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A construction of real weight functions for certain orthogonal polynomials in two variables

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Abstract

H.L. Krall and I.M. Sheffer considered the problem of classifying certain second-order partial differential equations having an algebraically complete, weak orthogonal bivariate polynomial system of solutions. Two of the equations that they considered are

$$\left(x^2+y\right)u_{xx}+2xyu_{xy}+y^2u_{yy}+gxu_x+g(y-1)u_y=\lambda u,$$

and

$$x^2u_{xx}+2xyu_{xy}+\left(y^2-y\right)u_{yy}+g(x-1)u_x+g(y-\gamma)u_y=\lambda u.$$

Even though they showed that these equations have a sequence of weak orthogonal polynomial solutions, they were unable to show that these polynomials were, in fact, orthogonal. The orthogonality of these two polynomial sequences was recently established by Kwon, Littlejohn, and Lee solving an open problem from 1967.

In this paper, we construct explicit weight functions for these two orthogonal polynomial sequences, using a method first developed by Littlejohn and then further developed by Han, Kim, and Kwon. Moreover, two additional partial differential equations were found by Kwon, Littlejohn,

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and Lee that have sequences of orthogonal polynomial solutions. These equations are given by

$$\begin{aligned}&\left(x^2-x\right)u_{xx}+2xyu_{xy}+y^2u_{yy}+(dx+e)u_x+(dy+h)u_y=\lambda u,\\&xu_{xx}+2yu_{xy}+(dx+e)u_x+(dy+h)u_y=\lambda u.\end{aligned}$$

In each of these examples, we also produce explicit orthogonalizing weight functions.

Keywords: Second-order partial differential equation; Bivariate orthogonal polynomials; Weak orthogonal polynomials; Polynomial killers; Weight functions

1. Introduction

In 1967, Krall and Sheffer [8] (see also [14]) considered, and solved, the problem of classifying all weak orthogonal polynomial solutions (see Definition 2.1) satisfying second-order partial differential equations of the form

$$\begin{aligned}L[u]:=A(x,y)u_{xx}+2B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y\\=\lambda u,\end{aligned}\tag{1.1}$$

where the eigenvalue parameter is a function of only the degree of the polynomial solution. Up to a complex linear change of variable, they showed that there are nine distinct weak orthogonal polynomial systems to the differential equation of the form (1.1). They found necessary and sufficient conditions for the existence of these polynomial solutions to the differential equation (1.1) and expressed these conditions in terms of an infinite system of recurrence relations for moments of the corresponding (weak) orthogonalizing moment functional. These conditions were later rewritten in a simpler form by Littlejohn [14] using the classical Lagrangian symmetry equations and the moment equations associated with (1.1). Although Krall and Sheffer identified all differential equations which have weak orthogonal polynomials as solutions, they only partially succeeded in showing that the polynomial solutions are, in fact, orthogonal.

In a recent paper, Kwon, Lee, and Littlejohn [10] explained the orthogonality for the weak orthogonal polynomials found by Krall and Sheffer. More specifically, the authors in [10] showed that the polynomial solutions $\{\phi_{m,n}(x,y)\}_{m,n=0}^{\infty}$ to the partial differential equations

$$x^2u_{xx}+2xyu_{xy}+\left(y^2-y\right)u_{yy}+g(x-1)u_x+g(y-\gamma)u_y=\lambda y,\tag{1.2}$$

$$\left(x^2+y\right)u_{xx}+2xyu_{xy}+y^2u_{yy}+gxu_x+g(y-1)u_y=\lambda u\tag{1.3}$$

are orthogonal. Furthermore, in [10], they found new second-order partial differential equations having weak orthogonal polynomials as solutions:

$$\left(x^2-x\right)u_{xx}+2xyu_{xy}+y^2u_{yy}+(dx+e)u_x+(dy+h)u_y=\lambda u,\tag{1.4}$$

$$xu_{xx}+2yu_{xy}+(dx+e)u_x+(dy+h)u_y=\lambda u.\tag{1.5}$$

These equations were omitted from the Krall–Sheffer classification since a complex change of variable will transform these equations into ones found by Krall and Sheffer. However, even though these changes of variable will preserve the orthogonality of the corresponding orthogonal polynomial solutions, the positive-definiteness of the orthogonalizing moment functionals will not be preserved; consequently, we feel that it is necessary to distinguish these equations and orthogonal polynomial solutions.

Because all these partial differential equations have polynomial solutions closely related to the Bessel polynomials, we will say that these polynomial solutions are of *Bessel type*.

This two-variable Krall–Sheffer classification mirrors the well-known one-dimensional classification due to S. Bochner in 1929 [2] (see also Lesky [12]), who determined, up to a complex linear change of variable, all sequences of polynomials which satisfy a second-order ordinary differential equation of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = \lambda y \tag{1.6}$$

and are orthogonal with respect to a bilinear form of the type

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\mu, \tag{1.7}$$

where μ is a real, finite, signed Borel measure with finite moments of all orders. In a more recent extension of this classical result, Kwon and Littlejohn [11] reconsidered this classification problem from a real linear change of variable point of view. This classification, which is crucial in the developments of this paper, will be further considered in Section 3 below.

In this paper, we construct real orthogonalizing weight functions for the orthogonal polynomials satisfying the partial differential equations given in (1.2)–(1.5). Consequently, orthogonalizing weights are now known for seven of the nine polynomial systems studied by Krall and Sheffer as well as the new polynomial systems satisfying the partial differential equations (1.4) and (1.5). We shed some light on a real orthogonalizing weight function for the orthogonal polynomials satisfying the partial differential equation (1.3) with $g > 0$ but, unfortunately, we are unable at this time to produce an explicit weight function for these polynomials when $g > 0$.

The contents of this paper are as follows. In Section 2, we recall the basic definitions and terminologies for orthogonal polynomials in two variables and review some of their basic facts. In Section 3, we revisit the one-dimensional classifications due to Bochner and others as well as review some key constructions of weight functions in this situation. Section 4 deals with a general constructive technique for two-variable orthogonalizing weights of bivariate orthogonal polynomial solutions to second-order partial differential equations. Lastly, in Section 5, we apply this general method to construct explicit weights for the orthogonal polynomial solutions to Eqs. (1.2)–(1.5).

