Journal of Algebra 345 (2011) 257–265

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Commutative Noetherian local rings whose ideals are direct sums of cyclic modules ☆

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ARTICLE INFO

Article history:
Received 30 March 2011
Available online 6 September 2011
Communicated by Luchezar L. Avramov

MSC:
primary 13C05, 13E05, 13F10
secondary 13E10, 13H99

Keywords:
Cyclic modules
Local rings
Köthe rings
Principal ideal rings

ABSTRACT

A theorem from commutative algebra due to Köthe and Cohen–Kaplansky states that, "a commutative ring R has the property that every R-module is a direct sum of cyclic modules if and only if R is an Artinian principal ideal ring". Therefore, an interesting natural question of this sort is "whether the same is true if one only assumes that every ideal is a direct sum of cyclic modules?" The goal of this paper is to answer this question in the case R is a finite direct product of commutative Noetherian local rings. The structure of such rings is completely described. In particular, this yields characterizations of all commutative Artinian rings with this property.

1. Introduction

It was shown by Köthe [11] that an Artinian commutative ring R has the property that every module is a direct sum of cyclic modules if and only if R is a principal ideal ring. Later Cohen and Kaplansky [4] obtained the following result: *"a commutative ring R has the property that every module is a direct sum of cyclic modules if and only if R is an Artinian principal ideal ring"*. (Recently, a generalization of Köthe's result and an analogue of the Cohen–Kaplansky theorem have been given by Behboodi et al. [2] for the noncommutative setting.) More generally, Griffith showed in [7, Theorem 4.3] that if R is a commutative ring and every R-module is a direct sum of finitely generated modules, then R is

☆ The research of the first author was in part supported by a grant from IPM (No. 90160034).
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.08.017

an Artinian principal ideal ring. Griffith asks in [7] whether the same is true if one only assumes that every module is a direct sum of countably generated modules. This question is answered by Warfield in [16, Theorem 2 and Theorem 3]. In fact, Warfield showed that if R is a commutative ring such that every R-module is a direct sum of indecomposable modules, then R is an Artinian principal ideal ring (so that every module is a direct sum of cyclic modules). Also, he showed that if R is a commutative ring and there is a cardinal number n such that every R-module is a summand of a direct sum of modules with n generators, then R is an Artinian principal ideal ring. Therefore, an interesting natural question of this sort is: "What is the class of commutative rings R for which every ideal is a direct sum of cyclic modules?" The goal of this paper is to answer this question in the case R is a finite direct product of commutative Noetherian local rings.

Throughout this paper, all rings are commutative with identity and all modules are unital. For a ring R we denote by $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ the set of prime ideals and the set of maximal ideals of R, respectively. Also, $\mathrm{Nil}(R)$ is the set of all nilpotent elements of R. We denote the classical Krull dimension of R by $\dim(R)$. Let X be either an element or a subset of R. The annihilator of X is the ideal $\mathrm{Ann}(X) = \{a \in R \mid aX = 0\}$. A ring R is *local* in case R has a unique maximal ideal. In this paper $(R, \mathcal{M})$ will be a local ring with maximal ideal $\mathcal{M}$. An R-module N is called *simple* if $N \neq (0)$ and it has no submodules except (0) and N. Also, a Köthe ring is a ring such that each R-module is a direct sum of cyclic modules.

It is shown that if every ideal of a Noetherian ring R is a direct sum of cyclic R-modules, then $\dim(R) \leqslant 1$ (see Corollary 2.7). Also, if a local ring $(R, \mathcal{M})$ has the property that every ideal of R is a direct sum of cyclic R-modules, then $\mathcal{M} = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$ where Λ is an index set, $w_\lambda \in R$ for each $\lambda \in \Lambda$, and at most 2 of Rw_λ's ($\lambda \in \Lambda$) are not simple (see Corollary 2.3). It is also shown that if a Noetherian local ring R has the property that every ideal of R is a direct sum of cyclic R-modules, then $|\mathrm{Spec}(R)| \leqslant 3$ (see Theorem 2.5). Moreover, in Theorem 2.11, we show that for a Noetherian local ring $(R, \mathcal{M})$ the following statements are equivalent:

(1) Every ideal of R is a direct sum of cyclic R-modules.
(2) $\mathcal{M} = Rw_1 \oplus \cdots \oplus Rw_n$ where $n \geqslant 1$ and at most 2 of $Rw_1, \ldots, Rw_n$ are not simple.
(3) There exists $n \geqslant 1$ such that every ideal of R is a direct sum of at most n cyclic R-modules.
(4) Every ideal of R is a summand of a direct sum of cyclic R-modules.

As a consequence, we obtain: if $R = R_1 \times \cdots \times R_k$, where each R_i $(1 \leqslant i \leqslant k)$ is a Noetherian local ring, then every ideal of R is a direct sum of cyclic R-modules if and only if each R_i satisfies the above equivalent conditions, so this yields characterizations of all commutative Artinian rings whose ideals are direct sum of cyclics (see Theorem 2.13).

