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Physics Letters B 566 (2003) 171–174

PHYSICS LETTERS B

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Causality and the Skyrme model

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Received 23 February 2003; received in revised form 7 March 2003; accepted 8 March 2003

Editor: P.V. Landshoff

Abstract

It is shown that despite the possibility of a breakdown of hyperbolicity, the $SU(2)$ Skyrme model can never exhibit bulk violations of Einstein causality because its energy momentum tensor satisfies the dominant energy condition. It also satisfies the strong energy condition. The Born–Infeld–Skyrme model also satisfies both the dominant and strong energy conditions.

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The Skyrme model continues to yield insights into pion and nucleon physics [1]. Initially, the static properties of the model were studied but time-dependent processes are also important. Their study led to the realization that during the time evolution the classical equations of motion, which are highly non-linear, may cease to be hyperbolic [2]. This is potentially worrying because although the equations of motion are relativistically covariant, their non-linear character makes it unclear whether Einstein causality is always maintained during the time evolution. The danger is particularly great during violent processes such as Skyrme–anti-Skyrme annihilation [4]. Presumably if Einstein causality were to fail one would have to give up the model, on the grounds that the various approximations made to obtain it from microphysics, for example, from QCD, can no longer be valid, since those models incorporate Einstein causality as a basic assumption. In this connection it is perhaps interest-

ing to recall the suggestion that superluminal motion of pions may be possible in hadronic fluids [3] when one loop effects at finite temperature are taken into account for the linear sigma model.

The point of this note is to show that, regardless of whether hyperbolicity breaks down, the energy–momentum tensor $T_{\mu\nu}$ of the $SU(2)$ Skyrme model satisfies the dominant energy condition [5] and hence, by a result of Hawking [5–7], causal behaviour is guaranteed.

The dominant energy condition is a pointwise condition on $T_{\mu\nu}$ and states that $-T_{\nu}^{\mu} t^{\nu}$ is future directed timelike or null for every future directed timelike vector t^{μ} . Equivalently $T_{\mu\nu} t^{\nu} s^{\mu} \geq 0$, for all pairs of future directed timelike vectors t^{μ} and s^{μ} . Another equivalent formulation is that in all local Lorentz frames $T_{00} \geq |T_{\mu\nu}|$ for all index pairs $\mu\nu$. Yet another formulation is that if we can diagonalize $T_{\mu\nu}$ relative to the spacetime metric $\eta_{\mu\nu}$, so that $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $T_{\mu\nu} = \text{diag}(T_{00}, T_{11}, T_{22}, T_{33})$, then $T_{00} \geq |T_{11}|$, etc. The quantities T_{11} , etc. are called ‘principle pressures’. Note that one cannot always diagonalize an arbitrary symmetric tensor relative to

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$\eta_{\mu\nu}$ because it is indefinite, but it is possible in the generic case.

The intuitive idea behind this condition is that if it holds, then relative to any local frame, the flow of energy and momentum is future directed timelike or null. In other words, locally matter flows no faster than light. Using this condition Hawking was able to show, for example, that if $T_{\mu\nu}$ vanishes at one time outside a compact set K , then $T_{\mu\nu}$ vanishes outside the future of K . In other words the local condition guarantees that bulk motion is never superluminal. In fact the main aim of Hawking's paper was rather broader. He showed that classical field theories satisfying the dominant energy condition do not permit particle creation ex nihilo even if the background metric is curved and time-dependent. In fact classical models of particle creation in external fields always seem to entail some loss of causality, for instance, particle and anti-particle appearing at spacelike separation [8]. In the quantum theory of course, there is no reason to expect that the dominant energy condition continues to hold and creation of particles can take place. In the case of the Skyrme model one still expects topological conservation laws to hold however. In the context of quantum gravity, this may lead to some constraints on the number of created Skyrmions and whether it is odd or even [9].

A remark that will prove useful in the sequel is that if $T_{\mu\nu}^1$ and $T_{\mu\nu}^2$ are two energy momentum tensors satisfying the dominant energy condition, then so does $aT_{\mu\nu}^1 + bT_{\mu\nu}^2$, for positive a and b . Put more formally, the set of energy momentum tensors satisfying the dominant energy condition form a Lorentz-invariant convex cone in the space of all second rank symmetric tensors.