2. Preliminaries

We denote the set of all polynomials in one variable x by $\mathcal{P}_1$ and the set of all polynomials in the two independent variables x and y by $\mathcal{P}_2$. By a bivariate polynomial system (PS), we mean a sequence $\{\phi_{n-j,j}(x, y) \mid n \in \mathbb{N}_0;\ j = 0, 1, \dots, n\}$ of polynomials with real coefficients such that $\deg \phi_{m,n} = m + n$ for all $m, n \in \mathbb{N}_0$ and $\{\phi_{n-j,j}\}_{j=0}^{n}$ is linearly independent modulo polynomials of degree $\leqslant n-1$. This linear independence is equivalent to the statement that the matrix $(a_i^{n,j})_{i,j=0}^n$ is nonsingular for each $n \in \mathbb{N}_0$, where $a_i^{n,j}$ $(0 \leqslant i \leqslant n)$ are the coefficients in the expression

$$\phi_{n-j,j}(x,y) = \sum_{i=0}^{n} a_i^{n,j} x^{n-i} y^i + \text{lower degree terms.}$$

For each $n \in \mathbb{N}_0$, it is convenient to view $\{\phi_{n-j,j}\}_{j=0}^n$ as the $(n+1)$-dimensional column vector $\Phi_n := (\phi_{n,0}, \phi_{n-1,1}, \dots, \phi_{0,n})^T$ and the PS $\{\phi_{n-j,j}(x,y) \mid n \in \mathbb{N}_0;\ j = 0, 1, \dots, n\}$ as the sequence $\{\Phi_n(x,y)\}_{n=0}^\infty$ of column vectors. Lastly, we say that the PS $\{\Phi_n(x,y)\}_{n=0}^\infty$ is monic if, for each $n \in \mathbb{N}_0$ and $0 \leqslant j \leqslant n$, $\phi_{n-j,j}(x,y) = x^{n-j}y^j +$ lower order terms.

For $i = 1, 2$, a linear mapping $\sigma : \mathcal{P}_i \to \mathbb{C}$ is called a *moment functional*. When $i = 1$, we refer the reader to [11] for various properties of one-variable moment functionals. The action of a moment functional σ on a polynomial $\phi \in \mathcal{P}_2$ is denoted by $\langle \sigma, \phi \rangle$ instead of the customary $\sigma(\phi)$. Similarly, we define the action of a moment functional σ on a matrix $Q = (Q_{i,j})$, where each $Q_{i,j} \in \mathcal{P}_2$, through the formula

$$\langle \sigma, Q \rangle = \big(\langle \sigma, Q_{i,j} \rangle\big).$$

For any moment functional σ, we define the first partial derivatives of σ, as moment functionals, by

$$\langle \sigma_x, \phi \rangle := -\langle \sigma, \phi_x \rangle, \quad \langle \sigma_y, \phi \rangle := -\langle \sigma, \phi_y \rangle \quad (\phi \in \mathcal{P}_2), \tag{2.1}$$

and define multiplication $\psi\sigma$, where $\psi \in \mathcal{P}_2$, to be the moment functional defined by

$$\langle \psi\sigma, \phi \rangle := \langle \sigma, \psi\phi \rangle \quad (\phi \in \mathcal{P}_2). \tag{2.2}$$

Definition 2.1. A PS $\{\Phi_n(x,y)\}_{n=0}^\infty$ is called a weak orthogonal polynomial system (WOPS) if there is a nonzero moment functional σ such that

$$\langle \sigma, \phi_{m,n}\phi_{k,l} \rangle = 0 \quad \text{if } (m,n) \neq (k,l).$$

In this case, we say that $\{\Phi_n(x,y)\}_{n=0}^\infty$ is a WOPS relative to σ. A WOPS $\{\Phi_n(x,y)\}_{n=0}^\infty$ relative to σ is called an orthogonal polynomial system (OPS) if $\langle \sigma, \phi_{m,n}\phi_{k,l} \rangle = K_{m,n}\delta_{m,k}\delta_{n,l}$ where each $K_{m,n}$ is a nonzero constant and $\delta_{m,k}$ denotes the Kronecker delta symbol. If $K_{m,n} > 0$ for each $m, n \geqslant 0$, then we say that $\{\Phi_n(x,y)\}_{n=0}^\infty$ is a positive-definite OPS relative to σ.

Definition 2.2. A moment functional σ is called quasi-definite (respectively positive-definite) if there is an OPS (respectively a positive-definite OPS) relative to σ.

The following algebraic characterization of orthogonality for polynomials in two variables is important to the study of orthogonal polynomials in more than one variable.

Theorem 2.1. *For a nonzero moment functional σ, the following conditions are equivalent*:

(i) *σ is quasi-definite*;
(ii) *there is a unique monic WOPS $\{\mathbb{P}_n(x, y)\}_0^\infty$ relative to σ*;
(iii) *there is a monic WOPS $\{\mathbb{P}_n(x, y)\}_0^\infty$ such that $H_n := \langle \sigma, \mathbb{P}_n \mathbb{P}_n^T \rangle$ is nonsingular for each $n \in \mathbb{N}_0$.*

Proof. See [7,8]. □

From Definitions 2.1 and 2.2, it follows that a PS $\{\Phi_n(x, y)\}_{n=0}^\infty$ is an OPS (respectively a positive-definite OPS) relative to σ if and only if $\langle \sigma, \Phi_m \Phi_n^T \rangle = H_n \delta_{m,n}$, where $H_n := \langle \sigma, \Phi_n \Phi_n^T \rangle$ is a nonsingular (respectively positive-definite) diagonal matrix.

For any PS $\{\Phi_n(x, y)\}_{n=0}^\infty$, there is a unique moment functional σ defined by the conditions

$$\langle \sigma, 1 \rangle = 1, \qquad \langle \sigma, \phi_{m,n} \rangle = 0 \quad (m + n \geqslant 1).$$

In this case, we call σ the *canonical moment functional* of a PS $\{\Phi_n(x, y)\}_{n=0}^\infty$. Note that if a PS $\{\Phi_n(x, y)\}_{n=0}^\infty$ is a WOPS relative to σ, then σ is necessarily a constant multiple of the canonical moment functional of $\{\Phi_n(x, y)\}_{n=0}^\infty$.

3. Results for the classical orthogonal polynomials

In this section, we review some fundamental results concerning the classification of (one-variable) sequences of polynomial solutions to second-order ordinary differential equations of the form (1.6) that are orthogonal with respect to bilinear forms of the type (1.7). In addition, we review an important technique of constructing orthogonalizing weights for these polynomials; these results are crucial for the constructions that will take place in the next section.

In an extension of the Bochner classification, Kwon and Littlejohn [11] showed that, up to a *real* linear change of variable, there are six distinct sequences of polynomial solutions to second-order differential equations of the form (1.6) that are orthogonal with respect to the bilinear form (1.7). These polynomial systems are the:

(1) Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, which satisfy the differential equation

$$\left(1 - x^2\right) y'' + \left[(\beta - \alpha) - (\alpha + \beta + 2)x\right] y' = \lambda y,$$

and are an OPS (respectively a positive-definite OPS) if and only if $-\alpha, -\beta, -\alpha - \beta - 1 \notin \mathbb{N}$ (respectively $\alpha, \beta > -1$).

(2) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$, which satisfy the differential equation

$$x y'' + (\alpha + 1 - x) y' = \lambda y,$$

and are an OPS (respectively a positive-definite OPS) if and only if $-\alpha \in \mathbb{N}$ (respectively $\alpha > -1$).