We note that two theorems from commutative algebra due to I.M. Isaacs and I.S. Cohen state that, to check whether every ideal in a ring is cyclic (resp. finitely generated), it suffices to test only the prime ideals (see [10, p. 8, Exercise 10] and [3, Theorem 2]). So this raises the natural question: "If every prime ideal of R is a direct sum of cyclics, can we conclude that all ideals are direct sum of cyclics?" This is not true in general. In fact, for each integer $n \geqslant 3$, we provide an example of an Artinian local ring $(R, \mathcal{M})$ such that $\mathcal{M}$ is a direct sum of n cyclic R-modules, but there exists a two generated ideal of R which is not a direct sum of cyclic R-modules (see Example 3.1). There exist non-Noetherian local rings R with $\dim(R) = 0$ such that every ideal of R is a direct sum of cyclic R-modules (see Example 3.2). There exist Artinian local rings R such that every ideal of R is a direct sum of at most 2 cyclic R-modules, but R is not a principal ideal ring (so R is not a Köthe ring) (see Example 3.3). Also, there exist Noetherian local rings R with $\dim(R) = 1$ such that every ideal of R is a direct sum of at most 2 cyclic R-modules, but R is not a principal ideal ring (see Example 3.6). Also, Example 3.8 shows that for a ring R the property that "every ideal is a direct sum of cyclics" is not a local property (see also, Remark 2.6). Finally, Example 3.9 shows that for a ring R the property that "every ideal is a direct sum of at most 2 cyclics" is not equivalent to the property that "every ideal is generated by at most two elements" (see Matlis' paper [13] for the two-generator problem for ideals).

2. Main results

First, by using Nakayama's lemma, we obtain the following lemma.

Lemma 2.1. *Let $(R, \mathcal{M})$ be a local ring and Rx be a nonzero summand of $\mathcal{M}$. Then Rx is a simple R-module if and only if $x^2 = 0$.*

The following proposition is crucial in our investigation.

Proposition 2.2. *Let $(R, \mathcal{M})$ be a local ring and $\mathcal{M} = Rx \oplus Ry \oplus Rz \oplus K$ where K is an ideal of R, $0 \neq x, y, z \in R$ and Rx, Ry, Rz are not simple R-modules. Then the ideal $J := R(x+y) + R(x+z)$ is not a direct sum of cyclic R-modules.*

Proof. Let $L = Rx \oplus Ry \oplus Rz$. Since Rx, Ry, Rz are not simple, by Lemma 2.1, x^2, y^2, z^2 are nonzero. We note that $L^2 = Rx^2 \oplus Ry^2 \oplus Rz^2 \subseteq J \subseteq L$, since $x^2 = x(x+y)$, $y^2 = y(x+y)$ and $z^2 = z(x+z)$. We claim that $y \in L \setminus J$, for if not, then there exist $r, s \in R$ such that $y = r(x+y) + s(x+z) = (r+s)x + ry + sz$. Thus $(r+s)x = (1-r)y = sz = 0$, and since both x and z are nonzero, $r+s, s \in \mathcal{M}$. Thus $r \in \mathcal{M}$ and so $1-r$ is unit, and hence $y = 0$, a contradiction. Also, $x + z \in J \setminus L^2$, for if not, then there exist $r_1, r_2, r_3 \in R$ such that $x + z = r_1x^2 + r_2y^2 + r_3z^2$ and this implies that $x = 0$, a contradiction. Therefore, $L^2 \subsetneqq J \subsetneqq L$.

Suppose, contrary to our claim, that J is a direct sum of cyclic R-modules. Thus $J = \bigoplus_{i=1}^m Rf_i$ where $0 \neq f_i = r_{i1}x + r_{i2}y + r_{i3}z$ and $r_{i1}, r_{i2}, r_{i3} \in R$ for each $1 \leqslant i \leqslant m$.

Since $x + y, x + z \in J$, there exist $s_i, t_i \in R$ for $1 \leqslant i \leqslant m$ such that

$$x + y = \sum_{i=1}^m s_i f_i = \left(\sum_{i=1}^m s_i r_{i1}\right)x + \left(\sum_{i=1}^m s_i r_{i2}\right)y + \left(\sum_{i=1}^m s_i r_{i3}\right)z,$$

$$x + z = \sum_{i=1}^m t_i f_i = \left(\sum_{i=1}^m t_i r_{i1}\right)x + \left(\sum_{i=1}^m t_i r_{i2}\right)y + \left(\sum_{i=1}^m t_i r_{i3}\right)z.$$

It follows that,

$$\left(1 - \sum_{i=1}^m s_i r_{i1}\right)x = \left(1 - \sum_{i=1}^m s_i r_{i2}\right)y = \left(\sum_{i=1}^m s_i r_{i3}\right)z = 0,$$

$$\left(1 - \sum_{i=1}^m t_i r_{i1}\right)x = \left(\sum_{i=1}^m t_i r_{i2}\right)y = \left(1 - \sum_{i=1}^m t_i r_{i3}\right)z = 0.$$

Since x, y and z all are nonzero, we conclude that

$$1 - \sum_{i=1}^m s_i r_{i1},\ 1 - \sum_{i=1}^m s_i r_{i2},\ 1 - \sum_{i=1}^m t_i r_{i3} \in \mathcal{M}.$$

Therefore, *"there exist $1 \leqslant i, j, k \leqslant m$ such that $r_{i1}, r_{j2}, r_{k3} \notin \mathcal{M}$"* $(*)$, for if not, then at least one of the above elements is unit in R, a contradiction.

