Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT•

An International Journal
computers & mathematics with applications

ELSEVIER Computers and Mathematics with Applications 46 (2003) 1827–1837
www.elsevier.com/locate/camwa

Nonnegative Solutions of Singular Boundary Value Problems with Sign Changing Nonlinearities

R. P. AGARWAL
Department of Mathematical Sciences, Florida Institute of Technology
Melbourne, FL 32901
agarwal@fit.edu

S. STANĚK
Department of Mathematical Analysis, Faculty of Science
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
stanek@risc.upol.cz

(Received October 2001; accepted November 2002)

Abstract—The paper presents sufficient conditions for the existence of positive solutions of the equation $x''(t) + q(t)f(t,x(t),x'(t)) = 0$ with the Dirichlet conditions $x(0) = 0$, $x(1) = 0$ and of the equation $(p(t)x'(t))' + p(t)q(t)f(t,x(t),p(t)x'(t)) = 0$ with the boundary conditions $\lim_{t\to 0^+} p(t)x'(t) = 0$, $x(1) = 0$. Our nonlinearity f is allowed to change sign and f may be singular at $x = 0$. The proofs are based on a combination of the regularity and sequential techniques and the method of lower and upper functions. © 2003 Elsevier Ltd. All rights reserved.

Keywords—Singular Dirichlet problem, Singular mixed problem, Positive solution, Sign changing nonlinearity.

1. INTRODUCTION

In this paper, we consider two singular boundary value problems (BVPs for short)

$$x''(t) + q(t)f(t,x(t),x'(t)) = 0, \tag{1}$$

$$x(0) = 0, \qquad x(1) = 0, \tag{2}$$

and

$$(p(t)x'(t))' + p(t)q(t)f(t,x(t),p(t)x'(t)) = 0, \tag{3}$$

$$\lim_{t\to 0^+} p(t)x'(t) = 0, \qquad x(1) = 0, \tag{4}$$

where our nonlinearity f is allowed to change sign and f may be singular at $x = 0$. Singular problems (1),(2) and (3),(4) have been discussed intensively in the literature (see, e.g., [1–4] and

The second author has been supported by Grant No. 201/01/1451 of the Grant Agency of Czech Republic and by the Council of Czech Government J14/98:153100011.

0898-1221/03/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi: 10.1016/S0898-1221(03)00406-1

references therein) usually when f is nonnegative (i.e., positone problems). This assumption has been overcome for BVP (1),(2) in [5–9], for BVP (1),(2) in [10], and for both BVPs in [11,12]. In [5–7,9,10], it is assumed that f is independent of x'. The case where f may depend on the x' variable was considered in [8,11,12].

The aim of this paper is to generalize some assumptions given in [6,11,12] for the solvability of BVPs (1),(2) and (3),(4) in the class of positive functions on $(0,1)$. The solvability is proved by a combination of the regularity and sequential techniques and the method of lower and upper functions. First, by a lower function α and a sequence $\{\beta_n\}$ of upper functions, we define a family of regular BVPs depending on $n \in \mathbb{N}$. Then, using the Schauder fixed-point theorem (see, e.g., [2,13]), we establish the existence of their solutions x_n which lie between α and β_n (Lemmas 1–3). The Arzelà-Ascoli theorem will then complete the solvability of BVP (1),(2) (Theorems 1 and 2) and BVP (3),(4) (Theorem 3).

We say that x is a *solution of BVP* (1),(2) if $x \in C^1([0,1]) \cap C^2((0,1))$, x satisfies the boundary conditions (2), and (1) holds on $(0,1)$.

A function x is said to be a *solution of BVP* (3),(4) if $x \in C^0([0,1]) \cap C^1((0,1))$, $px' \in C^0([0,1]) \cap C^1((0,1))$, x satisfies the boundary conditions (4) and (3) holds on $(0,1)$.

From now on, $\|x\| = \max\{|x(t)| : t \in [0,1]\}$ denotes the norm in the Banach space $C^0([0,1])$ and the derivative on the right (respectively, on the left) of a function x at a point t, we will denote by $x'_+(t)$ (respectively, $x'_-(t)$).

Throughout the paper, we will use the following assumptions.

(H_1) $q \in C^0((0,1))$, $q > 0$ on $(0,1)$.

(H_2) $f \in C^0([0,1] \times (0,\infty) \times \mathbb{R})$.

(H_3) There exists $\alpha \in C^0([0,1]) \cap C^1((0,1))$ having the second derivative on $(0,1)$ with $\alpha(0) = \alpha(1) = 0$, $\alpha > 0$ on $(0,1)$ such that $\sup\{|\alpha'(t)| : t \in (0,1)\} < \infty$, $\alpha''(t) + q(t)f(t,\alpha(t),\alpha'(t)) \geq 0$ for $t \in (0,1)$ and for a decreasing sequence $\{\bar{t}_n\} \subset (0,1)$, $\lim_{n\to\infty} \bar{t}_n = 0$ and an increasing sequence $\{t_n\} \subset (0,1)$, $\lim_{n\to\infty} t_n = 1$, $\bar{t}_1 < t_1$, we have (for $n \in \mathbb{N}$)

$$\begin{aligned} \alpha(t) \leq \alpha(\bar{t}_n), &\quad \text{for } t \in [0,\bar{t}_n], \quad \alpha(t_n) \geq \alpha(t), \quad \text{for } t \in [t_n,1], \\ f(\bar{t}_n,\alpha(\bar{t}_n),0) \geq 0, &\quad f(t_n,\alpha(t_n),0) \geq 0. \end{aligned} \tag{5}$$