(3) Bessel polynomials $\{B_n^{(d,e)}(x)\}_{n=0}^{\infty}$, which satisfy the differential equation

$$x^2y'' + (dx + e)y' = \lambda y, \tag{3.1}$$

and are an OPS if and only if $e \neq 0$ and $-(d+1) \notin \mathbb{N}$, but are never a positive-definite OPS.

(4) Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$, which satisfy the differential equation

$$y'' - 2xy' = \lambda y,$$

and are a positive-definite OPS.

(5) Twisted Hermite polynomials $\{\check{H}_n(x)\}_{n=0}^{\infty}$, which satisfy the differential equation

$$y'' + 2xy' = \lambda y,$$

and are an OPS but are not a positive-definite OPS.

(6) Twisted Jacobi polynomials $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, which satisfy the differential equation

$$\bigl(1 + x^2\bigr)y'' + \bigl((\alpha + \beta + 2)x + i(\alpha - \beta)\bigr)y' = \lambda y,$$

and are an OPS if and only if $-(\alpha + \beta + 1) \notin \mathbb{N}_0$ and $\bar{\alpha} = \beta$ but are never a positive-definite OPS.

In [9] (see also [13]), the following result is obtained:

Theorem 3.1. *Suppose that the second-order differential equation* (1.6) *has a sequence of polynomial solutions* $\{p_n(x)\}_{n=0}^{\infty}$, *where* $\deg(p_n) = n$ *for each* $n \in \mathbb{N}_0$. *Then* $\{p_n(x)\}_{n=0}^{\infty}$ *is an orthogonal polynomial sequence with respect to a distribution* $w \in \mathcal{D}'(\mathbb{R})$ *if and only if each of the following conditions is satisfied*:

(i) *w acts on the set $\mathcal{P}_1$ of all one-variable polynomials,*
(ii) $\langle w, 1\rangle \neq 0$,
(iii) *w satisfies the first-order differential equation* (*called the weight equation*)

$$a_2(x)w' + \bigl(a_2'(x) - a_1(x)\bigr)w = u \quad (x \in \mathbb{R}), \tag{3.2}$$

where $u \in \mathcal{D}'(\mathbb{R})$ is a distribution that acts on $\mathcal{P}_1$ and satisfies $\langle u, x^n\rangle = 0$ for all $n \in \mathbb{N}_0$.

Remark 3.1. The weight equation (3.2) is equivalent to the following infinite system of moment (recurrence) equations:

$$(an + d)w_{n+1} + (bn + e)w_n + cnw_{n-1} = 0 \quad (n \in \mathbb{N}_0); \tag{3.3}$$

here $a_2(x) = ax^2 + bx + c$, $a_1(x) = dx + e$ and $w_n = \langle w, x^n\rangle$ for each $n \in \mathbb{N}_0$.

Remark 3.2. A distribution u, with the properties given in Theorem 3.1, is called a *polynomial killer*. Any nontrivial function $g : \mathbb{R} \to \mathbb{R}$ satisfying $\int_{\mathbb{R}} x^n g(x)\,dx = 0$ $(n \in \mathbb{N}_0)$ generates a polynomial killer u through the formula

$$\langle u, \varphi\rangle = \int_{\mathbb{R}} \varphi(x)g(x)\,dx \quad \bigl(\varphi \in \mathcal{D}(\mathbb{R})\bigr),$$

where $\mathcal{D}(\mathbb{R})$ is the space of infinitely differentiable, real-valued functions with compact support. For example, Stieltjes [17] showed that the function

$$g(x)=\begin{cases} e^{-x^{1/4}}\sin(x^{1/4}), & x\geqslant 0,\\ 0, & x<0, \end{cases} \tag{3.4}$$

generates a polynomial killer (see also [18, p. 126] as well as [3, Chapter 2, Section 6]); that is to say,

$$\int_{\mathbb{R}} x^n g(x)\,dx=0 \quad (n\in\mathbb{N}_0).$$

Remark 3.3. The general Bessel polynomials $\{B_n^{(d,e)}(x)\}_{n=0}^{\infty}$ seem to depend on two parameters d and e. In fact, a change of variable $t=2x/e$ transforms the differential equation (3.1) into the differential equation

$$t^2Y''+(dt+2)Y'=n(n+d-1)Y. \tag{3.5}$$

We denote the polynomial solutions of Eq. (3.5) by $\{y_n^{d-2}(t)\}_{n=0}^{\infty}$. These two notations for Bessel polynomials are used interchangeably throughout this paper.

In [9], the authors effectively use Theorem 3.1, with the polynomial killer generated by (3.4), to construct a weight function for the simple Bessel polynomials $\{y_n^0(x)\}$. Maroni, in [15], extended the range of the technique used in [9] to general Bessel polynomials $\{y_n^a(x)\}$ for $a\geqslant 12(2/\pi)^4-2$. For the general Bessel case, the corresponding weight equation (3.2) is

$$x^2w'-(ax+2)w=u, \tag{3.6}$$

where u is a nonzero polynomial killer.

We remark that a general technique for constructing weight functions for orthogonal polynomials in one variable has been developed by Duran [5]. Consequently, from his constructive technique, it is possible to construct orthogonalizing weight functions for the general Bessel polynomials when $a\in\mathbb{R}$ satisfies $-(a+1)\notin\mathbb{N}$. We remark, however, that the weight functions that Duran obtains for the Bessel polynomials are not as explicit as the ones obtained by Kwon et al. [9] or Maroni in [15].

4. Real weight functions for bivariate orthogonal polynomials of Bessel type

Suppose that for a (possibly signed) weight function $w_1(x)$ on $[a,b]$,

(i) $x^m w_1(x)\in L^1(a,b)$ for each $m\in\mathbb{N}_0$ and
(ii) $\{p_n(k;x)\}_{n=0}^{\infty}$ is an OPS relative to the weight function $\rho^{2k+1}(x)w_1(x)$ on the interval (a,b) for each $k\in\mathbb{N}_0$, where $\rho(x)$ is a positive function on the interval (a,b).

Also, suppose $\{q_n(y)\}_{n=0}^{\infty}$ is an OPS with respect to the (possibly signed) weight function $w_2(y)$ on the interval (c, d) such that

$$\rho^k(x)q_k\big(y/\rho(x)\big) \quad \text{is a polynomial in } x \text{ and } y \text{ of degree } k \text{ for each } k \in \mathbb{N}_0. \tag{4.1}$$

To guarantee the validity of (4.1), we assume that

(a) $\rho(x)$ is a polynomial of degree $\leqslant 1$ *or*
(b) $\rho(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$ ($\beta^2 - 4\alpha\gamma \neq 0$) and $\{q_n(y)\}_{n=0}^{\infty}$ is a symmetric OPS (that is, $q_n(-y) = (-1)^n q_n(y)$ for each $n \in \mathbb{N}_0$).