Without loss of generality, we need only to consider the following three cases:

Case 1: $r_{11}, r_{12}, r_{13} \notin \mathcal{M}$. Suppose that $m \geqslant 2$. Since $f_2 f_1 = 0$ and $x^2, y^2, z^2 \neq 0$, we conclude that $r_{21}, r_{22}, r_{23} \in \mathcal{M}$. Note that $\mathcal{M} = Rx \oplus Ry \oplus Rz \oplus K$ implies that $f_2 = r'_{21}x^2 + r'_{22}y^2 + r'_{23}z^2$ where $r'_{21}, r'_{22}, r'_{23} \in R$. Thus $f_2 = (r'_{21}r_{11}^{-1}x + r'_{22}r_{12}^{-1}y + r'_{23}r_{13}^{-1}z)f_1$, a contradiction. Therefore, $m = 1$ and so $J = Rf_1$. Since $x + y \in J$, there exists $s \in R$ such that $x + y = sf_1 = sr_{11}x + sr_{12}y + sr_{13}z$. It follows that $(1 - sr_{11})x = (1 - sr_{12})y = sr_{13}z = 0$. Thus $1 - sr_{11}, 1 - sr_{12}, sr_{13} \in \mathcal{M}$, since $x, y, z \neq 0$. On the other hand, $sr_{13} \in \mathcal{M}$ implies that $s \in \mathcal{M}$, and so $1 - sr_{12}$ is unit and hence $y = 0$, a contradiction.

Case 2: $r_{11}, r_{12} \in \mathcal{M}$ and $r_{13} \notin \mathcal{M}$. Thus $f_1 = r'_{11}x^2 + r'_{12}y^2 + r_{13}z$ for some r'_{11}, $r'_{12} \in R$. Since $x^2, y^2 \in J$, so $z \in J$. Therefore, $x + y$, $x + z \in J$ implies that $x, y \in J$ and so $J = L$, a contradiction.

Case 3: $r_{11} \in \mathcal{M}$ and $r_{12}, r_{13} \notin \mathcal{M}$. Then there exists an $f_i = r_{i1}x + r_{i2}y + r_{i3}z$ such that $r_{i1} \notin \mathcal{M}$ (see the above fact $(*)$). Since $f_i f_1 = 0$, then $r_{i2}, r_{i3} \in \mathcal{M}$. Similarly to Case 2, we obtain a contradiction.

Thus the ideal $J = R(x+y) + R(x+z)$ is not a direct sum of cyclic R-modules. □

The following is now immediate.

Corollary 2.3. *Let $(R, \mathcal{M})$ be a local ring. If every ideal of R is a direct sum of cyclic R-modules, then $\mathcal{M} = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$ where Λ is an index set, $w_\lambda \in R$ for each $\lambda \in \Lambda$, and at most 2 of Rw_λ's $(\lambda \in \Lambda)$ are not simple.*

As a consequence of Corollary 2.3, we obtain the following interesting theorem. First, we need the following lemma.

Lemma 2.4. *(See Kaplansky [9, Theorem 12.3].) A commutative Noetherian ring R is a principal ideal ring if and only if every maximal ideal of R is principal.*

Theorem 2.5. *Let $(R, \mathcal{M})$ be a Noetherian local ring. If every ideal of R is a direct sum of cyclic R-modules, then* $\dim(R) \leqslant 1$ *and* $|\operatorname{Spec}(R)| \leqslant 3$.

Proof. By Corollary 2.3, $\mathcal{M} = \bigoplus_{i=1}^n Rw_i$ where $n \in \mathbb{N}$, $w_i \in R$ $(1 \leqslant i \leqslant n)$ and at most 2 of $Rw_1, \dots, Rw_n$ are not simple. If $\mathcal{M}^2 = (0)$, then $\dim(R) = 0$ and $\operatorname{Spec}(R) = \{\mathcal{M}\}$. Let $\mathcal{M}^2 \neq (0)$ and $\mathcal{M}$ be cyclic. Then by Lemma 2.4, R is a principal ideal ring and so $\dim(R) \leqslant 1$. If $\dim(R) = 0$, then R is Artinian and so $\operatorname{Spec}(R) = \{\mathcal{M}\}$. Suppose that $\dim(R) = 1$. Let $\mathcal{M} = Rx$ where $x \in R$ and $P \in \operatorname{Spec}(R) \setminus \{\mathcal{M}\}$. Since $P \subsetneqq \mathcal{M} = Rx$, so $P = Px$ and so by Nakayama's lemma $P = (0)$. Thus R is a principal ideal domain and $\operatorname{Spec}(R) = \{(0), \mathcal{M}\}$.

Now, we can assume that $\mathcal{M}^2 \neq (0)$ and $\mathcal{M}$ is not cyclic. Thus $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{i=3}^n Rw_i)$ where $x, y \neq 0$ and for each i, Rw_i is simple (i.e., $w_i^2 = 0$ by Lemma 2.1). Suppose that $P \in \operatorname{Spec}(R) \setminus \{\mathcal{M}\}$. If x, $y \in \operatorname{Nil}(R)$, then $\mathcal{M}$ is the only prime ideal of R (since $w_i^2 = 0$ for each $3 \leqslant i \leqslant n$). Hence $\dim(R) = 0$ and $\operatorname{Spec}(R) = \{\mathcal{M}\}$. Suppose that x, $y \notin \operatorname{Nil}(R)$. Since $xy = 0$, $x \in P$ or $y \in P$. If $x \in P$, then $P = Rx \oplus (P \cap Ry) \oplus (\bigoplus_{i=3}^n Rw_i)$. Also, $Ry \cap P = Py$, since $y \notin P$. Thus $P = Rx \oplus Py \oplus (\bigoplus_{i=3}^n Rw_i)$ and hence $Py = Py^2 = RyPy$, so by Nakayama's lemma $Py = 0$. Thus $P = Rx \oplus (\bigoplus_{i=3}^n Rw_i)$. If $y \in P$, then we conclude similarly that $P = Ry \oplus (\bigoplus_{i=3}^n Rw_i)$. On the other hand, since x, $y \notin \operatorname{Nil}(R)$, there exist $P_1, P_2 \in \operatorname{Spec}(R) \setminus \{\mathcal{M}\}$ such that $x \in P_1$, $y \notin P_1$ and $x \notin P_2$, $y \in P_2$. Therefore, $\operatorname{Spec}(R) = \{\mathcal{M}, Rx \oplus (\bigoplus_{i=3}^n Rw_i), Ry \oplus (\bigoplus_{i=3}^n Rw_i)\}$ and $\dim(R) = 1$. Finally, without loss of generality, we can assume that $x \in \operatorname{Nil}(R)$ and $y \notin \operatorname{Nil}(R)$. Since $w_i^2 = 0$ for each $3 \leqslant i \leqslant n$, $Rx \oplus (\bigoplus_{i=3}^n Rw_i) \subseteq P$. Thus $P = Rx \oplus (P \cap Ry) \oplus (\bigoplus_{i=3}^n Rw_i)$. Similarly as in the previous case we obtain $P = Rx \oplus (\bigoplus_{i=3}^n Rw_i)$. Therefore, $\operatorname{Spec}(R) = \{\mathcal{M}, Rx \oplus (\bigoplus_{i=3}^n Rw_i)\}$ and $\dim(R) = 1$. □