(H_4) With $\{\bar{t}_n\}$ and $\{t_n\}$ given in (H_3), for each $n \in \mathbb{N}$, there exists $\beta_n \in C^0([0,1]) \cap C^1((0,1))$ having the second derivative on $(0,1)$ such that

$$\begin{aligned} \beta_n(t) \geq \alpha(\bar{t}_n), &\quad \text{for } t \in [0,\bar{t}_n), \quad \beta_n(t) \geq \alpha(t_n), \quad \text{for } t \in (t_n,1], \\ \beta_n \geq \alpha, &\quad \text{on } [\bar{t}_n,t_n], \end{aligned} \tag{6}$$

$$\beta_n''(t) \leq \begin{cases} -q(t)f(\bar{t}_n,\beta_n(t),\beta_n'(t)), & \text{for } t \in (0,\bar{t}_n), \\ -q(t)f(t,\beta_n(t),\beta_n'(t)), & \text{for } t \in [\bar{t}_n,t_n], \\ -q(t)f(t_n,\beta_n(t),\beta_n'(t)), & \text{for } t \in (t_n,1), \end{cases}$$

and

$$L = \sup\{\|\beta_n\| : n \in \mathbb{N}\} < \infty, \qquad S = \sup\{|\beta_n'(t)| : t \in (0,1),\ m \in \mathbb{N}\} < \infty.$$

(H_5) $p \in C^0((0,1))$, $p > 0$ on $(0,1)$.

(H_6) There exists $\alpha \in C^0([0,1]) \cap C^1((0,1))$ with $p\alpha'$ continuous on $[0,1]$ and differentiable on $(0,1)$, $\lim_{t\to 0^+} p(t)\alpha'(t) \geq 0$, $\alpha(1) = 0$, $\alpha > 0$ on $[0,1)$ such that $(p(t)\alpha'(t))' + p(t)q(t)f(t,\alpha(t),p(t)\alpha'(t)) \geq 0$ for $t \in (0,1)$ and for a decreasing sequence $\{\bar{t}_n\} \subset (0,1)$, $\lim_{n\to\infty} \bar{t}_n = 0$ and an increasing sequence $\{t_n\} \subset (0,1)$, $\lim_{n\to\infty} t_n = 1$, $\bar{t}_1 < t_1$, inequalities (5) hold.

(H_7) With $\{\bar{t}_n\}$ and $\{t_n\}$ given in (H_6), for each $n \in \mathbb{N}$, there exists $\beta_n \in C^0([0,1]) \cap C^1((0,1))$ with $p\beta_n'$ continuous on $[0,1]$ and differentiable on $(0,1)$, $\lim_{t\to 0^+} p(t)\beta_n'(t) \le 0$ such that inequalities (6) and

$$(p(t)\beta_n'(t))' \le \begin{cases} -p(t)q(t)f(\bar{t}_n, \beta_n(t), p(t)\beta_n'(t)), & \text{for } t \in (0, \bar{t}_n), \\ -p(t)q(t)f(t, \beta_n(t), p(t)\beta_n'(t)), & \text{for } t \in [\bar{t}_n, t_n], \\ -p(t)q(t)f(t_n, \beta_n(t), p(t)\beta_n'(t)), & \text{for } t \in (t_n, 1), \end{cases}$$

hold for $n \in \mathbb{N}$ and

$$L_* = \sup\{\|\beta_n\| : n \in \mathbb{N}\} < \infty, \qquad S_* = \sup\{\|p\beta_n'\| : t \in (0,1)\ m \in \mathbb{N}\} < \infty.$$

REMARK 1. From the inequalities $\alpha(t) \le \alpha(\bar{t}_n)$ for $t \in [0, \bar{t}_n]$ and $\alpha(t_n) \ge \alpha(t)$ for $t \in [t_n, 1]$ which appear in (H_3) and (H_6), we deduce that $\alpha'(\bar{t}_n) \ge 0$ and $\alpha'(t_n) \le 0$ for $n \in \mathbb{N}$.

2. SOLVABILITY OF BVP (1),(2)

Let Assumptions (H_2)–(H_4) (or (H_2) and (H_5)–(H_7)) be satisfied. For each $n \in \mathbb{N}$, let $f_n \in C^0([0,1] \times \mathbb{R}^2)$, $\bar{f}_n \in C^0([0,1] \times (0,\infty) \times \mathbb{R})$, and $\Lambda_n \in C^0([0,1])$ be defined by the formulae

$$f_n(t,x,y) = \begin{cases} f(\bar{t}_n, \alpha(\bar{t}_n), y) + \phi(\alpha(\bar{t}_n) - x), & \text{for } (t,x,y) \in [0,\bar{t}_n) \times (-\infty, \alpha(\bar{t}_n)) \times \mathbb{R}, \\ f(\bar{t}_n, x, y), & \text{for } (t,x,y) \in [0,\bar{t}_n) \times [\alpha(\bar{t}_n), \beta_n(t)] \times \mathbb{R}, \\ f(\bar{t}_n, \beta_n(t), y) + \phi(\beta_n(t) - x), & \text{for } (t,x,y) \in [0,\bar{t}_n) \times (\beta_n(t), \infty) \times \mathbb{R}, \\ f(t, \alpha(t), y) + \phi(\alpha(t) - x), & \text{for } (t,x,y) \in [\bar{t}_n, t_n] \times (-\infty, \alpha(t)) \times \mathbb{R}, \\ f(t,x,y), & \text{for } (t,x,y) \in [\bar{t}_n, t_n] \times [\alpha(t), \beta_n(t)] \times \mathbb{R}, \\ f(t, \beta_n(t), y) + \phi(\beta_n(t) - x), & \text{for } (t,x,y) \in [\bar{t}_n, t_n] \times (\beta_n(t), \infty) \times \mathbb{R}, \\ f(t_n, \alpha(t_n), y) + \phi(\alpha(t_n) - x), & \text{for } (t,x,y) \in (t_n, 1] \times (-\infty, \alpha(t_n)) \times \mathbb{R}, \\ f(t_n, x, y), & \text{for } (t,x,y) \in (t_n, 1] \times [\alpha(t_n), \beta_n(t)] \times \mathbb{R}, \\ f(t_n, \beta_n(t), y) + \phi(\beta_n(t) - x), & \text{for } (t,x,y) \in (t_n, 1] \times (\beta_n(t), \infty) \times \mathbb{R}, \end{cases} \tag{7}$$

where

$$\phi(u) = \begin{cases} -1, & \text{for } u < -1, \\ u, & \text{for } |u| \le 1, \\ 1, & \text{for } u > 1, \end{cases} \tag{8}$$

