3-Transposition Groups with Non-central Normal 2-Subgroups*

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1. INTRODUCTION

A normal set $D$ of 3-transpositions in the group $G$ is a $G$-invariant set $D$ of elements of order 2 such that, for all $d$ and $e$ in $D$, the order of the product $de$ is 1, 2, or 3. If $G$ is generated by a normal set of 3-transpositions, we often say that $G$ is a 3-transposition group. Such groups were introduced and studied by Fischer [6, 7] who classified all finite 3-transposition groups with no non-trivial normal, solvable subgroups. His work was of great importance in the classification of finite simple groups.

Fischer [6, 7] proved that non-central normal solvable subgroups must essentially be either 2-groups or 3-groups. (See (2.2) below.) For geometric reasons the present author studied 3-transposition groups in which normal 2-subgroups possibly appeared. In [9, 10] the 3-transposition groups with trivial center and normal 2-subgroups were classified up to isomorphism without the assumption of finiteness but subject to a restriction on 3-generated subgroups.

This paper is designed to supplement those parts of the earlier papers concerned with normal 2-subgroups. Starting with their results we give a nearly complete classification of the finite, center-free 3-transposition groups which have normal 2-groups. We also briefly discuss the situation for normal 3-subgroups.

Much of this work was done while the author was on sabbatical leave at Oxford University in 1983–1984. Professor Fischer then pointed out that the results overlap considerably with those presented by François Zara in his thesis [16]. (See also [17].) The approach presented here is rather different from that of Zara. We are primarily interested in the isomorphism type of $G/Z(G)$. Zara instead considers the question: for a given 3-trans-

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position group $H$ (the possibilities being given by Fischer's work), how complicated can the solvable core $S(G)$ be of a finite 3-transposition group $G$ with $G/S(G) \cong H$? Thus we deal with centers only when necessary, while Zara studies them with care. On the other hand, Zara constructs certain types of subgroups $S(G)$ but is not primarily concerned with finding all possible $S(G)$. He also does not consider whether or not the extension of $S(G)$ by $H$ must split. Zara's approach is largely through the study of Coxeter style presentations of the groups and, as much as possible, handles normal 2-subgroups and normal 3-subgroups uniformly. Our approach is more characteristic sensitive, and we are interested mainly in 2-subgroups.

The most familiar example of a class of 3-transpositions is the transposition (i.e., 2-cycle) class of the symmetric group. More generally, the reflection class of any spherical Coxeter group with simply laced diagram (type $A - D - E$) is a 3-transposition class. In particular $W(D_n) \cong 2^{n-1}:S_n$ gives an example of a 3-transposition group with a non-central normal 2-subgroup. In general if $A$ is a subset of the 3-transposition class $D$ of the group $G$, then the diagram of $A$ is the graph with nodes the members of $A$ and edges connecting those pairs from $A$ with product of order 3. The subgroup of $G$ generated by $A$ must then be a quotient of $W(A)$, the Coxeter group with diagram $A$.

Suppose $A$ has affine diagram $\Phi$, the extended diagram associated with the simply laced spherical diagram $\Phi$ containing $n$ nodes. The group $W(\Phi)$ is then the split extension by $W(\Phi)$ of the root lattice $A \cong \mathbb{Z}^n$. The members of $A$ are images of reflections on $A$. Therefore, the homomorphism of $W(\Phi)$ onto $\langle A \rangle$ must contain in its kernel either $2A$ or $3A$. We write $W_p(\Phi)$ for $W(\Phi)/pA$, where $p$ is either 2 or 3. The quotient $W_p(\Phi)$ is a 3-transposition group with non-central normal elementary abelian $p$-subgroup $V = A/pA$. If $p = 2$, then the members of $A$ induce transvections on $V$, while if $p = 3$ they act as reflections.

The work of Fischer [6, 7] showed that generically a finite 3-transposition group with no non-central solvable normal subgroup is either a transvection group in characteristic 2 or a reflection group in characteristic 3. The only exceptions lead to the three sporadic Fischer groups. Correspondingly when solvable normal subgroups appear, we expect transvection groups acting on normal 2-subgroups or reflection groups acting on normal 3-subgroups.

In this paper we focus upon the case of normal 2-subgroups. Fischer's results indicate that we should consider transvection generated subgroups of symplectic groups over $GF(2)$ and unitary groups over $GF(4)$. It is possible to distinguish these two cases. In a 3-transposition group $G$, a subset with diagram a triangle must generate a homomorphic image of $W_2(\tilde{A}_2) \cong S_4$ or of $W_3(\tilde{A}_2) \cong SU_3(2)'$. (See result (A.1) of the appendix.) We say that a 3-transposition group is of symplectic type if each triangle
generates a homomorphic image of $S_4$. In [9, 10] we characterized and classified all center-free 3-transposition groups of symplectic type as transvection generated subgroups of symplectic groups over $GF(2)$. Thus our arguments concentrate on groups of unitary type—groups which contain a triangle that generates a subgroup $SU_3(2)'$.

For all the theorems stated in this section we assume:

**HYPOTHESIS.** The group $G = \langle D \rangle$ is finite with $D$ a conjugacy class of 3-transpositions. Furthermore, $Z(G) = 1$, and $F(G) = F^*(G) = O_2(G)$.

The first theorem presents the generic result.

**THEOREM 1.** Set $V = O_2(G)$ and $\bar{G} = G/V$. Then we have one of:

1. $\bar{G} \cong S_n$, $O_{2n}^-(2)$, or $Sp_{2n}(2)$; the subgroup $V$ is elementary abelian and a direct sum of irreducible $GF(2)$-transvection modules for $\bar{G}$; and the extension of $V$ by $\bar{G}$ splits;

2. $\bar{G} \cong Z_3 \cdot O_6^-(3)$, $SU_n(2)$ with $n \neq 3, 5, 7$, or $Z_3^{n-1} : S_n$; the subgroup $V$ is elementary abelian and a direct sum of irreducible $GF(4)$-transvection modules for $\bar{G}$; and the extension of $V$ by $\bar{G}$ splits;

3. $\bar{G} \cong SU_3(2)'$, $SU_5(2)$, or $SU_7(2)$.

Here $O_6^-(3)$ is the extension by a reflection of the simple orthogonal group for a space $GF(3)^6$ of Witt index 2. The group $Z_3 \cdot O_6^-(3)$ is then a non-split central extension of $Z_3$ by $O_6^-(3)$ and occurs as a subgroup of $SU_6(2)$ generated by transvections [6, 16.1.2; 11, RT7]. The group $Z_3^{n-1} : S_n$ is best thought of as that subgroup of degree $n$ monomial matrices over $GF(4)$ which is generated by unitary transvections.

We deal with the three exceptional unitary groups in turn.

**THEOREM 2.** Assume $\bar{G} = G/O_2(G) \cong SU_3(2)'$, and let $V$ be the module $O_2(G)/\Phi(O_2(G))$. Then the extension of $O_2(G)$ by $\bar{G}$ splits, and the module $V$ is a direct sum of irreducible $GF(4)$-transvection modules for $\bar{G}$.

**THEOREM 3.** Set $V = O_2(G)$, and assume that $\bar{G} = G/V \cong SU_5(2)$. Then the extension of $V$ by $\bar{G}$ splits. The subgroup $V$ is elementary abelian. As a $\bar{G}$-module it is a direct sum of copies of the irreducible module $V_{10}$ of $GF(2)$-dimension 10 or of the reducible, indecomposable module $V_{20}$ of $GF(2)$-dimension 20. Here $V_{10}$ is the usual $GF(4)$-transvection module, and the reducible module $V_{20}$ is a non-split extension of $V_{10}$ by itself.

**THEOREM 4.** Set $V = O_2(G)$, and assume that $\bar{G} = G/V \cong SU_7(2)$. Then the subgroup $V$ is elementary abelian and a direct sum of irreducible
GF(4)-transvection modules for $G$. For each such $V$ there are exactly two possibilities for the extension $G$ of $V$ by $\overline{G}$, one split and the other non-split.

The final theorem observes that each of the earlier groups is indeed a 3-transposition group and that the generating class is unique.

**Theorem 5.** Let $G$ be a group appearing in the conclusion to one of the Theorems 1–4. If $G/O_2(G) \cong SU_3(2)'$, assume further that $\Phi(O_3(G)) = 1$. Then $G$ is generated by a unique conjugacy class of 3-transpositions. In particular when the extension of $V$ by $\overline{G}$ splits, the 3-transposition class contains the transvection class of every complement to $V$ in $\overline{G}$. In the remaining case of a non-split extension of $V$ by $SU_7(2)$, the 3-transposition class is composed of transvection preimages.

We give a brief summary of the paper. In the second section we present some basic results about 3-transposition groups. We also present our general hypothesis and notation in (2.6). (In particular, $\overline{G} = G/O_2(G)$.) The next section discusses the important special case of the group $SU_3(2)'$. With this in hand we are able in the fourth section to consider those groups in which $O_3(\overline{G})$ is non-central. It is perhaps worth noting that to this point Fischer's fundamental classification result [6, 7] is not needed. However, the next step is to use Fischer's work to identify the remaining possibilities for $\overline{G}$. The sixth section gives a characterization of the unitary transvection modules in terms of a property related to 3-transpositions. This allows us to prove the main theorems. The body of the paper is concluded with an eighth section composed of remarks. In particular we briefly discuss 3-transposition groups with non-central normal 3-subgroups. The results of this paper together with those of Zara give a nearly complete description of the finite 3-transposition groups. The section of remarks mentions the most significant remaining questions. (The possibility of successfully extending our methods to the normal 3-subgroup case was suggested by the work of Zara, and in Section 8 we make use of one of his results. All other sections of this paper are independent of [16, 17].) The final section is an appendix which gives presentations for certain of the groups concerned in terms of their 3-transposition classes and properties.

Basic discussion of 3-transposition groups can be found in [7] or [10]. Unexplained group theoretic concepts can be found in [1]. Throughout the paper we shall only be concerned with finite groups.

2. **Basics**

(2.1) **Proposition.** Let $G = \langle D \rangle$ with $D$ a conjugacy class of 3-transpositions. Then $Z(G) = Z_3(G)$, and $D \cap dZ(G) = d, d \in D$. 
3-TRANSPOSITION GROUPS

Theorem. Let \( G = \langle D \rangle \) with \( D \) a conjugacy class of 3-transpositions. We have one of:

1. \( F^*(G) = Z(G)O_2(G) \);
2. \( F^*(G) = Z(G)O_3(G) \);
3. \( F^*(G) = Z(G)E(G) \), and \( E(G) \) is quasisimple.