Remark 2.6. One can easily see that, if R is a ring all of whose ideals are direct sums of cyclic modules, then for each $P \in \operatorname{Spec}(R)$ the localization R_P has this property. But the converse is not true in general (see Example 3.8).

The following corollary shows that the first part of Theorem 2.5 is still true if we drop the assumption "R is local".

Corollary 2.7. *Let R be a Noetherian ring. If every ideal of R is a direct sum of cyclic R-modules, then* $\dim(R) \leqslant 1$.

Proof. Assume that R is Noetherian and every ideal of R is a direct sum of cyclic R-modules. Suppose, contrary to our claim, that $\dim(R) \geqslant 2$. Then there exists a chain $P'' \subsetneqq P' \subsetneqq P$ of prime ideals of R. By Remark 2.6, every ideal of R_P is a direct sum of cyclic R_P-modules. Thus by Theorem 2.5, $\dim(R_P) \leqslant 1$. But, $P''_P \subsetneqq P'_P \subsetneqq P_P$ is a chain of prime ideals of R_P, a contradiction. Thus $\dim(R) \leqslant 1$. □

Remark 2.8. A local Artinian principal ideal ring is called a *special principal ring* and has an extremely simple ideal structure: there are only finitely many ideals, each of which is a power of the maximal ideal. A principal ideal ring R can be written as a direct product $\prod_{i=1}^{n} R_i$, where each R_i is either a principal ideal domain or a special principal ring (see [17, p. 245, Theorem 33]). □

The following theorem is an analogue of Kaplansky's theorem (see Lemma 2.4).

Theorem 2.9. *For a Noetherian local ring* $(R, \mathcal{M})$ *the following statements are equivalent*:

(1) $\mathcal{M}$ *is a direct sum of at most two nonzero cyclic* R*-modules.*
(2) *Every ideal of* R *is a direct sum of at most two nonzero cyclic* R*-modules.*

Proof. (1) $\Rightarrow$ (2). If $\mathcal{M}$ is cyclic, then by Kaplansky's theorem (see Lemma 2.4), R is a principal ideal ring. Thus we assume that $\mathcal{M} = Rx \oplus Ry$ where $x, y \in R \setminus \{0\}$. Therefore, the only maximal ideal of $R/\mathrm{Ann}(x)$ (resp. $R/\mathrm{Ann}(y)$) is cyclic. By Kaplansky's theorem, both $R/\mathrm{Ann}(x)$ and $R/\mathrm{Ann}(y)$ are principal ideal rings. Now we assume that I is an ideal of R. If $I \subseteq Rx$ or $I \subseteq Ry$, then I is cyclic. Thus without loss of generality we can assume that, $I \nsubseteq Rx$, $I \nsubseteq Ry$ and $(0) \subsetneqq I \subsetneqq \mathcal{M}$. Now we proceed by cases.

Case 1: $x \in \mathrm{Nil}(R)$ or $y \in \mathrm{Nil}(R)$. Without loss of generality, we can assume that $x \in \mathrm{Nil}(R)$. Thus there exists $k \in \mathbb{N}$ such that $x^k = 0$ and $x^{k-1} \neq 0$. Since $\mathcal{M} = \mathrm{Ann}(Rx^i/Rx^{i+1})$ for each i $(1 \leqslant i \leqslant k-1)$, the chain $(0) \subsetneqq Rx^{k-1} \subsetneqq \cdots \subsetneqq Rx^2 \subsetneqq Rx$ is a composition series for Rx. Thus $R/\mathrm{Ann}(x)$ is an Artinian local ring and so by Remark 2.8, it is a special principal ring. Now the R-module isomorphism $Rx \cong R/\mathrm{Ann}(x)$ implies that $(0), Rx, Rx^2, \ldots, Rx^{k-1}$ are all submodules of Rx.

Subcase 1: $I \cap Rx = (0)$. Thus $I \cong (I \oplus Rx)/Rx \subseteq R/Rx$. Also the only maximal ideal of the ring R/Rx is cyclic and so by Kaplansky's theorem, it is a principal ideal ring. It follows that I is a cyclic R-module.