$$\bar{f}_n(t,x,y) = \begin{cases} f(\bar{t}_n, x, y), & \text{for } (t,x,y) \in [0,\bar{t}_n) \times (0,\infty) \times \mathbb{R}, \\ f(t,x,y), & \text{for } (t,x,y) \in [\bar{t}_n, t_n] \times (0,\infty) \times \mathbb{R}, \\ f(t_n, x, y), & \text{for } (t,x,y) \in (t_n, 1] \times (0,\infty) \times \mathbb{R}, \end{cases} \tag{9}$$

and

$$\Lambda_n(t) = \begin{cases} \alpha(\bar{t}_n), & \text{for } t \in [0, \bar{t}_n), \\ \alpha(t), & \text{for } t \in [\bar{t}_n, t_n], \\ \alpha(t_n), & \text{for } t \in (t_n, 1]. \end{cases} \tag{10}$$

Consider the auxiliary regular BVPs

$$x''(t) + q(t)f_n(t, x(t), x'(t)) = 0, \tag{11}$$

$$x(0) = \alpha(\bar{t}_n), \qquad x(1) = \alpha(t_n), \tag{12}$$

depending on $n \in \mathbb{N}$.

We say that x is a *solution of BVP* (11),(12) if $x \in C^1([0,1]) \cap C^2((0,1))$, x satisfies the boundary conditions (12), and (11) holds on $(0,1)$.

LEMMA 1. *Let Assumptions (H_1)–(H_4),*

(A_1) $|f(t,x,y)| \le (g(x)+h(x))\psi(|y|)$ *for* $(t,x,y) \in [0,1] \times (0,L] \times \mathbb{R}$, *where* $g \in C^0((0,L])$ *is nonincreasing and positive on* $(0,L]$, $h \in C^0([0,L])$ *is nonnegative and nondecreasing on* $[0,L]$, *and* $\psi \in C^0([0,\infty))$ *is positive and nondecreasing on* $[0,\infty)$, *and*

(A_2)
$$\int_0^1 q(t)(g(\alpha(t))+h(L))\,dt < \int_\mu^\infty \frac{1}{\psi(s)}\,ds,$$

be satisfied with $\mu = \sup\{|\alpha(t_n) - \alpha(\bar{t}_n)| : n \in \mathbb{N}\}$.

Then, for each $n \in \mathbb{N}$, *there exists a solution* x *of BVP (11),(12) such that*

$$\Lambda_n(t) \le x(t) \le \beta_n(t), \quad |x'(t)| \le K, \qquad \text{for } t \in [0,1], \tag{13}$$

where Λ_n *is given by (10) and the positive constant* K *satisfies the inequality*

$$\int_0^1 q(t)(g(\alpha(t))+h(L))\,ds \le \int_\mu^K \frac{1}{\psi(s)}\,ds. \tag{14}$$

PROOF. Fix $n \in \mathbb{N}$ and let x be a solution of BVP (11),(12). We first prove that $\Lambda_n \le x$ on $[0,1]$. Suppose that the last inequality is not true. Then,

$$\max\{\Lambda_n(t) - x(t) : t \in [0,1]\} = \Lambda_n(t_0) - x(t_0) > 0, \tag{15}$$

where $t_0 \in (0,1)$ since $\Lambda_n(0) - x(0) = \Lambda_n(1) - x(1) = 0$. The next part of the proof is divided into four cases.

CASE 1. Let $t_0 \in (0,\bar{t}_n) \cup (t_n,1)$. Then, $x'(t_0) = \Lambda_n'(t_0) = 0$ and if $t_0 \in (0,\bar{t}_n)$ (the case where $t_0 \in (t_n,1)$ can be treated quite analogously) then

$$x''(t_0) = -q(t_0)\left[f\left(\bar{t}_n, \alpha\left(\bar{t}_n\right), 0\right) + \phi\left(\alpha\left(\bar{t}_n\right) - x(t_0)\right)\right] < -q(t_0) f\left(\bar{t}_n, \alpha\left(\bar{t}_n\right), 0\right) \le 0.$$

Hence, $(\Lambda_n(t) - x(t))''_{t=t_0} = -x''(t_0) > 0$, contrary to (15).

CASE 2. Let $t_0 = \bar{t}_n$. Then, $x'_-(t_0) \le 0$ and $x'_+(t_0) \ge \alpha'(t_0) \ge 0$ (see Remark 1). Thus, $x'(t_0) = 0$, and so

$$x''(t_0) = -q(t_0)[f(t_0,\alpha(t_0),0) + \phi(\alpha(t_0) - x(t_0))] < -q(t_0) f(t_0,\alpha(t_0),0) \le 0. \tag{16}$$

Hence, $\Lambda_n - x$ is decreasing on a left neighbourhood of $t = t_0$, contrary to (15).

CASE 3. Let $t_0 \in (\bar{t}_n, t_n)$. Then, $x'(t_0) = \Lambda_n'(t_0)$ $(= \alpha'(t_0))$ and

$$\begin{aligned} x''(t_0) &= -q(t_0)[f(t_0,\alpha(t_0),\alpha'(t_0)) + \phi(\Lambda_n(t_0) - x(t_0))] \\ &< -q(t_0) f(t_0,\alpha(t_0),\alpha'(t_0)) \le \alpha''(t_0) = \Lambda_n''(t_0), \end{aligned}$$

whence $(\Lambda_n(t) - x(t))''_{t=t_0} > 0$, contrary to (15).

CASE 4. Let $t_0 = t_n$. Then, $x'_-(t_0) \le \alpha'(t_0) \le 0$ (see Remark 1) and $x'_+(t_0) \ge 0$ which yields $x'(t_0) = 0$, and consequently, (16) holds. Hence, $\Lambda_n - x$ is increasing on a right neighbourhood of $t = t_0$, contrary to (15).

