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Orthogonality of Jack polynomials in superspace

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Abstract

Jack polynomials in superspace, orthogonal with respect to a "combinatorial" scalar product, are constructed. They are shown to coincide with the Jack polynomials in superspace, orthogonal with respect to an "analytical" scalar product, introduced in [P. Desrosiers, L. Lapointe, P. Mathieu, Jack polynomials in superspace, Comm. Math. Phys. 242 (2003) 331–360] as eigenfunctions of a supersymmetric quantum mechanical many-body problem. The results of this article rely on generalizing (to include an extra parameter) the theory of classical symmetric functions in superspace developed recently in [P. Desrosiers, L. Lapointe, P. Mathieu, Classical symmetric functions in superspace, J. Algebraic Combin. 24 (2006) 209–238].

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1. Introduction

Jack polynomials, $J_\lambda(x; 1/\beta)$, are symmetric functions of commutative indeterminates $x = (x_1, \dots, x_N)$ that generalize the elementary ($\beta = \infty$), monomial ($\beta = 0$), Schur ($\beta = 1$), and

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zonal ($\beta = 1/2$) symmetric functions. First introduced in statistics by Jack [9], they were later studied in algebraic combinatorics, in particular by Kadell [10], Macdonald [13,14], Stanley [21], and Knop and Sahi [11].

The standard definition of the (monic) Jack polynomials is the following [14]: they are the unique functions such that

$$(1)\quad J_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}(\beta) m_\mu \qquad \text{and} \qquad (2)\quad \langle\langle J_\lambda | J_\mu \rangle\rangle_\beta \propto \delta_{\lambda,\mu}, \tag{1.1}$$

where λ and μ stand for partitions of size not larger than N, m_λ is the monomial symmetric function, and $\mu < \lambda$ means that the latter partition is larger than the former in the dominance ordering. The scalar product involved in (2) is of a combinatorial nature. On the basis of power-sum symmetric functions, it is defined as

$$\langle\langle p_\lambda | p_\mu \rangle\rangle_\beta := \beta^{-\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}, \quad \text{where} \quad z_\lambda = \prod_i i^{m_i} m_i! \quad \text{if } \lambda = \left(1^{m_1} 2^{m_2} \cdots\right). \tag{1.2}$$

However, alternative characterizations of the Jack polynomials exist. For instance, when the indeterminate x_j is a complex number lying on the unit circle and β is a nonnegative real number, one can introduce another scalar product [13]:

$$\langle f(x) | g(x) \rangle_{\beta,N} = \prod_{1\leqslant j\leqslant N} \frac{1}{2\pi \mathrm{i}} \oint \frac{dx_j}{x_j} \prod_{\substack{1\leqslant k,l\leqslant N\\ k\neq l}} \left(1 - \frac{x_k}{x_l}\right)^\beta \overline{f(x)} g(x), \tag{1.3}$$

where the bar denotes the complex conjugation. Then, it can be shown that Jack polynomials are the unique symmetric functions that satisfy

$$(1)\quad J_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}(\beta) m_\mu \qquad \text{and} \qquad (2')\quad \langle J_\lambda | J_\mu \rangle_{\beta,N} \propto \delta_{\lambda,\mu}. \tag{1.4}$$

This analytical scalar product is rooted in the characterization of the Jack polynomials in terms of an eigenvalue problem; that is, as the common eigenfunctions of N independent and commuting differential operators that are self-adjoint with respect to the scalar product (1.3). These operators are in fact the conserved quantities of a well-known N-body problem in quantum mechanics, the trigonometric Calogero–Moser–Sutherland model. Every orthogonal and symmetric wave function of this model is proportional to a particular Jack polynomial [7,12].

In this work, we provide an extension to the theory of classical symmetric functions in superspace [6] that leads to a definition of Jack polynomials in superspace similar to (1.1). By superspace, we refer to a collection of variables $(x,\theta) = (x_1,\dots,x_N,\theta_1,\dots,\theta_N)$, called respectively bosonic and fermionic (or anticommuting or Grassmannian), and obeying the relations

$$x_i x_j = x_j x_i, \quad x_i\theta_j = \theta_j x_i \quad \text{and} \quad \theta_i\theta_j = -\theta_j\theta_i \quad \left(\Rightarrow \quad \theta_i^2 = 0\right). \tag{1.5}$$

A function in superspace (or superfunction for short) is a function of all these variables. It is said to be symmetric if it is invariant under the simultaneous interchange of $x_i \leftrightarrow x_j$ and $\theta_i \leftrightarrow \theta_j$ for any i, j. Symmetric polynomials in superspace are naturally indexed by superpartitions [3],

$$\Lambda := \left(\Lambda^a; \Lambda^s\right) = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N), \tag{1.6}$$

where Λ^a is a partition with distinct parts (one of them possibly equal to zero), and Λ^s is an ordinary partition. Every symmetric polynomial in x and θ can be written as a linear combination of the following monomial functions [3]:

$$m_\Lambda = {\sum}'_{\sigma \in S_N} \theta_{\sigma(1)} \cdots \theta_{\sigma(m)} x_1^{\Lambda_{\sigma(1)}} \cdots x_m^{\Lambda_{\sigma(m)}} x_{m+1}^{\Lambda_{\sigma(m+1)}} \cdots x_N^{\Lambda_{\sigma(N)}}, \tag{1.7}$$

where the prime indicates that the summation is restricted to distinct terms. Power sums with m fermions are given by [6]

$$p_\Lambda := \tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \cdots p_{\Lambda_N} \quad \text{with } p_n := m_{(n)} \text{ and } \tilde{p}_k := m_{(k;0)}. \tag{1.8}$$

In the article, we define a simple extension of the combinatorial scalar product (1.2):

$$\langle\langle p_\Lambda | p_\Omega \rangle\rangle_\beta := (-1)^{m(m-1)/2} z_\Lambda(\beta) \delta_{\Lambda,\Omega}, \quad z_\Lambda(\beta) := \beta^{-\ell(\Lambda)} z_{\Lambda^s} \tag{1.9}$$

where Λ is of the form (1.6) and where $\ell(\Lambda)$ is the length of Λ (given by the length of Λ^s plus m).

Jack polynomials in superspace were presented in [5] as the orthogonal eigenfunctions of a supersymmetric generalization of the quantum mechanical N-body problem previously mentioned [2,20]. In this case, the analytical scalar product reads [3]

$$\begin{aligned} &\left\langle A(x,\theta) \middle| B(x,\theta) \right\rangle_{\beta,N} \\ &\quad = \prod_{1 \leqslant j \leqslant N} \frac{1}{2\pi \mathrm{i}} \oint \frac{dx_j}{x_j} \int d\theta_j \theta_j \prod_{\substack{1 \leqslant k,l \leqslant N \\ k \neq l}} \left(1 - \frac{x_k}{x_l}\right)^\beta \overline{A(x,\theta)} B(x,\theta), \end{aligned} \tag{1.10}$$

where the "bar conjugation" is defined such that $\bar{x}_j = 1/x_j$ and $\overline{(\theta_{i_1} \cdots \theta_{i_m})} \theta_{i_1} \cdots \theta_{i_m} = 1$. Our main result here is that these Jack polynomials in superspace are also orthogonal with respect to the scalar product (1.9); i.e., the two scalar products are compatible. The following theorem is an alternative formulation of this statement.

Theorem 1. *There exists a unique family of functions* $\{J_\Lambda\colon \sum_i \Lambda_i < N\}$ *such that*

$$(1) \quad J_\Lambda = m_\Lambda + \sum_{\Omega < \Lambda} c_{\Lambda\Omega}(\beta) m_\Lambda,$$

$$(2) \quad \langle\langle J_\Lambda | J_\Omega \rangle\rangle_\beta \propto \delta_{\Lambda,\Omega} \quad \forall \Lambda, \Omega \qquad \text{or} \qquad (2') \quad \langle J_\Lambda | J_\Omega \rangle_{\beta,N} \propto \delta_{\Lambda,\Omega} \quad \forall \Lambda, \Omega,$$

where the ordering involved in the triangular decomposition is the Bruhat ordering on superpartition that will be defined in the next section.

The article is organized as follows. In Section 2, we summarize the theory of symmetric functions in superspace developed in [6]. We obtain a one-parameter deformation of the latter construction in Section 3. Section 4 is essentially a review of relevant results concerning our previous (analytical) construction of Jack polynomials in superspace. It is shown in Section 5 that these polynomials are also orthogonal with respect to the product (1.9). Direct non-trivial limiting cases (i.e., special values of the free parameter or particular superpartitions) of this connection are presented in Section 6. This section also contains a discussion of a duality transformation on the Jack superpolynomials, as well as a conjectured expression for their normalization constant. We present, as a concluding remark (Section 7), a precise conjecture concerning the existence of Macdonald polynomials in superspace.

In this work, we have relied heavily on the seminal paper [21], and on Section VI.10 of [14], without always giving these references complete credit in the bulk of the paper.

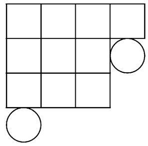
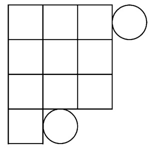
Remark 2. The terms "superanalogs of Jack polynomials," "super-Jack polynomials" and "Jack superpolynomials" have also been used in the literature for somewhat different polynomials. In [18], superanalogs of Jack polynomials designated the eigenfunctions of the CMS Hamiltonian constructed from the root system of the Lie superalgebra $su(m, N-m)$ (recall that to any root system corresponds a CMS model [15]). The same objects are called super-Jack polynomials in [19]. But we stress that such a Hamiltonian does not contain anticommuting variables, so that the resulting eigenfunctions are quite different from our Jack superpolynomials. Notice also that in [3,4], we used the term "Jack superpolynomials" for eigenfunctions of the supersymmetric extension of the trigonometric Calogero–Moser–Sutherland model that decompose triangularly in the monomial basis. However, these are not orthogonal. The construction of orthogonal Jack superpolynomials was presented in [5] and from now on, when we refer to "Jack superpolynomials," or equivalently, "Jack polynomials in superspace," we refer to the orthogonal ones.

2. Notation and background

$\Lambda \vdash (n|m)$ indicates that the superpartition $\Lambda = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_N)$ is of bosonic degree $n = |\Lambda| = \Lambda_1 + \cdots + \Lambda_N$ and of fermionic degree $m = \underline{\overline{\Lambda}}$ respectively (observe that the bosonic and fermionic degree refer to the respective degrees in x and θ of m_Λ). To every superpartition Λ, we can associate a unique partition Λ^* obtained by deleting the semicolon and reordering the parts in non-increasing order. A superpartition $\Lambda = (\Lambda^a; \Lambda^s)$ can be viewed as the partition Λ^* in which every part of Λ^a is circled. If a part $\Lambda^a{}_j = b$ is equal to at least one part of Λ^s, then we circle the leftmost b appearing in Λ^*. We shall use $C[\Lambda]$ to denote this special notation.

To each Λ, we associate the diagram, denoted by $D[\Lambda]$, obtained by first drawing the Ferrers' diagram associated to $C[\Lambda]$, that is, by drawing a diagram with $C[\Lambda]_1$ boxes in the first row, $C[\Lambda]_2$ boxes in the second row and so forth, all rows being left justified. If, in addition, the integer $C[\Lambda]_j = b$ is circled, then we add a circle at the end of the b boxes in the jth row.

This representation offers a very natural way to define a conjugation operation. The conjugate of a superpartition Λ, denoted by Λ', is obtained by interchanging the rows and the columns in the diagram $D[\Lambda]$. We can thus write $D[\Lambda'] = (D[\Lambda])^{\mathrm{t}}$ where t stands for the transposition operation. For instance, we have

$D\big([3,0;4,3]\big) =$  $\Longrightarrow$ $D\big([3,0;4,3]\big)^{\mathrm{t}} =$  (2.1)

meaning that $(3,0;4,3)' = (3,1;3,3)$.

