t)=0$ implies $C_1=D_1=0$, and so

$$0=p(u/t)=p\left(\frac{\begin{pmatrix}(B_2+C_2y/x+D_2x+E_2y^2/x\\ \quad +F_2y)(C_3+D_3x^2/y+E_3x/y+F_3x)\end{pmatrix}}{(A_4+B_4x+C_4y+D_4x^2+E_4y^2+F_4xy)(E_1y/x+F_1)}\right)$$

implies $D_3=0$ and $\sigma(x)=(E_1y+F_1x)/(E_3y+F_3x)=t$. Again from $p(t)=0$ we deduce now that $F_1=0$, $E_1\neq 0$. Then,

$$1=p\left(\frac{u^2}{t^5}\right)=p\left(\frac{\begin{pmatrix}(B_2+C_2y/x+D_2x+E_2y^2/x\\ \quad +F_2y)^2(E_3y/x+F_3)^5\end{pmatrix}}{(A_4+B_4x+C_4y+D_4x^2+E_4y^2+F_4xy)^2E_1^5y^5/x^7}\right)$$

together with $B_2 \neq 0$ allows us to conclude that $F_3 = 0$, i.e., $\sigma(x) = E_1/E_3$. Absurd.

The case $A_4 = 0$ can be handled in a similar way; so, from now on, we suppose $A_3 \neq 0$, $A_4 \neq 0$.

Now, for each rational number θ, we compute

$$\frac{u}{t^\theta} = \frac{\begin{pmatrix}(B_2 x + C_2 y + D_2 x^2 + E_2 y^2 + F_2 xy) \\ \times (A_3 + B_3 x + C_3 y + D_3 x^2 + E_3 y^2 + F_3 xy)^\theta\end{pmatrix}}{\begin{pmatrix}(B_1 x + C_1 y + D_1 x^2 + E_1 y^2 + F_1 xy)^\theta \\ \times (A_4 + B_4 x + C_4 y + D_4 x^2 + E_4 y^2 + F_4 xy)\end{pmatrix}}.$$

For $\theta = 1$, we have $0 = p(u/t) = B_2 A_3/B_1 A_4$, so $B_2 = 0$. For $\theta = 2$, $0 = p(u/t^2) = D_2 A_3^2/B_1^2 A_4$ implies $D_2 = 0$.

Consequently, $p(u/t^\theta) = ((C_2 \cdot A_3^\theta)/(A_4 \cdot B_1^\theta)) \cdot p(y/x^\theta)$ and, if $\theta = \frac{5}{2}$, we conclude that

$$C_2 A_3^{5/2} = A_4 B_1^{5/2}. \tag{1}$$

Next we must compute $(u^2 - t^5)^{p'}/t^{q'} = (M_1^{p'} \cdot M)/(N \cdot L \cdot N_1^{q'})$, where

$$\begin{aligned} M_1 &= y^2(C_2 + E_2 y + F_2 x)^2 (A_3 + B_3 x + C_3 y + D_3 x^2 + E_3 y^2 + F_3 xy)^5 \\ &\quad - (B_1 x + C_1 y + D_1 x^2 + E_1 y^2 + F_1 xy)^5 \\ &\quad \cdot (A_4 + B_4 x + C_4 y + D_4 x^2 + E_4 y^2 + F_4 xy)^2, \\ M &= (A_3 + B_3 x + C_3 y + D_3 x^2 + E_3 y^2 + F_3 xy)^{q'}, \\ N_1 &= (B_1 x + C_1 y + D_1 x^2 + E_1 y^2 + F_1 xy), \\ N &= (A_3 + B_3 x + C_3 y + D_3 x^2 + E_3 y^2 + F_3 xy)^5, \\ L &= (A_4 + B_4 x + C_4 y + D_4 x^2 + E_4 y^2 + F_4 xy)^2. \end{aligned}$$

Then, using (1),

$$\begin{aligned} &p\left(\frac{(u^2 - t)^5)^{p'}}{t^{q'}}\right) \\ &\quad = (A_3^{q'}/A_3^5 \cdot A_4^2)(C_2^2\, A_3^5/B_1^{q'})\, p\left(\frac{(y^2 - x^5 + U)^{p'}}{x^{q'}}\right), \end{aligned} \tag{2}$$

where, obviously, the degrees of monomials which appear in U are bounded.