From the one-variable polynomial sequences $\{p_n(k; x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$, we can define a new bivariate sequence of polynomials $\{\Phi_n(x, y)\}_{n=0}^{\infty}$ through the formula

$$\phi_{n-k,k}(x, y) = p_{n-k}(k; x)\rho^k(x)q_k\big(y/\rho(x)\big) \quad (0 \leqslant k \leqslant n). \tag{4.2}$$

Theorem 4.1. *Let* $\{\Phi_n(x, y)\}_{n=0}^{\infty}$ *be the polynomial sequence defined by* (4.2). *Then* $\{\Phi_n(x, y)\}_{n=0}^{\infty}$ *is a PS*; *moreover,* $\{\Phi_n(x, y)\}_{n=0}^{\infty}$ *is an OPS with respect to the* (*possibly signed*) *weight function*

$$w(x, y) = w_1(x)w_2\big(y/\rho(x)\big) \tag{4.3}$$

on the domain $R = \{(x, y) \mid a < x < b, c\rho(x) < y < d\rho(x)\}$.

Proof. To prove this theorem, we show that for $m, n \in \mathbb{N}_0$ and $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$,

$$\begin{aligned} &\iint_R \phi_{m-i,i}(x, y)\phi_{n-j,j}(x, y)w(x, y)\,dx\,dy \\ &\quad = \delta_{m,n}\delta_{i,j}\int_a^b p_{m-i}^2(i; x)\rho^{2i+1}(x)w_1(x)\,dx \int_c^d q_i^2(y)w_2(y)\,dy. \end{aligned} \tag{4.4}$$

We first show that

$$p_{m-i}(i; x)p_{n-j}(j; x)\rho^{i+j+1}(x)w_1(x) \in L^1(a, b). \tag{4.5}$$

Using the notation $\|f(x)\|_2 = |\int_a^b f^2(x)\rho(x)w_1(x)\,dx|^{1/2}$, we see from condition (i) and the Cauchy–Schwarz inequality that

$$\begin{aligned} &\left|\int_a^b p_{m-i}(i; x)\rho^{i+j}(x)p_{n-j}(j; x)\rho(x)w_1(x)\right| \\ &\quad \leqslant \big\|\rho^i(x)p_{m-i}(i; x)\big\|_2\big\|\rho^j(x)p_{n-j}(j; x)\big\|_2 \\ &\quad = \left|\int_a^b p_{m-i}^2(i; x)\rho^{2i+1}(x)w_1(x)\,dx\right|^{1/2}\left|\int_a^b p_{n-j}^2(j; x)\rho^{2j+1}(x)w_1(x)\,dx\right|^{1/2} \\ &\quad < \infty. \end{aligned}$$

Define the transformation $T : R \to D = (a,b) \times (c,d)$ by

$$T(x,y) = \bigl(x, y/\rho(x)\bigr).$$

Then T is a bijection of R onto D with the Jacobian determinant $|J_T(x,y)| = 1/\rho(x)$. Since, by (4.5), $p_{m-i}(i;x)p_{n-j}(j;x)\rho^{i+j+1}(x)w_1(x) \in L^1(a,b)$ and $q_i(y)q_j(y)w_2(y) \in L^1(c,d)$, we have, by change of variables,

$$\begin{aligned}
&\int_a^b p_{m-i}(i;x)p_{n-j}(j;x)\rho^{i+j+1}(x)w_1(x)\,dx \int_c^d q_i(y)q_j(y)w_2(y)\,dy \\
&\quad = \iint_D F(x,y)\,dx\,dy \\
&\quad = \iint_R (F\circ T)(x,y)\bigl|J_T(x,y)\bigr|\,dx\,dy \\
&\quad = \iint_R p_{m-i}(i;x)p_{n-j}(j;x)\rho^{i+j}(x)w_1(x)q_i \\
&\qquad \times \bigl(y/\rho(x)\bigr)q_j\bigl(y/\rho(x)\bigr)w_2\bigl(y/\rho(x)\bigr)\,dx\,dy \\
&\quad = \iint_R \phi_{m-i,i}(x,y)\phi_{n-j,j}(x,y)w(x,y)\,dx\,dy,
\end{aligned}$$

where $F(x,y) = p_{m-i}(i;x)p_{n-j}(j;x)\rho^{i+j+1}(x)w_1(x)q_i(y)q_j(y)w_2(y)$. Then we have (4.4) from the orthogonality of $\{p_n(k;x)\}_{n=0}^{\infty}$ and $\{q_n(y)\}_{n=0}^{\infty}$, which completes the proof of the theorem. □

Remark 4.1. In [19], Xu considered a situation similar to Theorem 4.1 under the stricter assumption that each $w_i(x)$ is positive and nondecreasing for each $i = 1, 2$. We note that Theorem 4.1 can be formulated using quasi-definite moment functionals and more orthogonal polynomials including the twisted Jacobi, twisted Hermite, and Bessel polynomials [10].

Remark 4.2. For the weight function $w(x,y)$ defined in (4.3), we have for $\phi \in \mathcal{P}_2$,

$$\bigl\langle w(x,y), \phi(x,y)\bigr\rangle = \int_a^b \rho(x)w_1(x)\left[\int_c^d \phi\bigl(x,\rho(x)y\bigr)w_2(y)\,dy\right]dx.$$

In particular, if we take $\phi(x,y) = x^m y^n$, we have the following expression for the moments of $w(x,y)$:

$$\begin{aligned}
w_{m,n} &:= \bigl\langle w(x,y), x^m y^n\bigr\rangle \\
&= \int_a^b x^m \rho^{n+1}(x)w_1(x)\,dx \int_c^d y^n w_2(y)\,dy \quad (m,n \in \mathbb{N}_0).
\end{aligned} \tag{4.6}$$

If at least one of these iterated integrals is not well defined, then we agree that these iterated integrals will be regularized (see [6]).

Consider the second-order partial differential equation

$$L[u] := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u, \tag{4.7}$$

where $A(x,y) = ax^2 + d_1x + e_1y + f_1$, $2B(x,y) = 2axy + d_2x + e_2y + f_2$, $C(x,y) = ay^2 + d_3x + e_3y + f_3$, $D(x) = gx + h_1$, $E(y) = gy + h_2$ and $\lambda_n = an(n-1) + gn$; the various letters refer to real constants.

Definition 4.1. (See [8]) The partial differential equation (4.7) is called admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$.

If a PS $\{\Phi_n(x,y)\}$ satisfies a partial differential equation of the form (4.7), then the canonical moment functional σ of $\{\Phi_n(x,y)\}_0^\infty$ satisfies

$$L^*[\sigma] := (A\sigma)_{xx} + 2(B\sigma)_{xy} + (C\sigma)_{yy} - (D\sigma)_x - (E\sigma)_y = 0. \tag{4.8}$$

Furthermore, (4.8) has a unique solution up to a constant multiple if (4.7) is admissible; for further information, see [8,14].