The Skyrme model is based on a map $\phi^A(t, \mathbf{x})$ from spacetime to a target model M with positive definite metric $G_{AB}(\phi)$, with $A = 1, 2, \dots, \dim M$. In the simplest case M is $SU(2) \equiv S^3$ with the bi-invariant or round metric. A more ambitious model takes $M = SU(3)$. The Lagrangian L and the energy–momentum tensor $T_{\mu\nu}$ are constructed from the pulled back metric $L_{\mu\nu} = G_{AB}(\phi)\partial_\mu\phi^A\partial_\nu\phi^B$. Generically we can diagonalize $L_{\mu\nu}$ relative to $\eta_{\mu\nu}$. The eigenvalues are necessarily non-negative so we write them as $T_{\mu\nu} = \text{diag}(\lambda_0^2, \lambda_1^2, \lambda_1^2, \lambda_3^2)$. The eigen-values $\lambda_1^2, \lambda_2^2, \lambda_3^2$ were introduced by Manton in the static case for which $\lambda_0^2 = 0$ [10].

Consider, to begin with, a non-linear sigma model without Skyrme term. This has

$$L^{\text{sigma}} = -\frac{1}{2}L^\mu{}_\mu, \quad T_{\mu\nu}^{\text{sigma}} = L_{\mu\nu} + \frac{1}{2}L^\tau{}_\tau\eta_{\mu\nu}. \quad (1)$$

In our special frame

$$T_{\mu\nu}^{\text{sigma}} = \text{diag} \frac{1}{2}(\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2, \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2, \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2). \quad (2)$$

By inspection, the sigma model energy–momentum tensor satisfies the dominant energy condition. Although this guarantees causality, it certainly does not guarantee non-singular evolution. Indeed it is known that finite energy initial data can collapse to a singularity in finite time [11]. One even has an explicit solution illustrating this collapse. If χ, α, β are polar coordinates on S^3 , and t, r, θ, ϕ polar coordinates on spacetime, this solution is given, for $t < 0$, by $\chi = 2 \tan^{-1}(-r/t)$ for $r \leq -t$ and $\chi = \frac{\pi}{2}$ for $r > -t$. This represents a one parameter family of inverse stereographic projections.

In order to prevent collapse, one adds an additional term, L^{Skyrme} to the Lagrangian. For $M = SU(2)$, the Skyrme term may be written in suitable units as

$$L^{\text{Skyrme}} = L_{\mu\nu}L^{\mu\nu} - L^\mu{}_\mu L^\nu{}_\nu. \quad (3)$$

The contribution to the energy momentum tensor is

$$T_{\mu\nu}^{\text{Skyrme}} = 4(L_{\mu\nu}L^\sigma{}_\sigma - L_{\mu\lambda}L_\nu{}^\lambda) + \eta_{\mu\nu}L^{\text{Skyrme}}. \quad (4)$$

In the adapted frame

$$T_{00}^{\text{Skyrme}} = 2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_0^2\lambda_1^2 + \lambda_0^2\lambda_2^2 + \lambda_0^2\lambda_3^2), \quad (5)$$

$$T_{11}^{\text{Skyrme}} = 2(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2) + 2\lambda_0^2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2). \quad (6)$$

Clearly $T_{00}^{\text{Skyrme}} \geq |T_{11}^{\text{Skyrme}}|$, which is sufficient to show that $T_{\mu\nu}^{\text{Skyrme}}$ satisfies the dominant energy condition, and hence by the convexity property, so does the complete energy–momentum tensor $T_{\mu\nu} = T_{\mu\nu}^{\text{sigma}} + T_{\mu\nu}^{\text{Skyrme}}$.

Since,

$$L^\mu{}_\mu = -T^{\text{sigma}\mu}{}_\mu,$$

$$L_{\mu\nu} = T^{\text{sigma}}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\text{sigma}\lambda}{}_\lambda, \quad (7)$$

one has the nice formula

$$T_{\mu\nu}^{\text{Skyrme}} = -4T_{\mu\lambda}^{\text{sigma}}T^{\text{sigma}\lambda}{}_\nu + \eta_{\mu\nu}T_{\lambda\rho}^{\text{sigma}}T^{\text{sigma}\lambda\rho}. \quad (8)$$

However this does not immediately suggest a less basis dependent proof that the dominant energy condition holds. An interesting question is whether the breakdown of hyperbolicity is associated with the boundary of the cone. More practically, it might be illuminating to calculate the eigen-values $\lambda_0^2, \lambda_1^2, \lambda_2^2, \lambda_3^2$ during the numerical evolution as a diagnostic tool.