So, since $p((u^2 - t^5)^{p'}/t^{q'})$ must equal 0, 1, or ∞ according to $p((y^2 - x^5)^{p'}/t^{q'})$, we conclude that $1 = p((y^2 - x^5)^2/x^{q'}) = p((y^2 - x^5 + U)^2/x^{q'})$, which implies, for a large enough choice of q, that the only non-zero coefficients are C_2, B_1, A_3, and A_4. Moreover, from Eq. (2), we have $A_3^q C_2^2 = A_4^2 \cdot B_1^q$.

Then $t=\lambda\cdot x$, $u=\mu y$, with $\mu, \lambda\in\mathbb{R}$, $\mu^2=\lambda^5$, and $\lambda^q=\mu^2$, and, with q being different from 5, we get

$$\lambda^5(1-\lambda^{q-5})=0, \qquad \text{with} \quad \lambda\neq 0.$$

Therefore $\lambda=\pm 1$. But $x\in P$ implies $\lambda x=t=\sigma(x)\in P$.

Thus $\lambda=1$. Finally, from $\mu^2=1$ and $\mu\cdot y=u=\sigma(y)\in P$ it follows that $\mu=1$, and the equality $\sigma=\mathrm{id}$ is proved.

2. A Characterization of Real Closed Fields

Given an ordered field (K, P), we will use K^2 to denote the set of squares in K. If $a, b\in K$ with $a<_P b$ we shall denote

$$(a, b)_P=\{x\in K: a<_P x<_P b\}$$
$$[a, b]_P=\{x\in K: a\leqslant_P x\leqslant_P b\}.$$

For any element $a\in K$, we write $|a|$ for the maximum of a and $-a$ with respect to P.

2.1. Definition. Let (K, P) be an ordered field.

(i) (K, P) verfies *Bolzano's theorem* if given $f\in K[x]$, $a, b\in K$ with $a<_P b$ and $f(a)\cdot f(b)<_P 0$, there exists $c\in(a, b)_P$ such that $f(c)=0$.

(ii) (K, P) verifies the *maximum principle* if, given $f\in K[x]$ and $a, b\in K$, $a<_P b$, there exists $c\in[a, b]_P$ such that $f(x)\leqslant_P f(c)$ for each $x\in[a, b]_P$.

Our goal in this section is to prove that each of the above properties characterizes real closed fields.

2.2. Theorem. *Let (K, P) be an ordered field. The following statements are equivalent*:

(1) *(K, P) verifies Bolzano's theorem.*

(2) *(K, P) is real closed.*

(3) *(K, P) verifies the maximum principle.*

Proof. $(1)\Rightarrow(2)$. It is enough to check:

(i) $P=K^2$.

(ii) If $f\in K[x]$ has odd degree, it has a root in K.

Let us see (i); let $a\in P$, $a\neq 0$, and consider $f(x)=x^2-a$. Since $f(0)\cdot f(a+1)=-a(a^2+a+1)<_P 0$, there exists $c\in K$ with $c^2=a$.

To show (ii) we set $f(x)=x^{2n+1}+a_1x^{2n}+\cdots+a_{2n+1}$. Then, if $u\in K$ is greater than $(2n+1)\cdot\max\{|a_j|: 1\leqslant j\leqslant 2n+1, 1\}$ with respect to P, it is easy to see that $f(u)\ {}_P\!>0$. Applying this argument to $g(x)=-f(-x)$, we find some $v\in K$ with $g(v)\ {}_P\!>0$. Consequently, if $w=-v$, we deduce that $f(u)\cdot f(w)<_P 0$ and apply Bolzano's theorem to find a root of f.

(2)$\Rightarrow$(3) It is well known (see [8, Ex. 3, p. 283]).

(3)$\Rightarrow$(1) Let $f(x)=\sum_{j=0}^n a_jx^j$ be a polynomial over K and $a, b\in K$, $a<_P b$, with $f(a)\cdot f(b)<_P 0$. Clearly, the formal derivative of $g(x)=\sum_{j=0}^n(a_j/j+1)x^{j+1}$ is f. By assumption, g attains its maximum in $c\in[a,b]_P$ and, applying the hypothesis to $-g$, g reaches its minimum in $d\in[a,b]_P$. We claim that either c or d belongs to $(a,b)_P$.