We now formulate the Bruhat ordering on superpartitions. Recall that two partitions λ and μ of n are such that λ dominates μ iff $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$ for all i. The Bruhat ordering on superpartitions of $(n|m)$ can then be described most simply as: $\Lambda \geqslant \Omega$ iff $\Lambda^* > \Omega^*$ or $\Lambda^* = \Omega^*$ and $\mathrm{sh}(D[\Lambda]) \geqslant \mathrm{sh}(D[\Omega])$, where $\mathrm{sh}(D[\Lambda])$ is the shape (including circles) of the diagram $D[\Lambda]$ (see [6] for the connection between this ordering and the usual Bruhat ordering on superpartitions). With this definition, it is then obvious that $\Lambda \geqslant \Omega$ iff $\Omega' \geqslant \Lambda'$.

We denote by $\mathscr{P}^{S_\infty}$ the ring of symmetric functions in superspace with coefficients in $\mathbb{Q}$. A basis for its subspace of homogeneous degree $(n|m)$ is given by $\{m_\Lambda\}_{\Lambda \vdash (n|m)}$ (now considered to be functions of an infinite number of variables). In this ring, the elementary e_n, homogeneous h_n, and power sum p_n symmetric functions possess fermionic counterparts which are obtained trough the following generating functions:

$$E(t,\tau) := \sum_{n=0}^{\infty} t^n (e_n + \tau \tilde{e}_n) = \prod_{i=1}^{\infty} (1 + t x_i + \tau \theta_i), \tag{2.2}$$

$$H(t,\tau) := \sum_{n=0}^{\infty} t^n (h_n + \tau \tilde{h}_n) = \prod_{i=1}^{\infty} \frac{1}{1 - t x_i - \tau \theta_i}, \tag{2.3}$$

$$P(t,\tau) := \sum_{n \geqslant 1} \big(t^n p_n + \tau n t^{n-1} \tilde{p}_{n-1}\big) = \sum_{i=1}^{\infty} \frac{t x_i + \tau \theta_i}{1 - t x_i - \tau \theta_i}, \tag{2.4}$$

where τ is an anticommuting parameter ($\tau^2 = 0$). To be more explicit, this leads to

$$\tilde{e}_n = m_{(0;1^n)}, \qquad \tilde{h}_n = \sum_{\Lambda \vdash (n|1)} (\Lambda_1 + 1) m_\Lambda, \qquad \tilde{p}_n = m_{(n;0)}. \tag{2.5}$$

This construction furnishes three multiplicative bases f_Λ of $\mathscr{P}^{S_\infty}$,

$$f_\Lambda := \tilde{f}_{\Lambda_1} \cdots \tilde{f}_{\Lambda_m} f_{\Lambda_{m+1}} \cdots f_{\Lambda_N}, \tag{2.6}$$

where f is either e, h or p.

With $(y_1, y_2, \ldots, \phi_1, \phi_2, \ldots)$ representing another set of bosonic and fermionic variables (with the additional understanding that $\phi_i \theta_j = -\theta_j \phi_i$), the generalized Cauchy formula is shown to satisfy

$$\prod_{i,j} (1 - x_i y_j - \theta_i \phi_j)^{-1} = \sum_\Lambda \overleftarrow{m_\Lambda(x,\theta)}\, \overrightarrow{h_\Lambda(y,\phi)} = \sum_\Lambda z_\Lambda^{-1} \overleftarrow{p_\Lambda(x,\theta)}\, \overrightarrow{p_\Lambda(y,\phi)}, \tag{2.7}$$

where

$$z_\Lambda := z_{\Lambda^s} = \prod_i i^{m_i} m_i! \quad \text{if } \Lambda^s = \left(1^{m_1} 2^{m_2} \cdots\right). \tag{2.8}$$

The arrows are used to encode signs resulting from reordering the fermionic variables: if the fermionic degree of a polynomial f in superspace is m, then $\overleftarrow{f} = (-1)^{m(m-1)/2} f$ and $\overrightarrow{f} = f$.

3. One-parameter deformation of the scalar product and the homogeneous basis

Let $\mathscr{P}^{S_\infty}(\beta)$ denote the ring of symmetric functions in superspace with coefficients in $\mathbb{Q}(\beta)$, i.e., rational functions in β. We first introduce the mapping,

$$\langle\!\langle \cdot | \cdot \rangle\!\rangle_\beta : \mathscr{P}^{S_\infty}(\beta) \times \mathscr{P}^{S_\infty}(\beta) \longrightarrow \mathbb{Q}(\beta) \tag{3.1}$$

defined by (1.9). This bilinear form can easily be shown to be a scalar product (using an argument similar to the one given in [6] in the case $\beta = 1$).

We next introduce an endomorphism that generalizes the involution $\hat\omega$ of [6], and which extends a known endomorphism in symmetric function theory. It is defined on the power sums as:

$$\hat\omega_\alpha(p_n) = (-1)^{n-1} \alpha p_n \quad \text{and} \quad \hat\omega_\alpha(\tilde p_n) = (-1)^n \alpha \tilde p_n, \tag{3.2}$$

where α is some unspecified parameter. This implies

$$\hat\omega_\alpha(p_\Lambda) = \omega_\Lambda(\alpha) p_\Lambda \quad \text{with} \quad \omega_\Lambda(\alpha) := \alpha^{\ell(\Lambda)} (-1)^{|\Lambda| - \underline{\overline{\Lambda}} + \ell(\Lambda)}. \tag{3.3}$$

Notice that $\hat\omega_1 \equiv \hat\omega$. This homomorphism is still self-adjoint, but it is now neither an involution ($\hat\omega_\alpha^{-1} = \hat\omega_{\alpha^{-1}}$) nor an isometry ($\|\hat\omega_\alpha p_\Lambda\|^2 = z_\Lambda(\beta/\alpha^2)$). Note also that

$$z_\Lambda(\beta)\omega_\Lambda(\beta) = z_\Lambda \omega_\Lambda \quad \text{and} \quad z_\Lambda(\beta)^{-1} \omega_\Lambda\left(\beta^{-1}\right) = z_\Lambda^{-1} \omega_\Lambda. \tag{3.4}$$

We now extend the Cauchy kernel introduced in (2.7).

Theorem 3. *One has*

$$K^\beta(x,\theta;y,\phi) := \prod_{i,j} \frac{1}{(1 - x_i y_j - \theta_i \phi_j)^\beta} = \sum_\Lambda z_\Lambda(\beta)^{-1} \overleftarrow{p_\Lambda(x,\theta)}\, \overrightarrow{p_\Lambda(y,\phi)}. \tag{3.5}$$

Proof. Starting from

$$\prod_{i,j} \frac{1}{(1 - x_i y_j - \theta_i \phi_j)^\beta} = \exp\left\{\beta \sum_{i,j} \ln\left[(1 - x_i y_j - \theta_i \phi_j)^{-1}\right]\right\}, \tag{3.6}$$

the above identity can be obtained straightforwardly by proceeding as in the proof of [6, Theorem 33]. $\square$

Remark 4. The inverse of K^β satisfies:

$$K(-x,-\theta;y,\phi)^{-\beta}=\prod_{i,j}(1+x_iy_j+\theta_i\phi_j)^\beta=\sum_\Lambda z_\Lambda(\beta)^{-1}\omega_\Lambda\overleftarrow{p_\Lambda(x,\theta)}\overrightarrow{p_\Lambda(y,\phi)},\tag{3.7}$$

which is obtained by using

$$p_\Lambda(-x,-\theta)=(-1)^{|\Lambda|+\overline{\Lambda}}p_\Lambda(x,\theta)\quad\text{and}\quad z_\Lambda(-\beta)=(-1)^{\ell(\Lambda)}z_\Lambda(\beta).\tag{3.8}$$

Notice also the simple relation between the kernel K of [6] (equal to K^β at $\beta=1$) and its β-deformation

$$K^\beta(x,\theta;y,\phi)=\hat\omega_\beta K(-x,-\theta;y,\phi)^{-1},\tag{3.9}$$

where it is understood that $\hat\omega_\beta$ acts either on (x,θ) or on (y,ϕ).

Corollary 5. *$K^\beta(x,\theta;y,\phi)$ is a reproducing kernel in the space of symmetric superfunctions with rational coefficients in β:*

$$\langle\!\langle K^\beta(x,\theta;y,\phi)\big|f(x,\theta)\rangle\!\rangle_\beta=f(y,\phi),\quad \textit{for all } f\in\mathscr{P}^{S_\infty}(\beta).\tag{3.10}$$

Paralleling the construction of the function g_n in Section VI.10 of [14], we now introduce a β-deformation of the bosonic and fermionic complete homogeneous symmetric functions, respectively denoted as $g_n(x)$ and $\tilde g_n(x,\theta)$ (the β-dependence being implicit). Their generating function is

$$G(t,\tau;\beta):=\sum_{n\geqslant 0}t^n\big[g_n(x)+\tau\tilde g_n(x,\theta)\big]=\prod_{i\geqslant 1}\frac{1}{(1-tx_i-\tau\theta_i)^\beta}.\tag{3.11}$$

Clearly, $g_n=h_n$ and $\tilde g_n=\tilde h_n$ when $\beta=1$. As usual, we define

$$g_\Lambda:=\tilde g_{\Lambda_1}\cdots\tilde g_{\Lambda_m}g_{\Lambda_{m+1}}\cdots g_{\Lambda_N}.\tag{3.12}$$

Proposition 6. *One has $K^\beta(x,\theta;y,\phi)=\sum_\Lambda\overleftarrow{m_\Lambda(x,\theta)}\overrightarrow{g_\Lambda(y,\phi)}$.*

Proof. The proof is similar to that of [6, Proposition 38]. □

Corollary 7. *One has*

$$g_n=\sum_{\Lambda\vdash(n|0)}z_\Lambda(\beta)^{-1}p_\Lambda\quad\textit{and}\quad\tilde g_n=\sum_{\Lambda\vdash(n|1)}z_\Lambda(\beta)^{-1}p_\Lambda.\tag{3.13}$$

Proof. On the one hand,

$$G(t,0;\beta)=\sum_{n\geqslant 0}t^ng_n(x)=K^\beta(x,0;y,0)\big|_{y=(t,0,0,\ldots)}.\tag{3.14}$$

The previous proposition and Theorem 3 imply

$$\sum_{n\geqslant 0} t^n g_n = \sum_{\lambda} t^{|\lambda|} z_\lambda(\beta)^{-1} p_\lambda \quad\Longrightarrow\quad g_n = \sum_{\lambda\vdash n} z_\lambda(\beta)^{-1} p_\lambda. \tag{3.15}$$

On the other hand,

$$\partial_\tau G(t,\tau;\beta) = \sum_{n\geqslant 0} t^n \tilde{g}_n(x,\theta) = K^\beta(x,\theta;y,\phi)\Big|_{\substack{y=(t,0,0,\ldots)\\ \phi=(-\tau,0,0,\ldots)}}. \tag{3.16}$$

Hence

$$\sum_{n\geqslant 0} t^n \tilde{g}_n = \sum_{\Lambda,\underline{\overline{\Lambda}}=1} t^{|\Lambda|} z_\Lambda(\beta)^{-1} p_\Lambda \quad\Longrightarrow\quad \tilde{g}_n = \sum_{\Lambda\vdash(n|1)} z_\Lambda(\beta)^{-1} p_\Lambda, \tag{3.17}$$

as claimed. □

Applying $\omega_{\beta^{-1}}$ on Eq. (3.13), simplifying with the help of (3.4), and then using [6]

$$e_n = \sum_{\Lambda\vdash(n|0)} z_\Lambda^{-1}\omega_\Lambda p_\Lambda \quad\text{and}\quad \tilde{e}_n = \sum_{\Lambda\vdash(n|1)} z_\Lambda^{-1}\omega_\Lambda p_\Lambda, \tag{3.18}$$

we get

$$\hat{\omega}_{\beta^{-1}}(g_n) = e_n \quad\text{and}\quad \hat{\omega}_{\beta^{-1}}(\tilde{g}_n) = \tilde{e}_n. \tag{3.19}$$

Or equivalently,

$$g_n = \hat{\omega}_\beta(e_n) \quad\text{and}\quad \tilde{g}_n = \hat{\omega}_\beta(\tilde{e}_n). \tag{3.20}$$

Lemma 8. *Let $\{u_\Lambda\}$ and $\{v_\Lambda\}$ be two bases of $\mathscr{P}^{S_\infty}$. Then*

$$K^\beta(x,\theta;y,\phi) = \sum_\Lambda \overleftarrow{u_\Lambda(x,\theta)}\,\overrightarrow{v_\Lambda(y,\phi)} \quad\Longleftrightarrow\quad \langle\langle \overleftarrow{u_\Lambda}|\overrightarrow{v_\Lambda}\rangle\rangle_\beta = \delta_{\Lambda,\Omega}. \tag{3.21}$$

Proof. The proof is identical to the one when the θ_i variables are not present (see [14, (I.4.6)]). □

This immediately implies the following.

