

Patching and Birationality

STEVEN E. LANDSBURG

Department of Mathematics, Colorado State University, Fort Collins, Colorado

Communicated by Richard G. Swan

Received September 13, 1991

1. INTRODUCTION

Our object of study will be a commutative square of commutative rings

$$\begin{array}{ccc}
 R & \xrightarrow{i_R} & R' \\
 \phi \downarrow & & \downarrow \phi' \\
 S & \xrightarrow{i_S} & S'
 \end{array} \tag{1}$$

in which R is always assumed to be noetherian.

1.1. *Milnor Patching.* A collection of *patching data* for (1) is a triple

$$(P_{R'}, P_S, \alpha: P_{R'} \otimes_{R'} S' \xrightarrow{\sim} S' \otimes_S P_S),$$

where $P_{R'}$ is a finitely generated projective R' -module, P_S is a finitely generated projective S -module, and α is an isomorphism of S' -modules.

The category of finitely generated projective R -modules maps functorially to the category of patching data via

$$F: P \mapsto (P \otimes_R R', S \otimes_R P, \alpha)$$

with α the obvious identification induced by $(p \otimes 1) \otimes 1 \mapsto 1 \otimes (1 \otimes p)$.

We say that (1) is a *Milnor Patching Diagram* if F is an equivalence of categories, with inverse equivalence given by

$$\begin{aligned}
 G: (P_{R'}, P_S, \alpha) \\
 \mapsto \{(p_{R'}, p_S) \mid p_{R'} \in P_{R'}, p_S \in P_S, \text{ and } \alpha(p_{R'} \otimes 1) = 1 \otimes p_S\}.
 \end{aligned}$$

The techniques of [M, Chap. 2] show that (1) is a Milnor Patching Diagram if and only if the following two statements hold:

- (a) (1) is a pullback in the category of commutative rings and
- (b) For any patching data (P_R, P_S, α) , the R -module $G(P_R, P_S, \alpha)$ is projective, and $F \circ G \approx 1$.

The techniques of [M, Chap. 2] show in addition that property (b) follows from

- (b') For every $n > 0$ and every $\alpha \in GL_n(S')$, there exists $k > 0$ and $\beta \in GL_k(S')$ and a factorization

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = A \cdot B$$

where A lifts to $GL_{n+k}(R')$ and B lifts to $GL_{n+k}(S)$.

In [PAI], we called (1) a *strong Milnor Patching Diagram* if it is a Milnor Patching Diagram and if (b') always holds with $k = n$ and $\beta = \alpha^{-1}$. In this paper, whenever we demonstrate that a diagram has the Milnor Patching property, we will do so by demonstrating that it has the strong Milnor Patching property; that is, we will show that it satisfies (a) and this strong form of (b').

1.2. Goals of This Paper. In [MPP], we gave a partial classification theorem for Milnor Patching Diagrams. There are two errors in that paper that should be noted. First, the reduction to the case where R is a domain is incorrect. Therefore, Ref. [MPP, Theorem 2.6] requires the additional hypothesis that R is a domain. Second, Ref. [MPP, Theorem 3.2] also requires an additional hypothesis, namely: R contains no maximal ideal M such that R_M is a regular local ring of dimension 2. Proposition 2.1 of the present paper is a corrected version of [MPP, Theorem 3.2], making use of that additional hypothesis.

At the same time, it is now possible to eliminate some of the hypotheses that are explicitly stated in [MPP]. For example, Ref. [MPP] imposes conditions on the height one ideals of R . For another example, Ref. [MPP] assumes that $S' = R' \otimes_R S$.

In the present paper, we will develop necessary and sufficient conditions for Milnor Patching that avoid these restrictive hypotheses. The resulting theorems generalize essentially all of the results in [MPP] that are correct as stated. Moreover, the conditions developed in the present paper are simpler and more intuitive than those of [MPP]. The same can be said of the techniques of proof.

Some of the results in this paper follow from those of [MPP]. However, this paper is self-contained and renders [MPP] obsolete.

For reasons that are explained in the paragraphs below, we will focus our attention on the case where R is a domain and R' is contained in the

quotient field of R . In this case we call R' a *birational extension* of R . Any such R' can be written in the form $R' = R[\{f_i/g_i\}]$, where i runs over a set of arbitrary cardinality. The multiplicative set D generated by the g_i will be called a set of *denominators* for R' . (Note that a single extension R' can have more than one set of denominators.)

Let $\phi: R \rightarrow S$ be a map. If the g_i can be chosen to be non-zero-divisors on S , then we call the set D a set of *allowable denominators* for ϕ and we define $S[R']$ to be that subring of the total quotient ring of S that is generated by S and all $\phi(f)/\phi(g)$. In this case we say that $S[R']$ is *defined*.