First, we prove that if $c\in\{a,b\}$, then

$$f(a)<_P 0,\ f(b)\ {}_P\!>0 \tag{*}$$

Otherwise, since $g'(a)=f(a)$ is positive, the same holds with

$$\varepsilon=\min\left\{\frac{1}{2}, g'(a)\cdot\left(1+\sum_{j=2}^{n+1}\frac{|g^{j)}(a)|}{j!}\right)^{-1}, b-a\right\},$$

and, if $x\in(a,a+\varepsilon)_P$, we have

$$\begin{aligned}\left|\sum_{j=2}^{n+1}\frac{g^{j)}(a)(x-a)^{j-1}}{j!}\right| &<_P \sum_{j=2}^{n+1}\frac{|g^{j)}(a)|}{j!}\varepsilon^{j-1}<_P\varepsilon\cdot\sum_{j=2}^{n+1}\frac{|g^{j)}(a)|}{j!}\\ &\leqslant_P g'(a)\left(1+\sum_{j=2}^{n+1}|g^{j}(a)|\right)^{-1}\cdot\sum_{j=2}^{n+1}\frac{|g^{j)}(a)|}{j!}<_P g'(a).\end{aligned}$$

So, since $g(x)-g(a)=(x-a)\sum_{j=1}^{n+1}(g^j(a)/j!)(x-a)^{j-1}$, for $x\in(a,a+\varepsilon)_P$, we have $[g(x)-g(a)](x-a)\ {}_P\!>0$, whence $c\neq a$.

The same argument above, replacing a by b, allows us to conclude that

$$[g(x)-g(b)](x-b)<_P 0$$

for $x\in(b-\eta,b)$, η positive with respect to P, and so $c\neq b$. Thus, (*) is proved.

Let us consider now $F=-f$, $G=-g$. If $c\in\{a,b\}$ we know from (*) that $F(a)\ {}_P\!>0$, $F(b)<_P 0$. Again using (*) we deduce that the maximum of G is $[a,b]_P$ is neither a nor b. This maximum is d. Thus we have proved that g reaches its maximum or its minimum in $t\in(a,b)_P$.

Then, if $g'(t) \neq 0$, the argument used before also shows that there exists $\varepsilon \;_P\!> 0$ such that

$$g'(t) \cdot [g(x) - g(t)] \cdot (x - t) \;_P\!> 0$$

for $x \in (t - \varepsilon, t + \varepsilon)_P$, and this is absurd. So $f(t) = g'(t) = 0$.

2.3. *Remark.* It has been proved [11] that there exist non-real closed fields which verify Rolle's theorem.

3. Fields with the Extension Property

In the Introduction we recalled the definition of EP fields (or fields with the extension property). It is clear that a field K is an EP if $\sigma(K) \subset K$ for every automorphism σ of $K(x)$. The following result is stated in [7]:

3.1. Proposition. *The following classes of fields are EP's*:

(1) *algebraically closed fields*;

(2) *algebraic extensions of* $\mathbb{Q}$;

(3) *euclidean fields*;

(4) *fields with a unique ordering, which are archimedean over* $\mathbb{Q}$.

Our goal in this section is to show that several different classes of fields are also EP's.

3.2. Proposition. *Every finite field is an EP.*

Proof. Let F be a finite field, $p = \operatorname{char} F$ and $\sigma \in \operatorname{Aut}(F(x))$. Since F is algebraic over its prime field $\mathbb{Z}_p$, σ fixes Z_p, and F is algebraically closed in $F(x)$, we deduce that $\sigma(F) \subset F$.

Statement (4) in Proposition 3.1 can be generalized in the following way:

3.3. Proposition. *Let F be a field with a unique ordering such that* $\operatorname{card} \operatorname{Hom}(F, F) < \operatorname{card} F$. *Then F is an EP.* (*Note that, if the unique ordering of F is archimedean, then* $\operatorname{Hom}(F, F) = \mathrm{id}$.)

We need a lemma before the proof of Proposition 3.3.