We are going to show that $x \le \beta_n$ on $[0,1]$. Suppose, on the contrary, that

$$\max\{x(t) - \beta_n(t) : t \in [0,1]\} = x(t_*) - \beta_n(t_*) > 0. \tag{17}$$

Since $x(0) - \beta_n(0) \le 0$ and $x(1) - \beta_n(1) \le 0$, we see that $t_* \in (0,1)$. Then, $x'(t_*) = \beta_n'(t_*)$ and

$$\begin{aligned} x''(t_*) &= -q(t_*)\left[f\left(\bar{t}_n, \beta_n(t_*), \beta_n'(t_*)\right) + \phi(\beta_n(t_*) - x(t_*))\right] \\ &> -q(t_*) f\left(\bar{t}_n, \beta_n(t_*), \beta_n'(t_*)\right) \ge \beta_n''(t_*), \end{aligned}$$

provided $t_* \in (0, \bar{t}_n)$,

$$\begin{aligned} x''(t_*) &= -q(t_*)[f(t_*, \beta_n(t_*), \beta_n'(t_*)) + \phi(\beta_n(t_*) - x(t_*))] \\ &> -q(t_*) f(t_*, \beta_n(t_*), \beta_n'(t_*)) \geq \beta_n''(t_*), \end{aligned}$$

provided $t_* \in [\bar{t}_n, t_n]$, and finally,

$$\begin{aligned} x''(t_*) &= -q(t_*)[f(t_n, \beta_n(t_*), \beta_n'(t_*)) + \phi(\beta_n(t_*) - x(t_*))] \\ &> -q(t_*) f(t_n, \beta_n(t_*), \beta_n'(t_*)) \geq \beta_n''(t_*), \end{aligned}$$

provided $t_* \in (t_n, 1)$. Hence, $(x(t) - \beta_n(t))''_{t=t_*} > 0$, contrary to (17).

We have verified that $\Lambda_n \leq x \leq \beta_n \leq L$ on $[0, 1]$, and consequently,

$$x''(t) + q(t)\bar{f}_n(t, x(t), x'(t)) = 0, \qquad t \in (0, 1),$$

where $\bar{f}_n$ is defined by (9).

Let $\|x'\| = |x'(\xi)|$ for some $\xi \in [0, 1]$. Since $\alpha(t_n) - \alpha(\bar{t}_n) = x(1) - x(0) = x'(\eta)$, where $\eta \in (0, 1)$, we see that $|x'(\eta)| \leq \mu$. Suppose that $\|x''\| = x'(\xi) > \mu$ (for $\|x'\| = -x'(\xi) > \mu$, we proceed similarly). Then, there exists $\nu \in [0, 1]$ such that $x'(\nu) = \mu$ and $x' > \mu$ on the open interval with the end points ν and ξ. Without loss of generality, we can assume that $\nu < \xi$. Then,

$$x''(t) \leq q(t)(g(x(t)) + h(x(t)))\psi(x'(t)) \leq q(t)(g(\alpha(t)) + h(L))\psi(x'(t)), \qquad t \in (\nu, \xi),$$

and integrating the inequality

$$\frac{x''(t)}{\psi(x'(t))} \leq q(t)(g(\alpha(t)) + h(L)), \qquad t \in (\nu, \xi),$$

from ν to ξ, we get

$$\int_\mu^{\|x'\|} \frac{1}{\psi(s)}\, ds \leq \int_\nu^\xi q(t)(g(\alpha(t)) + h(L))\, dt \leq \int_0^1 q(t)(g(\alpha(t)) + h(L))\, dt.$$

Hence, $\|x'\| \leq K$ by (14). We have proved that any solution x of BVP (11),(12) satisfies inequalities (13).

Let $V = \max\{K, S, \sup\{|\alpha'(t)| : t \in (0, 1)\}\}$,

$$y^* = \begin{cases} V + 1, & \text{for } y > V + 1, \\ y, & \text{for } |y| \leq V + 1, \\ -V - 1, & \text{for } y < -V - 1, \end{cases}$$

and

$$f_n^*(t, x, y) = f_n(t, x, y^*), \qquad (t, x, y) \in [0, 1] \times \mathbb{R}^2.$$

Now, it is easy to check from the Schauder fixed-point theorem that the BVP

$$x''(t) + q(t) f_n^*(t, x(t), x'(t)) = 0 \tag{18}$$

has a solution $x \in C^1([0, 1]) \cap C^2((0, 1))$. Of course, x satisfies (13) and from the definition of f_n^*, it follows that x is a solution of BVP (11),(12). ∎

LEMMA 2. *Let Assumptions (H_1)–(H_4),* $\sup\{q(t) : t \in (0,1)\} = Q < \infty$,

(B_1) $|f(t,x,y)| \le (r(t) + w(t)g(x) + h(x) + |y|)\psi(|y|)$ *for* $(t,x,y) \in [0,1] \times (0,L] \times \mathbb{R}$, *where* $w, r \in C^0([0,1])$ *are nonnegative,* $g \in C^0((0,L])$ *is nonincreasing and positive on* $(0,L]$, $h \in C^0([0,L])$ *is nonnegative and nondecreasing on* $[0,L]$, *and* $\psi \in C^0([0,\infty))$ *is positive and nondecreasing on* $[0,\infty)$, *and*

(B_2)

$$L + h(L) + \int_0^1 (r(t) + w(t)g(\alpha(t)))\,dt < \frac{1}{Q}\int_\mu^\infty \frac{1}{\psi(s)}\,ds,$$

where $\mu = \sup\{|\alpha(t_n) - \alpha(\bar{t}_n)| : n \in \mathbb{N}\}$ *is satisfied.*

Then, for each $n \in \mathbb{N}$, *there exists a solution* x *of BVP (11),(12) satisfying inequalities (13) where* K *is a positive constant such that*

$$L + h(L) + \int_0^1 (r(t) + w(t)g(\alpha(t)))\,ds \le \frac{1}{Q}\int_\mu^K \frac{1}{\psi(s)}\,ds. \tag{19}$$