Proof. See \([7, (2.1.1)]\).

Lemma. Let \( G = \langle D \rangle \) with \( D \) a conjugacy class of 3-transpositions. Set \( \overline{G} = G/O_2(G) \), and select \( d \in D \).

1. \( \langle D \cap C_G(d) \rangle = \overline{\langle D \cap C_G(d) \rangle} \);
2. \( [O_2(G), d, \langle D \cap C_G(d) \rangle] = 1 \);

Proof. The first part is clear as 3-transpositions which commute modulo a 2-group must commute. The second part follows as \([O_2(G), d]\) is abelian.

Lemma. Let \( G = \langle D \rangle \) with \( D \) a conjugacy class of 3-transpositions. Set \( \overline{D} = \langle D \rangle \) and \( Q = [N, G] \). For \( d \in D \), let \( R(d) = \langle e \in D \mid C_D(d) = C_D(e) \rangle \) and \( Q(d) = \langle d^e \mid e \in D, C_D(d) = C_D(e) \rangle \).

1. \( R(d) = \langle D \cap dN \rangle \) is elementary abelian;
2. \( Q(d) = [N, d] \) has index 2 in \( R(d) \), and \( Q = \langle Q(d) \mid d \in D \rangle \);
3. \([R(d), Q(e)] \leq R(d), \) for all \( d, e \in D \).

Proof. The first part is immediate from the previous lemma. For the rest, see \([7, (2.1.3)]\).

Lemma. Let \( G = \langle D \rangle \) with \( D \) a conjugacy class of 3-transpositions. Set \( \overline{G} = G/O_2(G) \). Assume that

(i) \( \langle C_D(d) \rangle \) is transitive on those \( \bar{e} \in \bar{D} \) with \( |\bar{d}\bar{e}| = 2 \);
(ii) \( \langle C_D(d) \rangle \) is transitive on those \( \bar{e} \in \bar{D} \) with \( |\bar{d}\bar{e}| = 3 \) and \( \bar{d} \notin O_3(\overline{G}) \);
(iii) there is a subset \( \bar{A} \) of \( \bar{D} \) with \( G = \langle \bar{A} \rangle \) whose diagram is connected and such that \( \bar{d} \notin O_3(\overline{G}) \), for each distinct pair \( \bar{d}, \bar{e} \in \bar{A} \).

Then there is an elementary abelian 2-group \( Y \) which is contained in \( Z(G) \) and in \([O_2(G), d]\), for every \( d \in D \), and such that the quotient \([O_2(G), G]/Y \) is elementary abelian.

Proof. The proof is essentially the same as that of \([10, (4.4) \) and (4.5)]\).
For most of the paper we shall be concerned with groups subject to the following:

\begin{align}
(2.6) \text{HYPOTHESIS AND NOTATION. } & \text{Assume that finite } G = \langle D \rangle, \text{ where } D \text{ is a conjugacy class of 3-transpositions, and that } F(G) = F^*(G) = O_2(G) \text{ with } Z(G) = 1. \text{ Set } N = O_2(G), \text{ and } \overline{G} = G/N. \\
\end{align}

Remember that a 3-transposition group is said to be of symplectic type if every triangle of its 3-transposition class generates a subgroup \( S_3 \) or \( S_4 \).

\begin{align}
(2.7) \text{THEOREM. } & \text{If } G \text{ satisfies Hypothesis 2.6 and is of symplectic type, then it is as in Theorem 1(1). The transvections of every complement to } N \text{ in } G \text{ belong to the unique generating class of 3-transpositions in } G. \\
& \text{Proof. } \text{This is contained in [10, Theorem 1].} \\
\end{align}

\begin{align}
(2.8) \text{PROPOSITION. } & \text{If } GF(2)^{2^n} \text{ is the natural } GF(4)-\text{module for } SU_n(2) \text{ with } n \geq 4, \text{ then } H^1(SU_n(2), GF(2)^{2^n}) = 0, \text{ for } n \geq 5, \text{ and is } GF(4) \text{ for } n = 4. \\
& \text{Proof. } \text{This is contained in [12, Theorem 2.14].} \\
\end{align}

3. Case \( SU_3(2)' \)

As discussed in the introduction, every triangle of \( D \) which does not lie in a subgroup \( S_3 \) generates either a subgroup \( W_3(\overline{A}_2) \cong S_4 \) or a subgroup \( W_3(\overline{A}_2) \cong SU_3(2)' \) (or its central factor \( PSU_3(2)' \)). Because of Theorem 2.7 we may concentrate our efforts upon those 3-transposition groups which contain such a subgroup. Hence, this first case although elementary is important.

\begin{align}
(3.1) \text{LEMMA. } & \text{Let } G = \langle D \rangle, \text{ where } D \text{ is a conjugacy class of 3-transpositions. Assume } a, b, c \in D \text{ with } H = \langle a, b, c \rangle \cong SU_3(2)' \text{ or } PSU_3(2)'). \text{ Choose an } x \in O_2(G) \text{ which is not stabilized by } H; \text{ say } x \neq x^a. \text{ Then } \langle a, b, c, ax \rangle \text{ is a split extension of } M \text{ by } H \cong SU_3(2)', \text{ where } M \text{ is either extraspecial of order } 2^7 \text{ or elementary abelian of order } 2^6. \text{ In either case } M/\Phi(M) \text{ is a natural } GF(4)-\text{transvection module for } H. \\
& \text{Proof. } \text{The subgroup } <a, b, c, ax^a> \text{ is a homomorphic image of the group } 2^{1+6}:SU_3(2)' \text{ of the result (A.3) from the appendix.} \\
& \text{Proof of Theorem 2. } \text{Choose a triangle } \{a, b, c\} \text{ from } D. \text{ Then } H = \langle a, b, c \rangle \text{ is a complement to } Z(G) O_2(G) \text{ by (A.1) of the Appendix.} \\
& \text{By (A.1) no non-trivial central extension of } SU_3(2)' \text{ is generated by a class of 3-transpositions, so } Z(G) \leq O_2(G) \text{ and } V \text{ has no trivial } SU_3(2)'-
module as quotient. Choose \( v = x\Phi(O_2(G)) \) a non-trivial element on \( V \). By (3.1) the submodule \( \langle v' \rangle \) is a natural module for \( H \) as required.

There are other valuable consequences of (3.1). In particular, it is important that a non-central extension of a 2-group by \( PSU_3(2)' \approx (\mathbb{Z}_2 \times \mathbb{Z}_2):\mathbb{Z}_2 \) cannot be generated by 3-transpositions. Using this, Fischer [7, (2.2.7)] proved:

(3.2) **Corollary.** Let \( H = \langle E \rangle \) with \( E \) a class of 3-transpositions, and assume that \( H \) satisfies \( F^*(H) = O_2(H) \) and \( H = O_{2,3,2}(H) \). Then \( H/O_2(H) \simeq S_3 \) or \( SU_3(2)' \).

Although it looks somewhat technical, the following further corollary to (3.1) is crucial to us in this paper.

(3.3) **Corollary.** Let \( G \) be as in Hypothesis 2.6. If \( G \) is not of symplectic type then, for \( d \in C \), 3 divides \( |C_G(d)\cap D'\cap C_G(d')| \).

**Proof:** Without loss of generality we may assume that \( V = O_2(G) \) is elementary abelian. If \( G \) is not of symplectic type, then by (3.1) there is a subgroup \( K - MH \) with \( d \in H \simeq SU_3(2)' \) acting naturally on \( M = V \cap K \simeq 2^6 \). In particular, \( Z = Z(H) \simeq \mathbb{Z}_3 \) is contained in \( C_G(d) \) but acts non-trivially on \( [M, d] \). By (2.3), \( Z \) is not contained in \( \langle D \cap C_G(d) \rangle \).

The next lemma provides part of Theorem 5.

(3.4) **Lemma.** Let \( G \) be as in Theorem 2. Then \( G/\Phi(O_2(G)) \) has a unique conjugacy class of 3-transpositions, and that class contains the transvections of every complement to \( V \) in \( G/\Phi(O_2(G)) \).

**Proof.** In \( G/\Phi(O_2(G)) \) there is a unique class of involutions in Sylow 3-normalizers. This class must be \( D\Phi(O_2(G))/\Phi(O_2(G)) \).

4. **Case \( O_3(G) \) Non-Central**

Let \( G \) satisfy Hypothesis 2.6. In this section we shall consider the case in which \( G/O_2(G) \) itself has a non-central solvable subgroup, a 3-group by (2.2). The section is devoted to a proof of the following theorem, which shows that in this case the main theorem, Theorem 1, is valid.

(4.1) **Theorem.** Assume that \( O_3(G) \) is not central in \( \bar{G} \). Then \( \bar{G} \simeq W_3(\bar{A}_{n-1}) \simeq \mathbb{Z}_3^{-1}:S_n \) for some \( n \). If \( n \neq 3 \), then \( N = Q \) is a direct sum of \( GF(4) \)-monomial (i.e., transvection) modules for \( \bar{G} \). The extension splits,
and the transvections of every complement belong to the 3-transposition class $D$.

Of course, the case $n = 3$ was handled in the previous section and in Theorem 2. In proving (4.1) we assume that $O_3(\bar{G})$ is not central in $\bar{G}$ and that $\bar{G} \neq SU_3(2)'$.

(4.2) **Lemma.** Let $d \in D$. The quotient $\bar{G}/O_3(\bar{G})$ is of symplectic type and $[d, O_3(\bar{G})]$ has order 3.

**Proof.** Otherwise within $\bar{G}$ we can find an $\bar{H} = \langle d^R \rangle$ with $\bar{H} = \langle d \rangle O_3(\bar{H})$ and $|O_3(\bar{H})| \geq 3^4$. This contradicts (3.2).

(4.3) **Lemma.** $\bar{G}$ is $W_3(\bar{A}_{n-1}) \approx Z_3^{n-1} : S_n$, for some $n \geq 4$.

**Proof.** By (4.2) the quotient $\bar{G}/O_3(\bar{G})$ is of symplectic type; so by [6, (6.2)] or [10] this quotient is either isomorphic to $S_n$, for some $n$, or contains a subgroup $W(D_4)$ generated by images of elements of $D$.