3.4. Lemma. *Let F be a field with a unique ordering, R a real closure of F. Let σ be an automorphism of $F(x)$, $a \in F$ with $\sigma(a) = f/g$, f, $g \in F[X]$ relatively prime. Then f and g have no roots in R.*

Proof. Since $\sigma(-a) = -\sigma(a)$, we can suppose that a is positive, so

$$a = a_1^2 + \cdots + a_n^2, \qquad a_i \text{ positive.}$$

Writing each a_i as a sum of squares and using 12.8 in [9], we write a as a sum of fourth powers and, repeating this argument,

$$a = \sum_{j=1}^{n(k)} a_{k_j}^{2^k} \qquad \text{for each } k \in \mathbb{N}.$$

Let us write $\sigma(a_{k_j}) = f_{k_j}/g_{k_j}$, $h_{k_j} = (\prod_{i=1}^{n(k)} g_{k_i})/g_{k_j}$, and $H = \sum_{j=1}^{n(k)} (f_{k_j} \cdot h_{k_j})^{2^k}$; we suppose that f_{k_j} and g_{k_j} are coprime.

Then

$$f \cdot \left(\prod_{j=1}^{n(k)} g_{k_j} \right)^{2^k} = g \cdot H. \tag{1}$$

We show first that, if $k \geqslant \deg(g)$, each g_{k_j} has no roots in R. Otherwise, let $c \in R$ be a root of g_{k_1} with multiplicity $n_1 \neq 0$, and let us denote by n_j the multiplicity of c as a root of g_{k_j}. Then, if n and m are the multiplicities of c as a root of f and g, respectively, we deduce from (1) that

$$\begin{aligned} n + 2^k \cdot \left(n_1 + \sum_{j=2}^{n(k)} n_j \right) & \\ &= m + (\text{multiplicity of } c \text{ as root of } H) \\ &\leqslant m + 2^k \cdot (\text{multiplicity of } c \text{ as root of } f_{k_1} \cdot h_{k_1}). \end{aligned}$$

If $f_{k_1}(c) = 0$, $\mathrm{Irr}(c, F)$ would be a common factor of f_{k_1} and g_{k_1}, which are coprime. Moreover, the multiplicity of c as a root of h_{k_1} equals $\sum_{j=2}^{n(k)} n_j$.

Consequently, $n + 2^k n_1 + 2^k \cdot \sum_{j=2}^{n(k)} n_j \leqslant m + 2^k \cdot \sum_{j=2}^{n(k)} n_j$, and so $2^k n_1 \leqslant m \leqslant \deg(g)$. Absurd.

In particular, if $t = \deg(g)$, each g_{t_j} has no "real" (in R) roots. Thus it follows from (1) that g has no real roots: for if $g(c) = 0$, $c \in R$, then $f(c) = 0$ and $\mathrm{Irr}(c, F)$ would be a common factor of f and g.

Finally, applying the argument above to $\sigma(1/a) = g/f$, we conclude that f has no roots in R.

Proof of 3.3. Let σ be an automorphism of $F(x)$. For each $a \in F$, the map $\sigma_a \colon F \to F$ which sends $b \in F$ to the value at a of the rational function $\sigma(b)$ is a well-defined homomorphism from 3.4.

Since $\operatorname{card} \mathrm{Hom}(F, F) < \operatorname{card} F$, there exists $\phi \in \mathrm{Hom}(F, F)$ and an infinite subset M of F such that $\phi = \sigma_a$ for every $a \in M$.

Let us prove that, for $b \in F$, $\sigma(b) \in F$. If we write $\sigma(b) = f/g$, $f, g \in K[x]$, we know that

$$\phi(b) = \sigma_a(b) = \frac{f(a)}{g(a)} = c_b \in F \qquad \text{for every } a \in M,$$

and so $M \subset \{\text{roots of } f - c_b g\}$.

Since M is infinite, $f = c_b \cdot g$ and $\sigma(b) = f/g = c_b \in F$.

Our next goal is to prove that quadratically closed fields and pythagorean fields are EP's. In fact, we establish a more general result.

3.5. Definition. Let $n \geqslant 2$ be a natural number, and let F be a field. We say that F is an *n-field* if $x^n - ax - 1$ has a root in F, for each $a \in F$.

3.6. *Remark.* Let us suppose that char $F \neq 2$; F is a 2-field if and only if $a^2 + 4 \in F^2$, $a \in F$, i.e., if and only if F is pythagorean. In particular, quadratically closed fields are 2-fields.

The result announced above says:

3.7. Proposition. *If F is an n-field for some n, then F is an EP.*

Before giving the proof of Proposition 3.7, we introduce some notation that will be used in the remainder of the paper.