PROOF. Fix $n \in \mathbb{N}$ and let x be a solution of BVP (11),(12). We can now proceed analogously to the proof of Lemma 1 to verify that $\Lambda_n \le x \le \beta_n \le L$ on $[0,1]$ and $|x'(\eta)| \le \mu$ for some $\eta \in (0,1)$. Let $\|x'\| = |x'(\xi)|$ with a $\xi \in [0,1]$ and let $\|x'\| > \mu$. Then, there exists $\nu \in [0,1]$ such that $|x'(\nu)| = \mu$ and $|x'| > \mu$ on the open interval with the end points ν and ξ. Without restriction of generality, we can assume that $\nu < \xi$ and $x' > \mu$ on $(\nu, \xi]$. Then,

$$\begin{aligned} x''(t) &\le q(t)(r(t) + w(t)g(\alpha(t)) + h(x(t)) + x'(t))\psi(x'(t)) \\ &\le Q(r(t) + w(t)g(\alpha(t)) + h(L) + x'(t))\psi(x'(t)), \end{aligned}$$

for $t \in (\nu, \xi)$ and integrating the inequality

$$\frac{x''(t)}{\psi(x'(t))} \le Q(r(t) + w(t)g(\alpha(t)) + h(L) + x'(t)), \qquad t \in (\nu, \xi),$$

over $[\nu, \xi]$, we have

$$\begin{aligned} \int_\mu^{\|x'\|} \frac{1}{\psi(s)}\,ds &\le Q\left(h(L)(\xi - \nu) + x(\xi) - x(\nu) + \int_\nu^\xi (r(t) + w(t)g(\alpha(t)))\,dt\right) \\ &\le Q\left(h(L) + L + \int_0^1 (r(t) + w(t)g(\alpha(t)))\,dt\right). \end{aligned}$$

Therefore, $\|x'\| \le K$ by (19), and consequently, any solution x of BVP (11),(12) satisfies inequalities (13). Now, by the Schauder fixed-point theorem, there is a solution x of BVP (11),(12) for which (13) holds. ∎

THEOREM 1. *Let Assumptions (H_1)–(H_4), (A_1), and (A_2) be satisfied. Then, there exists a solution x of BVP (1),(2) such that*

$$\alpha(t) \le x(t) \le L, \quad |x'(t)| \le K, \qquad \text{for } t \in [0,1], \tag{20}$$

where K is given in Lemma 1.

PROOF. By Lemma 1, for each $n \in \mathbb{N}$, there exists a solution x_n of BVP (11),(12) satisfying the inequalities

$$\Lambda_n(t) \le x_n(t) \le \beta_n(t) \le L, \quad |x_n'(t)| \le K, \qquad \text{for } t \in [0,1]. \tag{21}$$

Hence, the sequence $\{x_n\}$ is bounded in $C^1([0,1])$ and since

$$|x_n'(t_1) - x_n'(t_2)| = \left| \int_{t_1}^{t_2} q(t)\bar{f}_n(t, x_n(t), x_n'(t))\, dt \right| \le \psi(K) \left| \int_{t_1}^{t_2} q(t)(g(\alpha(t)) + h(L))\, dt \right|,$$

for $t_1, t_2 \in [0,1]$ and $n \in \mathbb{N}$, we deduce that $\{x_n'(t)\}$ is equicontinuous on $[0,1]$. The Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ converging to x in $C^1([0,1])$. Then, $x \in C^1([0,1])$, $x(0) = x(1) = 0$, $x \ge \alpha$, and $|x'| \le K$ on $[0,1]$. In addition,

$$\lim_{n\to\infty} f_{k_n}\left(t, x_{k_n}(t), x_{k_n}'(t)\right) = \lim_{n\to\infty} \bar{f}_{k_n}\left(t, x_{k_n}(t), x_{k_n}'(t)\right) = f(t, x(t), x'(t)),$$

for $t \in (0,1)$ and $|\bar{f}_{k_n}(t, x_{k_n}(t), x_{k_n}'(t))| \le (g(\alpha(t)) + h(L))\psi(K)$ for $t \in (0,1)$ and $n \in \mathbb{N}$. Let $\xi \in (0,1)$. Taking the limit in the equalities

$$x_{k_n}'(t) = x_{k_n}'(\xi) - \int_\xi^t q(s)\bar{f}_{k_n}\left(s, x_{k_n}(s), x_{k_n}'(s)\right) ds, \qquad t \in [0,1],$$

as $n \to \infty$, we have

$$x'(t) = x'(\xi) - \int_\xi^t q(s) f(s, x(s), x'(s))\, ds, \qquad t \in [0,1].$$

Hence, $x \in C^1([0,1]) \cap C^2((0,1))$ and x satisfies (1) on $(0,1)$. We have proved that x is a solution of (1),(2) satisfying (20). ∎

EXAMPLE 1. Consider the differential equation

$$x'' + t(1-t)\left(\frac{1}{x} + x' - \lambda^2\right) = 0, \tag{22}$$

where $\lambda \ne 0$ is a constant. Set $\varepsilon = (1/18)(\sqrt{\lambda^4 + 144} - \lambda^2)$. Then, (H_3) and (H_4) hold with $\alpha(t) = \varepsilon t(1-t)$, $\beta_n(t) = \beta = \max\{1, (\lambda^2\varepsilon)/4\}/\lambda^2$ and, for instance, $\bar{t}_n = 1/(4n)$, $t_n = 1 - 1/(4n)$. Applying Theorem 1 with $q(t) = t(1-t)$, $L = \max\{1, (\lambda^2\varepsilon)/4\}/\lambda^2$, $\mu = 0$, $g(x) = 1/x$, $h(x) = 1 + \lambda^2$, and $\psi(u) = 1 + u$, there exists a solution of BVP (22),(2) such that

$$\varepsilon t(1-t) \le x(t) \le \frac{\max\left\{1, \left(\lambda^2\varepsilon\right)/4\right\}}{\lambda^2}, \qquad t \in [0,1].$$

THEOREM 2. *Let Assumptions (H_1)–(H_4), (B_1), and (B_2) be satisfied and let* $\sup\{q(t) : t \in (0,1)\} < \infty$. *Then, there exists a solution x of BVP (1),(2) satisfying inequalities (20), where K is given in Lemma 2.*