Theorem 4.2. (See [7,14]) *For an OPS $\{\Phi_n(x,y)\}$ relative to a moment functional σ, the following statements are equivalent*:

(i) $\{\Phi_n(x,y)\}$ *satisfies the partial differential equation* (4.7).
(ii) *The moments $\{\mu_{m,n}\}$ of σ, defined by $\mu_{m,n} := \langle\sigma, x^m y^n\rangle$ for each $m,n \in \mathbb{N}_0$, satisfy the recurrence relations*

$$\begin{aligned} 0 = B_{m,n} &:= 2\big[a(m+n)+g\big]\mu_{m,n+1} + me_2\mu_{m-1,n+1} + [md_2 + 2ne_3]\mu_{m,n} \\ &\quad + mf_2\mu_{m-1,n} + 2nf_3\mu_{m,n-1} + 2nd_3\mu_{m+1,n-1} \end{aligned} \tag{4.9}$$

$$\begin{aligned} 0 = C_{m,n} &:= 2\big[a(m+n)+g\big]\mu_{m+1,n} + [2md_1 + ne_2]\mu_{m,n} + nd_2\mu_{m+1,n-1} \\ &\quad + 2mf_1\mu_{m-1,n} + nf_2\mu_{m,n-1} + 2me_1\mu_{m-1,n+1}. \end{aligned} \tag{4.10}$$

(iii) *σ satisfies the moment equations*

$$M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0, \tag{4.11}$$

$$M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0. \tag{4.12}$$

(iv) *$L[\cdot]\sigma$ is formally symmetric on polynomials in the sense that*

$$\big\langle L[p]\sigma, q\big\rangle = \big\langle L[q]\sigma, p\big\rangle \quad \text{for all } p,q \in \mathcal{P}_2.$$

Theorem 4.3. *Let $\{\Phi_n(x,y)\}$ be an OPS satisfying the admissible second-order partial differential equation* (4.7). *Then a nontrivial distribution $w \in D'(\mathbb{R}^2)$ is an orthogonalizing weight of $\{\Phi_n(x,y)\}$ if and only if w has the following properties*:

(i) $\langle w, 1\rangle \neq 0$;
(ii) *w is a distribution which acts on $\mathcal{P}_2$, the space of bivariate polynomials*;
(iii) *for $i = 1, 2$, $M_i[w]$ is a polynomial killer in two variables*; *that is to say,*

$$\big\langle M_i[w], x^m y^n\big\rangle = 0 \quad \textit{for each } m, n \in \mathbb{N}_0\ (i = 1, 2).$$

Proof. ($\Rightarrow$) Let w be an orthogonalizing weight for an OPS $\{\Phi_n(x, y)\}_{n=0}^{\infty}$ satisfying a partial differential equation of the form (4.7). Clearly, (i) and (ii) are obvious. Using Definitions 2.1 and 2.2, part (iii) follows from the observations that

$$-2\big\langle M_2[w], x^m y^n\big\rangle = B_{m,n} = 0,$$

and

$$-2\big\langle M_1[w], x^m y^n\big\rangle = C_{m,n} = 0,$$

where $B_{m,n}$ and $C_{m,n}$ are defined in (4.9) and (4.10).

($\Leftarrow$) This is clear from Theorem 4.2. □

In the following, we show how to construct polynomial killers in two variables using one-variable polynomial killers.

Theorem 4.4. *Let $g : (0, \infty) \to \mathbb{R}$ be a nontrivial function such that $x^n g(x) \in L^1(0, \infty)$ for each $n \in \mathbb{N}_0$ and*

$$\int_0^\infty x^n g(x)\, dx = 0 \quad (n \in \mathbb{N}_0);$$

that is, g generates a polynomial killer. Let $h : (c, d) \to \mathbb{R}$ be any function such that $ph \in L^1(c, d)$ for all $p \in \mathcal{P}_1$. Then

$$K_1(x, y) = g(x)h(y/x), \qquad K_2(x, y) = g(x)(y/x)h(y/x)$$

generate two-dimensional polynomial killers $u_i \in \mathcal{D}'(\check{R})$ in $\check{R} = \{(x, y) \mid x > 0,\ cx < y < dx\}$ in the sense that

$$\big\langle u_i, p(x, y)\big\rangle := \iint_{\check{R}} p(x, y)K_i(x, y)\, d\check{R} = 0 \quad (p \in \mathcal{P}_2;\ i = 1, 2).$$

Proof. Let $\check{D} = \{(x, y) \mid x > 0,\ c < y < d\}$ and $S : \check{R} \to \check{D}$ be the transformation defined by

$$S(x, y) = (x, y/x).$$

Then S is a C^1-bijection of $\check{R}$ onto $\check{D}$ and $|J_S(x, y)| = 1/x$. Since $x^{m+n+1} g(x) \in L^1(0, \infty)$ and $y^n h(y) \in L^1(c, d)$, we have $x^{m+n+1} y^n g(x)h(y) \in L^1(\check{D})$ and

$$0 = \int_0^\infty x^{m+n+1} g(x)\, dx \int_c^d y^n h(y)\, dy = \iint_{\check{D}=S(\check{R})} x^{m+n+1} y^n g(x)h(y)\, dx\, dy$$

$$= \iint_{\check{R}} x^m y^n g(x)h(y/x)\,dx\,dy = \iint_{\check{R}} x^m y^n K_1(x,y)\,dx\,dy.$$

That is, $K_1(x,y)$ generates a polynomial killer in $\check{R}$.

Similarly, we have

$$0 = \int_0^\infty x^{m+n+1} g(x)\,dx \int_c^d y^{n+1} h(y)\,dy = \iint_{\check{D}=S(\check{R})} x^{m+n+1} y^{n+1} g(x)h(y)\,dx\,dy$$
$$= \iint_{\check{R}} x^{m-1} y^{n+1} g(x)h(y/x)\,dx\,dy = \iint_{\check{R}} x^m y^n K_2(x,y)\,dx\,dy,$$

and the proof is complete. □

We now recall a classical result, extending Boas' moment theorem (see [3, pp. 74–75]), on the multi-dimensional moment problem [1] which states that for any sequence $\{\mu_\alpha\}_{|\alpha|=0}^\infty$ of real numbers, there exists a signed Borel measure μ on $\mathbb{R}^k$ such that

$$\mu_\alpha = \int_{\mathbb{R}^k} \mathbf{x}^\alpha\,d\mu(\mathbf{x}) \quad \left(\alpha \in \mathbb{N}_0^k;\ \mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k\right), \tag{4.13}$$

where $\mathbb{N}_0^k := \{(\alpha_1, \alpha_2, \ldots, \alpha_k) \mid \alpha_i \geqslant 0,\ \alpha_i \in \mathbb{N}_0\}$, $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $|\alpha| = \sum_{i=1}^k \alpha_i$. From this fact, the orthogonality for any OPS $\{\Phi_n(\mathbf{x})\}$ in k variables can be written in integral form as

$$\int_{\mathbb{R}^k} \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{x})\,d\mu(\mathbf{x}) = K_\alpha \delta_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{N}_0^k,$$

where μ is a signed Borel measure on $\mathbb{R}^k$, K_α is a nonzero constant for each $\alpha \in \mathbb{N}_0^k$ and $\delta_{\alpha,\beta}$ is the Kronecker Dirac delta function defined by

$$\delta_{\alpha,\beta} = \begin{cases} 0 & (\alpha \neq \beta), \\ 1 & (\alpha = \beta). \end{cases}$$