3.8. *Notation.* Given a field F, an automorphism σ of $F(x)$, and an element $a \in F$, we write $\sigma(a) = f_a/g_a$, with f_a, g_a *coprime* polynomials in $F[x]$, and we denote by δ the map

$$\delta\colon F \to \mathbb{Z}\colon a \to \delta(a) = \deg f_a + \deg g_a.$$

Proof of Proposition 3.7. For each $a \in F$ let us choose $b \in F$ such that $b^n - ab - 1 = 0$. Then $a = b^m - 1/b$, $m = n - 1$, and so

$$f_a/g_a = f_b^m/g_b^m - g_b/f_b = \frac{f_b^n - g_b^n}{f_b \cdot g_b^m}.$$

Since f_b and g_b are coprime, the same holds with $f_b^n - g_b^n$ and $f_b \cdot g_b^m$. Therefore $\deg f_a = \deg(f_b^n - g_b^n)$ and

$$\deg g_a = \deg f_b \cdot g_b^m = \deg f_b + m \cdot \deg g_b.$$

Consequently,

$$\delta(a) = \delta(b) + (m-1) \deg g_b + \deg(f_b^n - g_b^n). \tag{2}$$

We consider, first, the case $n > 2$. If F is not an EP, then $M = \{a \in F\colon \sigma(a) \notin F\} \neq \varnothing$. Let $a \in M$ be such that $\delta(a) = \min\{\delta(c)\colon c \in M\}$, and let b denote, as above, a root of $x^n - ax - 1$.

Obviously $b \in M$, because $\sigma(a) = \sigma(b)^m - 1/\sigma(b)$. Now, from (2) and since $m - 1 > 0$, and $\delta(a) \leqslant \delta(b)$, we conclude that $\deg g_b = 0 = \deg(f_b^n - g_b^n)$. But this means that f_b, $g_b \in F$, hence $\sigma(b) \in F$, a contradiction.

Now consider $n = 2$. Since $\sigma(1/a) = 1/\sigma(a)$, if F is not an EP, we deduce that

$$L = \{a \in F\colon \sigma(a) \notin F, \deg f_a \geqslant \deg g_a\} \neq \varnothing.$$

Let $a \in L$ be such that $\delta(a) = \min\{\delta(c): c \in L\}$. Since the roots of $x^2 - ax - 1$ are b and $-1/b$, one of them belongs to L, e.g., $b \in L$. Now, by (2) and since $\delta(a) \leqslant \delta(b)$, we have $\deg(f_b^2 - g_b^2) = 0$ and so

$$f_b^2 = c + g_b^2, \qquad c \in F.$$

Consequently, $f_a/g_a \cdot f_b/g_b = \sigma(a \cdot b) = \sigma(b^2 - 1) = c/g_b^2$, and

$$\deg f_a \geqslant \deg g_a, \qquad \deg f_b \geqslant \deg g_b \qquad \text{since } a, b \in L.$$

This shows that $g_b^2 \in F$ and also that $f_b^2 = c + g_b^2 \in F$. Then $g_b, f_b \in F$, i.e., $\sigma(b) \in F$. This is absurd because $b \in L$.

The proof above leads us to the following:

3.9. COROLLARY. *Let (F, v) be a henselian field whose residue class field R is real closed. Then F is an EP.*

Proof. Let A be the valuation ring of v. Clearly, it suffices to show that $\sigma(a) \in F$ for every $\sigma \in \operatorname{Aut} F(x)$, $a \in A$, $a < 0$, in the order induced in F by R. Then, if $\mathfrak{m}$ is the maximal ideal of A,

$$x^3 - (a + \mathfrak{m})x - (1 + \mathfrak{m})$$

has a simple root in R, and so there exists $b \in A$ such that

$$a^3 - ab - 1 = 0.$$

If F is not an EP, the set

$$M = \{c \in A: c < 0, \sigma(c) \notin F\}$$

is not empty. We choose $a \in M$ such that $\delta(a) \leqslant \delta(c)$ for each $c \in M$, and $b \in A$ with $a^3 - ab - 1 = 0$ as above. Then $d = b$ or $d = -b$ verifies $\delta(d) < \delta(a)$, $d \in A$. This is absurd.

3.10. *Remark.* (a) Using 3.8 and [4], we conclude that if a formally real field verifies Rolle's theorem for any ordering, then it is an EP.

(b) From 3.8 and Theorem 24 in [3], the generalized real closure of a field with respect to an ordering of high level is an EP.

(c) If F is an intersection of real closed fields, then F is pythagorean. So, F is an EP.