Our first goal is to demonstrate that it is interesting to consider the class of diagrams (1) in which R' is a birational extension of R . Our second goal is to determine the conditions under which such a diagram satisfies Milnor Patching.

The first goal is addressed by Theorem 1.3 below, while the second is addressed by Theorem 3.3, which is the main theorem of the paper.

THEOREM 1.3. *Suppose that (1) is a Milnor Patching Diagram and that R contains no maximal ideal M such that R_M is a regular local ring of dimension 2. Then:*

- (i) *There is a split surjection $R' \otimes_R S \twoheadrightarrow S'$.*
- (ii) *Suppose in addition that R , R' and S are domains, and that the map from (i) above is an isomorphism. Then at least one of R' and S is a birational extension of R . (In applications, we will always assume without loss of generality that R' is a birational extension.)*
- (iii) *Suppose conversely that R is a domain, that R' is a birational extension of R , and that $S[R']$ is defined. Let D be a set of allowable denominators. Then there are surjective maps*

$$R' \otimes_R S \twoheadrightarrow S' \twoheadrightarrow S[R']$$

and the kernels of these maps are all D -torsion. In particular, if there is no D -torsion in $R' \otimes_R S$, then both of the maps are isomorphisms.

The proof will occupy Section 2.

Theorem 1.1 suggests that for many Milnor Patching Diagrams, R' is a birational extension of R and $S' = S[R']$. In Section 3, we will examine this class of diagrams and state our main theorem (Theorem 3.3) which gives necessary and sufficient conditions for such diagrams to satisfy Milnor Patching. Sections 4–6 contain the proof of Theorem 3.3. Section 7 contains some counterexamples and open questions.

I thank Ray Heitmann for listening to results in progress.

2. SOME CONSEQUENCES OF MILNOR PATCHING

In this section we will develop some consequences of Milnor Patching for an arbitrary diagram (1). The culmination will be the proof of Theorem 1.3.

PROPOSITION 2.1. *Suppose that (1) is a Milnor Patching Diagram. Suppose also that R contains no maximal ideal M such that R_M is a regular local ring of dimension 2. Let $J \subset R$ be any ideal such that $JR' = R'$ and $JS = S$. Then $J = R$.*

Proof. Assuming the contrary, we replace J with a maximal ideal containing J , so we take J to be maximal.

Let $J = (j_1, \dots, j_n)$. Consider the following diagram, in which the left column is defined by taking kernels:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K & \longrightarrow & R^n & \xrightarrow{(j_1, \dots, j_n)} & R & \longrightarrow & R/J \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K_R \times K_S & \longrightarrow & (R')^n \times S^n & \longrightarrow & R' \times S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K_{S'} & \longrightarrow & (S')^n & \longrightarrow & S' & \longrightarrow & 0
 \end{array}$$

Note that exactness of the second and third columns is a consequence of the pullback property, which is in turn a consequence of Milnor Patching. Exactness of the first column follows.

Now $K_R, K_S,$ and $K_{S'}$ are projective over $R', S,$ and S' , so that K is projective over R by Milnor Patching. This implies that the projective dimension of R/J as an R -module is at most 2. Projective dimension 2 is ruled out by hypothesis; therefore the projective dimension is at most 1. It follows that J is projective as an R -module, so that R_J is a DVR.

Let $j \in J$ map to a generator for the maximal ideal in R_J . Then j maps to units in $R' \otimes_R R_J$ and $S \otimes_R R_J$, and consequently (by the pullback property for (1)) to a unit in R_J . In other words, J is the unit ideal in R .

Q.E.D.

Remark. The hypothesis “ R contains no maximal ideal M such that R_M is a regular local ring of dimension 2” can be replaced by the hypothesis

“the addition map $R' \times S \rightarrow S'$ is onto” since in that case we have a diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R & \xrightarrow{(j_1, \dots, j_n)} & R'' & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R' \times S & \longrightarrow & (R')'' \times S'' & \longrightarrow & M_1 \times M_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S' & \longrightarrow & (S')'' & \longrightarrow & M_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

in which the final column is defined by taking cokernels. M is projective by Milnor patching, so the top row splits, which proves that (j_1, \dots, j_n) generates the unit ideal.