By Theorem 4.3, in order to find a real-valued weight function for an OPS $\{\Phi_n(x,y)\}$ satisfying a second-order partial differential equation (4.7), we need to solve the system of nonhomogeneous weight equations

$$(Aw)_x + (Bw)_y - Dw = u_1, \tag{4.14}$$
$$(Bw)_x + (Cw)_y - Ew = u_2, \tag{4.15}$$

where each u_i $(i = 1, 2)$ is a two-dimensional polynomial killer. Conversely, if we can find a distribution w, which acts on $\mathcal{P}_2$, satisfying Eqs. (4.14) and (4.15), then w is a real weight for an OPS $\{\Phi_n(x,y)\}_{n=0}^\infty$.

Similar to the situation with one-variable classical orthogonal polynomials, the weight function for a positive-definite bivariate OPS can be found by simultaneously solving (4.14) and (4.15) with $u_i \equiv 0$ in the distributional sense. In the case of a quasi-definite (specifically, non positive-definite) OPS, however, the simultaneous distributional solution of (4.14) and (4.15) with $u_i \equiv 0$ does not necessarily produce a real-valued weight for an OPS (see [14]).

Before discussing explicit weight functions for the four bivariate OPS's of Bessel type for the partial differential equations (1.2)–(1.4), and (1.5), we discuss further pertinent information concerning the Bessel polynomials $\{y_n^a(x)\}$ that we will require in the constructions below.

For $-(a+1) \notin \mathbb{N}$, Favard's theorem [3, pp. 21–22] and a result of Duran [4] guarantees that the Bessel polynomials $\{y_n^a(x)\}$ are orthogonal on the real line with respect to signed weight function from the Schwarz class $\mathcal{S}(\mathbb{R})$; furthermore, any such weight function must necessarily satisfy the distributional differential equation

$$x^2 w' - (ax+2)w = u,$$

where u is a polynomial killer. For $a \geqslant 12(2/\pi)^4 - 2 \simeq -0.02893$, Maroni constructed the following explicit weight function $w_a(x)$ for $\{y_n^a(x)\}$:

$$w_a(x) = \begin{cases} 0, & x \leqslant 0, \\ x^a \exp(-2/x) \int_x^\infty t^{-a-2} \exp(2/t - t^{1/4}) \sin(t^{1/4})\, dt, & x > 0, \end{cases}$$

this result generalizes the work of Kwon et al. [9].

Let $\varpi^{(a,b)}(t)$ be a distributional weight in Schwartz space for the Bessel polynomials $\{B_n^{(a,b)}(t)\}_{n=0}^\infty$ satisfying the following nonhomogeneous differential equation:

$$\left(t^2 \varpi^{(a,b)}(t)\right)' = [at+b]\varpi^{(a,b)}(t) + G^{(a,b)}(t), \tag{4.16}$$

where $G^{(a,b)}(t)$ is a nonzero polynomial killer with $a \neq 0, -1, \ldots$ and $b \neq 0$. Such a distribution $\varpi^{(a,b)}$ does exist (see Section 3). The moments $\{u_n^{(a,b)}\}_{n=0}^\infty$ of $\varpi^{(a,b)}(t)$ satisfy the recurrence relation

$$(n+a)u_{n+1}^{(a,b)} + bu_n^{(a,b)} = 0 \quad (n \geqslant 0), \tag{4.17}$$

where $u_n^{(a,b)} := \langle \varpi^{(a,b)}, t^n \rangle = \int_0^\infty t^n \varpi^{(a,b)}(t)\, dt$. In fact, we have

$$u_n^{(a,b)} = \frac{\Gamma(a)(-b)^n}{\Gamma(n+a)}.$$

5. Examples

We are now in position to construct orthogonalizing weights for the polynomial solutions to Eqs. (1.2)–(1.5) for certain ranges of the equations' parameters.

Example 5.1. We first consider the second-order partial differential equation defined in (1.2) where we assume that $g \notin \{1, 0, -1, \ldots\}$. As mentioned earlier, Krall and Sheffer showed that this equation has at least a WOPS of solutions. In fact, as shown in [10], Eq. (1.2) has an OPS $\{\Phi_n(x,y)\}_{n=0}^\infty$ of Bessel type as solutions if $g+n \neq 0$ and $g\gamma + n \neq 0$ for each $n \in \mathbb{N}_0$. In fact, the polynomial solutions, corresponding to the eigenvalue parameter $\lambda = \lambda_n = n(n-1-g)$, are specifically given by

$$\phi_{n-k,k}(x,y) = B_{n-k}^{(g+2k,-g)}(x)x^k L_k^{(g\gamma-1)}(gy/x) \quad (n \in \mathbb{N}_0,\ 0 \leqslant k \leqslant n),\ x, y > 0, \tag{5.1}$$

where $\{B_{n-k}^{(g+2k,-g)}(x)\}$ and $\{L_k^{g\gamma-1}(y)\}$ are Bessel and Laguerre polynomials, respectively (see Section 3). For $g, \gamma > 0$, we define the function $w(x,y)$ on $D = \{(x,y) \mid x, y > 0\}$ by

$$w(x,y) = \varpi^{(g-1,-g)}(x)e^{-g\frac{y}{x}}x^{1-g\gamma}y^{g\gamma-1}. \tag{5.2}$$

Then

$$\begin{aligned} M_1\big[w(x,y)\big] &= \big(x^2w\big)_x + (xyw)_y - (gx-g)w \\ &= G^{(g-1,-g)}(x)e^{-g\frac{y}{x}}x^{1-g\gamma}y^{g\gamma-1}, \\ M_2\big[w(x,y)\big] &= (xyw)_x + \big(\big(y^2-y\big)w\big)_y - (gy-g\gamma)w \\ &= G^{(g-1,-g)}(x)e^{-g\frac{y}{x}}x^{2-g\gamma}y^{g\gamma-2} \end{aligned}$$

are both polynomial killers in $D = \{(x,y) \mid x, y > 0\}$ by Theorem 4.4. Thus $w(x,y)$ is a real-valued weight function for the OPS $\{\Phi_n(x,y)\}_{n=0}^\infty$ defined by (5.1).