Corollary 9. *The set $\{g_\Lambda\}_\Lambda$ constitutes a basis of $\mathscr{P}^{S_\infty}(\beta)$ dual to that of the monomial basis in superspace; that is,*

$$\langle\langle \overleftarrow{g_\Lambda}|\overrightarrow{m_\Omega}\rangle\rangle_\beta = \delta_{\Lambda,\Omega}. \tag{3.22}$$

We will need in the next section to make explicit the distinction between an infinite and a finite number of variables. Therefore, we also let

$$\langle\!\langle \cdot|\cdot \rangle\!\rangle_{\beta,N} : \mathscr{P}^{S_N}(\beta) \times \mathscr{P}^{S_N}(\beta) \longrightarrow \mathbb{Q}(\beta) \tag{3.23}$$

where $\mathscr{P}^{S_N}$ is the restriction of $\mathscr{P}^{S_\infty}$ to N variables, defined by requiring that the bases $\{g_\Lambda\}_{\ell(\Lambda)\leqslant N}$ and $\{m_\Lambda\}_{\ell(\Lambda)\leqslant N}$ be dual to each other:

$$\langle\!\langle \overleftarrow{g_\Lambda} | \overrightarrow{m_\Omega} \rangle\!\rangle_{\beta,N} := \delta_{\Lambda,\Omega}, \tag{3.24}$$

whenever $\ell(\Lambda)$ and $\ell(\Omega)$ are not larger than N. From this definition, it is thus obvious that

$$\big\langle\!\big\langle f_1^{(N)} \big| f_2^{(N)} \big\rangle\!\big\rangle_{\beta,N} = \langle\!\langle f_1 | f_2 \rangle\!\rangle_\beta \tag{3.25}$$

if f_1 and f_2 are elements of the ring of symmetric functions in superspace of bosonic degrees smaller than N, and if $f_1^{(N)}$ and $f_2^{(N)}$ are their respective restriction to N variables. This is because f_1 and $f_1^{(N)}$ (respectively f_2 and $f_2^{(N)}$) then have the same expansion in terms of the g and m bases. Note that with this definition, we have that

$$K^{\beta,N} = \sum_{\ell(\Lambda)\leqslant N} g_\Lambda(x,\theta) m_\Lambda(y,\phi), \tag{3.26}$$

where $K^{\beta,N}$ is the restriction of K^β to N variables and where (x,θ) and (y,ϕ) stand respectively for $(x_1,\dots,x_N,\theta_1,\dots,\theta_N)$ and $(y_1,\dots,y_N,\phi_1,\dots,\phi_N)$.

We complete this section by displaying a relationship between the g-basis elements and the bases of monomials and homogeneous polynomials.

Proposition 10. *Let $n_\Lambda! := n_{\Lambda^s}(1)! n_{\Lambda^s}(2)! \cdots$, where $n_{\Lambda^s}(i)$ is the multiplicity of i in Λ^s, and*

$$\binom{\beta}{n} := \frac{(\beta)_n}{n!}, \quad (\beta)_n := \beta(\beta-1)\cdots(\beta-n+1). \tag{3.27}$$

Then

$$g_n = \sum_{\Lambda\vdash(n|0)} \prod_i \binom{\beta+\Lambda_i-1}{\Lambda_i} m_\Lambda = \sum_{\Lambda\vdash(n|0)} \frac{(\beta)_{\ell(\Lambda)}}{n_\Lambda!} h_\Lambda, \tag{3.28}$$

$$\tilde{g}_n = \sum_{\Lambda\vdash(n|1)} (\beta+\Lambda_1) \prod_i \binom{\beta+\Lambda_i-1}{\Lambda_i} m_\Lambda = \sum_{\Lambda\vdash(n|1)} \frac{(\beta)_{\ell(\Lambda)}}{n_\Lambda!} h_\Lambda. \tag{3.29}$$

Proof. We start with the generating function (3.11). The product on the right-hand side can also be written as

$$\begin{aligned} &\prod_{i\geqslant 1} \sum_{k\geqslant 0} (-1)^k \binom{-\beta}{k} (tx_i+\tau\theta_i)^k \\ &\quad = \prod_{i\geqslant 1} \left[\sum_{k\geqslant 0} \binom{\beta+k-1}{k} (tx_i)^k + \tau\theta_i \sum_{k\geqslant 1} k \binom{\beta+k-1}{k} (tx_i)^{k-1} \right]. \end{aligned} \tag{3.30}$$

After some easy manipulations, (3.11) then becomes

$$G(t,\tau;\beta)=\sum_{n\geqslant 0}t^n\Bigg[\sum_{\lambda\vdash n}\prod_i\binom{\beta+\lambda_i-1}{\lambda_i}m_\lambda+\tau\sum_{\Lambda\vdash(n|1)}(\beta+\Lambda_1)\prod_i\binom{\beta+\Lambda_i-1}{\Lambda_i}m_\Lambda\Bigg] \tag{3.31}$$

and the first equality in the two formulas (3.28) and (3.29) are seen to hold.

To prove the remaining two formulas, we use the generating function of the homogeneous symmetric functions and proceed as follows:

$$\begin{aligned}\prod_i(1-tx_i-\tau\theta_i)^{-\beta}&=\Bigg(1+\sum_{m\geqslant 1}t^m h_m+\tau\sum_{n\geqslant 0}t^n\tilde{h}_n\Bigg)^\beta\\&=\sum_{k\geqslant 0}\binom{\beta}{k}\Bigg(\sum_{m\geqslant 1}t^m h_m+\tau\sum_{n\geqslant 0}t^n\tilde{h}_n\Bigg)^k\\&=\sum_{n\geqslant 0}\sum_{\lambda\vdash n}t^n\frac{(\beta)_{\ell(\lambda)}}{n_\lambda!}h_\lambda+\tau\sum_{m\geqslant 0}\sum_{\lambda\vdash m}t^m\frac{(\beta)_{\ell(\lambda)+1}}{\lambda!}h_\lambda\sum_{n\geqslant 0}t^n\tilde{h}_n\\&=\sum_{n\geqslant 0}t^n\Bigg[\sum_{\Lambda\vdash(n|0)}\frac{(\beta)_{\ell(\Lambda)}}{n_\Lambda!}h_\Lambda+\tau\sum_{\Lambda\vdash(n|1)}\frac{(\beta)_{\ell(\Lambda)}}{n_{\Lambda^s}!}h_\Lambda\Bigg]\end{aligned} \tag{3.32}$$

from which the desired expressions can be obtained. □

4. Jack polynomials in superspace: Analytical characterization

We review the main properties of Jack superpolynomials as they were defined in [5]. The section is completed with the presentation of a technical lemma to be used in Section 6. All the results of this section are independent of those of Section 3.

First, we define a scalar product in $\mathscr{P}$, the ring of polynomials in superspace in N variables. Given

$$\Delta(x)=\prod_{1\leqslant j<k\leqslant N}\left[\frac{x_j-x_k}{x_jx_k}\right], \tag{4.1}$$

$\langle\cdot|\cdot\rangle_{\beta,N}$ is defined (for β a positive integer) on the basis elements of $\mathscr{P}$ as

$$\langle\theta_I x^\lambda|\theta_J x^\mu\rangle_{\beta,N}=\begin{cases}\text{C.T.}[\Delta^\beta(\bar{x})\Delta^\beta(x)\bar{x}^\mu x^\lambda] & \text{if } I=J,\\ 0 & \text{otherwise,}\end{cases} \tag{4.2}$$

where $\bar{x}_i=1/x_i$, and where C.T.$[E]$ stands for the constant term of the expression E. (This is another form of the scalar product (1.10). More precisely, the latter is the analytic deformation of the former for all values of β.) This gives our first characterization of the Jack superpolynomials.

Proposition 11. [5] *There exists a unique basis* $\{J_\Lambda\}_\Lambda$ *of* $\mathscr{P}^{S_N}$ *such that*

$$(1)\quad J_\Lambda = m_\Lambda + \sum_{\Omega<\Lambda} c_{\Lambda\Omega}(\beta) m_\Lambda \qquad \textit{and} \qquad (2')\quad \langle J_\Lambda | J_\Omega\rangle_{\beta,N} \propto \delta_{\Lambda,\Omega}.$$

In order to present the other characterizations, we need to introduce the Dunkl–Cherednik operators (see [1] for instance):

$$\mathcal{D}_j := x_j\partial_{x_j} + \beta\sum_{k<j}\mathcal{O}_{jk} + \beta\sum_{k>j}\mathcal{O}_{jk} - \beta(j-1), \tag{4.3}$$

where

$$\mathcal{O}_{jk} = \begin{cases} \frac{x_j}{x_j-x_k}(1-K_{jk}), & k<j, \\ \frac{x_k}{x_j-x_k}(1-K_{jk}), & k>j. \end{cases} \tag{4.4}$$

Here K_{jk} is the operator that exchanges the variables x_j and x_k:

$$K_{jk} f(x_j, x_k, \theta_j, \theta_k) = f(x_k, x_j, \theta_j, \theta_k). \tag{4.5}$$

The Dunkl–Cherednik operators can be used to define two families of operators that preserve the elements of homogeneous degree $(n|m)$ of $\mathscr{P}^{S_N}$:

$$\mathcal{H}_r := \sum_{j=1}^N \mathcal{D}_j^r \quad \text{and} \quad \mathcal{I}_s := \frac{1}{(N-1)!}\sum_{\sigma\in S_N} \mathcal{K}_\sigma\big(\theta_1\partial_{\theta_1}\mathcal{D}_1^s\big)\mathcal{K}_\sigma^{-1}, \tag{4.6}$$

for $r \in \{1,2,3,\dots,N\}$ and $s \in \{0,1,2,\dots,N-1\}$ and where $\mathcal{K}_\sigma$ is built out of the operators $\mathcal{K}_{jk}$ that exchange $x_j \leftrightarrow x_k$ and $\theta_j \leftrightarrow \theta_k$ simultaneously:

$$\mathcal{K}_{i,i+1} := \kappa_{i,i+1} K_{i,i+1} \quad \text{where} \quad \kappa_{ij} f(x_i, x_j, \theta_i, \theta_j) = f(x_i, x_j, \theta_j, \theta_i). \tag{4.7}$$

The operators $\mathcal{H}_r$ and $\mathcal{I}_s$ are mutually commuting when restricted to $\mathscr{P}^{S_N}$; that is,

$$[\mathcal{H}_r, \mathcal{H}_s]f = [\mathcal{H}_r, \mathcal{I}_s]f = [\mathcal{I}_r, \mathcal{I}_s]f = 0 \quad \forall r, s, \tag{4.8}$$

where f represents an arbitrary polynomial in $\mathscr{P}^{S_N}$. Since they are also symmetric with respect to the scalar product $\langle\cdot|\cdot\rangle_\beta$ and have, when considered as a whole, a non-degenerate spectrum, they provide our second characterization of the Jack superpolynomials.