We claim that $\bar{G}/O_3(\bar{G}) \approx S_n$. Assume, for a contradiction, that there is a set of four members from the class $DO_3(\bar{G})/O_3(\bar{G})$ having diagram $D_4$. Lift these to a subset $\bar{E}$ of $\bar{D}$ again with diagram $D_4$. Within the subgroup $\langle \bar{E} \rangle$ it is possible to find a fifth element $\bar{e}$ which together with $\bar{E}$ gives a diagram of type $\bar{D}_4$. For suitable $g \in O_3(\bar{G})$, the subset $\{ \bar{E}, \bar{e}^g \}$ of $\bar{D}$ has diagram $\bar{D}_4$ and generates a subgroup $\bar{H} \approx W_3(\bar{D}_4)$. This group can be thought of as $GF(3)^4$ extended by the group of those $GF(3)$-monomial matrices of degree 4 which contain an even number of 1's. Therefore in $\bar{H}$ the centralizer of an element of $\bar{D}$ is generated by members of $\bar{D}$. As $\bar{H}$ is not of symplectic type, this contradicts (3.3) and so demonstrates the claim.

By (4.2) we have $[\bar{e}, O_3(\bar{G})] = 3$, for each $\bar{e} \in \bar{D}$. Thus a generating set of $\bar{G}/O_3(\bar{G})$ with diagram $A_{n-1}$ can be lifted into a subset of $\bar{D}$ with diagram $\bar{A}_{n-1}$ which generates $\bar{G}$. Hence, $\bar{G}$ is a quotient of $W_3(\bar{A}_{n-1})$.

This completes a proof of the lemma, except possibly when $n$ is a multiple of 3. In those cases $W_3(\bar{A}_{n-1})$ has a normal central subgroup of order 3 which might be mapped to the identity of $\bar{G}$. Again (3.3) applies to show that this possibility does not occur.

(4.4) **Lemma.** The extension of $N$ by $\bar{G} \approx W_3(\bar{A}_{n-1})$ is split, and the Coxeter generators of every complement to $N$ in $\bar{G}$ are contained in $D$. The subgroup $N = [N, G]$ is an elementary abelian 2-group.

**Proof.** We can construct a complement by lifting a subset of $\bar{D}$ with diagram $\bar{A}_{n-1}$ to a subset of $\bar{D}$ also having diagram $\bar{A}_{n-1}$.

That $[N, G]$ is elementary abelian follows from (2.5) where the generating set $\bar{A}$ can be taken, for instance, to have type $\bar{A}_{n-1}$. As
\( \bar{G} \simeq W_3(\mathbb{A}_{n-1}) \), the quotient \( \bar{G} \) has no non-trivial central extension which is generated by 3-transpositions. This forces \( N = [N, G] \).

By (2.2) the submodule \( C_N(O_3(\bar{G})) \) is central in \( G \) and so is trivial. Sylow's theorem and the action of \( O_3(\bar{G}) \) now prove again that the extension is split and additionally show that all complements to \( N \) in \( G \) are conjugate to the one already found and generated by members of \( D \). This completes the proof of the lemma.

We establish some notation for use in the rest of the section. Set \( \bar{V} = O_3(\bar{G}) \simeq \mathbb{Z}_3^{n-1} \). For each \( i \in \bar{D} \), let
\[
S(i) = \langle i^{\bar{V}} \rangle = \langle i \rangle[\bar{V}, i] \simeq S_3.
\]
Furthermore, let \( \Sigma = \{S(i)^G\} \).

\textbf{(4.5) Lemma.} (1) \( \bar{G} \) acts on \( \Sigma \) as \( \bar{G}/\bar{V} \) acts on its transposition class \( \{i^{\bar{V}} | i \in \bar{D}\} \).

(2) If distinct \( \sigma, \tau \in \Sigma \) with \( [\sigma, \tau] \neq 1 \), then \( \langle \sigma, \tau \rangle \simeq SU_3(2)' \).

(3) For \( \sigma = S(\bar{s}) \in \Sigma \), \( N_G(\sigma) = \sigma C_G(\bar{s}) = \sigma(\bar{D} \cap C_G(\bar{s}))Z \), where \( Z = Z(\langle \sigma, \tau \rangle) \), for any \( \tau \in \Sigma \) with \( [\sigma, \tau] \neq 1 \).

\textit{Proof.} These all can be calculated within \( \bar{G} \). The first is clear. The second can be proven using (3.1). The last follows from a Frattini argument and the centralizer structure already discussed in (4.3).

\textbf{(4.6) Lemma.} The module \( N \) is a direct sum of \( GF(4) \)-monomial modules for \( \bar{G} \).

\textit{Proof.} By (4.4) elementary abelian \( N \) is equal to \( [N, G] \), so in fact \( N = \langle [N, \bar{s}]^G \rangle \), for any fixed \( s \in \bar{D} \). The result will follow when we have shown that, for each \( w \in [N, \bar{s}] \), the submodule \( M = \langle w^G \rangle \) is a \( GF(4) \)-monomial module for \( \bar{G} \).

Set \( \sigma = S(\bar{s}) \in \Sigma \). Choose \( \bar{i} \in \bar{D} \) with \( [\bar{s}, \bar{i}] \neq 1 \) and \( S(\bar{s}) \neq S(\bar{i}) \). Let \( \tau = S(\bar{i}) \). Set \( \bar{r} = i\bar{s}i \) and \( \rho = S(\bar{r}) \).

By (4.5) we have \( \bar{H} = \langle \sigma, \bar{i} \rangle = \langle \sigma, \tau \rangle \simeq SU_3(2)' \). By Theorem 2 as \( \bar{H} \)-module \( N \) is a direct sum of \( C_N(\bar{H}) = C_N(O_3(\bar{H})) \) and 6-dimensional natural modules for \( SU_3(2)' \) (forming \( [N, O_3(\bar{H})] \)). In particular there is a natural submodule \( U \) of order \( 2^6 \) such that \( U = [U, \sigma] \) of order \( 2^4 \). Set \( W(\sigma) = W \). By (2.3) and (4.5)(3) the group \( W(\sigma) \) is invariant under \( N_G(\sigma) \). So for each \( \alpha \in \Sigma \), we see that \( W(\alpha) \), given by
\[
W(\alpha) = W^g, \quad \text{for} \quad \alpha = \sigma^g,
\]
is well defined. The subspaces \( W(\tau) \) and \( W(\rho) \) are then contained in \( U \). Therefore, we have \( |W(\sigma) \cap W(\tau)| = 4 \) and \( \langle W(\sigma), W(\tau) \rangle \geq W(\rho) \).
As members of $\Sigma$ correspond to transpositions of $G/P \simeq S_n$, there is a subset $\Delta$ composed of $n - 1$ members of $\Sigma$ such that $G = \langle \Delta \rangle$ and $[\alpha, \beta] \neq 1$, for each pair $\alpha, \beta \in \Delta$. Therefore, $M$ has order at most $2^{2n}$. We prove the lemma by induction on $n \geq 4$. Lemma 3.1 provides the necessary initialization.

Inside $G$ choose a subgroup $R \simeq W_3(A_{n-2}) \simeq Z_3^{n-2}:S_{n-1}$. By induction, as $K$-module, $M$ is a direct sum of a $G(4)$-monomial $K$-module $U = [M, O_3(K)]$ and a trivial module $X = C_M(O_3(K))$ of order at most 4. In fact $X$ has order equal to 4, because $V \leq N(O_3(K))$ and $C_n(V)$ is trivial. The stabilizer of $X$ is thus $\tilde{K}V \simeq Z_3^{n-1}:S_{n-1}$ of index $n$ in $\tilde{G}$. As $G$-module, $M$ is then uniquely determined up to isomorphism as the induced module $X^{\tilde{G}}_{K,F}$. This completes the lemma.

Theorem 4.1 now is a consequence of the lemmas of this section.

5. IDENTIFYING $\tilde{G}$

Again let $G$ satisfy Hypothesis 2.6. In this section we identify the factor $\tilde{G} = G/O_3(G)$. We prove:

(5.1) Theorem. We have one of the following:

(1) $G$ has symplectic type;
(2) $O_3(\tilde{G})$ is not central in $\tilde{G}$;
(3) $\tilde{G} \simeq SU_n(2)$, for some $n \geq 4$;
(4) $\tilde{G} \simeq Z_3 \cdot O_6$ (3).

If $G \neq SU_3(2)'$, then $N = O_3(G)$ is elementary abelian.

Note that the groups which occur in the first two parts of the theorem have been determined in the earlier sections of the paper. The groups $\tilde{G}$ of the remaining two parts are the ones expected. This is clearly true for $SU_n(2)$, but the last group $\tilde{G}$ must also arise as $Z_3 \cdot O_6$ (3) is a subgroup of $SU_6(2)$ generated by transvections.

In the balance of the section assume that $G$ is not of symplectic type and that $O_3(\tilde{G})$ is central in $\tilde{G}$. As a direct consequence of the results of Fischer [6, 7] we have:

(5.2) Theorem. $G/Z(\tilde{G})$ is isomorphic to one of the following:

(1) $PSU_n(2)$ for some $n \geq 4$;
(2) $O_n^{\pm \mu}$ (3), for some $\mu = \pm$, $n \geq 6$;
(3) \( \text{Fi}_{22}, \text{Fi}_{23}, \text{or} \ \text{Fi}_{24}; \)

(4) \( O_8^+(2) : S_3 \) or \( O_8^+(3) : S_3. \)

Here in Fischer's notation \( O_{n^+}(3) \) is the extension by one of the two reflection classes, indexed by \( \pi \), of the simple orthogonal group for a space \( \text{GF}(3)^n \) of Witt type \( \mu. \)

The lemmas of this section present the proof of Theorem 5.1. This consists mainly of considering the cases provided by (5.2) and deleting those which cannot occur. Note that, aside from some of the remarks in Section 8 below, this is the only place where we use Fischer's results.

(5.3) Lemma. If \( \bar{G}/Z(\bar{G}) \cong \text{PSU}_n(2) \) with \( n \geq 4 \), then \( \bar{G} \cong \text{SU}_n(2). \)

Proof. By [8] the odd part of the Schur multiplier of \( \text{PSU}_n(2) \) is \( \mathbb{Z}C_3J \). Therefore, either \( \bar{G} \) is \( \text{SU}_n(2) \) or \( n \equiv 0 \pmod{3} \) and \( \bar{G} \) is \( \text{PSU}_n(2). \) In \( \text{SU}_n(2) \) the centralizer of a transvection has the shape \( 2^{1+2(n-2)}GU_{n-2}(2) \)

\( = 2^{1+2(n-2)}SU_{n-2}(2).Z_3. \) When \( n \equiv 0 \pmod{3} \) the centralizer of a transvection image in \( \text{PSU}_n(2) \) is \( 2^{1+2(n-2)}SU_{n-2}(2). \) In this case with \( n \geq 6 \) the group is not of symplectic type, but the centralizer of a 3-transposition is generated by its 3-transpositions. This contradicts (3.3).