STANDING HYPOTHESIS. *In order to apply Proposition 2.1, I will assume throughout Section 2 that R contains no maximal ideal M such that R_M is a regular local ring of dimension 2. In order to avoid repetitiveness, I will not explicitly restate the assumption each time it is used. However, it is worth remarking that I know of no counterexample to the assertion that this hypothesis can be eliminated whenever it is used.*

COROLLARY 2.2. *Suppose that (1) is a Milnor Patching Diagram. Let $M \subset R$ be any maximal ideal and consider the localized diagram*

$$\begin{array}{ccc}
 R_M & \longrightarrow & R' \otimes_R R_M \\
 \downarrow & & \downarrow \\
 S \otimes_R R_M & \longrightarrow & S' \otimes_R R_M.
 \end{array} \tag{2}$$

Let $J \subset R_M$ be any ideal such that $J \cdot (R' \otimes_R R_M) = R' \otimes_R R_M$ and $J \cdot (S \otimes_R R_M) = S \otimes_R R_M$. Then $J = R_M$.

Proof. Write $J = (j_1/1, \dots, j_n/1)$ for some $j_i \in R$. Let $J' \subset R$ be the ideal (j_1, \dots, j_n) . The hypothesis guarantees that there exist $m_1, m_2 \in M$ such that $(J', m_1)R' = R'$ and $(J', m_2)S = S$. It follows from this and Proposition 2.1 that $(J', m_1, m_2) = R$. Consequently some element of J' is not contained in M , which is what is needed. Q.E.D.

PROPOSITION 2.3. *Suppose that (1) is a Milnor Patching Diagram. Then the addition map*

$$R' \times S \rightarrow S'$$

is onto.

Proof. Choose $s' \in S'$. Choose free modules of rank 2 over R' and S , choose bases for these free modules, and consider the projective R -module P that results when these modules are patched along the S' -isomorphism represented by the matrix

$$\alpha = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \in GL_2(S').$$

Let M be any maximal ideal in R . Write α_M for the image of the matrix α in $GL_2(S' \otimes_R R_M)$. Since P_M is a free R_M module, it follows that α_M splits as the product of matrices that lift to elements of $GL_2(R' \otimes_R R_M)$ and $GL_2(S \otimes_R R_M)$. Write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 1 & s'/1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \tag{3}$$

for some

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \phi'(GL_2(R' \otimes_R R_M))$$

and

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix} \in i_S(GL_2(S \otimes_R R_M)).$$

Note that the pullback property for (1) implies the pullback property for the localized diagram (2). From this and Eq. (3), we see that $A \in R' \otimes_R R_M$ and $E \in S \otimes_R R_M$ lift simultaneously to $A_0 \in R_M$; similarly C and G lift simultaneously to $C_0 \in R_M$. Moreover, the image of the ideal (A_0, C_0) generates the unit ideal in both $R' \otimes_R R_M$ and $S \otimes_R R_M$; consequently (by Corollary 2.2), A_0 and C_0 generate the unit ideal in R_M . Write

$$\tilde{r}A_0 + \tilde{r}'C_0 = 1 \tag{4}$$

for some $\tilde{r}, \tilde{r}' \in R_M$.

From Eq. (3) we have

$$\phi'(A) \cdot (s'/1) + B = E \in i_S(S \otimes_R R_M)$$

$$\phi'(C) \cdot (s'/1) + D = F \in i_S(S \otimes_R R_M).$$

Using Eq. (4), this yields

$$s'/1 = (\tilde{r}E + \tilde{r}'F) - (\tilde{r}B + \tilde{r}'D) \in i_S(S \otimes_R R_M) + \phi'(R' \otimes_R R_M).$$

But the maximal ideal at which we localized was arbitrary, so

$$s' \in i_S(S) + \phi'(R')$$

as needed.

Q.E.D.

PROPOSITION 2.4. *Suppose that (1) is a Milnor Patching Diagram. Let P be any prime ideal of R , $k = k(P)$ the residue field of R at P , $k_T = k \otimes_R T$ for $T = R', S$, or S' , and d_T the dimension of k_T as a vector space over k . Then either*

- (i) $d_{R'} + d_S = d_{S'}$
- (ii) $d_{R'} + d_S = d_{S'} + 1$.

Moreover, if P is either the zero ideal or an intersection of maximal ideals, then (ii) holds.

Proof. From Proposition 2.3 and the pullback property, we have an exact sequence

$$0 \rightarrow R \rightarrow R' \oplus S \rightarrow S' \rightarrow 0 \tag{5}$$

and consequently an exact sequence

$$k \rightarrow k_{R'} \oplus k_S \rightarrow k_{S'} \rightarrow 0. \tag{6}$$

It follows immediately that if $k \rightarrow k_{R'} \oplus k_S$ is not injective then (i) holds, and if $k \rightarrow k_{R'} \oplus k_S$ is injective then (ii) holds.

In case $P = 0$, k is flat over R . Since (6) results from tensoring (5) over R with k , it follows that $k \rightarrow k_{R'} \oplus k_S$ is injective and (ii) holds.