Proposition 12. [5] *The Jack superpolynomials* $\{J_\Lambda\}_\Lambda$ *are the unique common eigenfunctions of the* $2N$ *operators* $\mathcal{H}_r$ *and* $\mathcal{I}_s$*, for* $r \in \{1,2,3,\dots,N\}$ *and* $s \in \{0,1,2,\dots,N-1\}$.

We will now define two operators that play a special role in our study:

$$\mathcal{H} := \mathcal{H}_2 + \beta(N-1)\mathcal{H}_1 - \text{cst} \quad \text{and} \quad \mathcal{I} := \mathcal{I}_1, \tag{4.9}$$

where $\text{cst} = \beta N(1 - 3N - 2N^2)/6$. When acting on symmetric polynomials in superspace, the explicit form of $\mathcal{H}$ is simply

$$\mathcal{H} = \sum_i (x_i \partial_{x_i})^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_{x_i} - x_j \partial_{x_j}) - 2\beta \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2} (1 - \kappa_{ij}). \tag{4.10}$$

The operator $\mathcal{H}$ is the Hamiltonian of the supersymmetric form of the trigonometric Calogero–Moser–Sutherland model (see Section 1); it can be written in terms of two fermionic operators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ as

$$\mathcal{H} = \mathcal{Q}\mathcal{Q}^\dagger + \mathcal{Q}^\dagger \mathcal{Q}, \tag{4.11}$$

where

$$\mathcal{Q} := \sum_i \theta_i x_i \partial_{x_i} \quad \text{and} \quad \mathcal{Q}^\dagger = \sum_i \partial_{\theta_i} \left(x_i \partial_{x_i} + \beta \sum_{j \neq i} \frac{x_i + x_j}{x_i - x_j} \right), \tag{4.12}$$

so that $\mathcal{Q}^2 = (\mathcal{Q}^\dagger)^2 = 0$. Physically, $\mathcal{Q}$ is seen as creating fermions while $\mathcal{Q}^\dagger$ annihilates them. A state (superfunction) which is annihilated by the fermionic operators is called supersymmetric. In the case of polynomials in superspace, the only supersymmetric state is the identity.

Remark 13. The Hamiltonian $\mathcal{H}$ has an elegant differential geometric interpretation as a Laplace–Beltrami operator. To understand this assertion, consider first the real Euclidean space $\mathbb{T}^N$, where $\mathbb{T} = [0, 2\pi)$. Then, set $x_j = e^{\mathrm{i}t_j}$ for $t_j \in \mathbb{T}$, and identify the Grassmannian variable θ_i with the differential form dt_i. This allows us to rewrite the scalar product (1.10) as a Hodge–de Rham product involving complex differential forms; that is,

$$\langle A(t,\theta) \big| B(t,\theta) \rangle_{\beta,N} \sim \int_{\mathbb{T}^N} \overline{A(t,dt)} \wedge *B(t,dt), \tag{4.13}$$

where the bar denotes the complex conjugation and where the Hodge duality operator $*$ is formally defined by

$$A(t,dt) \wedge *B(t,dt) = C_{\beta,N} \prod_{i<j} \sin^{2\beta}\left(\frac{t_i - t_j}{2}\right) \sum_k \sum_{i_1 < \cdots < i_k} A_{i_1,\ldots,i_k} B_{i_1,\ldots,i_k} dt_1 \wedge \cdots \wedge dt_N, \tag{4.14}$$

for some constant $C_{\beta,N}$. Hence, we find that the fermionic operators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ can be respectively interpreted as the exterior derivative and its dual: $\mathcal{Q} \sim -\mathrm{i}d$ and $\mathcal{Q}^\dagger \sim \mathrm{i}d^*$. Thus

$$\mathcal{H} = \Delta := dd^* + d^*d. \tag{4.15}$$

In consequence, the Jack superpolynomials can be viewed as symmetric, homogeneous, and orthogonal eigenforms of a Laplace–Beltrami operator. This illustrates the known connection between supersymmetric quantum mechanics and differential geometry [8,22].

If the triangularity of the Jack polynomial J_Λ with respect to the monomial basis is imposed, requiring that it be a common eigenfunction of $\mathcal{H}$ and $\mathcal{I}$ is sufficient to define it. This is our third characterization of the Jack superpolynomials.

Theorem 14. [5] *The Jack polynomials in superspace $\{J_\Lambda\}_\Lambda$ form the unique basis of $\mathscr{P}^{S_N}(\beta)$ such that*

$$\mathcal{H}(\beta)J_\Lambda = \varepsilon_\Lambda(\beta)J_\Lambda, \quad \mathcal{I}(\beta)J_\Lambda = \epsilon_\Lambda(\beta)J_\Lambda \quad \textit{and} \quad J_\Lambda = m_\Lambda + \sum_{\Omega<\Lambda} c_{\Lambda\Omega}(\beta)m_\Omega. \tag{4.16}$$

The eigenvalues are given explicitly by

$$\varepsilon_\Lambda(\beta) = \sum_{j=1}^{N}\big[\big(\Lambda_j^*\big)^2 + \beta(N+1-2j)\Lambda_j^*\big], \tag{4.17}$$

$$\epsilon_\Lambda(\beta) = \sum_{i=1}^{m}\big[\Lambda_i - \beta m(m-1) - \beta\#_\Lambda\big], \tag{4.18}$$

where $\#_\Lambda$ denotes the number of pairs (i,j) such that $\Lambda_i < \Lambda_j$ for $1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant N$.

When no Grassmannian variables are involved, that is when $\overline{\underline{\Lambda}} = 0$, our characterizations of the Jack superpolynomials specialize to known characterizations of the Jack polynomials that can be found for instance in [21]. However, in the usual case there is a more common characterization of the Jack polynomials in which the scalar product appearing in Proposition 11 is replaced by the scalar product (1.9). As already announced, this more combinatorial characterization can be extended to the supersymmetric case. But before turning to the analysis of the behavior of J_Λ with respect to the combinatorial scalar product, we present a lemma concerning properties of the eigenvalues $\varepsilon_\Lambda(\beta)$ and $\epsilon_\Lambda(\beta)$.

Lemma 15. *Let $\Lambda \vdash (n|m)$ and write $\lambda = \Lambda^*$. Let also $\varepsilon_\Lambda(\beta)$ and $\epsilon_\Lambda(\beta)$ be the eigenvalues given in Theorem* 14. *Then*

$$\varepsilon_\Lambda(\beta) = 2\sum_j j\big(\lambda'_j - \beta\lambda_j\big) + \beta n(N+1) - n, \tag{4.19}$$

$$\epsilon_\Lambda(\beta) = \big|\Lambda^a\big| - \beta\big|\Lambda'^a\big| - \beta\frac{m(m-1)}{2}. \tag{4.20}$$

Proof. The first formula is known (see [21] for instance). As for the second one, we consider

$$\#_\Lambda = \sum_{i=1}^{m}\#_{\Lambda_i}, \tag{4.21}$$

where $\#_{\Lambda_i}$ denotes the number of parts in Λ^s bigger than Λ_i. But from the definition of the conjugation, we easily find that

$$\#_{\Lambda_i} = \Lambda'_{m+1-i} + 1 - i, \tag{4.22}$$

so that

$$\#_\Lambda = \sum_{i=1}^{m}\big(\Lambda_i' + 1 - i\big) = \big|\Lambda'^{a}\big| + \frac{m(m-1)}{2}, \tag{4.23}$$

from which the second formula follows. □

5. Combinatorial orthogonality of the Jack superpolynomials

In terms of the scalar product (1.9), we can directly check the self-adjointness of our eigenvalue-problem defining operators, $\mathcal{H}$ and $\mathcal{I}$.

Proposition 16. *The operators $\mathcal{H}$ and $\mathcal{I}$ defined in* (4.9) *are, when $N \to \infty$, self-adjoint* (*symmetric*) *with respect to the scalar product $\langle\langle\cdot|\cdot\rangle\rangle_\beta$ defined in* (1.9)*.*

Proof. We first rewrite the limit as $N \to \infty$ of $\mathcal{H}$ and $\mathcal{I}$ in terms of power sums. Since these differential operators are both of order two, it is sufficient to determine their action on the products of the form $p_m p_n$, $\tilde{p}_m p_n$ and $\tilde{p}_m \tilde{p}_n$. Direct computations give

$$\begin{aligned}\mathcal{H} = {} & \sum_{n\geqslant 1}\big[n^2 + \beta n(N-n)\big](p_n\partial_{p_n} + \tilde{p}_n\partial_{\tilde{p}_n}) + \beta\sum_{n,m\geqslant 1}\big[(m+n)p_m p_n \partial_{p_{m+n}} + 2m p_n \tilde{p}_m \partial_{\tilde{p}_{n+m}}\big] \\ & + \sum_{n,m\geqslant 1} mn\big[p_{m+n}\partial_{p_n}\partial_{p_m} + 2\tilde{p}_{n+m}\partial_{\tilde{p}_m}\partial_{p_n}\big]\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\mathcal{I} = {} & \sum_{n\geqslant 0}(1-\beta)(n\tilde{p}_n\partial_{\tilde{p}_n}) + \frac{\beta}{2}\sum_{m,n\geqslant 0}\tilde{p}_m\tilde{p}_n\partial_{\tilde{p}_m}\partial_{\tilde{p}_n} \\ & + \sum_{m\geqslant 0, n\geqslant 1}\big[n\tilde{p}_{m+n}\partial_{\tilde{p}_m}\partial_{p_n} + \beta p_n \tilde{p}_m \partial_{\tilde{p}_{m+n}}\big].\end{aligned} \tag{5.2}$$

Note that these equations are valid when N is either infinite or finite. In the latter case, the sums over the terms containing $\tilde{p}_m$ and p_n are respectively restricted such that $m \leqslant N-1$ and $n \leqslant N$.

Then, letting $A^\perp$ denote the adjoint of a generic operator A with respect to the scalar product (1.9), it is easy to check that

$$\beta p_n^\perp = n\partial_{p_n} \quad \text{and} \quad \beta\tilde{p}_n^\perp = \partial_{\tilde{p}_n}. \tag{5.3}$$

Hence, comparing the three previous equations, we obtain that $\mathcal{H}^\perp = \mathcal{H}$ and $\mathcal{I}^\perp = \mathcal{I}$. For these calculations, we observe that $(ab)^\perp = b^\perp a^\perp$, even when a and b are both fermionic. □

In order to demonstrate the orthogonality of the Jack superpolynomials with respect to the scalar product (1.9), the most natural path consists in establishing the self-adjointness of all the operators $\mathcal{H}_n$ and $\mathcal{I}_n$. But proceeding as for $\mathcal{H}$ and $\mathcal{I}$ above, by trying to reexpress them in terms of p_n, $\tilde{p}_n$ and their derivatives, seems hopeless. An indirect line of attack is mandatory.