Subcase 2: $I \cap Rx \neq (0)$. Since $R/\mathrm{Ann}(x)$ is a special principal ring, so $I \cap Rx = Rx^t$ for some $1 \leqslant t \leqslant k-1$. If $t = 1$, then $I = (I \cap Ry) \oplus Rx$. Since $I \nsubseteq Rx$, $(0) \neq I \cap Ry \subseteq Ry$ and since $R/\mathrm{Ann}(y)$ is a principal ideal ring, we conclude that $I \cap Ry$ is cyclic and so I is a direct sum of two nonzero cyclic R-modules. Now assume that $1 < t \leqslant k-1$. Then every element of I is of the form $ay + bx^{t-1}$ for some $a, b \in R$ (if $ay + bx^l \in I$ where $l < t-1$ and $b \notin \mathcal{M}$, then $x^{t-1} \in I \cap Rx$, a contradiction). Set

$$J = \{ay \mid ay + bx^{t-1} \in I, \text{ for some } b \in R\}.$$

Then J is an ideal of R and $J \subseteq Ry$. Thus $J = Ra_0y$ for some $a_0 \in R$ and there exists $b_0 \in R$ such that $a_0y + b_0x^{t-1} \in I$. Let $z_0 = a_0y + b_0x^{t-1}$. If $b_0 \in \mathcal{M}$, then $z_0 = a_0y + b_0'x^t$ for some $b_0' \in R$. Since $Rx^t \subseteq I$, so $a_0y \in I$ and so $J \oplus Rx^t \subseteq I$. Now let $ay + bx^{t-1} \in I$. Since $ay \in J$, $bx^{t-1} \in I$ and so $b \in \mathcal{M}$. Thus there exist $a', b' \in R$ such that $ay + bx^{t-1} = a'a_0y + b'x^t$ and hence $I = J \oplus Rx^t$ (that is direct sum of two nonzero cyclic R-modules). Now let $b_0 \notin \mathcal{M}$. We claim that $I = Rz_0$. Clearly $Rz_0 \subseteq I$. On the other hand if $z = ay + bx^{t-1} \in I$, then $ay = a'a_0y$ for some $a' \in R$ and so $z = a'a_0y + bx^{t-1}$. Thus

$$z - a'z_0 = a'a_0y + bx^{t-1} - a'(a_0y + b_0x^{t-1}) = (b - a'b_0)x^{t-1} \in I \cap Rx = Rx^t,$$

and hence,

$$z - a'z_0 = rx^t = rb_0^{-1}x(a_0y + b_0x^{t-1}) = rb_0^{-1}xz_0 \in Rz_0$$

for some $r \in R$. It follows that $z \in Rz_0$. Therefore, $I = Rz_0$.

Case 2: $x, y \notin \mathrm{Nil}(R)$. We claim that every element of I is of the form $rx^n + sy^m$ where $n, m \in \mathbb{N}$ and $r, s \in (R \setminus \mathcal{M}) \cup \{0\}$. Let $z \in I$. Then $z = r_1x + s_1y$ where $r_1, s_1 \in R$. If $r_1x \neq 0$, then $r_1x = rx^n$ for some $n \in \mathbb{N}$ and $r \notin \mathcal{M}$, for if not, we have, $r_1x = r_ix^i$ for each $i \in \mathbb{N}$ where $r_i = r_{i+1}x + s_{i+1}y \in \mathcal{M}$. Thus $r_1x \in \bigcap_{i=1}^{\infty} Rx^i$, a contradiction (since by Krull's intersection theorem $\bigcap_{i=1}^{\infty} Rx^i = (0)$). If $s_1y \neq 0$, we conclude similarly that $s_1y = sy^m$ where $s \notin \mathcal{M}$. Thus $z = rx^n + sy^m$ where $r, s \in (R \setminus \mathcal{M}) \cup \{0\}$.

Since $I \nsubseteq Rx$ and $I \nsubseteq Ry$, there exists an element $ax^{n_1} + by^{m_1} \in I$ such that $n_1, m_1 \in \mathbb{N}$ and $a, b \notin \mathcal{M}$. Let k, l be the smallest natural numbers such that $x^k, y^l \in I$. If $I = Rx^k \oplus Ry^l$, the proof is complete. If not, there exists an element $w = cx^{t_1} + dy^{t_2} \in I \setminus (Rx^k \oplus Ry^l)$ where $c, d \in (R \setminus \mathcal{M}) \cup \{0\}$. It is easy to check that $t_1 = k-1$, $t_2 = l-1$ and $c, d \neq 0$. We show that $I = Rw$. Clearly $Rx^k \oplus Ry^l \subsetneqq Rw \subseteq I$. Let $u \in I$. Then $u = rx^{k_1} + sy^{l_1}$ where $r, s \in (R \setminus \mathcal{M}) \cup \{0\}$. If $u \in Rx^k \oplus Ry^l$, then $u \in Rw$. Thus we can assume that $u \in I \setminus (Rx^k \oplus Ry^l)$. We conclude similarly that $k_1 = k-1$ and $l_1 = l-1$. Therefore,

$$u - rc^{-1}w = \left(rx^{k-1} + sy^{l-1}\right) - rc^{-1}\left(cx^{k-1} + dy^{l-1}\right) = \left(s - rc^{-1}d\right)y^{l-1} \in I.$$

We must have $s - rc^{-1}d \in \mathcal{M}$ and so $s = rc^{-1}d + u_1x + u_2y$ where $u_1, u_2 \in R$. Therefore,

$$u = rx^{k-1} + \left(rc^{-1}d + u_1x + u_2y\right)y^{l-1} = rc^{-1}\left(cx^{k-1} + dy^{l-1}\right) + u_2y^l = \left(rc^{-1} + u_2d^{-1}y\right)w \in Rw.$$

Thus $I = Rw$. Therefore, in any case every ideal of R is a direct sum of at most two cyclic R-modules.