PROOF. By Lemma 2, for each $n \in \mathbb{N}$ there exists a solution x_n of BVP (11),(12) for which inequalities (21) hold. Now, we can argue as in the proof of Theorem 1 to prove the assertions of our theorem. ∎

EXAMPLE 2. Consider the differential equation

$$x'' + \left(\left(\frac{t(1-t)}{x}\right)^2 + x' - t - \lambda^2\right)(1 + x') = 0, \tag{23}$$

where λ is a constant. Then, (H_3) and (H_4) hold with $\alpha(t) = t(1-t)/(5+\lambda^2)$, $\beta_n(t) = \beta = 1$ and, for instance, $\bar{t}_n = 1/(4n)$, $t_n = 1 - 1/(4n)$. Applying Theorem 2 with $Q = 1$, $L = 1$, $\mu = 0$, $r(t) = t$, $w(t) = t^2(1-t)^2$, $g(x) = 1/x^2$, $h(x) = \lambda^2$, and $\psi(u) = 1 + u$, we see that there exists a solution of BVP (23),(2) such that

$$\frac{t(1-t)}{5+\lambda^2} \le x(t) \le 1, \qquad t \in [0,1].$$

3. SOLVABILITY OF BVP (3),(4)

Let Assumptions (H_1), (H_2), (H_6), and (H_7) be satisfied. Consider the family of regular BVPs

$$(p(t)x'(t))' + p(t)q(t)f_n(t, x(t), p(t)x'(t)) = 0, \tag{24}$$

$$\lim_{t\to 0^+} p(t)x'(t) = 0, \qquad x(1) = \alpha(t_n), \tag{25}$$

depending on $n \in \mathbb{N}$, where f_n is defined by (7).

We say that x is a *solution of BVP* (24),(25) if $x \in C^0([0,1]) \cap C^1((0,1))$, $px' \in C^0([0,1]) \cap C^1((0,1))$, x satisfies the boundary conditions (25), and (24) holds on $(0,1)$.

LEMMA 3. *Let Assumptions (H_1), (H_2), (H_5)–(H_7), (A_1) with $L = L_*$,*

$$\int_0^1 p(t)q(t)(g(\alpha(t)) + h(L_*))\,dt < \int_0^\infty \frac{1}{\psi(s)}\,ds \tag{26}$$

and

$$\int_0^1 \frac{1}{p(t)} \int_0^s p(v)q(v)(g(\alpha(v)) + h(L_*))\,dv\,ds < \infty, \tag{27}$$

be satisfied. Then, for each $n \in \mathbb{N}$, there exists a solution x of BVP (24),(25) such that

$$\Lambda_n(t) \le x(t) \le \beta_n(t), \quad |p(t)x'(t)| \le V, \qquad \text{for } t \in [0,1], \tag{28}$$

where Λ_n is given by (10) and the positive constant V satisfies the inequality

$$\int_0^1 p(t)q(t)(g(\alpha(t)) + h(L_*))\,dt \le \int_0^V \frac{1}{\psi(s)}\,ds. \tag{29}$$

PROOF. Fix $n \in \mathbb{N}$ and let x be a solution of BVP (24),(25). We are going to show that x satisfies (28). Suppose that

$$\max\{\Lambda_n(t) - x(t) : t \in [0,1]\} = \Lambda_n(t_0) - x(t_0) > 0, \tag{30}$$

where $t_0 \in [0,1)$ since $\Lambda_n(1) - x(1) = 0$. Then, four cases occur.

CASE 1. Let $t_0 = 0$. Then,

$$\begin{aligned}\lim_{t\to 0^+} \left(f\left(\bar{t}_n, \alpha\left(\bar{t}_n\right), p(t)x'(t)\right) + \phi\left(\alpha\left(\bar{t}_n\right) - x(t)\right)\right) &= f(\bar{t}_n, \alpha(\bar{t}_n), 0) + \phi(\Lambda_n(0)) - x(0)) \\ &\ge \phi(\Lambda_n(0)) - x(0)) > 0,\end{aligned}$$

and so

$$(p(t)x'(t))' = -p(t)q(t)\left[f\left(\bar{t}_n, \alpha\left(\bar{t}_n\right), p(t)x'(t)\right) + \phi\left(\Lambda_n\left(\bar{t}_n\right) - x(t)\right)\right] < 0,$$

on a right neighbourhood $\mathcal{U} \subset (0, \bar{t}_n)$ of $t = 0$. Hence, px' is decreasing on $\mathcal{U}$ and from

$$\lim_{t\to 0^+} p(t)x'(t) = 0,$$

we deduce that $x' < 0$ on $\mathcal{U}$, contrary to (30).

CASE 2. Let $t_0 \in (0, \bar{t}_n) \cup (t_n, 1)$. Then, $x'(t_0) = \Lambda_n'(t_0) = 0$. Suppose that $t_0 \in (0, \bar{t}_n)$ (the case $t_0 \in (t_n, 1)$ can be treated analogously). Then,

$$\begin{aligned}(p(t)x'(t))'_{t=t_0} &= -p(t_0)q(t_0)\left[f\left(\bar{t}_n, \alpha\left(\bar{t}_n\right), 0\right) + \phi(\Lambda_n(t_0) - x(t_0))\right] \\ &< -p(t_0)q(t_0)f\left(\bar{t}_n, \alpha\left(\bar{t}_n\right), 0\right) \le 0.\end{aligned}$$

Hence, $x' < 0$ on a right neighbourhood of $t = t_0$, contrary to (30).

CASE 3. Let $t_0 \in \{\bar{t}_n, t_n\}$. If $t_0 = t_n$, then $x'_+(t_0) \geq 0$ and $x'_-(t_0) \leq \alpha'(t_0) \leq 0$ (see Remark 1). Hence, $x'(t_0) = \alpha'(t_0) = 0$ and

$$\begin{aligned}(p(t)x'(t))'_{t=t_0} &= -p(t_0)q(t_0)[f(t_0, \alpha(t_0), 0) + \phi(\alpha(t_0) - x(t_0))] \\ &< -p(t_0)q(t_0)f(t_0, \alpha(t_0), 0) \leq (p(t)\alpha'(t))'_{t=t_0}.\end{aligned}$$

Consequently, $(x - \alpha)' > 0$ on a left neighbourhood of $t = t_0$, contrary to (30). For the case $t_0 = \bar{t}_n$, the proof is similar.