Let us first recall that the conserved operators (4.6) can all be expressed in terms of the Dunkl–Cherednik operators defined in (4.3). The $\mathcal{D}_i$'s commute among themselves:

$$[\mathcal{D}_i, \mathcal{D}_j] = 0. \tag{5.4}$$

They obey the Hecke relations (correcting a misprint in Eq. (25) of [5])

$$\mathcal{D}_i K_{i,i+1} - K_{i,i+1}\mathcal{D}_{i+1} = \beta. \tag{5.5}$$

We will also need the following commutation relations:

$$[\mathcal{D}_i, x_i] = x_i + \beta\Bigg(\sum_{j<i} x_i K_{ij} + \sum_{j>i} x_j K_{ij}\Bigg), \tag{5.6}$$

while if $i \neq k$,

$$[\mathcal{D}_i, x_k] = -\beta x_{\max(i,k)} K_{ik}. \tag{5.7}$$

The idea of the proof of the orthogonality is the following: in a first step, we show that the conserved operators $\mathcal{H}_n$ and $\mathcal{I}_n$ are self-adjoint with respect to the scalar product (1.9) and then we demonstrate that this implies the orthogonality of the J_Λ's. The self-adjointness property is established via the kernel: showing that $F = F^\perp$ is the same as showing that

$$F^{(x)} K^{\beta,N} = F^{(y)} K^{\beta,N}, \tag{5.8}$$

where $K^{\beta,N}$ is the restriction of K^β defined in Theorem 3 to N variables, and where $F^{(x)}$ (respectively $F^{(y)}$) stands for the operator F in the variable x (respectively y). In order to prove this for our conserved operators $\mathcal{H}_n$ and $\mathcal{I}_n$, we need to establish some results on the action of symmetric monomials in the Dunkl–Cherednik operators acting on the following expression:

$$\tilde{\Omega} := \prod_{i=1}^{N} \frac{1}{(1 - x_i y_i)} \prod_{i,j=1}^{N} \frac{1}{(1 - x_i y_j)^\beta}, \tag{5.9}$$

as well as some modification of $\tilde{\Omega}$. For that matter, we recall a result of Sahi [17]:

Proposition 17. *The action of the Dunkl–Cherednik operators $\mathcal{D}_j$ on $\tilde{\Omega}$ defined by* (5.9) *satisfies*:

$$\mathcal{D}_j^{(x)} \tilde{\Omega} = \mathcal{D}_j^{(y)} \tilde{\Omega}. \tag{5.10}$$

Before turning to the core of our argument, we establish the following lemma.

Lemma 18. *Given a set $J = \{j_1, \ldots, j_\ell\}$, denote by x_J the product $x_{j_1} \cdots x_{j_\ell}$. Suppose $x_J = K_\sigma x_I$ for some $\sigma \in S_N$ such that $K_\sigma F K_{\sigma^{-1}} = F$. Then*

$$\frac{1}{x_I} F^{(x)} x_I \tilde{\Omega} = \frac{1}{y_I} F^{(y)} y_I \tilde{\Omega} \quad \Longrightarrow \quad \frac{1}{x_J} F^{(x)} x_J \tilde{\Omega} = \frac{1}{y_J} F^{(y)} y_J \tilde{\Omega}. \tag{5.11}$$

Proof. The proof is straightforward and only uses the simple property $K_\sigma^{(x)}\tilde{\Omega} = K_{\sigma^{-1}}^{(y)}\tilde{\Omega}$. To be more precise, we have

$$\begin{aligned}\frac{1}{x_J}F^{(x)}x_J\tilde{\Omega} &= K_\sigma^{(x)}\frac{1}{x_I}F^{(x)}x_I K_{\sigma^{-1}}^{(x)}\tilde{\Omega} = K_\sigma^{(y)}K_\sigma^{(x)}\frac{1}{x_I}F^{(x)}x_I\tilde{\Omega} = K_\sigma^{(y)}K_\sigma^{(x)}\frac{1}{y_I}F^{(y)}y_I\tilde{\Omega} \\ &= K_\sigma^{(y)}\frac{1}{y_I}F^{(y)}y_I K_{\sigma^{-1}}^{(y)}\tilde{\Omega} = \frac{1}{y_J}F^{(y)}y_J\tilde{\Omega}. \qquad \square \end{aligned} \tag{5.12}$$

We are now ready to attack the main proposition.

Proposition 19. *The mutually commuting operators $\mathcal{H}_n$ and $\mathcal{I}_n$ satisfy*

$$\mathcal{H}_n^{(x)}K^{\beta,N} = \mathcal{H}_n^{(y)}K^{\beta,N} \quad \textit{and} \quad \mathcal{I}_n^{(x,\theta)}K^{\beta,N} = \mathcal{I}_n^{(y,\phi)}K^{\beta,N}, \tag{5.13}$$

with $K^{\beta,N}$ the restriction to N variables of the kernel K^β defined in Theorem 3.

Proof. We first expand the kernel as follows:

$$K^{\beta,N} = K_0\prod_{i,j}\left(1+\beta\frac{\theta_i\phi_j}{(1-x_iy_j)}\right) \tag{5.14}$$

$$= K_0\left\{1+\beta e_1\left(\frac{\theta_i\phi_j}{(1-x_iy_j)}\right)+\cdots+\beta^N e_N\left(\frac{\theta_i\phi_j}{(1-x_iy_j)}\right)\right\} \tag{5.15}$$

where K_0 stands for $K^{\beta,N}(x,y,0,0)$, i.e.,

$$K_0 := \prod_{i,j=1}^{N}\frac{1}{(1-x_iy_j)^\beta}, \tag{5.16}$$

and where $e_\ell(u_{i,j})$ is the elementary symmetric function e_ℓ in the variables

$$u_{i,j} := \frac{\theta_i\phi_j}{(1-x_iy_j)}, \quad i,j=1,\dots,N. \tag{5.17}$$

Note that, in these variables, the maximal possible elementary symmetric function is e_N given that $\theta_i^2=\phi_i^2=0$. In the following, we will use the compact notation $I^-=\{1,\dots,i-1\}$ and $I^+=\{i,\dots,N\}$ (and similarly for $J^\pm$), together with $w_{I^-}=w_1\cdots w_{i-1}$ and $w_{I^+}=w_i\cdots w_N$.

The action of the operators on K^β can thus be decomposed into their action on each monomial in this expansion. Now observe that K_0 is invariant under the exchange of any two variables x or any two variables y. Therefore, if an operator F is such that $\mathcal{K}_\sigma F\mathcal{K}_\sigma^{-1}=F$ for all $\sigma\in S_N$, and such that

$$F^{(x,\theta)}v_{I^-}K_0 = F^{(y,\phi)}v_{I^-}K_0 \quad \text{with } v_i := u_{i,i} \tag{5.18}$$

for all $i=1,\dots,N+1$, then we immediately have by symmetry that $F^{(x,\theta)}K^\beta = F^{(y,\phi)}K^\beta$. We will use this observation in the case of $\mathcal{H}_n$ and $\mathcal{I}_n$.

We first consider the case $F = \mathcal{H}_n$. Recall from (4.6) that $\mathcal{H}_n = p_n(\mathcal{D}_i)$ is such that $\mathcal{K}_\sigma \mathcal{H}_n \mathcal{K}_\sigma^{-1} = \mathcal{H}_n$ (see [5]). Since $\mathcal{H}_n$ does not depend on the fermionic variables, we thus have to prove from the previous observation that

$$\mathcal{H}_n^{(x)} \frac{1}{(1-xy)_{I^-}} K_0 = \mathcal{H}_n^{(y)} \frac{1}{(1-xy)_{I^-}} K_0, \tag{5.19}$$

or equivalently

$$\mathcal{H}_n^{(x)} (1-xy)_{I^+} \tilde{\Omega} = \mathcal{H}_n^{(y)} (1-xy)_{I^+} \tilde{\Omega}, \tag{5.20}$$

for all $i = 1, \ldots, N+1$ (the case $i = N+1$ corresponds to the empty product).

The underlying symmetry of the problem implies the result will follow from showing

$$y_{J^+} \mathcal{H}_n^{(x)} x_{J^+} \tilde{\Omega} = x_{J^+} \mathcal{H}_n^{(y)} y_{J^+} \tilde{\Omega}, \tag{5.21}$$

for $j \geqslant i$, or equivalently,

$$\frac{1}{x_{J^+}} \mathcal{H}_n^{(x)} x_{J^+} \tilde{\Omega} = \frac{1}{y_{J^+}} \mathcal{H}_n^{(y)} y_{J^+} \tilde{\Omega}. \tag{5.22}$$

This follows from Lemma 18 which assures us that all the different terms can be obtained from these special ones.

Now, instead of analyzing the family $\mathcal{H}_n = p_n(\mathcal{D}_i)$, it will prove simpler to consider the equivalent family $e_n(\mathcal{D}_i)$. We will first show the case $e_N(\mathcal{D}_i)$; that is,

$$\frac{1}{x_{J^+}} \mathcal{D}_1^{(x)} \cdots \mathcal{D}_N^{(x)} x_{J^+} \tilde{\Omega} = \frac{1}{y_{J^+}} \mathcal{D}_1^{(y)} \cdots \mathcal{D}_N^{(y)} y_{J^+} \tilde{\Omega}. \tag{5.23}$$

Let us concentrate on the left-hand side. We note that

$$\frac{1}{x_{J^+}} \mathcal{D}_1^{(x)} \cdots \mathcal{D}_N^{(x)} x_{J^+} \tilde{\Omega} = \frac{1}{x_{J^+}} \mathcal{D}_1^{(x)} x_{J^+} \cdots \frac{1}{x_{J^+}} \mathcal{D}_N^{(x)} x_{J^+} \tilde{\Omega}. \tag{5.24}$$

It thus suffices to study each term $(x_{J^+})^{-1} \mathcal{D}_j x_{J^+}$ separately. In each case we find that

$$\mathcal{D}_k x_{J^+} = x_{J^+} \tilde{\mathcal{D}}_k. \tag{5.25}$$

The form of $\tilde{\mathcal{D}}$ depends upon j and k. There are two cases:

$$\begin{aligned} k < j: &\quad \tilde{\mathcal{D}}_k = \mathcal{D}_k - \beta \sum_{\ell=j}^{N} K_{\ell,k}, \\ k \geqslant j: &\quad \tilde{\mathcal{D}}_k = \mathcal{D}_k + 1 + \beta \sum_{\ell=1}^{j-1} K_{\ell,k} \end{aligned} \tag{5.26}$$

which can be easily checked using (5.6) and (5.7). We can thus write

$$\frac{1}{x_{J^+}}\mathcal{D}_1^{(x)}\cdots\mathcal{D}_N^{(x)}x_{J^+}\tilde{\Omega}=\tilde{\mathcal{D}}_1^{(x)}\cdots\tilde{\mathcal{D}}_N^{(x)}\tilde{\Omega}. \tag{5.27}$$

Using Proposition 17 and $K_{ij}^{(x)}\tilde{\Omega}=K_{ij}^{(y)}\tilde{\Omega}$, the rightmost term $\tilde{\mathcal{D}}_N^{(x)}$ can thus be changed into $\tilde{\mathcal{D}}_N^{(y)}$. Since it commutes with the previous terms (i.e., it acts on the variables y while the others act on x), we have

$$\begin{aligned}\tilde{\mathcal{D}}_1^{(x)}\cdots\tilde{\mathcal{D}}_{N-1}^{(x)}\tilde{\mathcal{D}}_N^{(y)}\tilde{\Omega}&=\tilde{\mathcal{D}}_N^{(y)}\tilde{\mathcal{D}}_1^{(x)}\cdots\tilde{\mathcal{D}}_{N-1}^{(x)}\tilde{\Omega}=\tilde{\mathcal{D}}_N^{(y)}\tilde{\mathcal{D}}_{N-1}^{(y)}\cdots\tilde{\mathcal{D}}_1^{(y)}\tilde{\Omega}\\&=\frac{1}{y_{J^+}}\mathcal{D}_N^{(y)}y_{J^+}\cdots\frac{1}{y_{J^+}}\mathcal{D}_1^{(y)}y_{J^+}\tilde{\Omega}\\&=\frac{1}{y_{J^+}}\mathcal{D}_N^{(y)}\cdots\mathcal{D}_1^{(y)}y_{J^+}\tilde{\Omega}=\frac{1}{y_{J^+}}\mathcal{D}_1^{(y)}\cdots\mathcal{D}_N^{(y)}y_{J^+}\tilde{\Omega},\end{aligned} \tag{5.28}$$

which is the desired result.