In case P is an intersection of maximal ideals, we prove (ii) by supposing the contrary; i.e., suppose that $k \rightarrow k_{R'} \oplus k_S$ is not injective (and consequently is the zero map). This means that there exist $t_1, t_2 \notin P$ such that $i_{R'}(t_1) \in PR'$ and $\phi(t_2) \in PS$. Put $t = t_1 t_2$. Then for any $x \in R$, $(P, 1 - xt)$ generates the unit ideal in both R' and S . By Proposition 2.1, it follows that $(P, 1 - xt) = R$ for any x , i.e., t is in every maximal ideal containing P and hence in P itself, a contradiction. Q.E.D.

COROLLARY 2.5. *Suppose that (1) is a Milnor Patching Diagram. Suppose also that $S' = R' \otimes_R S$ and that the maps from R' and S to S' are the obvious ones. Let P be any prime in R . Then in the notation of 2.4, either*

- (i) $k_{R'} = k_S = 0$ or
- (ii) at least one of the maps $k \rightarrow k_T$ ($T = R', S$) is an isomorphism.

Moreover, if P is either the zero ideal or an intersection of maximal ideals, then (ii) holds.

Proof. In the notation of Proposition 2.4, the hypothesis implies that $d_{S'} = d_{R'} d_S$. The surjectivity on the right side of (6) shows that $d_{R'}$ and d_S cannot both exceed 1. Now statement 2.4(i) implies $d_{R'} = d_S = 0$, which is Corollary 2.5(i), and statement 2.4(ii) implies $d_{R'} = 1$ or $d_S = 1$, which is Corollary 2.5(ii). Q.E.D.

COROLLARY 2.6. *Suppose that (1) is a Milnor Patching Diagram. Suppose also that $S' = R' \otimes_R S$ and that the maps from R' and S to S' are the obvious ones. Finally, assume that R, R' and S are integral domains. Then at least one of R' and S is contained in the quotient field of R .*

Proof. Apply Corollary 2.5 to the prime $P=0$. Q.E.D.

PROPOSITION 2.7. *Suppose that (1) is a Milnor Patching Diagram. Suppose also that R is a domain with quotient field K and that the map $K \rightarrow K \otimes_R R'$ is an isomorphism. Then the map $S \rightarrow S'$ is injective.*

Proof. Applying Proposition 2.4 to the prime $P=0$ and noting that $d_{R'} = 1$, we have $d_S = d_{S'}$. In other words the map $K \otimes_R S \rightarrow K \otimes_R S'$ is an isomorphism.

Thus if s maps to $0 \in S'$, then there exists a non-zero $a \in R$ with $as = 0 \in S$. Note also that the pullback property implies the existence of $r \in R$ such that $\phi(r) = s \in S$ and $i_{S'}(R) = 0 \in R'$. It follows that ar maps to zero in both S and R' and so $ar = 0$, whence $r = 0$ and $s = 0$. Q.E.D.

COROLLARY 2.8. *Suppose that (1) is a Milnor Patching Diagram. Suppose also that R is a domain with quotient field K and that R' is a birational extension of R with set of denominators D . Then the kernel of the map $R' \otimes_R S \rightarrow S$ consists entirely of D -torsion.*

Proof. Let ξ be in the kernel. Consider the sequence of maps

$$S \rightarrow R' \otimes_R S \rightarrow S'$$

Note that for some $d \in D$, $d \cdot \xi$ lifts to S . By Proposition 2.7, $d \cdot \xi = 0$. Q.E.D.

2.9. *Proof of 1.3.* Statement (i) is an immediate consequence of Proposition 2.3 and the pullback property. Statement (ii) is Corollary 2.6. Statement (iii) follows from Corollary 2.8, once it is noted that under the hypotheses of the statement, the kernel of the map $R' \otimes_R S \rightarrow S[R']$ is precisely equal to the set of all D -torsion in $R' \otimes_R S$. Q.E.D.

Remark. The hypothesis that R is noetherian is used only to ensure the finite generation of J in the proof of Proposition 2.1. The noetherian hypothesis can be dropped at the cost of complicating the hypothesis that R contains no maximal ideal M such that R_M is a regular local ring of dimension 2. The new hypothesis would replace the phrase “maximal ideal M ” with “ideal M that is maximal among finitely generated ideals.” Proposition 2.1 would then require that J be finitely generated, which suffices for the application in the proof of Proposition 2.3.

3. BIRATIONAL EXTENSIONS

Corollary 2.6 suggests that it is interesting to study the class of diagrams (1) in which R is a domain and R' is a birational extension of R . The corollary tells us that many Milnor Patching Diagrams are necessarily of this form. For expositional reasons, we will restrict attention to a slightly smaller class of diagrams, which we now describe.