(5.4) Lemma. If \( \bar{G}/Z(\bar{G}) \cong O_{n^+}(3), \) for some \( \mu = \pm, \pi = \pm, n \geq 6, \) then \( \bar{G} \cong Z_3 \cdot O^-_6(3). \) (As \( O^{-+}_6(3) \cong O^{-+}_6(3), \) the notation can be abbreviated somewhat.)

Proof. These groups are not of symplectic type. The centralizer of a reflection (i.e., 3-transposition) in each of these groups is generated by reflections. Thus by (3.3) when one of these groups occurs as \( \bar{G}/Z(\bar{G}), \) it must be accompanied by an element of order 3 from its multiplier. By [2, 8], this forces \( \bar{G} \cong Z_3 \cdot O^-_6(3) \) or \( Z_3 \cdot O^+_{-+}(3) \) (a perfect central extension of the simple group \( \Omega_3(3)) \). This last case cannot occur because a transposition centralizer is \( Z_2 \times Z_3 \cdot O^-_6(3) \) and is again generated by 3-transpositions.

(5.5) Lemma. We cannot have \( \bar{G}/Z(\bar{G}) \) isomorphic to one of \( \text{Fi}_{22}, \text{Fi}_{23}, \text{Fi}_{24}, \) \( O_8^+(2) : S_3, \) or \( O_8^+(3) : S_3. \)

Proof. If \( Z(\bar{G}) = 1, \) then the centralizer of a 3-transposition is generated by 3-transpositions in each of these groups. As they are not of symplectic type, (3.3) shows they do not occur. The only one of these groups which has a 3 in its multiplier is \( \text{Fi}_{22} \) [8]. By [2] or [6, 14.2 and 17.2.3] the centralizer of a 3-transposition in \( \text{Fi}_{22} \) is isomorphic to \( Z_2 \cdot \text{PSU}_6(2) \). If \( \bar{G} \) is to be isomorphic to \( Z_3 \cdot \text{Fi}_{22}, \) then by (5.3) a transposition centralizer
must be $Z_2 \cdot SU_d(2)$ and so is generated by transpositions. Again this does not occur by (3.3).

(5.6) **Lemma.** $O_2(G)$ is elementary abelian.

**Proof.** Set $Q = [O_2(G), G]$ and $\tilde{G} = G/Q$. By (2.4) and Lemmas 5.3-5.5 of this section, $Q$ is elementary abelian. (Actually the specific results of these earlier lemmas are not required for this. The more elementary results [7, Sect. 3] are enough.)

The group $\tilde{G}$ is a 2-central extension of $\tilde{G}$ which is generated by 3-transpositions. Using [2.8] we find that it is a quotient of one of

$Z_2 \times SU_n(2)$ \quad (n \geq 4), \quad Z_2 \times ((Z_2 \times Z_2) \cdot SU_6(2)), \quad Z_6 \cdot O_6^-(3)$.

In all cases, $O_2(Z(\tilde{G}))$ is contained in $\langle \tilde{D} \cap C_0(d) \rangle$, for each $d \in \tilde{D}$. Therefore, by (2.3) $O_2(Z(\tilde{G}))$ acts trivially on $Q$. The subgroup $N = O_2(G)$ must then be a central extension of $Q$ by an elementary abelian 2-group of order at most 8. The characteristic subgroups $\Phi(N)$ and $N'$ are both of order at most 8, and so both are central. Now the general hypothesis (2.6) forces $\Phi(N)N' = 1$, as required.

6. **Identifying the Module**

In this section we characterize the modules which can lie within the normal 2-subgroup of our 3-transposition group in terms of the property of Lemma 2.3(2). For this section only we adopt the following:

**Hypothesis.** The group $H$ is isomorphic to $SU_n(2)$ for $n \geq 4$ or to $Z_3 \cdot O_6^-(3)$ ($n = 6$), and $T = t^H$ is the transvection (i.e., 3-transposition) class of $H$. Set $C = C_H(t)$ and $P = \langle T \cap C \rangle$. Let $V$ be a GF(2)$H$-module with $[V, t, P] = 1$.

(6.1) **Proposition.** If $H \neq SU_5(2)$, then $C/P \cong Z_3$. If $H \cong SU_5(2)$, then $C/P \cong A_4$. In any event, $P$ is transitive on $T - (T \cap C)$.

(6.2) **Lemma.** Let $U$ be $C$-submodule of $V$ with $[U, P] = 1$. Then $|\langle U^H, t \rangle| \leq |U|$.

**Proof.** Fix an $h \in H$ with $[U^h, t] \neq 1$. We show $\langle U^H, t \rangle = [U^h, t]$, from which the lemma follows.

Let $g \in H$. If $[t, t^g] = 1$, then

$[U^g, t] = [U, t^{g^{-1}}] \leq [U, P]^g = 1 \leq [U^h, t]$. 

Now assume \([t, t^g] \neq 1\). By (6.1) there is a \(p \in P\) with \(t^{gp} = t^h\). Here \(gp = ch\) with \(c = (gp)h^{-1} \in C\). Then

\[
[U^g, t] = [U^g, t]^p = [U^{gp}, t^p] = [U^{ch}, t] = [U^h, t].
\]

Thus, for all \(g \in H\), we have \([U^g, t] \leq [U^h, t]\), as required.

(6.3) THEOREM. Either \(H \cong SU_3(2)\) or \([V, H]/C_4(H)\) is a direct sum of natural \(GF(4)\)-modules for \(H\).

Proof. Without loss of generality, \(V = [V, H]/C_4(H)\). Let \(U\) be an indecomposable \(C\)-submodule of \(V\) with \([U, P] = 1\). Thus \(U\) is either \(GF(2)\) or \(GF(2)^2\).

In the first case, either \(U\) is trivial for all \(H\) or by (6.2) the elements of \(T\) induce \(GF(2)\)-transvections on \((U^g)^H\). As \(H\) has subgroups \(SU_3(2)'\) generated by members of \(T\), this cannot happen. So in the first case \(U\) is a trivial \(H\)-module, against assumption.

Therefore, \(U \cong GF(2)^2\). Set \(W = \langle U^h \rangle\). Then \(W\) is (isomorphic to) a quotient of the induced module \(U^h_1\). Indeed by hypothesis \(W\) is a quotient of \(\tilde{W} = U^h_1/\langle [U^h_1, t, P]^{H'} \rangle\). On the other hand, the natural module \(GF(2)^{2n}\) is also a quotient of \(\tilde{W}\).

As \(H\)-module, \(\tilde{W}\) satisfies \([\tilde{W}, t, P] = 1\). Additionally \(\tilde{W} = \langle \tilde{U}^h \rangle\). Thus by (6.2), \(|[\tilde{W}, t]| \leq |\tilde{U}| = 4\).

If \(H \neq SU_3(2)\), then \(H\) is generated by \(n\) members of \(T\) by [6, 11, Theorem 4.9]. Therefore, \(W, \tilde{W}\), and the natural module \(GF(2)^{2n}\) all have the same dimension and are all isomorphic irreducible modules.

If \(H \cong SU_3(2)\), then \(H\) is generated by five members of \(T\), and \(\tilde{W}\) of \(GF(2)\)-dimension 10 is the non-split extension of a trivial \(GF(4)\)-module by a natural module (see (2.8) or [11, Theorems II, T8 and 4.9]). As by assumption \(W\) has no trivial submodule, even in this case \(W\) is isomorphic to the natural, irreducible module \(GF(2)^{2n}, n = 4\).

To finish the proof of the lemma we need only note that \([V, t] = 1\) and that, by assumption, \(V = \langle [V, t]^H \rangle\).

(6.4) COROLLARY. If \(H\) is not isomorphic to \(SU_3(2)\) or \(SU_5(2)\), then \(V\) is a direct sum of trivial modules and natural modules.

Proof. The natural module \(GF(2)^{2n}\) is self-dual, and so the result follows by (2.8).

(6.5) COROLLARY. Let \(G\) satisfy Hypothesis 2.6 with \(\tilde{G}\) isomorphic to \(SU_n(2)\) for \(n \geq 4\) \((n \neq 5)\) or \(Z_3, O_6^-\) (3). Then \([N, G]\) is a direct sum of natural \(GF(4)\)-transvection modules for \(\tilde{G}\). If \(\tilde{G} \cong SU_4(2)\), then \(N = [N, G]\).
Proof. The first conclusion follows directly from the theorem, (2.3), and Theorem 5.1. The second is then a consequence of (2.8).

We now consider the modules for \( H \cong SU_5(2) \).

(6.6) Lemma. Assume that \( H \cong SU_5(2) \). Then \( V \) is a direct sum of \( C_V(H) \) and \([V, H]\). Every composition factor of \([V, H]\) is a natural module \( V_{10} = GF(2)^{10} \) for \( H \).

Proof. The hypothesis is inherited by sections of \( V \), so by (2.8) it is enough to show that irreducible \( V \) is either a trivial module or \( V_{10} \).

Let \( U \) be an irreducible \( C \)-submodule of \([V, t]\). The subgroup \( R = O^3(C) \) is trivial on \( U \), because it contains \( P \) as a normal subgroup of index 4 which acts trivially. Therefore, \( U \) is one of the two irreducible \( C/R \)-modules \( GF(2) \) or \( GF(2)^2 \). The proof now proceeds as that of (6.3). If \( U \) is trivial for \( R \), then \( V \) is trivial for \( H \). Otherwise \( V \) and \( V_{10} \) are both quotients of, and therefore equal to, the dimension 10 module \( \tilde{V} = U \uparrow^C_R / \langle [U \uparrow^C_R, t, R] \rangle \).

We conclude that the irreducible module \( V = \langle U^H \rangle \) is either a trivial module or \( V_{10} \).

(6.7) Lemma. Let \( U_2 \) be the non-faithful but irreducible \( A_4 \)-module \( GF(2)^2 \); and let \( U_4 \) be the indecomposable, faithful \( A_4 \)-module \( GF(2)^4 = GF(4)^2 \) which comes from \( A_4 \) as a Sylow 2-normalizer in \( SL_2(4) \). Let \( U \) be a \( GF(2)A_4 \)-module on which a Sylow 3-subgroup has no fixed points. Then \( U \) is a direct sum of copies of \( U_2 \) and \( U_4 \).