At this point, we have only considered a single conserved operator, namely $e_N(\mathcal{D}_i)$. But by replacing $\mathcal{D}_i$ with $\mathcal{D}_i+t$ in $e_N(\mathcal{D}_i)$, we obtain a generating function for all the operators $e_n(\mathcal{D}_i)$. Since to prove

$$e_N\big(\mathcal{D}_i^{(x)}+t\big)K^{\beta,N}=e_N\big(\mathcal{D}_i^{(y)}+t\big)K^{\beta,N}$$

simply amounts to replacing $\tilde{\mathcal{D}}_i$ by $\tilde{\mathcal{D}}_i+t$ in the previous argument, we have completed the proof of $\mathcal{H}_n^{(x)}K^{\beta,N}=\mathcal{H}_n^{(y)}K^{\beta,N}$.

For the case of $\mathcal{I}_n$, we start with the expression given in (4.6) which readily implies that $\mathcal{K}_\sigma\mathcal{I}_n\mathcal{K}_\sigma^{-1}=\mathcal{I}_n$. Therefore, from the observation surrounding formula (5.18), and because the derivative $\theta_1\partial_{\theta_1}$ annihilates the K_0 term in the expansion of $K^{\beta,N}$, we only need to show that

$$\mathcal{I}_n^{(x,\theta)}v_{I^-}K_0=\mathcal{I}_n^{(y,\phi)}v_{I^-}K_0, \tag{5.29}$$

for $i=2,\dots,N+1$. Up to an overall multiplicative factor, the only contributing part in $\mathcal{I}_n$, when acting on v_{I^-}, is

$$\mathcal{O}_n:=\mathcal{D}_1^n+K_{12}\mathcal{D}_1^nK_{12}+\cdots+K_{1,i-1}\mathcal{D}_1^nK_{1,i-1}. \tag{5.30}$$

It thus suffices to show that

$$\mathcal{O}_n^{(x)}(1-xy)_{I^+}\tilde{\Omega}=\mathcal{O}_n^{(y)}(1-xy)_{I^+}\tilde{\Omega}. \tag{5.31}$$

Once more, we can use Lemma 18 since $\mathcal{O}_n$ commutes with $K_{k,\ell}$ for $k,\ell\geqslant i$. Thus, we only need to check that for $j\geqslant i$,

$$\frac{1}{x_{J^+}}\mathcal{O}_n^{(x)}x_{J^+}\tilde{\Omega}=\frac{1}{y_{J^+}}\mathcal{O}_n^{(y)}y_{J^+}\tilde{\Omega}. \tag{5.32}$$

Since the $K_{1\ell}$'s act trivially on the variables x_j for $j > \ell$, the previous relation reduces to proving

$$\frac{1}{x_{J^+}}\big[\mathcal{D}_1^n\big]^{(x)} x_{J^+}\tilde{\Omega} = \frac{1}{y_{J^+}}\big[\mathcal{D}_1^n\big]^{(y)} y_{J^+}\tilde{\Omega}. \tag{5.33}$$

The left-hand side takes the form

$$\frac{1}{x_{J^+}}\big[\mathcal{D}_1^n\big]^{(x)} x_{J^+}\tilde{\Omega} = \left\{\frac{1}{x_{J^+}}\mathcal{D}_1^{(x)} x_{J^+}\right\}^n \tilde{\Omega}. \tag{5.34}$$

We then only have to evaluate $(x_{J^+})^{-1}\mathcal{D}_1^{(x)} x_{J^+}$. The result is given by the first case in (5.26) (since $j > 1$) . The proof is completed as follows:

$$\left\{\frac{1}{x_{J^+}}\mathcal{D}_1^{(x)} x_{J^+}\right\}^n \tilde{\Omega} = \big[\tilde{\mathcal{D}}_1^{(x)}\big]^n \tilde{\Omega} = \big[\tilde{\mathcal{D}}_1^{(y)}\big]^n \tilde{\Omega} = \frac{1}{y_{J^+}}\big[\mathcal{D}_1^n\big]^{(y)} y_{J^+}\tilde{\Omega}. \qquad \square \tag{5.35}$$

As previously mentioned, the proposition has the following corollary.

Corollary 20. *The operators $\mathcal{H}_r$ and $\mathcal{I}_s$ defined in* (4.6) *are self-adjoint* (*symmetric*) *with respect to the scalar product $\langle\langle\cdot|\cdot\rangle\rangle_{\beta,N}$ given in* (3.24).

This immediately gives our main result.

Theorem 21. *The Jack superpolynomials $\{J_\Lambda\}_\Lambda$ are orthogonal with respect to the combinatorial scalar product*; *that is,*

$$\langle\langle J_\Lambda | J_\Omega\rangle\rangle_\beta \propto \delta_{\Lambda,\Omega}. \tag{5.36}$$

Proof. The fact that in N variables $\langle\langle J_\Lambda | J_\Omega\rangle\rangle_{\beta,N} \propto \delta_{\Lambda,\Omega}$ is a consequence of Corollary 20 and Proposition 12, which says that the Jack superpolynomials are the unique common eigenfunctions of the $2N$ operators appearing in Corollary 20. Given that the expansion coefficients of the Jack superpolynomials in terms of supermonomials do not depend on the number of variables N [5], the theorem then follows from (3.25). $\square$

Remark 22. That the Jack superpolynomials are orthogonal with respect to the analytical and combinatorial scalar products is certainly remarkable given their rather different nature. Even in the absence of fermionic variables, the orthogonality of the Jack polynomials with respect to both scalar products is a highly non-trivial observation. In that case, one can provide a partial rationale for the compatibility between the two scalar products, by noticing their equivalence in the following two circumstances [10,14]:

$$\langle f|g\rangle_{\beta=1,N} = \langle\langle f|g\rangle\rangle_{\beta=1,N} \quad (m=0) \tag{5.37}$$

(see, e.g., [14, VI.9, Remark 2]) and

$$\lim_{N\to\infty}\frac{\langle f|g\rangle_{\beta,N}}{\langle 1|1\rangle_{\beta,N}} = \langle\langle f|g\rangle\rangle_\beta \quad (m=0) \tag{5.38}$$

(see, e.g., [14, VI.9, (9.9)]) for f, g, two arbitrary symmetric polynomials. In superspace, when $m \neq 0$, this compatibility between the two products is even more remarkable since the limiting-case equivalences (5.37) and (5.38) are simply lost. This is most easily seen by realizing that, after integration over the fermionic variables, we obtain

$$\langle p_\lambda \tilde{p}_n | p_\mu \tilde{p}_r \rangle_{\beta,N} = \langle p_\lambda | p_\mu p_{r-n} \rangle_{\beta,N}, \quad r > n, \tag{5.39}$$

and thus the power sums cannot be orthogonal for any value of N and β. This shows that the connection between the two scalar products is rather intricate.

Corollary 23. *The following statements are direct consequences of the orthogonality property of the Jack polynomials in superspace.*

1. *The Jack polynomials in superspace $\{J_\Lambda\}_\Lambda$ form the unique basis of $\mathscr{P}^{S_\infty}$ such that*

$$\begin{array}{lll} 1.1. & J_\Lambda = m_\Lambda + \sum_{\Omega<\Lambda} c_{\Lambda\Omega}(\beta) m_\Lambda & (\textit{triangularity}); \\ 1.2. & \langle\langle J_\Lambda | J_\Omega \rangle\rangle_\beta \propto \delta_{\Lambda,\Omega} & (\textit{orthogonality}). \end{array} \tag{5.40}$$

2. *Let K^β be the reproducing kernel defined in Theorem* 3. *Then,*

$$K^\beta(x,\theta;y,\phi) = \sum_{\Lambda\in \mathrm{SPar}} j_\Lambda(\beta)^{-1} \overleftarrow{J_\Lambda(x,\theta)}\,\overrightarrow{J_\Lambda(y,\phi)}, \tag{5.41}$$

where

$$j_\Lambda(\beta) := \langle\langle \overleftarrow{J_\Lambda} | \overrightarrow{J_\Lambda} \rangle\rangle_\beta. \tag{5.42}$$

3. *Let $\{g_\Lambda\}_\Lambda$ be the basis, defined in* (3.12)*, dual to that of the monomials with respect to the combinatorial scalar product. Then, the Jack superpolynomials expand upper triangularly in this basis*:

$$J_\Lambda = \sum_{\Omega\geqslant\Lambda} u_{\Lambda\Omega}(\beta) g_\Omega, \quad \textit{with } u_{\Lambda\Lambda}(\beta) \neq 0. \tag{5.43}$$

Proof. 1. We have seen that the Jack polynomials in superspace satisfy 1.1 and 1.2. To prove unicity, suppose $\{\tilde{J}_\Lambda\}_\Lambda$ satisfies 1.1 and 1.2. It was shown in [5] that the operators $\mathcal{H}$ and $\mathcal{I}$ act triangularly on the monomial basis. Thus, $\mathcal{H}$ and $\mathcal{I}$ also act triangularly on the basis $\{\tilde{J}_\Lambda\}_\Lambda$. Furthermore, from Proposition 16, they are self-adjoint with respect to the combinatorial scalar product. Hence, we must conclude from the orthogonality of $\{\tilde{J}_\Lambda\}_\Lambda$ that $\tilde{J}_\Lambda$ is an eigenfunction of $\mathcal{H}$ and $\mathcal{I}$, from which Theorem 14 implies that $\tilde{J}_\Lambda = J_\Lambda$.

2. The proof is similar to that of Lemma 8 (see also Section VI.2 of [14]).

3. Suppose that $\langle\langle J_\Lambda | J_\Omega \rangle\rangle_\beta \propto \delta_{\Lambda,\Omega}$, and let $J_\Lambda = \sum_{\Omega\in\mathcal{S}} u_{\Lambda\Omega} g_\Omega$, where $\mathcal{S}$ is some undefined set. If Λ is not the smallest element of $\mathcal{S}$, then there exists at least one element Γ of $\mathcal{S}$ that does not dominate any other of its elements. In this case, we have

$$\langle\langle J_\Lambda | J_\Gamma \rangle\rangle_\beta = \sum_{\Omega\in\mathcal{S}} u_{\Lambda\Omega}(\beta) \sum_{\Delta\leqslant\Gamma} c_{\Gamma\Delta}(\beta) \langle\langle g_\Omega | m_\Delta \rangle\rangle_\beta. \tag{5.44}$$

Since Γ does not dominate any element of $\mathcal{S}$, the unique non-zero contribution in this expression is that of $u_{\Lambda\Gamma}(\beta)c_{\Gamma\Gamma}(\beta)\langle\langle g_\Gamma, m_\Gamma\rangle\rangle_\beta = u_{\Lambda\Gamma}(\beta)$. Since this term is non-zero by supposition, we have the contradiction $0 = \langle\langle J_\Lambda | J_\Gamma\rangle\rangle_\beta = u_{\Lambda\Gamma}(\beta) \neq 0$. □

Actually, it can be shown that all statements of Corollary 23 and Theorem 27 below are not only consequences of Proposition 21 but are equivalent to it.