Assumptions and Notation 3.1. Let R be an integral domain, $I \subset R$ an ideal, g a non-zero element of R , and $R' = R[I/g]$ the subring of the quotient field of R that is generated by R and all f/g with $f \in I$. (Note that any birational extension that is finitely generated as an algebra is of this form.)

Let $R \xrightarrow{\phi} S$ be a ring homomorphism such that $\phi(g)$ is not a zero divisor on S . Let $S[I/g] = S[R']$ be the subring of the total quotient ring of S that is generated by S and all $\phi(f)/\phi(g)$ with $f \in I$.

Let $\phi' : R[I/g] \rightarrow S[I/g]$ be the obvious induced map.

We will consider the diagram

$$\begin{array}{ccc}
 R & \longrightarrow & R[I/g] \\
 \phi \downarrow & & \downarrow \phi' \\
 S & \longrightarrow & S[I/g]
 \end{array} \tag{7}$$

and will determine the conditions under which it satisfies Milnor Patching.

Further Notation 3.2. Consider the sequence of R -modules

$$(I, g)/(g) \rightarrow (I, g)^2/(g^2) \rightarrow (I, g)^3/(g^3) \rightarrow \dots,$$

where each arrow is induced by multiplication by g . Let A_0 be the direct limit

$$A_0 = \varinjlim_k (I, g)^k/(g^k).$$

More generally, fix a non-negative integer n , consider the sequence

$$(I, g)^{n+1}/(g^{4n+1}) \rightarrow (I, g)^{n+2}/(g^{4n+2}) \rightarrow (I, g)^{n+3}/(g^{4n+3}) \rightarrow \dots,$$

where each arrow is multiplication by g , and let A_n be the direct limit

$$A_n = \varinjlim_k (I, g)^{n+k}/(g^{4n+k}).$$

Similarly, we consider the sequence

$$(I, g)S/gS \rightarrow (I, g)^2S/g^2S \rightarrow (I, g)^3S/g^3S \rightarrow \dots,$$

where again each arrow is induced by multiplication by g . Let B_0 be the direct limit

$$B_0 = \varinjlim_k (I, g)^kS/g^kS.$$

More generally, fix a non-negative integer n , consider the sequence

$$(I, g)^{n+1}S/g^{4n+1}S \rightarrow (I, g)^{n+2}S/g^{4n+2}S \rightarrow (I, g)^{n+3}S/g^{4n+3}S \rightarrow \dots,$$

where each arrow is multiplication by g , and let B_n be the direct limit

$$B_n = \varinjlim_k (I, g)^{n+k}S/g^{4n+k}S.$$

There are maps $A_n \rightarrow B_n$ induced by the map $R \rightarrow S$.

Here is the main theorem:

THEOREM 3.3. *Consider the following three conditions:*

- (a) *The maps $A_n \rightarrow B_n$ are isomorphisms for all n .*
- (b) *The diagram (7) is a Milnor Patching Diagram.*
- (c) *The map $A_0 \rightarrow B_0$ is an isomorphism.*

The following statements holds:

- (i) (a) *implies* (b).
- (ii) *Suppose that R contains no maximal ideal M such that R_M is a regular local ring of dimension 2. Then (b) implies (c).*
- (iii) *Suppose that $I=(f)$ is a principal ideal and g is a non-zero-divisor on S/fS . Then (c) implies (a). Thus, if we add the hypothesis of statement (ii), then the three conditions (a), (b), and (c) are all equivalent.*

The proof will span three sections. In Section 4 we prove part (ii). In Section 5 we prove part (i). In Section 6, we prove part (iii). The assumption that R is noetherian will be used only in the proof of (ii).

4. NECESSARY CONDITIONS FOR MILNOR PATCHING

In this section we will develop some consequences of Milnor Patching. The culmination will be a proof of Theorem 3.3(ii).

The modules A_0 and B_0 that appear in this section are as defined in Notation 3.2.

PROPOSITION 4.1. *Diagram (7) is a pullback square if and only if the map $A_0 \rightarrow B_0$ is injective.*

Proof. Note first that because g is a non-zero-divisor on both R and S , the injectivity of $A_0 \rightarrow B_0$ is equivalent to the injectivity of $(I, g)^k/(g^k) \rightarrow (I, g)^k S/g^k S$ for all k .

Suppose first that the diagram is a pullback, let $r \in (I, g)^k R$, and suppose that $\phi(r) \in g^k S$. Write

$$\phi(r) = g^k s$$

for some $s \in S$. Then $r/g^k \in R[I/g]$ and $s \in S$ have the same image in $S[I/g]$ so r/g^k lifts to R ; in other words $r \in g^k R$ as needed.