Proof. An elementary calculation with the permutation module for \( A_4 \) shows that cyclic \( U \) must be one of the two given modules. The result follows easily.

(6.8) Lemma. Assume that \( H \cong SU_5(2) \). Then \( V \) is a direct sum of \( C_V(H) \) and \([V, H]\). The module \([V, H]\) is a direct sum of copies of the irreducible module \( V_{10} \) or the indecomposable module \( V_{20} \cong V_{10} \cdot V_{10} \) for \( H \).

Proof. By (6.6) we may assume that \( V = [V, H] \) and that \( U = [V, t] \) is a module for \( C/P \cong A_4 \) on which a Sylow 3-subgroup acts without fixed points. By (6.7) the module \( U \) is a direct sum of \( C/P \)-submodules \( X \) isomorphic to \( U_2 \) or to \( U_4 \). In the first case \( \langle X^H \rangle \) is isomorphic to \( V_{10} \) by (6.6). If \( X \) is isomorphic to \( U_4 \), then \( \langle X^H \rangle \) is of \( GF(2) \)-dimension at most 20 by (6.2), in which case it is a non-split extension of \( V_{10} \) by \( V_{10} \), again by (6.6).

Remark. This lemma leaves in doubt the actual existence of such a module \( V_{20} \). The relations might collapse the induced module more than expected. We shall later see that this is not the case.
7. PROOF OF THE MAIN THEOREMS

We have already proven Theorem 2. Furthermore, we have nearly completed a proof of Theorem 1. The groups of symplectic type are given by Theorem 2.7. Next Theorem 4.1 furnishes all those groups satisfying Hypothesis 2.6 which have a non-central normal 3-subgroup in $G$, and all the remaining possibilities for $G$ are given in Theorem 5.1. For each of these other than $SU_4(2)$ and $SU_5(2)$, Theorem 6.3 completely describes $N$. The main result of this section is Theorem 7.3, which shows that the extension of $N$ by $G$ must split in all other cases except for $G \simeq SU_7(2)$. At the same time the 3-transposition class $D$ will be located unambiguously. This will complete the proof of Theorem 1 except for $G \simeq SU_7(2)$. The exceptional cases $SU_4(2)$, $SU_5(2)$, and $SU_7(2)$ are then handled in the rest of the section, completing the proof of Theorem 1 and providing proofs of Theorems 3, 4, and 5.

We begin with two propositions, one aimed at Theorem 1 and the other primarily at Theorem 5.

(7.1) PROPOSITION. Let $C$ be a central extension of the elementary abelian 2-group $X$ by $2^{2m} : SU_m(2)$ $(m \geq 5)$ with $C$ generated by the conjugacy class $E$ of 3-transpositions. Then $|X| \leq 4$.

Proof: Set $Q = [O_2(C), C]$. Then by (2.4) the group $Q$ is generated the subgroups $Q(e) = [O_2(C), e]$, for $e \in E$, where $Q(e)$ has index 2 in $R(e) = \langle E \cap eO_2(C) \rangle$. By (2.1) we have $|E \cap eO_2(C)| = 4$, so $|R(e)| \leq 16$ and $|Q(e)| \leq 8$. In particular $|Q(e) \cap X| \leq 2$.

By (2.5) there is a central subgroup $Y \leq Q(e)$ such that $O_2(C)/Y$ is elementary abelian. The previous paragraph shows that we may take $Y = Y \cap X$ to have order at most 2. Next by (6.5), $C/Y$ must be a quotient of $Z_2 \times (2^{2m} : SU_m(2))$. This forces $|X| \leq 4$.

Remark. In fact it is not difficult to prove that $C$ of (7.1) must be a quotient of $Z_2 \times (2^{1 + 2m} : SU_m(2))$, where $2^{1 + 2m}$ is extraspecial of order $2^{1 + 2m}$.

(7.2) PROPOSITION. If $V$ is a direct sum of natural modules $GF(2)^{2m}$ for $K \simeq SU_m(2)$, $m \geq 4$, and $H$ is the split extension of $V$ by $K$, then $H$ has a unique conjugacy class of 3-transpositions.

Proof: The group $H$ is a quotient of a transvection generated subgroup of a unitary group, so it contains a class of 3-transpositions. If $V$ is trivial, then uniqueness was observed by Fischer [6, 7]. Therefore, in the general case, any 3-transposition class must meet and contain generators of $M \cdot V$, the split extension of $V$ by the monomial subgroup $M \simeq Z_3^{m-1} : S_m$ of $K$. By (4.1) the subgroup $M \cdot V$ contains a unique class of 3-transpositions.
(7.3) **Theorem.** Assume Hypothesis 2.6. If \( \bar{G} \) is isomorphic to \( Z_3 \cdot O_6 \) (3) or to \( SU_n(2) \), for \( n \geq 4 \) with \( n \neq 4, 5, 7 \), then the extension of \( N \) by \( \bar{G} \) is split.

**Proof.** By (6.5) the module \( N \) is a direct sum of \( m \) natural \( GF(4) \)-transvection modules \( GF(2)^{2n} \) for \( \bar{G} \), where if \( \bar{G} \simeq Z_3 \cdot O_6 \) (3) we take \( n = 6 \).

If \( \bar{G} \) is isomorphic to \( Z_3 \cdot O_6 \) (3) or to \( SU_n(2) \) with \( n \equiv 0 \) (mod 3), then \( \bar{G} \) contains a central element of order 3 acting without fixed points on \( N \). The existence of a complement (indeed a unique conjugacy class of complements) is therefore immediate by Sylow's theorem. We prove the existence of a complement for the remaining groups \( SU_n(2) \) by induction on \( n \). Initialization is provided by the cases \( n = 6, 9 \) just discussed. Assume \( n \neq 4, 5, 6, 7, 9 \).

First assume that \( m = 1 \), so that \( N \simeq GF(2)^{2n} \). Let \( d \in D \). Then \( C = \langle D \cap C_G(d) \rangle \) has the structure
\[
(2^2.2^{3(n-2)}).2^1 + 2^{(n-2)} \cdot SU_{n-2}(2).
\]
Here the submodule \( C_N(d) \) is of type \( 2^2.2^{3(n-2)} \) and itself contains \([N, d]\) of type \( 2^2 \). Let \( t \) be an element of order 3 which centralizes \( d \) chosen so that \( C_G(d) = \bar{C}\langle t \rangle \). For instance, we may choose \( t \) to generate the center of some subgroup \( SU_3(2) \) which contains \( d \).

By (6.3) and induction, \( C/Z(C) \) is isomorphic to the split extension of \( GF(2)^{2(n-2)} \oplus GF(2)^{2(n-2)} \) by \( SU_{n-2}(2) \). Thus there is a \( t \)-invariant subgroup \( K \) of \( C \) generated by \( Z(C) \) together with elements of \( D \) and such that \( K \cap N = Z(C) \cap [N, d] \) and \( K/(K \cap N) \simeq 2^{1 + 2(n-2)} \cdot SU_{n-2}(2) \). Set \( P = \langle D \cap K \rangle \). By (7.1) we know that \( P \cap N = Z(P) \cap N \) has order at most 2. However, the element \( t \) is irreducible on \([N, d] \) of order 4. Thus in fact \( P \cap N = 1 \). The subgroup \( P \) has odd index in \( G \), so the extension splits by Gashütz' theorem [1, (10.4)].

Now suppose that \( m \geq 2 \), and let \( M \) be a submodule of \( N \) which is isomorphic to \( GF(2)^{2n} \). By a second induction (on \( m \)), \( G/M \) is the split extension of \( N/M \) by a subgroup \( S/M \simeq SU_n(2) \) generated by members of the class \( DM/M \). But then the case \( m = 1 \) applies to the preimage \( S \) in \( G \) to produce a complement to \( N \) in \( G \), completing both inductions and the proof of the theorem.

We now deal with the exceptional unitary groups \( SU_4(2), SU_5(2), \) and \( SU_7(2) \). The next lemma completes the proof of Theorem 1.

(7.4) **Lemma.** Assume Hypothesis 2.6 with \( \bar{G} \simeq SU_4(2) \). Then \( N = [N, G] \) is a direct sum of natural modules \( GF(2)^8 \) for \( \bar{G} \), and the extension of \( N \) by \( \bar{G} \) splits.

**Proof.** By (5.1) the subgroup \( N \) is elementary abelian, and by (6.5) its subgroup \([N, G]\) is a direct sum of natural modules. By (A.2) of Appendix
there are 5 members of \(D\) which generate a subgroup \(H\) of \(G\) isomorphic either to \(SU_2(2)\) or to \(Z_2 \times SU_2(2)\). Note that \(Z(H)\) is contained in \(N\) and is centralized by \(\langle N, H \rangle = G\) whereas \(Z(G) = 1\) by hypothesis. Thus in fact \(H \cong SU_2(2)\), and the extension splits with the transvections of a complement contained within \(D\). If \(N \neq [N, G]\), then \(G/[N, G] \cong Z_2 \times SU_2(2)\). However, \(H = \langle D \cap H \rangle\) is perfect, so \(G\) is perfect as well.

**Proof of Theorem 3.** Assume Hypothesis 2.6 with \(G \cong SU_2(2)\). By (6.5) we know \(N = [N, G]\). Furthermore, by (2.3) and (6.8) the module \(N\) is a direct sum of indecomposable modules \(V_{10}\) or \(V_{20}\) of \(GF(2)\)-dimension 10 and 20, respectively. By (A.4) there is a subset of five elements of \(D\) which generate a complement to \(N\) in \(G\).

**Proof of Theorem 4.** Assume Hypothesis 2.6 with \(\bar{G} \cong SU_7(2)\). By (6.5) we know that \(N = [N, G]\) is a direct sum of \(m\) natural \(GF(4)\)-transvection modules. To prove Theorem 4 we must decide to what extent the extension splits.

Choose seven elements \(a, b, c, d, e, f, g \in D\) such that the corresponding transvections \(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g} \in \bar{D}\) have centers \(a, b, c, d, e, f, g\) as given in the proof of (A.5) from Appendix. We wish to select additional elements \(h\) and \(x\) of \(D\) (with the centers \(h\) and \(x\) of (A.5)) such that all the relations of (A.5) are valid in \(K = \langle a, b, c, d, e, f, g, h, x \rangle\). First note that \(S = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g} \rangle \cong SU_2(2)\). Further choose any \(y \in D\) so that the transvection \(\bar{y}\) has center \(x\). Then \(\langle \bar{f}, \bar{y} \rangle \cong S_3 \cong SU_2(2)\) centralizes \(S\).