(2) ⇒ (1) is clear. □

Lemma 2.10. *(See Warfield [15, Proposition 3].) Let R be a local ring and N an R-module. If $N = \bigoplus_{\lambda \in \Lambda} R/I_\lambda$ where each I_λ is an ideal of R, then every summand of N is also a direct sum of cyclic R-modules, each isomorphic to one of the R/I_λ.*

The following main theorem is an answer to the question: "What is the class of Noetherian local rings R for which every ideal is a direct sum of cyclic modules?" Also, this theorem is an analogue of Kaplansky's theorem.

Theorem 2.11. *Let $(R, \mathcal{M})$ be a Noetherian local ring. Then the following statements are equivalent:*

(1) *Every ideal of R is a direct sum of cyclic R-modules.*
(2) *$\mathcal{M} = Rw_1 \oplus \cdots \oplus Rw_n$ where $n \geqslant 1$ and at most 2 of $Rw_1, \ldots, Rw_n$ are not simple.*
(3) *There exists $n \geqslant 1$ such that every ideal of R is a direct sum of at most n cyclic R-modules.*
(4) *Every ideal of R is a summand of a direct sum of cyclic R-modules.*

Proof. (1) ⇒ (2) is by Corollary 2.3.

(2) ⇒ (3). The proof is by induction on n. If $n = 1$ or 2, then by Theorem 2.9, every ideal of R is a direct sum of at most n cyclic R-modules. Thus we can assume that $\mathcal{M} = Rx \oplus Ry \oplus Rw_3 \oplus \cdots \oplus Rw_n$ where $n \geqslant 3$, $x, y \in R$ and $Rw_3, \ldots, Rw_n$ are simple R-modules. Suppose that I is an ideal of R. We need to consider the following two cases.

Case 1: $Rw_n \nsubseteq I$ (i.e., $Rw_n \cap I = (0)$). Set $R' = R/Rw_n$. Obviously, R' is a Noetherian local ring with maximal ideal $\mathcal{M}' = \mathcal{M}/Rw_n$. If $n = 3$, then $\mathcal{M}' = R'(x + Rw_n) \oplus R'(y + Rw_n)$, and if $n > 3$, then $\mathcal{M}' = R'(x + Rw_n) \oplus R'(y + Rw_n) \oplus (\bigoplus_{i=3}^{n-1} R'(w_i + Rw_n))$. Since $Rw_n \cap I = (0)$, $Rw_n \subseteq \mathrm{Ann}(I)$ and so $I \cong (I \oplus Rw_n)/Rw_n$ as R'-modules. By the induction assumption, $(I \oplus Rw_n)/Rw_n$ is a direct sum of at most $n-1$ cyclic R'-modules. From this we deduce that I is also a direct sum of at most $n-1$ cyclic R-modules.

Case 2: $Rw_n \subseteq I$. Then $I = (I \cap X) \oplus R_{w_n}$, where $X = Rx \oplus Ry \oplus Rw_1 \oplus \cdots \oplus R_{w_{n-1}}$. Then $Rw_n \nsubseteq (I \cap X)$, so we apply Case 1 to $I \cap X$. Therefore, $I \cap X$ is a direct sum of at most $n-1$ cyclic R-modules, i.e., I is a direct sum of at most n cyclic R-modules.

(3) ⇒ (4) is clear.

(4) ⇒ (1) is by Lemma 2.10. □

Remark 2.12. Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a nonzero ring. One can easily see that, the ring R has the property that its ideals are direct sum of cyclic R-modules if and only if for each i the ring R_i has this property.

We are thus led to the following strengthening of Theorem 2.11. In particular, this theorem yields characterizations of all commutative Artinian rings whose ideals are direct sum of cyclic modules.

Theorem 2.13. *Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a Noetherian local ring with maximal ideal $\mathcal{M}_i$ $(1 \leqslant i \leqslant k)$. Then the following statements are equivalent*:

(1) *Every ideal of R is a direct sum of cyclic R-modules.*
(2) *For each i, $\mathcal{M}_i = R_i w_{1_i} \oplus \cdots \oplus R_i w_{n_i}$ where $n_i \geqslant 1$ and at most 2 of $R_i w_{1_i}, \ldots, R_i w_{n_i}$ are not simple.*
(3) *There exists $n \geqslant 1$ such that every ideal of R is a direct sum of at most n cyclic R-modules.*
(4) *Every ideal of R is a summand of a direct sum of cyclic R-modules.*

Proof. The proof is straightforward by Theorem 2.11 and Remark 2.12. □

We conclude this section with the following proposition that is an analogue of "Invariant Basis Number (IBN)" of free modules over commutative rings. First, we need the following lemma.