Suppose conversely that the maps $(I, g)^k/(g^k) \rightarrow (I, g)^k S/g^k S$ are all injective. Suppose that $r' \in R[I/g]$ and $s \in S$ have the same image in $S[I/g]$. Write $r' = r/g^k$ for some positive integer k and $r \in (I, g)^k$. Then r maps into $g^k S$ and so must be in $g^k R$. In other words, $r/g^k \in R$. Moreover, r/g^k must map to $s \in S$ (using the fact that g is not a zero-divisor on S). This establishes the required pullback property. Q.E.D.

PROPOSITION 4.2. *In diagram (7), the following conditions are equivalent:*

- (a) *The addition map $R[I/g] \times S \rightarrow S[I/g]$ is onto.*
- (b) *The map $A_0 \rightarrow B_0$ is onto.*

Proof. First suppose (a). Let $s \in (I, g)^k S$. Then $s/g^k \in S[I/g]$ so by Proposition 2.3 we can write

$$\frac{s}{g^k} = \frac{r}{g^{k+m}} + \tilde{s}$$

for some $r/g^{k+m} \in R[I/g]$ (i.e., $r \in (I, g)^{k+m}$) and $\tilde{s} \in S$.

Multiplying through by g^{k+m} , we find that $g^m s$ lifts to $(I, g)^{k+m}$ modulo $g^{k+m} S$, which is exactly what is required to prove (b).

The same argument works in reverse to show that (b) implies (a).

Q.E.D.

COROLLARY 4.3. *In Theorem 3.3, statement (ii) is true.*

Proof. Milnor Patching implies the pullback property, which, by Proposition 4.1, implies that $A_0 \rightarrow B_0$ is injective. Under the hypotheses of the statement, Propositions 2.3 and 4.2 combine to show that $A_0 \rightarrow B_0$ is surjective also. Q.E.D.

Remark. If g is allowed to be a zero-divisor on S , Proposition 4.2 remains true (though Proposition 4.1 might not).

5. SUFFICIENT CONDITIONS FOR MILNOR PATCHING

PROPOSITION 5.1. *Suppose in diagram (7) that the map $A_0 \rightarrow B_0$ is an isomorphism and that the maps $A_n \rightarrow B_n$ are surjective for all $n \geq 1$. Then (7) is a Milnor Patching Diagram.*

Proof. By Proposition 4.1, the isomorphism $A_0 \rightarrow B_0$ implies that (7) is a pullback diagram. It therefore suffices to establish condition (b') of Section 1.

Because g is a non-zero-divisor on S , we can safely identify elements of S with their images in $S[I/g]$, and will freely do so.

Let $u \in GL_m(S[I/g])$. Write

$$\begin{aligned} u &= \alpha/g^n \\ u^{-1} &= \beta/g^n \end{aligned}$$

for some positive integer n , and some $m \times m$ matrices α and β with entries in $(I, g)^n S$.

By the surjectivity of $A_n \rightarrow B_n$, there exists a positive integer k , elements r_1 and r_2 in $(I, g)^{n+k}$, and elements s_1 and s_2 in S such that

$$\begin{aligned} g^k \alpha &= r_1 + g^{4n+k} s_1 \\ g^k \beta &= r_2 + g^{4n+k} s_2. \end{aligned}$$

Now note that we can factor

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \phi'(A) \cdot B,$$

where

$$A = \begin{pmatrix} (1 - r_1 r_2 / g^{2n+2k}) / g^n & r_1 / g^{n+k} \\ -r_2 / g^{n+k} & g^n \end{pmatrix} \in GL_{2m}(R[I/g])$$

and

$$B = \begin{pmatrix} \alpha & -1 + g^{2n} s_1 \beta \\ 1 - g^{2n} s_2 \alpha & \beta s_1 \beta + g^{2n} s_2 - g^{4n} s_2 s_1 \beta \end{pmatrix} \in GL_{2m}(S).$$

To check that A and B are invertible, one can explicitly write down their inverses:

$$A^{-1} = \begin{pmatrix} g^n & -r_1 / g^{n+k} \\ r_2 / g^{n+k} & (1 - r_2 r_1 / g^{2n+2k}) / g^n \end{pmatrix}$$

and

$$B^{-1} = \begin{pmatrix} \beta s_1 \beta + g^{2n} s_2 - g^{4n} \beta s_1 s_2 & 1 - g^{2n} \beta s_1 \\ -1 + g^{2n} \alpha s_2 & \alpha \end{pmatrix}. \quad \text{Q.E.D.}$$

Remark. The proof of Proposition 5.1 should be compared to the proof of the Generalized Analytic Isomorphism Theorem in [PAI]. The matrices in [PAI] appear more complicated because they allow for the possibility that the map $S \rightarrow S'$ is not injective. By imitating the full proof from [PAI], Proposition 5.1 can be generalized to the case where g is permitted to be a zero-divisor on S .