By (A.4) the subgroup \(S = \langle a, b, c, d, e, f, g \rangle\) is isomorphic either to \(SU_2(2)\) or to \(Z_2 \times SU_2(2)\). Indeed as \(\langle \bar{S}, \bar{g}, \bar{y} \rangle\) has no fixed points in \(N\), the group \(S\) must be \(SU_2(2)\). Set \(h = ((ac) \bar{b} + (ad) \bar{c})^2 \in S\). Then \(h\) is conjugate to \(a, b, c, d, e, f, g\) in \(S\), has center \(h\), and so satisfies the expected relations with respect to \(a, b, c, d, e, f, g\).

A particular consequence is that \(\{b, c, d, e, f, g, h\}\) has a diagram of type \(E_7\). In the subgroup which this set generates, we use the recipe given in (A.5) to locate a conjugate \(x \in D\) which extends the set to a diagram of type \(E_7\). As the centers \(a\) and \(x\) are perpendicular, the 3-transpositions \(a\) and \(x\) commute.

Now we have all the relations of (A.5) satisfied, so \(K\) is either \(SU_2(2)\) or a non-split extension \(2^{14} : SU_2(2)\). First suppose that \(K \cong SU_2(2)\). Then \(G\) is the split extension of \(N\) by \(K\), and \(D\) is the unique 3-transposition class in \(G\) by (7.2). Using (2.8) we can count the number of complements to \(N\) in \(G\). We then discover that every seven set of the 3-transposition class \(D\) which is mapped to \(\{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\} \leq \bar{D}\) must generate one of these complements.

Now suppose \(K \cong 2^{14} : SU_2(2)\). The seven set \(\{a, b, c, d, e, f, g\}\) of \(D\) does not generate a complement to \(N\) in \(G\), so by the previous paragraph the extension is non-split. On the other hand \(K/O_2(K)\) is a complement to...
$N/O_2(K)$ in $G/O_2(K)$, so that extension is split. Thus any non-split $G$ must be isomorphic to the split extension of $N/O_2(K)$ by $K$ with $O_2(K)$ acting trivially on $N/O_2(K)$. This completes the theorem.

There are really two parts to Theorem 5: existence and uniqueness. For the most part uniqueness can be proven from results already presented. For existence the following proposition is of help.

(7.5) Proposition. Let the group $H$ contain the normal elementary abelian 2-subgroup $W$. Assume that $\tilde{H} = H/W \cong SU_{2m}(2)$, for some $m \geq 4$, and that every composition factor in $W$ is a natural module $GF(2)^{2m}$ for $H$. Let $d$ be an involution of $H$ with $\tilde{d}$ a transvection of $\tilde{H}$, and suppose that

(i) $dW = dH \cap dW$ and
(ii) $[W, d]$ a TI-set in $H$.

Then $dH$ is a conjugacy class of 3-transpositions which generates $H$.

Proof. Let $e \in dH$. We must show that $|de|$ is 1, 2, or 3. If $d = e$, then $|de|$ is 1. Assume $d \neq e$.

First suppose that $dW = eW$. Then $d$ and $e$ are involutions of the same coset and so commute; $|de|$ is 2. Now suppose $\tilde{d}$ and $\tilde{e}$ are distinct but commute. Then $|de|$ is 2 or 4. However, $(de)^2 \in [W, d] \cap [W, e]$ by (i). Then by (ii) we have $(de)^2 = 1$, and $|de|$ is 2.

Finally suppose that $|d\tilde{e}| = 3$. Then $|de|$ is 3 or 6. By the previous paragraph, we can find a subgroup $S \leq C_H(\langle d, e \rangle)$ with $\tilde{S} \cong SU_{m-2}(2)$. The subgroup $\tilde{S} \times \langle \tilde{d}, \tilde{e} \rangle$ of $SU_m(2)$ acts on $W$ without fixed points, so the element $(de)^3$ of $W$ is 1. Thus in this case $|de|$ is 3, completing the argument.

Proof of Theorem 5. Let $G$ be a group appearing in the conclusion of one of earlier theorems. Let $\tilde{G} = G/O_2(G)$, and let $V$ be the $\tilde{G}$-module $O_2(G)/\Phi(O_2(G))$. For the groups of symplectic type, existence and uniqueness of a class of 3-transpositions is part of Theorem 2.7. For $\tilde{G} \cong Z_3^{-1} : S_n$ they were proven as part of Theorem 4.1.

Now suppose that $G = Z_3 \cdot O_6^-(3)$ and $V$ is a direct sum on $GF(4)$-transvection modules $GF(2)^{12}$. Existence is of no concern, since by (7.2) a class of 3-transpositions is found through restriction from a split extension of $V$ by $SU_6(2)$. As remarked earlier, Sylow's theorem proves that there is a unique conjugacy class of complements to $V$, namely the subgroups $C_G(Z)$ as $Z$ runs through the Sylow 3-subgroups of $O_2(G)$. By (3.3) each element of any generating class of 3-transpositions must centralize at least one such $Z$. Therefore, each generating 3-transposition class must meet and generate each complement, so uniqueness follows by the corresponding result in $Z_3 \cdot O_6^-(3)$. 

It remains to consider \(G\) unitary. We first prove existence of a generating class of 3-transpositions. Choose \(d\) a transvection of a complement when the extension of \(V\) by \(G\) splits and otherwise choose \(d\) to be a Coxeter generator of a subgroup \(2^{14} \cdot SU_7(2)\). In any split extension, no unexpected fusion can occur within cosets of the normal subgroup; so \(d^V = d^G \cap dV\). If \(V\) is a direct sum of natural modules for \(G\), then certainly \([V, d]\) is a TI-set. Thus we have existence by (7.5) except possibly when \(n = 5, 7\). (Alternatively as noted in the proof of (7.2) the non-exceptional groups \(G\) may be found as a sections of transvection generated subgroups of unitary groups of larger dimension.)

Suppose \(G \simeq SU_5(2)\). As the extension is split, \(d^V = d^G \cap dV\). By the construction in Section 6, a transvection of \(SU_5(2)\) has commutator of order 16 in any summand \(V_{20}\) of \(V\). As \(SU_5(2)\) is generated by five of its transvections (see (A.4)), the commutator of \(d\) on such a summand must be a TI-set; so \([V, d]\) is a TI-set. (Note that we are still not sure that a module \(V_{20}\) exists; nevertheless, we know enough about such a module to do the present calculation.) We now may apply (7.5) to conclude that \(d^G\) is a generating class of 3-transpositions in \(G\).

Suppose next that \(G \cong SU_7(2)\). As \(V\) is a direct sum of natural modules, \([V, d]\) is a TI-set. In the proof of Theorem 4 we saw that \(G\) is a split extension of a direct sum of natural modules by either \(SU_7(2)\) or by \(2^{14} \cdot SU_7(2)\). In either case, no unexpected fusion of \(d\) occurs, since by (A.5) none occurs within \(2^{14} \cdot SU_7(2)\). Therefore, (7.5) applies, and \(d^G\) is a generating class of 3-transpositions.

This completes existence arguments in all cases where \(G\) is unitary. Uniqueness holds by (7.2) provided we are not in the exceptional cases involving \(SU_5(2)\) and \(SU_7(2)\). Even in these cases, the argument of (7.2) can be made effectively. If we restrict to the extension of \(V\) by the monomial subgroup of \(G\), then we still find a subgroup which must be generated by members of any 3-transposition class which generates \(G\). But now this subgroup is the split extension by a monomial group of a direct sum of natural modules, and so we already know that its 3-transposition class is uniquely determined. Therefore, that of \(G\) is as well.

This completes the proof of the theorem.

Remark. With Theorem 5 in hand we may now firmly assert the existence of the indecomposable module \(V_{20}\) for \(SU_5(2)\). Indeed consider the group \(2^{14} \cdot SU_5(2)\) constructed in (A.5). This is a 3-transposition group by Theorem 5. As in the proof of Theorem 7.3, the weak closure of a 3-transposition in its centralizer has the form

\[
(2^2 \cdot 2^{10}) \cdot 2^{1+10} \cdot SU_5(2),
\]

and has a central quotient \(2^{20} \cdot SU_5(2)\). If the module \(2^{20}\) in this quotient is
Then the argument of (7.3) would continue to prove that $SU_3(2)$ splits off of $2^{14}$. As this is not the case, the module must in fact be $V_{20}$. Of course, this is a very roundabout proof of existence for this module. Zara has given a direct construction [16, 6.48].

8. REMARKS

8.1. Although we have made use of Fischer’s important results from [6, 7], we could actually have made do with various weaker versions of his classification. Indeed the following would probably have been sufficient.

(8.1) **Theorem.** Let $G = \langle D \rangle$ with $D$ a conjugacy class of 3-transpositions. Assume $F^*(G)$ is a finite quasi-simple group. Let $d \in D$. Then either $|C_G(d)/\langle D \cap C_G(d) \rangle| \leq 3$ or $F^*(G) \simeq SU_3(2)$.

8.2. Two things are needed for a complete description of 3-transposition groups with non-central normal 2-subgroups. The first is an understanding of centers. Zara [16] studies certain central extensions carefully. The question of possible centers is in fact more about multipliers and covering groups than about 3-transposition groups. In order for a class to remain 3-transpositions in a central extension, it must lift to a class of involutions with the same cardinality and such that commuting members still commute after lifting.

The other unresolved situation for normal 2-subgroups concerns groups with $G/O_2(G) \simeq SU_3(2)'$ in which $\Phi(O_2(G))$ is not central. Zara [16, 4.124] has constructed 3-transposition groups which are the split extensions of 2-groups with class 3 by $SU_3(2)'$.

8.3. Zara [16, 17] treats both 2-groups and 3-groups. Using methods similar to those of the earlier sections of this paper, it is possible to produce counterparts for groups with normal 3-subgroups to many of our results concerning 2-subgroups.

Normal 3-subgroups behave differently from 2-subgroups in two main ways. First, in dealing with normal 2-subgroups we were able to prove that, except in one specific case, the Frattini subgroup $\Phi([O_3(G), G])$ is contained within $Z(G)$. In the normal 3-subgroup case this is no longer true with such precision. In particular, Zara [16, Prop. 4.13] gave examples of 3-transposition groups $G$ with $G/O_3(G)$ a symmetric group (of any degree) and $O_3(G) = [O_3(G), G]$ of class 4. Zara was however able to show, for center-free 3-transposition groups with non-symmetric quotients, that $[O_3(G), G]$ is elementary abelian [16, Théorème 5.20].