6. Further properties

6.1. Duality

In this subsection, we show that the homomorphism $\hat{\omega}_\beta$, defined in Eq. (3.2), has a simple action on Jack superpolynomials. To avoid any confusion, we make explicit the β dependence by writing $J_\Lambda^{(1/\beta)}$.

Remark 24. The rationale for this notation is to match the one used in [14] when $m = 0$:

$$J_\Lambda^{(1/\beta)}(x,\theta) = J_{\Lambda^s}^{(1/\beta)}(x) = J_{\Lambda^s}^{(\alpha)}(x),$$

where $\alpha = 1/\beta$. (Similarly, in our previous works [3,5], we denoted $J_\Lambda^{(1/\beta)}$ by $J_\Lambda(x,\theta;1/\beta)$ to keep our definition similar to the usual form introduced by Stanley [21] as $J_\lambda(x;\alpha)$ when $m = 0$.) We stress however, that when we need to make explicit the β-dependence of $j_\Lambda, \mathcal{H}$ and $\mathcal{I}$, we write $j_\Lambda(\beta)$, $\mathcal{H}(\beta)$ and $\mathcal{I}(\beta)$ respectively.

Proposition 25. *One has*

$$\mathcal{H}(\beta)\hat{\omega}_\beta J_\Lambda^{(\beta)} = \varepsilon_{\Lambda'}(\beta)\hat{\omega}_\beta J_\Lambda^{(\beta)} \quad \textit{and} \quad \mathcal{I}(\beta)\hat{\omega}_\beta J_\Lambda^{(\beta)} = \epsilon_{\Lambda'}(\beta)\hat{\omega}_\beta J_\Lambda^{(\beta)}. \tag{6.1}$$

Proof. Let us rewrite the special form of the operator $\mathcal{H}(\beta)$ appearing in the proof of Proposition 16 as

$$\mathcal{H}(\beta) = \sum_{n\geqslant 1}\big[n^2 + \beta n(N-n)\big]\hat{A}_n + \sum_{m,n\geqslant 1}(\beta\hat{B}_{m,n} + \hat{C}_{m,n}), \tag{6.2}$$

with

$$\begin{aligned}
\hat{A}_n &= p_n\partial_{p_n} + \tilde{p}_n\partial_{\tilde{p}_n},\\
\hat{B}_{m,n} &= (m+n)p_m p_n\partial_{p_{m+n}} + 2mp_n\tilde{p}_m\partial_{\tilde{p}_{n+m}},\\
\hat{C}_{m,n} &= mn(p_{m+n}\partial_{p_n}\partial_{p_m} + 2\tilde{p}_{n+m}\partial_{\tilde{p}_m}\partial_{p_n}).
\end{aligned} \tag{6.3}$$

From these definitions, we get

$$\hat{\omega}_{1/\beta}\hat{A}_n = \hat{A}_n\hat{\omega}_{1/\beta}, \quad \hat{\omega}_{1/\beta}\hat{B}_{m,n} = -\frac{1}{\beta}\hat{B}_{m,n}\hat{\omega}_{1/\beta} \quad \text{and} \quad \hat{\omega}_{1/\beta}\hat{C}_{m,n} = -\beta\hat{C}_{m,n}\hat{\omega}_{1/\beta}. \tag{6.4}$$

These relations imply

$$\begin{aligned}\hat\omega_{1/\beta}\mathcal{H}(\beta)\hat\omega_\beta &= \sum_{n\geqslant 1}\big[n^2+\beta n(N-n)\big]\hat A_n - \sum_{m,n\geqslant 1}(\hat B_{m,n}+\beta\hat C_{m,n})\\ &= (1+\beta)N\sum_{n\geqslant 1} n\hat A_n - \beta\mathcal{H}(1/\beta).\end{aligned}$$

Now, considering $\sum_{n\geqslant 1} n\hat A_n m_\Lambda = |\Lambda| m_\Lambda$ and Lemma 15, we obtain

$$\hat\omega_{1/\beta}\mathcal{H}(\beta)\hat\omega_\beta J_\Lambda^{(\beta)} = \varepsilon_{\Lambda'}(\beta) J_\Lambda^{(\beta)} \tag{6.5}$$

as claimed. The relation involving $\mathcal{I}(\beta)$ is proved in a similar way. □

For the next theorem, we will need the following result from [6]:

Proposition 26. *Let Λ be a superpartition and Λ' its conjugate. Then*

$$\overleftarrow{e_\Lambda} = m_{\Lambda'} + \sum_{\Omega<\Lambda'} N_\Lambda^\Omega m_\Omega, \quad \textit{with } N_\Lambda^\Omega \in \mathbb{Z}. \tag{6.6}$$

Theorem 27. *The homomorphism $\hat\omega_\beta$ is such that*

$$\hat\omega_{1/\beta}\overrightarrow{J_\Lambda^{(1/\beta)}} = j_\Lambda(\beta)\overleftarrow{J_{\Lambda'}^{(\beta)}}, \tag{6.7}$$

with $j_\Lambda(\beta)$ defined in (5.42).

Proof. Let us first prove that $\hat\omega_\beta J_\Lambda^{(\beta)} \propto J_{\Lambda'}^{(1/\beta)}$. From the third point of Corollary 23, we know that $J_\Lambda^{(1/\beta)} = \sum_{\Omega\geqslant\Lambda} u_{\Lambda\Omega}(\beta) g_\Omega$. But Eq. (3.19) implies $\hat\omega_{1/\beta} g_\Lambda = e_\Lambda$. Hence,

$$\hat\omega_{1/\beta}\big(J_\Lambda^{(1/\beta)}\big) = \sum_{\Omega\geqslant\Lambda} u_{\Lambda\Omega}(\beta) e_\Omega = \sum_{\Omega\geqslant\Lambda} u_{\Lambda\Omega}(\beta)\sum_{\Gamma\leqslant\Omega'} N_\Omega^\Gamma \overleftarrow{m_\Gamma} = \sum_{\Gamma\leqslant\Lambda'} v_{\Lambda\Gamma}(\beta)\overleftarrow{m_\Gamma}, \tag{6.8}$$

where we have used (3.19), Proposition 26 and the fact that $\Omega\geqslant\Lambda \Leftrightarrow \Omega'\leqslant\Lambda'$. Further, since $N_\Lambda^{\Lambda'} = 1$ and $u_{\Lambda\Lambda}(\beta)\neq 0$, we have $v_{\Lambda\Lambda'}\neq 0$. Now, from Proposition 25, $\hat\omega_{1/\beta}(J_\Lambda^{(1/\beta)})$ is an eigenfunction of $\mathcal{H}(1/\beta)$ and $\mathcal{I}(1/\beta)$ with eigenvalues $\varepsilon_{\Lambda'}(1/\beta)$ and $\epsilon_{\Lambda'}(1/\beta)$ respectively. The triangularity we just obtained ensures from Theorem 14, that $\hat\omega_{1/\beta}(J_\Lambda^{(1/\beta)})$ is proportional to $J_{\Lambda'}^{(\beta)}$.

Again from Proposition 26, we know that $m_\Lambda = (-1)^{m(m-1)/2} e_{\Lambda'} +$ higher terms, so that

$$J_\Lambda^{(1/\beta)} = (-1)^{m(m-1)/2} e_{\Lambda'} + \text{higher terms}. \tag{6.9}$$

Moreover, from Eq. (3.19), we get

$$\hat\omega_\beta J_\Lambda^{(1/\beta)} = (-1)^{m(m-1)/2} g_{\Lambda'} + \text{higher terms}. \tag{6.10}$$

But the proportionality proved above implies

$$\hat\omega_{1/\beta}\overrightarrow{J_\Lambda^{(1/\beta)}} = A_\Lambda(\beta)\overleftarrow{J_{\Lambda'}^{(\beta)}} = A_\Lambda(\beta)\overleftarrow{m_{\Lambda'}} + \text{lower terms}, \tag{6.11}$$

for some constant $A_\Lambda(\beta)$. Finally, considering the duality between g_Λ and m_Λ, we obtain

$$\begin{aligned}(-1)^{m(m-1)/2} j_\Lambda(\beta) &= \langle\!\langle J_\Lambda^{(1/\beta)} \,|\, J_\Lambda^{(1/\beta)} \rangle\!\rangle_\beta \\ &= \langle\!\langle \hat\omega_\beta J_\Lambda^{(1/\beta)} \,|\, \hat\omega_{1/\beta} J_\Lambda^{(1/\beta)} \rangle\!\rangle_\beta \\ &= \langle\!\langle (-1)^{m(m-1)/2} \overrightarrow{g_{\Lambda'}} \,|\, A_\Lambda(\beta) \overleftarrow{m_{\Lambda'}} \rangle\!\rangle_\beta \\ &= (-1)^{m(m-1)/2} A_\Lambda(\beta) \end{aligned} \tag{6.12}$$

as desired. □

6.2. Limiting cases

In Section 5, we have proved that the Jack superpolynomials are orthogonal with respect to the combinatorial scalar product. This provides a direct link with the classical symmetric functions in superspace. Other links, less general but more explicit, are presented in this section, from the consideration of J_Λ for special values of β or for particular superpartitions.

Proposition 28. *For $\Lambda = (n)$ or $(n;0)$, one has (using the notation of Proposition 10):*

$$J_{(n)} = \frac{n!}{(\beta+n-1)_n} g_n \quad \textit{and} \quad J_{(n;0)} = \frac{n!}{(\beta+n)_{n+1}} \tilde g_n. \tag{6.13}$$

Proof. Since $J_{(0;1^n)} = m_{(0;1^n)} = \tilde e_n$, we have on the one hand $\hat\omega_\beta(J_{(0;1^n)}) = \tilde g_n$ from (3.19). On the other hand, from Proposition 25, $\hat\omega_\beta(J_{(0;1^n)})$ is an eigenfunction of $\mathcal{H}(\beta)$ and $\mathcal{I}(\beta)$ with eigenvalues $\varepsilon_{(n;0)}(\beta)$ and $\epsilon_{(n;0)}(\beta)$ respectively. Since $(n;0)$ is the highest partition with one fermion in the order on superpartitions, we have from Theorem 14, that there exists a unique eigenfunction of $\mathcal{H}$ and $\mathcal{I}$ with such eigenvalues. We must thus conclude that $\tilde g_n$ is also proportional to $J_{(n;0)}$. Looking at Proposition 10 and considering that the coefficient of $m_{(n;0)}$ in $J_{(n;0)}$ needs to be equal to one, we obtain $(\beta+n)_{n+1} J_{(n;0)} = n!\tilde g_n$. The relation between $J_{(n)}$ and g_n is well known and can be proved in a similar way. □

Corollary 29. *For $\Lambda = (n)$ or $(n;0)$, the combinatorial norm of J_Λ is*

$$\langle\!\langle J_{(n)} | J_{(n)} \rangle\!\rangle_\beta = \frac{n!}{(\beta+n-1)_n} \quad \textit{and} \quad \langle\!\langle J_{(n;0)} | J_{(n;0)} \rangle\!\rangle_\beta = \frac{n!}{(\beta+n)_{n+1}}. \tag{6.14}$$