It should also be noted that while [PAI] makes the assumption that $S' = R' \otimes_R S$, the arguments there are actually valid under the more general circumstance that S' is any quotient of $R' \otimes_R S$. In view of Proposition 2.3 of the present paper, this is the greatest generality that can be hoped for.

COROLLARY 5.2. *In Theorem 3.3, statement (i) is true.*

6. EQUIVALENT CONDITIONS FOR MILNOR PATCHING

In this section we will prove statement 3.3(iii), completing the proof of Theorem 3.3.

PROPOSITION 6.1. *Let R be a ring, and let f and g be elements that form a regular sequence on R . Then $R[f/g] \approx R[X]/(gX - f)$.*

Proof. There is an obvious surjection $R[X] \rightarrow R[f/g]$, taking X to f/g . Let

$$p(X) = a_0 + a_1 X + a_2 X^2 + \dots + a_n X^n$$

represent an element of the kernel. We will show that p is in the ideal generated by $gX - f$, using induction on n .

Note first that the case $n = 0$ is trivial. In the general case, we have $a_0 g^n + a_1 g^{n-1} f + \dots + a_n f^n = 0$. Since g is not a zero-divisor mod f , we can conclude that $a_0 = f a'_0$ for some a'_0 . Then

$$p(X) = a'_0 \cdot (gX - f) + X \cdot q(X),$$

where $q(X)$ has degree $n - 1$.

Because X does not map to a zero-divisor in $R[f/g]$, it follows that $q(X)$ is in the kernel and consequently in the ideal generated by $(gX - f)$ by induction, which suffices to complete the proof. Q.E.D.

COROLLARY 6.2. *Under the assumptions of Proposition 6.1, the ring $R[f/g]/(gR[f/g])$ is a polynomial ring in one variable over $R/(f, g)$.*

COROLLARY 6.3. *Suppose that $\phi: R \rightarrow S$ is a ring homomorphism, that f and g are elements of R that form a regular sequence on both R and S , and that the map*

$$R[f/g] \times S \rightarrow S[f/g] \tag{8}$$

is a surjection. Then the map

$$R/(f, g) \rightarrow S/(f, g)S$$

is a surjection also.

Proof. Use an overbar to denote reduction mod (f, g) . Reduce (8) mod g and apply Corollary 6.2 to show that the natural map

$$\bar{R}[X] \times \bar{S} \rightarrow \bar{S}[X]$$

is onto, from which the corollary follows. Q.E.D.

PROPOSITION 6.4. *In the notation of Section 3, suppose that the map $A_0 \rightarrow B_0$ is surjective and that the map $R/(I, g) \rightarrow S/(I, g)S$ is surjective. Then the maps $A_n \rightarrow B_n$ are surjective for all n .*

Proof. Given non-negative integers n and k and given $s \in (I, g)^{n+k}S$, we will show:

(*) For all $t > 0$, there exists $m \geq 0$ such that $g^m s \in (I, g)^{n+k+m} + g^{n+k+m+t}S$.

Taking $t = 3n$ in (*) will yield the result. For $t = 0$, (*) simply asserts the surjectivity of $A_0 \rightarrow B_0$, which is hypothesized. Now suppose that (*) is true for t , so that we can write

$$g^m s = r + g^{n+k+m+t} s', \quad r \in (I, g)^{n+k+m}, s' \in S.$$

Since $R/(I, g)$ maps onto $S/(I, g)$, we may write

$$s' = r' + s'', \quad r' \in \phi(R), s'' \in (I, g)S.$$

Since A_0 maps onto B_0 , there exists an l such that

$$g^l s'' = r'' + g^{l+1} s''', \quad r'' \in \phi((I, g)^l), s''' \in S.$$

Putting this together yields

$$\begin{aligned} g^{m+l} s &= g^l r + g^{n+k+m+l+t} s' \\ &= g^l r + g^{n+k+m+l+t} r' + g^{n+k+m+l} r' + g^{n+k+m+l+t+1} s'' \\ &\in (I, g)^{n+k+m+l} + g^{n+k+m+l+t+1} S \end{aligned}$$

so that (*) holds for $t + 1$, on taking $m + l$ for the new value of m . Q.E.D.