The second difference between the characteristic 2 and 3 cases is the
natural appearance of non-split extensions as generic examples when considering normal 3-subgroups. As we shall see below in (8.3), if one extension (in particular the split extension) is generated by 3-transpositions then all extensions are generated by 3-transpositions. In the normal 2-subgroup case the $SU_3(2)$ example is exceptional. More typically there are non-split extensions of natural $GF(2)$-modules by symplectic or symmetric groups which are not generated by 3-transpositions (see [4, 5]).

For the most part the quotients $\bar{G} = G/O_3(G)$ which arise are $GF(3)$-reflection groups—the groups $O_{n}^{\mu}(3)$ (with centers of order 2 restored where appropriate) and the Weyl groups $W(A_n)$, $W(D_n)$, and $W(E_n)$. For these groups, the non-trivial composition factors in $O_3(G)$ are copies of the natural reflection module. The only additional quotients $\bar{G}$ which can arise are $Z_2 \times SU_3(2)$ and its subgroup $2^{1+6} : SU_3(2)'$. The module for $Z_2 \times SU_3(2)$ has $GF(3)$-dimension 10, with the 3-transpositions of $Z_2 \times SU_3(2)$ having commutator dimension 2. This example is associated with the representation of $Z_2 \times SU_3(2)$ as a quaternionic reflection group [3, Case U] and can also be found using the Leech lattice mod 3. Nevertheless the observation that this module gives rise to a 3-transposition group seems to be due to Zara [16, Prop. 6.38].

Our approach to the study of normal 3-subgroups is basically the same as our approach to the normal 2-subgroup problem. In particular we first identify the factor $\bar{G} = G/O_3(G)$. The primary tool for doing so is the case $p=2$, $q=3$ of the following elegant result of Zara [17, Prop. 9.13]. (Note that the case $p=3$, $q=2$ is contained in (4.3)).

(8.2) Proposition. Let $\{p, q\} = \{2, 3\}$. A non-trivial extension of a $q$-group by the group $W_p(\bar{D}_4)$ or its central quotient $W_p^*(\bar{D}_4) = W_p(\bar{D}_4)/Z(W_p(\bar{D}_4))$ cannot be generated by a conjugacy class of 3-transpositions.

Consider now the case where $O_{3,2}(G)/O_3(G)$ is not central in $\bar{G}$. By Zara’s proposition (8.2), the 3-transposition class of the factor group $G/O_{3,2}(G)$ contains no diagram $D_4$; so our earlier results (applied to $\bar{G}$) imply that $G/O_{3,2}(G)$ is either $S_n$ or $Z_n^{-1} : S_n$, for some $n \geq 3$. Again using Zara’s proposition, it is not difficult to show that, in the first event, $\bar{G} \simeq W(D_n)$ and, in the second, of necessity $n = 3$ and $\bar{G} \simeq 2^{1+6} : SU_3(2)'$. As already mentioned, all these possibilities do occur.

The next case is that of groups in which $O_{3,2}(G)/O_3(G)$ is central in $\bar{G}$. With the remarks of the previous paragraph in hand, examination of the 2-local subgroups of the groups on Fischer’s list shows that here the only possibilities for $\bar{G}/Z(\bar{G})$ are those which are expected—the reflection groups
and $SU_3(2)$. This was noted by Zara [16, 17]. The cases with $G/Z(G)$ equal to $\Omega_8^+(2): S_3$ or $\Omega_8^+(3): S_3$ were handled by Zara in [16, Théorème 8.19] but not in [17]. These cannot occur. Both groups have 3-transposition subgroups $\Omega_8^+(2): S_3$. By examining diagrams of type $\bar{E}_8$, subgroups $O_8^+(2)$ of $\bar{G}/Z(\bar{G})$ lift to subgroups $W(\bar{E}_8) \leq W_3(\bar{E}_8)$ within $G$. The group $W(\bar{E}_8)$ does not admit a triality automorphism, so neither group $\Omega_8^+(p): S_3$ can arise as $G/Z(G)$.

The modules which appear as composition factors $V$ within $O_3(G)$ can be studied as factors of induced modules in a similar fashion to that of Section 6. Here the defining equations are again of the form $[V, d, e] = 1$, for commuting 3-transpositions $d$ and $e$, but now with the restriction that the 3-transpositions must be distinct: $d \neq e$. (This explains why $G/Z(G) \cong SU_5(2)$ must lift to $\bar{G} \cong Z_2 \times SU_5(2)$.) The result is that, with the exceptions noted already, each non-trivial $V$ must be a $GF(3)$-reflection module for $G$.

It remains to discuss the nature of the extension $G$ of $O_3(G)$ by $\bar{G}$. If $\bar{G}$ is symmetric, then a complement can be generated by a set of 3-transpositions with diagram $A_n$; so the extension splits. Therefore, by Zara’s result [16, Théorème 5.20] we may assume that $[O_3(G), G]$ is a $GF(3)G$-module.

(8.3) THEOREM. Let $H$ be a group with $O_3(H) \leq Z(H)$ which is generated by the conjugacy class $T$ of 3-transpositions. Further let $R$ be an elementary abelian 3-group and $\varphi: H \to \text{Aut}(R)$ a homomorphism. Consider a collection of extensions $\Phi_i$, $i \in I$,

$$\Phi_i: 1 \to R \to G_i \to H \to 1$$

with the action of $H$ on $R$ specified by $\varphi$. Then $T$ lifts to a conjugacy class $D_j$ of 3-transpositions in $G_j$, for some $j \in I$, if and only if $T$ lifts to a conjugacy class $D_i$ of 3-transpositions in $G_i$, for each $i \in I$.

Proof: As $R$ is a 3-group, the group $\langle d \rangle R$, for any involution $d$, contains a unique class of involutions. Thus in any event the candidate class $D_i$ is uniquely determined by $t \in T$.

If $|H| = 2$, then all extensions split; so the result is trivial. One direction of the problem is clear in general. Assume now that $|H| > 2$ and that $D_j$ is a class of 3-transpositions in the group $G_j$, for some fixed but arbitrary $j \in I$.

Let $S_4 \cong K \leq H$, and set $Q = O_3(K)$. We first claim that $K$ is trivial on $C_R(Q)$. Let $M$ be the preimage of $K$ in $G_j$ and $A$ a Sylow 2-subgroup of $O_{3,2}(M)$. Consider $N = N_M(A)$. Then $N \cap R = C_R(A) = C_R(Q)$, and $N/N \cap R \cong S_4$.

Here $\langle D_j \cap N \rangle$ has the non-central normal 2-subgroup $A$. By Fischer’s result (2.2) the normal 3-subgroup $\langle [C_R(A), d] \mid d \in D_j \cap N \rangle$ is central in
\( \langle D_j \cap N \rangle \). Therefore, for each \( d \in D_j \cap N \), we have \([C_R(A), d] = [C_R(A), d, d] = 1\). Thus in fact \( N = C_R(A) \times K_\ell \), where \( K_\ell = \langle D_j \cap N \rangle \cong S_4 \). That is, \( K \) acts trivially on \( C_R(A) = C_R(Q) \), as claimed.

Consider now \( d, e \) in the class \( D_i \) of the group \( G_t \). We must prove that \( |de| \) is 1, 2, or 3. Let \( dR = r \) and \( eR = s \), members of \( T \). (Note that we allow the possibility \( r = s \).) As \( O_3(H) \) is central, we may assume that \( r, s \in K \). Let \( L \) be the complete preimage of \( K \) in \( G \), and \( B \) a Sylow 2-subgroup of \( O_{3,3}(L) \).

By previous argument \( K \) is trivial on \( C_R(Q) = C_R(B) \), and so \( N_L(B) = C_R(B) \times K_\ell \) with \( K_\ell \) generated by members of \( D_i \) and isomorphic to \( S_4 \). In particular, \( \Phi_\ell \) splits when restricted to \( K \). As \( \Phi_\ell \) splits when restricted to \( K \) as well, we have \( L \) isomorphic to the subgroup \( M \) above of \( G \). Here \( D_i \cap L \) is the unique involution class not in \( L' \), just as \( D_j \cap M \) is the unique involution class not in \( M' \). Because \( D_j \cap M \) is a 3-transposition class in \( M \), also \( D_i \cap L \) is a 3-transposition class in \( L \). In particular, \( d, e \in D_i \cap L \) have product of order 1, 2, or 3, as required.

The theorem shows that, for normal 3-subgroups, the generation of an extension by a class of 3-transpositions conveys no additional cohomological information. In fact in most cases \( G \) splits off generically. For the Weyl groups of type \( A-D-E \) the Coxeter presentation provides a complement. For \( 2^{1+6} : SU_3(2) ' \) and \( Z_2 \times SU_3(2) \) we can instead use (A.3) and (A.4) to construct complements. This leaves the orthogonal groups in characteristic 3. The results of Küsefoglu \[13, 14]\ show that non-split extensions of a natural module by an orthogonal group can only appear for \( G \) one of \( O_{3,3}(3) \) or \( O_{3,3}(3) \). Such non-split extensions do in fact occur and can be found as 3-local subgroups of \( Fi_{24} \) and the Monster \[2\].

### A. APPENDIX: SOME PRESENTATIONS

We give Coxeter presentations for certain groups in terms of their 3-transposition class. By a Coxeter presentation of a group we mean a presentation of the group as a specified quotient of the Coxeter group with the given diagram. Note that (with a single exception) all the relations given here are of "3-transposition" type. That is, each Coxeter diagram is simply laced and all the additional relations state that a product of two conjugates of the Coxeter generators has order dividing 2 or dividing 3. The single exceptional relation is the one in Proposition A.5 which states that the product \( h \) defined is in fact a conjugate of the generating elements.

These presentations were verified using Leonard Soicher's coset enumeration program *enum* on a Sun 3/60. Each of the first four presentations was also checked by Virotte Ducharme and Zara \[15, 16\] without
the aid of a computer. (Some of the coset enumerations mentioned here are small enough to be done by hand but would be tedious.)

(A.1) PROPOSITION. The following relations on the generators $a$, $b$, and $c$ give a presentation of $SU_3(2)'$ as the factor $W_3(\mathbb{A}_2)$ of the Coxeter group $W(\mathbb{A}_2)$.

\[
(a^b c)^3 = 1.
\]

Proof. See [7, (1.5)].