Next, we claim that if $g > 0$ and $\gamma < 0$ with $g\gamma \notin \{0, -1, -2, \ldots\}$, then the regularization $w_+(x,y)$ of (5.2) defined by

$$\big\langle w_+(x,y), \phi(x,y)\big\rangle = \int_0^\infty x\varpi^{(g-1,-g)}(x)\big\langle y_+^{g\gamma-1}, e^{-gy}\phi(x,xy)\big\rangle\,dx \tag{5.3}$$

is a real weight for an OPS $\{\Phi_n(x,y)\}_{n=0}^\infty$ in (5.1). (See [6] or [16] for the definition of the regularization $y_+^{g\gamma-1}$ of $y^{g\gamma-1}$.)

If we set

$$v_n := \big\langle y_+^{g\gamma-1}, e^{-gy}y^n\big\rangle \quad (n \in \mathbb{N}_0),$$

then the sequence $\{v_n\}_{n=0}^\infty$ satisfies the recurrence relation

$$gv_{n+1} - (n+g\gamma)v_n = 0 \quad (n \in \mathbb{N}_0), \tag{5.4}$$

and the moments of $w_+(x,y)$, by (4.6), can be written as

$$w_{m,n} := \big\langle w_+(x,y), x^my^n\big\rangle = u_{m+n+1}^{(g-1,-g)}v_n$$

since $\rho(x) = x$. Using the definition (2.1) for the derivatives of moment functionals, we see that by using (4.17) and (5.4),

$$\begin{aligned} \big\langle M_1\big[w_+(x,y)\big], x^my^n\big\rangle &= \big\langle\big(x^2w_+\big)_x + (xyw_+)_y - (gx-g)w_+, x^my^n\big\rangle \\ &= -(m+n+g)\big\langle w_+(x,y), x^{m+1}y^n\big\rangle + g\big\langle w_+(x,y), x^my^n\big\rangle \\ &= \big[-(m+n+g)u_{m+n+2}^{(g-1,-g)} + gu_{m+n+1}^{(g-1,-g)}\big]v_n \\ &= 0, \end{aligned}$$

and

$$\begin{aligned}
\langle M_2[w_+(x,y)], x^m y^n\rangle &= \langle (xyw_+)_x + ((y^2-y)w_+)_y - (gy - g\gamma)w_+, x^m y^n\rangle \\
&= -(m+n+g)\langle w_+(x,y), x^m y^{n+1}\rangle \\
&\quad + (n+g\gamma)\langle w_+(x,y), x^m y^n\rangle \\
&= -(m+n+g)u_{m+n+2}^{(g-1,-g)} v_{n+1} + (n+g\gamma)u_{m+n+1}^{(g-1,-g)} v_n \\
&= \frac{(n+g\gamma)}{g}\left[-(m+n+g)u_{m+n+2}^{(g-1,-g)} + g u_{m+n+1}^{(g-1,-g)}\right] v_n \\
&= 0.
\end{aligned}$$

Consequently, $M_i[w_+(x,y)]$ $(i=1,2)$ is a polynomial killer. Thus, for $g>0$, $\gamma<0$ and $g\gamma \notin \{0,-1,-2,\ldots\}$, we see from Theorem 4.3 that $w_+(x,y)$ is a real weight for the OPS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ defined in (5.1).

Example 5.2. We now consider the second-order partial differential equation defined in (1.3), where we assume $g \notin \{1,0,-1,\ldots\}$. Krall and Sheffer showed that this equation has at least a WOPS as solutions but failed to show that these polynomial solutions are orthogonal. In [10], the authors showed that (1.3) has an OPS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ of Bessel type of solutions if $g+n \neq 0$ for each $n \in \mathbb{N}_0$; more precisely, this OPS is explicitly defined in $D = \{(x,y) \mid x, y > 0\}$ by

$$\phi_{n-k,k}(x,y) = \begin{cases} B_{n-k}^{(g+2k,-g)}(y)y^k H_k\left(\sqrt{-\frac{g}{2}}\frac{x}{y}\right) & \text{if } g<0, \\ B_{n-k}^{(g+2k,-g)}(y)y^k \check{H}_k\left(\sqrt{\frac{g}{2}}\frac{x}{y}\right) & \text{if } g>0, \end{cases} \tag{5.5}$$

where $\{B_{n-k}^{(g+2k,-g)}(y)\}$ and $\{H_k(x)\}$ are Bessel and Hermite polynomials, respectively. For each $n \in \mathbb{N}_0$ and $0 \leqslant k \leqslant n$, $\phi_{n-k,k}(x,y)$ satisfies (1.3) when $\lambda = \lambda_n = n(n-1+g)$. For $g<0$, we see from Theorem 4.3 that the function $w(x,y)$ defined by

$$w(x,y) = e^{\frac{gx^2}{2y^2}} \varpi^{(g-1,-g)}(y)$$

is a real-valued weight function for the OPS in (5.5) since

$$M_1[w(x,y)] = ((x^2+y)w)_x + (xyw)_y - gxw = \frac{x}{y} G^{(g-1,-g)}(y) e^{\frac{gx^2}{2y^2}}$$

and

$$M_2[w(x,y)] = (xyw)_x + (y^2 w)_y - (gy-g)w = G^{(g-1,-g)}(y) e^{\frac{gx^2}{2y^2}}$$

are both polynomial killers in $D = \{(x,y) \mid x, y > 0\}$ by Theorem 4.4. In the case that $g>0$, we are unable, at this time, to find a real weight function for this OPS since the real weight function for the twisted Hermite polynomial is unknown at this time.

We remark that the constructions in Examples 5.1 and 5.2 answer open questions posed by Krall and Sheffer in 1967 (see [8] and [14]).

Lastly, we construct real-valued weights for the two OPS's of Bessel type discovered recently in [10].

Example 5.3. Consider the partial differential equation defined in (1.4). In this case, this equation has an OPS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ of Bessel type as solutions if $d+n\neq 0$, $e-n\neq 0$, $d+e+n\neq 0$ for each $n\in\mathbb{N}_0$ and $h\neq 0$, defined explicitly by

$$\phi_{n-k,k}(x,y)=P_{n-k}^{(d+e+2k-1,-e-1)}(2x-1)(1-x)^k B_k^{(d+e,h)}\big(y/(1-x)\big) \\ (0\leqslant k\leqslant n); \tag{5.6}$$

here $\{P_{n-k}^{(d+e+2k-1,-e-1)}(x)\}$ and $\{B_k^{(d+e,h)}(y)\}$ are, respectively, Jacobi and Bessel polynomials. The eigenvalue parameter corresponding to $\phi_{n-k,k}(x,y)$ is given by $\lambda=\lambda_n=n(n-1+d)$. We define a distribution w_1 on the interval $[0,1]$ by