COROLLARY 6.5. *In diagram (7), suppose that $I = (f)$ is a principal ideal and that (f, g) is a regular sequence on both R and S . Suppose (in the notation of Section 3) that the map $A_0 \rightarrow B_0$ is onto. Then the maps $A_n \rightarrow B_n$ are onto for all $n \geq 0$.*

Proof. By Proposition 4.2, the map $R[f/g] \times S \rightarrow S[f/g]$ is onto. By Corollary 6.3, the map $R/(f, g) \rightarrow S/(f, g)S$ is also onto. By Proposition 6.4, this suffices. Q.E.D.

COROLLARY 6.6. *Statement (iii) of Theorem 3.3 is true.*

Proof. Under the hypotheses of the statement, we need to show that the maps $A_n \rightarrow B_n$ are all isomorphisms. Surjectivity is established by Corollary 6.5, and injectivity is an easy consequence of injectivity for $A_0 \rightarrow B_0$. Q.E.D.

6.7. Discussion. One would like to show, in Theorem 3.3, that (c) implies (a) in circumstances more general than those of statement 3.3(iii). A variety of special results along these lines are available. The arguments of this section will apply whenever $R[I/g]/gR[I/g]$ and $S[I/g]/gS[I/g]$ are appropriately graded; this happens in many circumstances other than the " $I = (f)$ with (f, g) a regular sequence" circumstances discussed here.

Here is another passing observation, though one of surely limited interest: If I is finitely generated, if $(I, g)/(g) \rightarrow (I, g)S/gS$ is onto, and if $\phi(R) + gS$ happens to be integrally closed in S , then $R/(I, g) \rightarrow S/(I, g)S$ is onto and so is each $A_n \rightarrow B_n$; hence (7) is a Milnor Patching Diagram. (Proof left as exercise.)

7. COUNTEREXAMPLES

Corollary 6.5 and the remarks following establish that in a variety of special circumstances, conditions (a), (b), and (c) of Theorem 3.3 are equivalent. The following example shows that the implication (c) \Rightarrow (b) can fail.

EXAMPLE 7.1. Let R be a UFD and let f and g be relatively prime, irreducible elements of R such that (f, g) is not the unit ideal. Then the diagram

$$\begin{array}{ccc} R & \longrightarrow & R[f/g] \\ \downarrow & & \downarrow \\ S = R[g/f] & \longrightarrow & R[f/g, g/f] \end{array}$$

satisfies condition (c) of Theorem 3.3 (that is, the map $A_0 \rightarrow B_0$ is an isomorphism), but it is not a Milnor Patching Diagram.

Proof. The verification that $A_0 \rightarrow B_0$ is an isomorphism is routine. (The UFD property is used only to establish injectivity.)

If the diagram is Milnor Patching, the unit $f/g \in R[f/g, g/f]$ must split as a product $u \cdot v$, where u is a unit in $R[f/g]$ and v is a unit in $R[g/f]$. But one checks easily that the only units in $R[f/g]$ and $R[g/f]$ are units in R , so such a factorization is impossible. Q.E.D.

One might suspect that the problem with Example 7.1 is related to the fact that $S' = R[f/g, g/f]$ is not equal to the tensor product $R[f/g] \otimes_R R[g/f]$. However, the following example establishes that this is not the only difficulty:

EXAMPLE 7.2. Let $R, f,$ and g be as in Example 7.1, and suppose that R is not 2-dimensional. Then the following diagram satisfies condition (c) of Theorem 3.3, but is not a Milnor Patching Diagram:

$$\begin{array}{ccc} R & \longrightarrow & R[f/g] \\ \downarrow & & \downarrow \\ R[g/f] & \longrightarrow & R[f/g] \otimes_R R[g/f] \end{array}$$

Proof. Let M be a maximal ideal containing (f, g) . Then the two vector spaces $(R[f/g]/(MR[f/g]))$ and $(R[g/f]/(MR[g/f]))$ are both infinite dimensional over R/M . If the diagram is Milnor Patching, this violates Corollary 2.5. Q.E.D.

The examples in this section eliminate the hope that all three conditions of Theorem 3.3 could be equivalent. However, I know of no counterexample to the assertion that conditions (a) and (b) are always equivalent. If this could be established in complete generality, it would go a long way toward completely characterizing all Milnor Patching Diagrams.

REFERENCES

- [M] J. MILNOR, "An Introduction to Algebraic K -Theory," Princeton Univ. Press, Princeton, NJ, 1971.
- [MPP] S. LANDSBURG, The Milnor patching property, *J. Algebra* **74** (1982), 112–123.
- [PAI] S. LANDSBURG, Patching and analytic isomorphisms, *Proc. Amer. Math. Soc.* **114** (1992), 637–639.