(A.2) PROPOSITION. The following relations on the generators $a$, $b$, $c$, $d$, and $e$ give a Coxeter presentation of $Z_2 \times SU_4(2)$ in terms of its 3-transposition class.

\[
(b^c d)^3 = 1.
\]

Proof. This is attributed by Fischer [6, p. 12] to Conway and McKay. Alternatively one can find 3-transpositions of $Z_2 \times SU_4(2)$ satisfying the given relations, thereby proving this group a homomorphic image of the presented group. Then an enumeration of the 960 cosets of the subgroup $\langle b, c, d \rangle \cong SU_3(2)'$ of order 54 (see (A.1)) within the presented group shows this homomorphism to be an isomorphism, because $Z_2 \times SU_4(2)$ has order $51840 = 960 \times 54$. (It might be more efficient to enumerate the 80 cosets of the subgroup $\langle a, b, c, d \rangle \cong Z_3^4 : S_4$.)

(A.3) PROPOSITION. The following relations on the generators $a$, $b$, $c$, and $d$ give a Coxeter presentation of the split extension of an extraspecial group $2^{1+6}$ by $SU_3(2)'$ in terms of its 3-transposition class. This is the subgroup of a transvection stabilizer in $SU_5(2)$ which is generated by transvections.

\[
(a^b)^3 = (a^d)^3 = (a^c d^a)^3 = 1.
\]
Proof. This presentation was also noted by Zara [16, 4.97]. The subgroup $2^{1+6} : SU_3(2)'$ generated by all transvections in a transvection stabilizer of $SU_3(2)$ satisfies these relations and so is a homomorphic image of the presented group. In particular, the subgroup $\langle a, b, c \rangle$ is isomorphic to $SU_3(2)'$ by (A.1). Next an enumeration of the $128 = 2^7$ cosets of the subgroup $\langle a, b, c \rangle$ within the presented group proves that the homomorphism is an isomorphism.

(A.4) Proposition. The following relations on the generators $a, b, c, d,$ and $e$ give a Coxeter presentation of $Z_2 \times SU_3(2)$ in terms of its 3-transposition class.

\[(a'b)^3 = (a'd)^3 = (ab'^d)^3 = 1.\]

Proof. This presentation was also noted by Virolette Ducharme [15, Prop. 4–16] and by Zara [16, 6.30].

Consider the following isotropic vectors of the usual unitary space $GF(4)^5$, where $\omega$ is a primitive element in $GF(4)$:

\[
\begin{align*}
a &= (1, 1, \omega, \omega^2, 0); & b &= (0, 0, 0, 1, 1); \\
c &= (0, 0, 1, 1, 0); & d &= (0, 1, 1, 0, 0); \\
e &= (1, 1, 0, 0, 0).
\end{align*}
\]

The transvections of $SU_3(2)$ with these vectors as centers satisfy the relations of the proposition. Therefore, $Z_2 \times SU_3(2)$ is a homomorphic image of the presented group.

Enumeration of the cosets of the subgroup $\langle a, b, c, d \rangle$ within the group $\langle a, b, c, d, e \rangle$ counts 3960 cosets. Therefore, by (A.3) the presented group has order at most that of $Z_2 \times SU_3(2)$, confirming isomorphism.

(A.5) Proposition. The following relations on the generators $a, b, c, d, e, f,$ and $g$ give a Coxeter presentation of an extension $K$ of an elementary abelian 2-group by $SU_3(2)$. The conjugacy class of the Coxeter generators contains $10836 = 4 \times 2709$ elements, 2709 being the number of transvections in $SU_3(2)$. The subgroup $O_3(K)$ is a natural module $GF(2)^{14}$ for $SU_3(2)$, and the extension of $O_3(K)$ by $SU_3(2)$ is non-split. (Note that $h$ and $x$ are not generators but are defined in terms of the other generators. The relations
suggested for \( x \) by the diagram are in fact consequences of the other relations.)

\[
(a^b)^3 = (a^c d)^3 = (a^b c^d)^3 = 1; \\
h = ((a c b)^{-2}(a c d)^2) ^2; \\
x = h e f a b c d e d c e d. \\
(ax)^2 = 1.
\]

**Proof.** A coset enumeration locates the \( 32508 = 12 \times 2709 \) cosets of the subgroup

\[ H = \langle a, b, c, d, e, f \rangle = \langle a, b, c, d, e, f, h \rangle \]

within the group

\[ K = \langle a, b, c, d, e, f, g, h, x \rangle. \]

Consider the following isotropic vectors from the usual unitary space \( GF(4) \):

\[
\begin{align*}
\mathbf{a} &= (1, 1, 1, 1, \omega, \bar{\omega}, 0); \\
\mathbf{b} &= (0, 0, 0, 0, 0, 1, 1); \\
\mathbf{c} &= (0, 0, 0, 0, 1, 1, 0); \\
\mathbf{d} &= (0, 0, 0, 1, 1, 0, 0); \\
\mathbf{e} &= (1, 0, 1, 0, 0, 0); \\
\mathbf{f} &= (0, 1, 0, 0, 1, 1, 1); \\
\mathbf{g} &= (0, 0, 0, 0, 0, 0, 0); \\
\mathbf{h} &= (0, 0, 0, 1, 1, 1, 1); \\
\mathbf{x} &= (0, 1, 1, 1, 1, 1). 
\end{align*}
\]

The transvections of \( SU_7(2) \) with these vectors as centers satisfy the relations of the proposition. Therefore, \( SU_7(2) \) is a homomorphic image of the presented group \( K \) with Coxeter generators being mapped to transvections. Note that \( \mathbb{Z}_2 \times SU_7(2) \) is not an image. Indeed by the diagram all the generators are conjugate in \( K \), and the relations force \( h \) to be in the subgroup generated by all squares; hence \( G \) is perfect.

Before proceeding, it is perhaps appropriate to motivate this presentation. It is an attempt to present \( SU_7(2) \) in terms of its class of 3-transpositions, that is, transvections. Its failure to succeed in this goal then reveals the non-split extension which itself is generated by 3-transpositions.
The diagram and the first line of relations show that the subgroup \( S = \langle a, b, c, d, e \rangle \) is either \( SU_3(2) \) or \( Z_2 \times SU_3(2) \) by (A.4). In fact the relation defining \( h \) as a member of \( S \) shows that we must have the first possibility. Next the diagram gives \( \langle b, c, d, e, f, g, h \rangle \) as an expected subgroup \( Sp_6(2) \) or \( Z_2 \times Sp_6(2) \cong W(E_7) \) of \( SU_3(2) \). The relation defining \( x \) then locates within this subgroup an element which extends the diagram of these generators to \( \tilde{E}_7 \). (Note that we are not really considering \( x \) to be part of the original diagram.) The final relation guarantees that the element \( a \) centralizes the correct portion of the symplectic subgroup.

Let \( M \) be the kernel of the homomorphism onto \( SU_7(2) \), and set \( \bar{K} = K/M \). Let \( Y \) be the conjugacy class of the Coxeter generators of \( K \).

Note that the subgroup \( H \) is contained in \( \langle C_K(x) \cap Y \rangle \). Within \( S \) we can find a subgroup \( Z \) of order 3 which centralizes \( h \) but is not contained in \( \langle C_K(h) \cap Y \rangle \) by consideration of \( \bar{K} \). Therefore,

\[
\langle H, Z^{efgcdebcdefedcd} \rangle \leq C_K(x)
\]

of index in \( K \) dividing 10836 = 32508/3. As \( \bar{K} \) contains 2709 transvections, we conclude that \( |Y| = 2709k \), where \( k = 1, 2, \) or \( 4 \); and for each \( y \in Y \), we have \( |Y \cap yM| = k \).

By [8] the only central extension of \( SU_7(2) \) which is a quotient of perfect \( K \) is \( SU_7(2) \) itself. In particular the case \( k = 1 \) cannot occur, as otherwise the index of \( H \) in \( K \) would equal \( 3 \times 2709 \).

We consider \( K \) as a permutation group on \( Y \), the kernel of this action being \( Z(K) \), a subgroup of \( M \). If \( k = 2 \), then \( M/Z(K) \) is an elementary abelian 2-group \( V \) on which \( \tilde{K} \approx SU_7(2) \) acts with \( \tilde{y} \) satisfying \( [[V, \tilde{y}]] = 2 \), for each \( y \in Y \). As \( SU_7(2) \) is not a \( GF(2) \)-transvection group, this cannot be the case. We must have \( k = 4 \); so \( |Y| = 4 \times 2709 = 10836 \) and \( |Y \cap yM| = 4 \).

The stabilizer of \( Y \cap xM \) induces a transitive subgroup of \( S_4 \) on \( Y \cap xM \) and contains \( C_K(x) \). Therefore, \( S \) acts trivially on \( Y \cap xM \), and as a consequence \( [M, y, w] = 1 \), whenever \( y, w \in Y \) with \( [\tilde{y}, \tilde{w}] = 1 \).

The action of \( K \) on \( Y \) now gives us an isomorphism of \( K/Z(K) \) into the wreathed product \( S_4 \wr S_{2709} \). Here \( M/Z(K) \) is the image in the base group \( (S_4)^{2709} \). As \( |M:C_M(y)| \) divides 4, the image group \( M/Z(K) \) must be a 2-group. Indeed by the earlier remarks on central extensions, \( M \) itself must be a 2-group; that is, \( M = O_2(K) \). By (6.5) the module \( M/\Phi(M) \) is a natural module \( GF(2)^{14} \) for \( \tilde{K} \approx SU_7(2) \). Every coset of \( \Phi(M) \) contains a unique member of \( Y \), so \( \Phi(M) \leq Z(K) \).

By (2.8), a split extension of \( 2^{14} \) by \( SU_7(2) \) has exactly \( 2^{14} \) complements. If the present extension were to split, then the seven generating transvections \( a, b, c, d, e, f, g \) would have to lie within and generate one of the complements, which is not the case. Therefore, the extension of \( 2^{14} \) by \( SU_7(2) \) is non-split.
Finally we note that $Z(K) = 1$. Otherwise by (2.8) there would be a quotient of $K$ having the form $2^{1+14} \cdot SU_7(2)$. This forces the automorphism group of the extraspecial group $2^{1+14}$ of Witt type $+\cdot$ to contain a non-split extension $2^{14} \cdot SU_7(2)$, where the action of $SU_7(2)$ is uniquely determined. But this subgroup must be a split extension, as can be seen within a transvection stabilizer of $SU_6(2)$.

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REFERENCES

5. U. DEMPWOLFF, On extensions of elementary abelian 2-groups by $\Sigma_n$, Glasnik Math. 34 (1979), 35–40.
6. B. FISCHER, Finite groups generated by 3-transpositions, University of Warwick Lecture Notes, Univ. of Warwick, 1969.