Proof. Using the previous proposition, we get

$$\begin{aligned}(n!)^2 \langle\!\langle g_n | g_n \rangle\!\rangle_\beta &= (\beta+n-1)_n^2 \langle\!\langle J_{(n)} | J_{(n)} \rangle\!\rangle_\beta, \\ (n!)^2 \langle\!\langle \tilde g_n | \tilde g_n \rangle\!\rangle_\beta &= (\beta+n)_{n+1}^2 \langle\!\langle J_{(n;0)} | J_{(n;0)} \rangle\!\rangle_\beta. \end{aligned} \tag{6.15}$$

From Proposition 10, we know that

$$n! g_n = (\beta+n-1)_n m_{(n)} + \cdots, \qquad n!\tilde g_n = (\beta+n)_{n+1} m_{(n;0)} + \cdots, \tag{6.16}$$

where the dots stand for lower terms in the order on superpartitions. Thus, considering Corollary 9, we get

$$\langle\!\langle g_n | g_n \rangle\!\rangle_\beta = \frac{(\beta+n-1)_n}{n!}, \qquad \langle\!\langle \tilde{g}_n | \tilde{g}_n \rangle\!\rangle_\beta = \frac{(\beta+n)_{n+1}}{n!} \tag{6.17}$$

and the proof follows. □

Theorem 30. *For $\beta = 0, 1$, or $\beta \to \infty$, the limiting expressions of $J_\Lambda^{(1/\beta)}$ are*

$$J_\Lambda^{(1/\beta)} \longrightarrow \begin{cases} m_\Lambda & \text{when } \beta \to 0, \\ \overleftarrow{e_{\Lambda'}} & \text{when } \beta \to \infty, \end{cases} \tag{6.18}$$

and

$$J_{(n)}^{(1)} = h_n \quad \text{and} \quad J_{(n;0)}^{(1)} = \frac{1}{n+1}\tilde{h}_n. \tag{6.19}$$

Proof. The case $\beta \to 0$ is a direct consequence of Theorem 14, given that $\mathcal{H}(\beta)$ and $\mathcal{I}(\beta)$ act diagonally on supermonomials in this limit. The second case is also obtained from the eigenvalue problem. Indeed, when $\beta \to \infty$, $\beta^{-1}\mathcal{H}(\beta)$ and $\beta^{-1}\mathcal{I}(\beta)$ behave as first order differential operators. Then, it is easy to get

$$\left[\lim_{\beta\to\infty} \frac{\mathcal{H}(\beta)}{\beta}\right] e_{\Lambda'} = \left[-2\sum_j j\lambda_j + n(N-1)\right] e_{\Lambda'} \quad \text{where } \lambda = \Lambda^* \tag{6.20}$$

(Λ^* being defined in Lemma 15) and

$$\left[\lim_{\beta\to\infty} \frac{\mathcal{I}(\beta)}{\beta}\right] e_{\Lambda'} = \left[-\left|\Lambda^a\right| - \frac{m(m-1)}{2}\right] e_{\Lambda'}. \tag{6.21}$$

These are the eigenvalues of J_Λ in the limit where $\beta \to \infty$ (cf. Lemma 15). The proportionality constant between $e_{\Lambda'}$ and J_Λ is fixed by Proposition 26 and Theorem 14. We have thus

$$\overleftarrow{e_{\Lambda'}} = \lim_{\beta\to\infty} \overrightarrow{J_\Lambda^{(1/\beta)}}. \tag{6.22}$$

Finally, we note that the property concerning h_n and $\tilde{h}_n$ is an immediate corollary of Proposition 28. □

6.3. Normalization

In this subsection, $\tilde{m}_\Lambda$ shall denote the augmented supermonomial:

$$\tilde{m}_\Lambda = n_\Lambda! m_\Lambda, \tag{6.23}$$

where $n_\Lambda!$ is defined in Proposition 10.

It is easy to see that the smallest superpartition of degree $(n|m)$ in the order on superpartitions is

$$\Lambda_{\min} := \big(\delta_m; 1^{\ell_{n,m}}\big), \tag{6.24}$$

where

$$\delta_m := (m-1, m-2, \ldots, 0), \quad \ell_{n,m} := n - |\delta_m| \quad \text{and} \quad |\delta_m| = \frac{m(m-1)}{2}. \tag{6.25}$$

Now, let $c_\Lambda^{\min}(\beta)$ stand for the coefficient of $\tilde{m}_{\Lambda_{\min}}$ in the monomial expansion of $J_\Lambda^{(1/\beta)}$. We will establish a relation between this coefficient and the norm of the Jack superpolynomials J_Λ.

Proposition 31. *The norm $j_\Lambda(\beta)$ defined in* (5.42)*, with $\Lambda \vdash (n|m)$, is*

$$j_\Lambda(\beta) = \beta^{-m-\ell_{n,m}} \frac{c_\Lambda^{\min}(\beta)}{c_{\Lambda'}^{\min}(1/\beta)}. \tag{6.26}$$

Proof. One readily shows that

$$m_{\Lambda_{\min}} = p_{\Lambda_{\min}} + \text{higher terms}. \tag{6.27}$$

Since $m_{\Lambda_{\min}}$ is the only supermonomial containing $p_{\Lambda_{\min}}$, we can write

$$J_\Lambda^{(1/\beta)} = c_\Lambda^{\min}(\beta) p_{\Lambda_{\min}} + \text{higher terms}. \tag{6.28}$$

Let us now apply $\hat{\omega}_{1/\beta}$ on this expression. Using Eq. (3.3) we get

$$\hat{\omega}_{1/\beta} J_\Lambda^{(1/\beta)} = \beta^{-m-\ell_{n,m}} (-1)^{m(m-1)/2} c_\Lambda^{\min}(\beta) p_{\Lambda_{\min}} + \text{higher terms}. \tag{6.29}$$

But if we apply $\hat{\omega}_{1/\beta}$ on $J_\Lambda^{(1/\beta)}$ by using first Theorem 27 to write it as $(-1)^{m(m-1)/2} j_\Lambda(\beta) J_{\Lambda'}^{(\beta)}$ and expand $J_{\Lambda'}^{(\beta)}$ using (6.28), we get instead

$$\hat{\omega}_{1/\beta} J_\Lambda^{(1/\beta)} = j_\Lambda(\beta)(-1)^{m(m-1)/2} c_{\Lambda'}^{\min}(1/\beta) p_{\Lambda_{\min}} + \text{higher terms}. \tag{6.30}$$

Here we have used the fact that $\Lambda_{\min}$, being the smallest superpartition of degree $(n|m)$ in the ordering on superpartitions, labels the smallest supermonomial in both the decomposition of J_Λ and $J_{\Lambda'}$. The result follows from the comparison of the last two equations. □

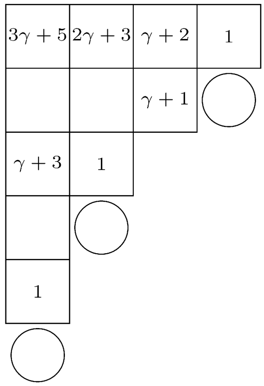
The coefficient $c_\Lambda^{\min}(\beta)$ appears from computer experimentation to have a very simple form. We will now introduce the notation needed to describe it. Recall (from the definition of conjugation in Section 2) that $D[\Lambda]$ is the diagram used to represent Λ. Given a cell s in $D[\Lambda]$, let $a_\Lambda(s)$ be the number of cells (including the possible circle at the end of the row) to the right of s. Let also $\ell_\Lambda(s)$ be the number of cells (not including the possible circle at the bottom of the column) below s. Finally, let Λ°, be the set of cells of $D[\Lambda]$ that do not appear at the same time in a row containing a circle *and* in a column containing a circle.

Conjecture 32. *The coefficient* $c_\Lambda^{\min}(\beta)$ *of* $\tilde{m}_{\Lambda_{\min}}$ *in the monomial expansion of* $J_\Lambda^{(1/\beta)}$ *is given by*

$$c_\Lambda^{\min}(\beta) = \frac{1}{\prod_{s\in\Lambda^\circ}(a_\Lambda(s)/\beta + \ell_\Lambda(s) + 1)} \tag{6.31}$$

with $\Lambda_{\min}$ *is defined in* (6.24).

For instance, if $\Lambda = (3,1,0;4,2,1)$, we can fill $D[\Lambda]$ with the values $(a_\Lambda(s)/\beta + \ell_\Lambda(s) + 1)$ corresponding to the cells $s \in \Lambda^\circ$. This gives (using $\gamma = 1/\beta$):



(6.32)

Therefore, in this case,

$$c_\Lambda^{\min}(\beta) = \frac{1}{(3/\beta+5)(2/\beta+3)(1/\beta+2)(1/\beta+1)(1/\beta+3)}. \tag{6.33}$$

Even though the Jack superpolynomials cannot be normalized to have positive coefficients when expanded in terms of monomials, we nevertheless conjecture they satisfy the following *integrality* property.

Conjecture 33. *Let*

$$J_\Lambda^{(1/\beta)} = c_\Lambda^{\min}(\beta) \sum_{\Omega\leqslant\Lambda} \tilde{c}_{\Lambda\Omega}(\beta)\tilde{m}_\Omega. \tag{6.34}$$

Then $\tilde{c}_{\Lambda\Omega}$ *is a polynomial in* $1/\beta$ *with integral coefficients.*

7. Outlook: Macdonald polynomials in superspace

In this work, we have highlighted the existence of a one-parameter (i.e., β) deformation of the scalar product as the key tool for defining Jack superpolynomials combinatorially. However, there also exists a two-parameter deformation (t and q) of the combinatorial scalar product. Again, this has a natural lift to the superspace, namely

$$\langle\langle \overleftarrow{p_\Lambda} | \overrightarrow{p_\Omega} \rangle\rangle_{q,t} := z_\Lambda(q,t)\delta_{\Lambda,\Omega}, \tag{7.1}$$

where

$$z_\Lambda(q,t) = z_\Lambda \prod_{i=1}^{m} \frac{1-q^{\Lambda_i+1}}{1-t^{\Lambda_i+1}} \prod_{i=m+1}^{\ell(\Lambda)} \frac{1-q^{\Lambda_i}}{1-t^{\Lambda_i}}, \quad m = \underline{\overline{\Lambda}}. \tag{7.2}$$

This reduces to the previous scalar product $\langle\langle\cdot|\cdot\rangle\rangle_\beta$ when $q = t^{1/\beta}$ and $t \to 1$. The generalized form of the reproducing kernel reads

$$\prod_{i,j} \frac{(tx_iy_j + t\theta_i\phi_j; q)_\infty}{(x_iy_j + \theta_i\phi_j; q)_\infty} = \sum_\Lambda z_\Lambda(q,t)^{-1} \overleftarrow{p_\Lambda(x,\theta)}\,\overrightarrow{p_\Lambda(y,\phi)}, \tag{7.3}$$

with $(a;q)_\infty := \prod_{n\geqslant 0}(1-aq^n)$.

Now, the scalar product (7.1) leads directly to a conjectured definition of Macdonald superpolynomials.

Conjecture 34. *In the space of symmetric superfunctions with rational coefficients in q and t, there exists a basis $\{M_\Lambda\}_\Lambda$, where $M_\Lambda = M_\Lambda(x,\theta;q,t)$, such that*

$$(1)\quad M_\Lambda = m_\Lambda + \sum_{\Omega<\Lambda} C_{\Lambda\Omega}(q,t) m_\Lambda \qquad \textit{and} \qquad (2)\quad \langle\langle \overleftarrow{M_\Lambda} | \overrightarrow{M_\Omega} \rangle\rangle_{q,t} \propto \delta_{\Lambda,\Omega}. \tag{7.4}$$

Note that in this context, the combinatorial construction cannot be compared with the analytical one since the corresponding supersymmetric eigenvalue problem has not been formulated yet. In other words, the proper supersymmetric version of the Ruijsenaars–Schneider model [16] is still missing.

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