CASE 4. Let $t_0 \in (\bar{t}_n, t_n)$. Then, $x'(t_0) = \alpha'(t_0)$ and

$$\begin{aligned}(p(t)x'(t))'_{t=t_0} &= -p(t_0)q(t_0)[f(t_0, \alpha(t_0), p(t_0)\alpha'(t_0)) + \phi(\alpha(t_0) - x(t_0))] \\ &< -p(t_0)q(t_0)f(t_0, \alpha(t_0), p(t_0)\alpha'(t_0)) \leq (p(t)\alpha'(t))'_{t=t_0}.\end{aligned}$$

Hence, again $(x - \alpha)' > 0$ on a left neighbourhood of $t = t_0$, contrary to (30).

We have proved that $\Lambda_n(t) \leq x(t)$ for $t \in [0,1]$. Suppose that $x \leq \beta_n$ on $[0,1]$ is not true. Then,

$$\max\{x(t) - \beta_n(t) : t \in [0,1]\} = x(t_*) - \beta_n(t_*) > 0, \tag{31}$$

and $t_* \in [0,1)$ since $x(1) = \alpha(t_n) \leq \beta_n(1)$.

(a) Let $t_* = 0$. If $\lim_{t\to 0^+} p(t)\beta'_n(t) < 0$ then $(\beta_n - x)' < 0$ on a right neighbourhood of $t = 0$, contrary to (31). Let $\lim_{t\to 0^+} p(t)\beta'_n(t) = 0$. Then,

$$\begin{aligned}\lim_{t\to 0^+} &(f(\bar{t}_n, \beta_n(t), p(t)x'(t)) - f(\bar{t}_n, \beta_n(t), p(t)\beta'_n(t)) + \phi(\beta_n(t) - x(t))) \\ &= f(\bar{t}_n, \beta_n(0), 0) - f(\bar{t}_n, \beta_n(0), 0) + \phi(\beta_n(0) - x(0)) < 0,\end{aligned}$$

and so from the inequality

$$\begin{aligned}p(t)(x(t) - \beta_n(t))' \geq -\int_0^t p(s)q(s)\,[f(\bar{t}_n, \beta_n(s), p(s)x'(s)) \\ - f(\bar{t}_n, \beta_n(s), p(s)\beta_n(s)) + \phi(\beta_n(s) - x(s))]\, ds,\end{aligned}$$

which is satisfied on any interval $(0, \varepsilon] \subset (0, \bar{t}_n)$ where $x > \beta_n$, we see that $(x - \beta_n)' > 0$ on a right neighbourhood of $t = 0$, contrary to (31).

(b) Let $t_* \in (0, \bar{t}_n] \cup [t_n, 1)$. Then, $x'(t_*) = \beta'_n(t_*)$ and we can assume that, for example, $t_* \in (0, \bar{t}_n]$ since the proof is similar for the case that $t_* \in [t_n, 1)$. From the inequalities

$$\begin{aligned}(p(t)x'(t))'_{t=t_*} &= -p(t_*)q(t_*)\,[f(\bar{t}_n, \beta_n(t_*), p(t_*)\beta'_n(t_*)) + \phi(\beta_n(t_*) - x(t_*))] \\ &> -p(t_*)q(t_*)f(\bar{t}_n, \beta_n(t_*), p(t_*)\beta'_n(t_*)) \geq (p(t)\beta'_n(t))'_{t=t_*},\end{aligned}$$

it follows that $(x - \beta_n)' > 0$ on a right neighbourhood of $t = t_*$, contrary to (31).

(c) Let $t_* \in (\bar{t}_n, t_n)$. Then, $x'(t_*) = \beta'_n(t_*)$ and

$$\begin{aligned}(p(t)x'(t))'_{t=t_*} &= -p(t_*)q(t_*)[f(t_*, \beta_n(t_*), p(t_*)\beta'_n(t_*)) + \phi(\beta_n(t_*) - x(t_*))] \\ &> -p(t_*)q(t_*)f(t_*, \beta_n(t_*), p(t_*)\beta'_n(t_*)) \geq (p(t)\beta'_n(t))'_{t=t_*}.\end{aligned}$$

Hence, $(x - \beta_n)' > 0$ on a right neighbourhood of $t = t_*$, contrary to (31).

Therefore, $x \leq \beta_n$ on $[0,1]$.

Let $\|px'\| = |(px')(\xi)|$. Since $\lim_{t\to 0^+} p(t)x'(t) = 0$, $\xi \in (0,1]$. By the inequalities $\Lambda_n \leq x \leq \beta_n \leq L_*$ on $[0,1]$, (H_1), (H_5), and (A_1) with $L = L_*$,

$$|(p(t)x'(t))'| \leq p(t)q(t)(g(\alpha(t)) + h(L_*))\psi(|p(t)x'(t)|), \qquad t \in (0,1).$$

Assume that $p(\xi)x'(\xi) > 0$ (the case where $p(\xi)x'(\xi) < 0$ can be considered similarly). Then, there exists $\nu \in [0,\xi)$ such that $\lim_{t\to\nu^+} p(t)x'(t) = 0$ and $px' > 0$ on $(\nu,\xi]$. Integrating the inequality

$$\frac{(p(t)x'(t))'}{\psi(p(t)x'(t))} \le p(t)q(t)(g(\alpha(t)) + h(L_*)),$$

from $t \in (\nu,\xi)$ to ξ and letting $t \to \nu^+$, we get

$$\int_0^{\|px'\|} \frac{1}{\psi(s)}\,ds \le \int_\nu^\xi p(t)q(t)(g(\alpha(t)) + h(L_*))\,dt$$
$$\le \int_0^1 p(t)q(t)(g(\alpha(t)) + h(L_*))\,dt,$$

and consequently, $\|px'\| < V$ which follows from (29). Hence, (28) is true.