From (5), (6) one

$$T_{00}^{\text{Skyrme}} + T_{11}^{\text{Skyrme}} + T_{22}^{\text{Skyrme}} + T_{33}^{\text{Skyrme}} \geq 0. \quad (9)$$

One easily checks from (2) that (9) also holds for the sigma model energy momentum tensor. It follows that $T_{\mu\nu}$ belongs to the relativistically invariant convex cone of symmetric second rank tensors for which

$$\left(T_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}T^\lambda{}_\lambda\right)t^\mu t^\nu \geq 0, \quad (10)$$

for all future directed timelike or null vectors t^μ . In other words, the strong energy condition [6] is also satisfied. Physically this means that the gravitational field of Skyrme matter is always attractive.

A different extension of the non-linear sigma model Lagrangian to include higher powers of first derivatives is the Born–Infeld–Skyrme model. This has

$$L^{\text{Born-Infeld}} = \sqrt{-\det \eta_{\mu\nu}} - \sqrt{-\det(\eta_{\mu\nu} + L_{\mu\nu})}. \quad (11)$$

One finds that

$$T_{00}^{\text{Born-Infeld}} = \frac{1}{\sqrt{1-\lambda_0^2}} \left(\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)(1+\lambda_3^2)} - 1 \right), \quad (12)$$

and

$$T_{11}^{\text{Born-Infeld}} = 1 - \frac{\sqrt{1-\lambda_0^2}}{1+\lambda_1^2} \sqrt{(1+\lambda_1^2)(1+\lambda_2^2)(1+\lambda_3^2)}. \quad (13)$$

Note that the arbitrary constant in the Lagrangian has been chosen to make it and the energy momentum tensor vanish for a constant field configuration. It follows from (12), (13) that $T_{00}^{\text{Born-Infeld}} \geq |T_{11}^{\text{Born-Infeld}}|$, and so the Born–Infeld–Skyrme energy–momentum tensor also satisfies the dominant energy condition. However, unlike the usual Skyrme model, a scaling argument reveals there are no smooth static solutions with positive energy.

Finally we note that it follows easily from (12) and (13) that the Born–Infeld–Skyrme energy momentum tensor also satisfies the strong energy condition. Thus neither Skyrme matter nor Born–Infeld–Skyrme matter is a suitable candidate for the dark energy currently expanding the universe. To violate the strong energy condition one may shift the arbitrary constant multiple of $g_{\mu\nu}$ in $T_{\mu\nu}$. This looks less arbitrary if one also introduces a potential as one does in the case of tachyons and considers

$$L = -V(\phi)\sqrt{\det(g_{\mu\nu} + L_{\mu\nu})}. \quad (14)$$

Now one finds that this can lead to a period of acceleration [12] but as it stands this is probably not a viable model either for the inflaton or for the dark energy.

Acknowledgements

Some of the ideas in this Letter emerged during unpublished work with Michael Green. I would like to thank him and Nick Manton for many helpful and enjoyable discussions.

References

- [1] T. Gisiger, M.B. Paranjape, Phys. Rep. 306 (1998) 109, hep-th/9812148.
- [2] W.Y. Crutchfield, J.B. Bell, J. Comput. Phys. 110 (1994) 234.
- [3] N. Bilić, H. Nikolić, hep-ph/0301275.
- [4] M.A. Halasz, R.D. Amado, Phys. Rev. D 63 (2001) 05020.
- [5] S.W. Hawking, Commun. Math. Phys. 18 (1970) 301.
- [6] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Spacetime, Cambridge Univ. Press, Cambridge, 1973.

- [7] B. Carter, gr-qc/0205010.
- [8] Ya.B. Zeldovich, in: J. Klauder (Ed.), *Magic Without Magic*, Freeman, San Francisco, CA, 1972.
- [9] G.W. Gibbons, *Class. Quantum Grav.* 10 (1993) L89.
- [10] N.S. Manton, *Commun. Math. Phys.* 111 (1987) 469.
- [11] S. Shatah, *Commun. Pure Appl. Math.* 41 (1988) 459.
- [12] G.W. Gibbons, *Phys. Lett. B* 537 (2002) 1.