$$\big\langle w_1,\phi(x)\big\rangle=\begin{cases}\int_0^1 x^{-e-1}(1-x)^{d+e-2}\phi(x)\,dx & \text{if } d+e>-1,\ e<0,\\ \langle W_p[x^{-e-1}(1-x)^{d+e-2}],\phi(x)\rangle & \text{otherwise,}\end{cases}$$

where $W_P[x^{-e-1}(1-x)^{d+e-2}]$ is the distribution obtained by regularizing the integral

$$\int_0^1 x^{-e-1}(1-x)^{d+e-2}\phi(x)\,dx$$

on the interval $[0,1]$ (see [16]). The moments $v_n:=\langle w_1,x^m\rangle$ of w_1 satisfy the recurrence relation

$$(n+d-1)v_{n+1}-(n-e)v_n=0,$$

and are given by

$$v_n=\frac{\Gamma(n-e)\Gamma(d+e-1)}{\Gamma(n+d-1)}\quad (n\in\mathbb{N}_0).$$

Now define the distribution w on $D=\{(x,y)\mid 0<x<1,\ y>0\}$ by

$$\big\langle w,\phi(x,y)\big\rangle=\big\langle w_1,(1-x)\big\langle\varpi^{(d+e,h)}(y),\phi(x,(1-x)y)\big\rangle\big\rangle.$$

Then a calculation shows that the moments of w are given by

$$\begin{aligned}w_{m,n}&=\big\langle w,x^my^n\big\rangle=\big\langle w_1(x),x^m(1-x)^{n+1}\big\rangle\big\langle\varpi^{(d+e,h)}(y),y^n\big\rangle\\&=\frac{\Gamma(m-e)\Gamma(d+e+n)}{\Gamma(m+n+d)}u_n^{(d+e,h)}.\end{aligned}\tag{5.7}$$

Consequently, from (5.7), we see that

$$\begin{aligned}\big\langle M_1[w],x^my^n\big\rangle&=\big\langle\big(\big(x^2-x\big)w\big)_x+(xyw)_y-(dx+e)w,x^my^n\big\rangle\\&=-\big[(m+n+d)w_{m+1,n}+(e-m)w_{m,n}\big]\\&=-(m+n+d)\frac{\Gamma(m-e+1)\Gamma(d+e+n)}{\Gamma(m+n+d+1)}v_n\\&\quad-(e-m)\frac{\Gamma(m-e)\Gamma(d+e+n)}{\Gamma(m+n+d)}v_n\\&=0,\end{aligned}$$

and

$$\begin{aligned}
\big\langle M_2[w], x^m y^n\big\rangle &= \big\langle (xyw)_x + \big(y^2 w\big)_y - (dy+h)w, x^m y^n\big\rangle \\
&= -(m+n+d)w_{m,n+1} - hw_{m,n} \\
&= -(m+n+d)\frac{\Gamma(m-e)\Gamma(d+e+n+1)}{\Gamma(m+n+d+1)}v_{n+1} \\
&\quad - h\frac{\Gamma(m-e)\Gamma(d+e+n)}{\Gamma(m+n+d)}v_n \\
&= -\frac{\Gamma(m-e)\Gamma(d+e+n)}{\Gamma(m+n+d)}\big[(d+e+n)v_{n+1} + hv_n\big] \\
&= 0.
\end{aligned}$$

Thus, by Theorem 4.3, w is a real weight function for the OPS defined in (5.6).

Example 5.4. We now consider the partial differential equation defined in (1.5). In [10], the authors showed that this equation has an OPS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ of Bessel type as solutions if $d \neq 0$, $e \notin \{1, 0, -1, \ldots\}$, and $h \neq 0$, defined explicitly by

$$\phi_{n-k,k}(x,y) = L_{n-k}^{(e+2k-1)}(-dx)x^k B_k^{(e,-h)}\left(\frac{y}{x}\right) \quad (0 \leqslant k \leqslant n); \tag{5.8}$$

here, as before, the polynomials involved in (5.8) are Laguerre and Bessel polynomials. In this case, the eigenvalue parameter λ corresponding to $\phi_{n-k,k}(x,y)$ is given by $\lambda = \lambda_n = nd$. Assume that $d < 0$ and define the function $w(x,y)$ on $D = \{(x,y) \mid x, y > 0\}$ by

$$w(x,y) = \begin{cases} x^{e-2}e^{dx}\varpi^{(e,-h)}(y/x) & \text{if } e > 1, \\ x_+^{e-2}e^{dx}\varpi^{(e,-h)}(y/x) & \text{if } e < 1 \text{ and } e \notin \{1, 0, -1, \ldots\}. \end{cases} \tag{5.9}$$

If we put

$$v_m := \begin{cases} \int_0^{\infty} x^{e-2}e^{dx}x^m\,dx & \text{if } e > 1, \\ \langle x_+^{e-2}, x^m e^{dx}\rangle & \text{if } e < 1 \text{ and } e \notin \{1, 0, -1, \ldots\}, \end{cases}$$

then we see that:

(a) $\{v_m\}_{m=0}^{\infty}$ satisfies the recurrence relation (for the Laguerre polynomials)

$$dv_{m+1} - (1-e-m)v_m = 0, \tag{5.10}$$

(b) $\{u_m^{(e,-h)}\}_0^{\infty}$ satisfies the recurrence relation (for the Bessel polynomials)

$$(n+e)u_{n+1}^{(e,-h)} - hu_n^{(e,-h)} = 0, \tag{5.11}$$

(c) $w_{m,n} := \langle w, x^m y^n\rangle = v_{m+n+1}u_n^{(e,-h)}$ by (4.6).

We claim that $w(x,y)$, defined in (5.9), is a real-valued weight function for the OPS defined in (5.8); from Theorem 4.3, it suffices to show that $M_i[w(x,y)]$ $(i = 1, 2)$ are polynomial killers in D. Using (5.10) and (5.11), we see that

$$\begin{aligned}\big\langle M_1\big[w(x,y)\big], x^m y^n\big\rangle &= -d\big\langle w, x^{m+1}y^n\big\rangle - (m+n+e)\big\langle w, x^m y^n\big\rangle\\ &= -dw_{m+1,n} - (m+n+e)w_{m,n}\\ &= \big[-dv_{m+n+2} - (m+n+e)v_{m+n+1}\big]u_n^{(e,-h)}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\big\langle M_2\big[w(x,y)\big], x^m y^n\big\rangle &= -m\big\langle w, x^{m-1}y^{n+1}\big\rangle - d\big\langle w, x^m y^{n+1}\big\rangle - h\big\langle w, x^m y^n\big\rangle\\ &= -mw_{m-1,n+1} - dw_{m,n+1} - hw_{m,n}\\ &= -(mv_{m+n+1} + dv_{m+n+2})u_{n+1}^{(e,-h)} - hv_{m+n+1}u_n^{(e,-h)}\\ &= 0.\end{aligned}$$

Consequently, $w(x,y)$ is a real-valued weight function for the OPS defined in (5.8).

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