The Schauder fixed-point theorem guarantees that BVP (24),(25) has a solution x and by the above consideration, we see that x satisfies inequalities (28). ∎

THEOREM 3. *Let Assumptions (H_1), (H_2), (H_5)–(H_7), (A_1) with $L = L_*$, (26), and (27) be satisfied. Then, there exists a solution x of BVP (3),(4) such that*

$$\alpha(t) \le x(t) \le L_*, \qquad \text{for } t \in [0,1]. \tag{32}$$

PROOF. By Lemma 3, for each $n \in \mathbb{N}$, there exists a solution x_n of BVP (24),(25) satisfying

$$\Lambda_n(t) \le x_n(t) \le \beta_n(t) \le L_*, \quad |p(t)x_n'(t)| \le V, \qquad \text{for } t \in [0,1]. \tag{33}$$

Consider the sequence $\{x_n(t)\}$. It follows from (33) that $\{x_n(t)\}$ and $\{p(t)x_n'(t)\}$ are uniformly bounded on $[0,1]$ and from the inequalities

$$|x_n(t_1) - x_n(t_2)| = \left|\int_{t_1}^{t_2} \frac{1}{p(t)} \int_0^t p(s)q(s)f_n(s,x_n(s),p(s)x_n'(s))\,ds\,dt\right|$$
$$\le \left|\int_{t_1}^{t_2} \frac{1}{p(t)} \int_0^t p(s)q(s)(g(\alpha(s)) + h(L_*))\,ds\,dt\right|,$$
$$|p(t_1)x_n'(t_1) - p(t_2)x_n'(t_2)| = \left|\int_{t_1}^{t_2} p(t)q(t)f_n(t,x_n(t),p(t)x_n'(t))\,dt\right|$$
$$\le \left|\int_{t_1}^{t_2} p(t)q(t)(g(\alpha(t)) + h(L_*))\,dt\right|,$$

for $t_1,t_2 \in [0,1]$ and $n \in \mathbb{N}$, we deduce that $\{x_n(t)\}$ and $\{p(t)x_n'(t)\}$ are equicontinuous on $[0,1]$. By the Arzelà-Ascoli theorem, going if necessary to subsequences, we can assume that $\{x_n(t)\}$ and $\{p(t)x_n'(t)\}$ are uniformly convergent on $[0,1]$, $\lim_{n\to\infty} x_n(t) = x(t)$, $\lim_{n\to\infty} p(t)x_n'(t) = \gamma(t)$. Clearly, $x(1) = 0$, $\alpha(t) \le x(t)$ for $t \in [0,1]$, $\lim_{t\to 0^+}\gamma(t) = 0$ and $\gamma(t) = p(t)x'(t)$ for $t \in (0,1)$. Letting $n \to \infty$ in the equalities

$$x_n(t) = \alpha(t_n) + \int_t^1 \frac{1}{p(s)}\Bigg[\chi_{[0,\bar t_n]}(s)\int_0^s p(v)q(v)f(\bar t_n,x_n(v),p(v)x_n'(v))\,dv$$
$$+\chi_{[\bar t_n,t_n]}(s)\int_0^s p(v)q(v)f(v,x_n(v),p(v)x_n'(v))\,dv$$
$$+\chi_{[t_n,1]}(s)\int_0^s p(v)q(v)f(t_n,x_n(v),p(v)x_n'(v))\,dv\Bigg]\,ds,$$

where $\chi_{[c_1,c_2]}$ stands for the characteristic function of the interval $[c_1,c_2] \subset [0,1]$, and using Lebesgue dominated convergence theorems, we have

$$x(t) = \int_t^1 \frac{1}{p(s)} \int_0^s p(v)q(v)f(v,x(v),p(v)x'(v))\,dv\,ds, \qquad t \in [0,1].$$

Therefore, $x \in C^0([0,1]) \cap C^1((0,1))$ and

$$(p(t)x'(t))' = -p(t)q(t)f(t,x(t),p(t)x'(t)), \qquad t \in (0,1).$$

Consequently, x is a solution of BVP (3),(4) satisfying (32). ∎

REFERENCES

1. R.P. Agarwal and D. O'Regan, A note on existence of negative solutions to singular semi-positone problems, *Nonlin. Anal.* **36**, 615–622, (1999).
2. D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific, Singapore, (1994).
3. S. Staněk, Positive solutions of singular positone Dirichlet boundary value problems, *Mathl. Comput. Modelling* **33** (4/5), 341–351, (2001).
4. S. Staněk, Positive solutions of singular Dirichlet and periodic boundary value problems, *Mathl. Comput. Modelling* **43** (6/7), 681–692, (2002).
5. R.P. Agarwal and D. O'Regan, Some new results for singular problems with sign changing nonlinearities, *J. Comput. Appl. Math.* **113**, 1–15, (2000).
6. R.P. Agarwal and D. O'Regan, Singular initial and boundary value problems with sign changing nonlinearities, *IMA J. Appl. Math.* **65**, 173–198, (2000).
7. A. Lomtatidze, Positive solutions of boundary value problems for second order differential equations with singular points, *Differentsial'nye Uravneniya* **23**, 1685–1692, (1987).
8. A. Lomtatidze and P. Torres, On a two-point boundary value problem for second order singular equations, (preprint).
9. J. Wang and W. Gao, A note on singular two-point boundary value problems, *Nonlin. Anal.* **39**, 281–287, (2000).
10. R. Kannan and D. O'Regan, Singular and nonsingular boundary value problems with sign changing nonlinearities, *J. Inequal. & Appl.* **5**, 621–637, (2000).
11. R.P. Agarwal, D. O'Regan and V. Lakshmikantham, An upper and lower solution approach for nonlinear singular boundary value problems with y' dependence, *Archives Inequal. Appl.* **1**, 119–135, (2003).
12. D. O'Regan and R.P. Agarwal, Singular problems: An upper and lower solution approach, *J. Math. Anal. Appl.* **251**, 230–250, (2000).
13. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, (1985).