Lemma 2.14. *(See [12, Lemma 1.1].) Let R be a ring and M be an R-module. If $\{e_i \mid i \in I\}$ is a minimal generating set of M where the cardinality I is infinite, then M cannot be generated by fewer than $|I|$ elements.*

Proposition 2.15. *Let R be a ring. Then the following statements are equivalent*:

(1) *R is a local ring.*
(2) *If $\bigoplus_{i=1}^{n} Rx_i \cong \bigoplus_{j=1}^{m} Ry_j$ where $n, m \in \mathbb{N}$ and Rx_i, Ry_j are nonzero cyclic R-modules, then $n = m$.*
(3) *If $\bigoplus_{i\in I} Rx_i \cong \bigoplus_{j\in J} Ry_j$ where I, J are index sets and Rx_i, Ry_j are nonzero cyclic R-modules, then $|I| = |J|$.*

Proof. (1) ⇒ (3). Suppose that R is a local ring with maximal ideal $\mathcal{M}$ and $\bigoplus_{i\in I} Rx_i \cong \bigoplus_{j\in J} Ry_j$ where I, J are index sets and Rx_i, Ry_j are nonzero cyclic R-modules. If I or J are infinite, then by Lemma 2.14, $|I| = |J|$. Thus we can assume that $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$, where $n, m \in \mathbb{N}$. Set $N = \bigoplus_{i=1}^{n} Rx_i$. Then

$$N/\mathcal{M}N \cong (Rx_1/\mathcal{M}x_1) \oplus (Rx_2/\mathcal{M}x_2) \oplus \cdots \oplus (Rx_n/\mathcal{M}x_n).$$

Also, $Rx_i \neq \mathcal{M}x_i$ for each $1 \leqslant i \leqslant n$, for if not, by Nakayama's lemma and since R is a local ring, we obtain $x_i = 0$, a contradiction. Thus $Rx_i/\mathcal{M}x_i \cong R/\mathcal{M}$ for each $1 \leqslant i \leqslant n$ and so $\text{v.dim}_{R/\mathcal{M}}(N/\mathcal{M}N) = n$. Now since $N \cong \bigoplus_{j=1}^{m} Ry_j$, by a similar argument, we obtain $\text{v.dim}_{R/\mathcal{M}}(N/\mathcal{M}N) = m$ and so $m = n$.

(3) ⇒ (2) is clear.

(2) ⇒ (1). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two maximal ideals of R and $\mathcal{M}_1 \neq \mathcal{M}_2$. Thus by the Chinese Remainder Theorem, $R/(\mathcal{M}_1 \cap \mathcal{M}_2) \cong R/\mathcal{M}_1 \oplus R/\mathcal{M}_2$, a contradiction. □

3. Examples

In this section some relevant examples and counterexamples are indicated. We begin with the following example that shows that there exist rings R such that every prime ideal of R is a direct sum of cyclic R-modules, but some of the ideals of R are not direct sum of cyclics. Furthermore, the following example shows that for each integer $n \geqslant 3$, there exists an Artinian local ring $(R, \mathcal{M})$ such that $\mathcal{M}$ is a direct sum of n cyclic R-modules, but there exists a two generated ideal of R such that it is not a direct sum of cyclic R-modules.

Example 3.1. Let F be a field, $n \geqslant 3$ and let R be the F-algebra with generators $x_1, x_2, \ldots, x_n$ subject to the relations

$$x_1^3 = x_2^3 = x_3^3 = x_k^2 = 0, \quad 4 \leqslant k \leqslant n \quad \text{and} \quad x_i x_j = 0 \quad \text{for } 1 \leqslant i \neq j \leqslant n$$

(i.e., $R \cong F[X_1, X_2, \dots, X_n]/\langle\{X_1^3, X_2^3, X_3^3, X_k^2, X_iX_j \mid 4 \leqslant k \leqslant n,\ 1 \leqslant i \neq j \leqslant n\}\rangle$). Then R is an Artinian local ring with maximal ideal $\mathcal{M} = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n$. Thus the only prime ideal of R is a direct sum of n cyclic R-modules. But by Proposition 2.2, the ideal $J = R(x_1 + x_2) + R(x_1 + x_3)$ is not a direct sum of cyclic R-modules.

The following example shows that there exist non-Noetherian local rings R with $\dim(R) = 0$ such that every ideal of R is a direct sum of cyclic R-modules.

Example 3.2. Let F be a field and R be the F-algebra with generators $\{x_i \mid i \in \mathbb{N}\}$ subject to the relations $x_ix_j = 0$ for $i, j \in \mathbb{N}$ (i.e., $R \cong F[\{X_i \mid i \in \mathbb{N}\}]/\langle\{X_iX_j \mid i, j \in \mathbb{N}\}\rangle$). The ring R is a non-Noetherian local ring with the maximal ideal $\mathcal{M} = \bigoplus_{i \in \mathbb{N}} Rx_i$. Since $\mathcal{M}^2 = (0)$, every proper ideal of R is an $R/\mathcal{M}$-module and so every ideal of R is a direct sum of cyclic R-modules. Clearly, $\dim(R) = 0$ and $\operatorname{Spec}(R) = \{\mathcal{M}\}$.

We recall that by Köthe [11] and Cohen and Kaplansky [4], a commutative ring R has the property that every module is a direct sum of cyclic modules if and only if R is an Artinian principal ideal ring. Next, we give an example of an Artinian local ring R such that every ideal of R is a direct sum of at most 2 cyclic R-modules, but R is not a principal ideal ring (so R is not a Köthe ring).

Example 3.3. Let F be a field, $n \geqslant 2$ and R be the F-algebra with generators x, y subject to the relations $x^n = y^n = xy = 0$ (i.e., $R \cong F[X, Y]/\langle X^n, Y^n, XY\rangle$). The ring R is a Noetherian local ring with maximal ideal $\mathcal{M} = Rx \oplus Ry$. Since $\mathcal{M}^n = (0)$, $\dim(R) = 0$ and so R is an Artinian local ring. Also, by Theorem 2.9, every ideal of R is a direct sum of at most 2 cyclic R-modules. Now, by Proposition 2.15, $\mathcal{M}$ is not cyclic, i.e., R is not a principal ideal ring, so it is not a Köthe ring.