The following result improves part (3) in Proposition 3.1:

3.11. PROPOSITION. *Let F be a field,* $\operatorname{char}(F) = 0$, *$\dot{F} = F \setminus \{0\}$ and $\dot{F}^2$ the multiplicative subgroup of non-zero squares. Then, if $[\dot{F} : \dot{F}^2]$ is finite, F is an EP. (Note that if F is euclidean, then $[\dot{F} : \dot{F}^2] = 2$.)*

Proof. If F is not an EP there exists $\sigma \in \operatorname{Aut} F(x)$ such that $M = \{a \in F: \sigma(a) \notin F, \deg f_a \geqslant \deg g_a\} \neq \varnothing$ (note that $\sigma(1/a) = 1/\sigma(a)$).

Then we choose an element $a \in M$ such that

$$\deg f_a \leqslant \deg f_b \qquad \text{for every} \quad b \in M.$$

Since $[\dot{F} : \dot{F}^2]$ is finite, we can choose $a_1, ..., a_m$ in F with $F = a_1 F^2 \cup \cdots \cup a_m F^2$.

Now we can find $i \in \{1, ..., m\}$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $q > p$, verifying $a + p \in a_i F^2$, $a + q \in a_i F^2$, and $\deg f_a = \deg(f_a + pg_a)$.

We set $d = a + p$. Clearly $\sigma(d) = (f_a + pg_a)/g_a$, whence $d \in M$. Now, if $r = q - p$, then $b = d/(d + r) = c^2$, $c \in F$, and $f_b/g_b = f_d/f_d + rg_d$, $f_b/g_b = f_c^2/g_c^2$. Consequently, $2 \deg f_c = \deg f_b = \deg f_d = \deg f_a$. Moreover $b \in M$ (and so $c \in M$) because $\deg f_b - \deg g_b = \deg f_d - \deg(f_d + rg_d) \geqslant 0$, since $d \in M$.

Finally, $c \in M$, $2 \deg f_c = \deg f_a$ imply, by our choice of a, that $f_a \in F$. Since $a \in M$ we also have $g_a \in F$. Thus $\sigma(a) \in F$, a contradiction.

3.12. EXAMPLE (following [6]). Let k be a field, Ω its algebraic closure, and $a_1, ..., a_m \in k \backslash k^2$. Let F be a maximal element in

$$\Sigma = \{\text{fields } L,\ k \subset L \subset \Omega: a_i \notin L^2, i = 1, ..., m\}$$

(with the terminology of [6], F is n-maximal with respect to the exclusion of $\sqrt{a_1}, ..., \sqrt{a_m}$). Then $[\dot{F} : \dot{F}^2] = n + 1$ and F is an EP from Proposition 3.11.

In particular, if k is not pythagorean, in the above construction we can take $a_i = 1 + f_i^2$, $f_i \in k$. Then, by replacing Ω by a real closure of k, we get a field F with a unique ordering which is not pythagorean but is an EP. Even more, if k is non-archimedean so is F.

Finally we establish a result which allows us to prove, for instance, that $\mathbb{Q}((x))$ is an EP.

3.13. PROPOSITION. *Let F be a field with an ordering P such that $P \subset \mathbb{Q} + F^2$. Then F is an EP.*

Proof. As always, if F is not an EP the set

$$M = \{a \in P: \sigma(a) \notin F\}$$

is not empty. Let us take $a \in M$ such that

$$\deg g_a \leqslant \deg g_b \qquad \text{for each} \quad b \in M.$$

Since $a = q + c^2$, $q \in \mathbb{Q}$, $c \in F$, we have $\deg g_a = 2 \deg g_c$, $c \in M$. Absurd.

The same proof above allows us to state:

3.14. COROLLARY. *Let F be a field, $F \subset \mathbb{Q} + F^2$. Then F is an EP.*

Some things remain unanswered about EP fields. For example:

Question 1. Is every field with a unique ordering an EP? Part (4) in 3.1 gives us an affirmative answer in the archimedean case.

We also know, from the first step in the proof of 1.4 in [7] and Lemma 3.3, that, if F admits a unique ordering and $\sigma \in \operatorname{Aut} F(x)$, then $\sigma(F)$ is contained in the holomorphy ring of $F(x)$ relative to F (see, for instance, [13] for a reference about holomorphy rings).

Question 2. What can be said about the behaviour of the extension property under algebraic, finite, or quadratic extensions?

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