Also, we will show below that there exist Noetherian local rings R with $\dim(R) = 1$ such that every ideal of R is a direct sum of at most 2 cyclic R-modules, but R is not a principal ideal ring. First, we need the following two lemmas.

Lemma 3.4. *(See Hungerford [8, Corollary 12].) Let R be a principal ideal ring. Then R is a direct sum of principal ideal domains if and only if it has no nonzero nilpotent elements.*

A ring R is said to be *indecomposable* if R cannot be decomposed into a direct product of two nonzero rings. Clearly, a ring R is indecomposable if and only if it has no nontrivial idempotents.

Lemma 3.5. *(See Tuganbaev [14, Lemma 16.6].) Let R be a ring. If $\bar{e}$ is an idempotent element in $R/\operatorname{Nil}(R)$, then there exists an idempotent pre-image for $\bar{e}$.*

Example 3.6. Let F be a field and R be the F-algebra with generators x, y subject to the relations $y^2 = xy = 0$ (i.e., $R \cong F[X, Y]/\langle XY, Y^2\rangle$). The ring R is an indecomposable Noetherian ring and $\mathcal{M} = Rx \oplus Ry$ is a maximal ideal of R. Since $Ry \subseteq \operatorname{Nil}(R)$ and $R/Ry \cong F[x]$, we conclude that $R/\operatorname{Nil}(R)$ is a principal ideal ring. Therefore, $\dim(R) = \dim(R/\operatorname{Nil}(R)) \leqslant 1$. Also, by Lemma 3.5, $R/\operatorname{Nil}(R)$ is an indecomposable ring and hence by Lemma 3.4, $R/\operatorname{Nil}(R)$ is a principal ideal domain. Thus $\operatorname{Nil}(R)$ is a prime ideal of R and since $\mathcal{M} = Rx \oplus Ry$ is a maximal ideal and x is not nilpotent, we conclude that $\dim(R) = 1$. Let $R' = R_{\mathcal{M}}$ be the localization of R at $\mathcal{M}$. Then R' is a Noetherian local ring with the maximal ideal $\mathcal{M}_{\mathcal{M}} = R'\bar{x} \oplus R'\bar{y}$ where $\bar{x} = \frac{x}{1}$ and $\bar{y} = \frac{y}{1}$. Thus by Theorem 2.9 or Theorem 2.11, every ideal of R' is a direct sum of at most 2 cyclic R'-modules. We note that every element of $R \setminus \mathcal{M}$ can be written as $a + rx + sy$, where a is a unit in R and $r, s \in R$. So $(a + rx + sy)y = ay$ is nonzero, and $(a + rx + sy)x = ax + rx^2$ is nonzero (since otherwise $x \in \mathcal{M}^2$, which would force $\mathcal{M}^2 = (0)$ by Nakayama's lemma, contrary to assumption). It follows that both $R'\bar{x}$ and $R'\bar{y}$ are nonzero. Thus by Proposition 2.15, the maximal ideal $\mathcal{M}_{\mathcal{M}}$ of R' is not cyclic, i.e., R' is not a principal ideal ring. Finally, it is easy to see $\operatorname{Spec}(R') = \{\mathcal{M}_{\mathcal{M}}, R'\bar{y}\}$ and $\dim(R') = 1$.

Also, we will show below that the converse of Remark 2.6, is not true in general. First, we need the following proposition.

Proposition 3.7. *Let R be a Von-Neumann regular ring (i.e., every finitely generated ideal of R is generated by an idempotent). Then every ideal of R is a direct sum of cyclic R-modules if and only if R is a hereditary ring (i.e., every ideal of R is projective R-module).*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) By [6, Proposition 2.6], is clear. □

The following example shows that for a ring R the property that "every ideal is a direct sum of cyclics" is not a local property.

Example 3.8. Let X be an uncountable set. Then $R := \mathcal{P}(X)$, the power set of X, is a ring under the multiplication $A \cdot B = A \cap B$ and addition $A + B = (A \cup B) \setminus (A \cap B)$ as symmetric difference for $A, B \subseteq X$; $0 = \emptyset$ and $1 = X$. Clearly, R is a Boolean ring, but R is not a hereditary ring (see [5, Example 13.4.3]). For each maximal ideal $\mathcal{M}$ of R, $R_{\mathcal{M}}$ is a field (see [1, p. 44, Exercise 10(ii)] or [6, Theorem 1.16]). Thus for each maximal ideal $\mathcal{M}$ of R, every ideal of $R_{\mathcal{M}}$ is a direct sum of cyclic $R_{\mathcal{M}}$-modules, but by Proposition 3.7, there exists an ideal of R such that it is not a direct sum of cyclic R-modules.

Finally, the following example shows that there exist Artinian (finite) local rings $(R, \mathcal{M})$ such that every ideal of R is generated by at most 2 elements, but $\mathcal{M}$ is not a direct sum of cyclic R-modules. Therefore, for a ring R the property that "every ideal is a direct sum of at most 2 cyclics" is not equivalent to the property that "every ideal is generated by at most two elements".

Example 3.9. Let R be the $\mathbb{Z}_2$-algebra with generators x, y subject to the relations $x^2 = y^2 = 0$ (i.e., $R \cong \mathbb{Z}_2[X, Y]/\langle X^2, Y^2\rangle$). Then R is a finite local ring with maximal ideal $M = \langle x, y\rangle = \{0, x, y, x + y, xy, x+xy, y+xy, x+y+xy\}$. It is easy to see that every ideal of R is generated by at most 2 elements, but $\mathcal{M}$ is not a direct sum of cyclic R-modules.

Acknowledgment

The authors would like to thank the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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