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Alternation Theorems for Functions of Several Variables

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1. INTRODUCTION

The classical Chebyshev alternation theorems characterize the best uniform approximation to a continuous real valued function F by functions f in a specified subspace M, by the oscillating nature of the difference $F(x)-f(x)$. For example, if M is a unisolvent linear space of functions on the closed interval $[0,1]$, if M has dimension N (e.g. M consists of the polynomials in x of degree at most $N-1$), and if $f\in M$ has the property that for some particular $N+1$ points $x_i\in[0,1]$,

$$F(x_i)-f(x_i)=(-1)^i\rho, \qquad i=1,2,\ldots,N+1,$$

where

$$\rho=\|F-f\|=\max_{0\leqslant x\leqslant 1}|F(x)-f(x)|,$$

then f is the unique best uniform approximation to F on $[0,1]$. (An interesting treatment may be found in [*6*].)

The crucial property used in the proof seems to be the unisolvence of M, which is equivalent to

(1) *If $f\in M$ and $x_1,x_2,\ldots,x_N$ are distinct points of* $[0,1]$ *such that $f(x_i)=0$ for $i=1,2,\ldots,N$, then $f\equiv 0$;*

as well as to

(2) *Given distinct points $x_1,x_2,\ldots,x_N$ in* $[0,1]$, *and real constants c_i, there exists a unique function $f\in M$ such that $f(x_i)=c_i$ for $i=1,2,\ldots,N$.*

The classical alternation theorems belong solely to the study of functions of one variable; the basic reason for this is probably the Mairhuber characterization theorem which shows that the notion of unisolvence is essentially restricted to functions of one variable (see [*4*], [*5*]). If X is a compact connected subset of R^n and if $C[X]$ contains a unisolvent linear subspace of finite dimension at least 2, then X is homeomorphic either to the unit interval or the unit circumference.

In spite of this, we shall in this note obtain some general alternation type theorems applying to *any* finite dimensional subspace M of $C[X]$ for X a cell in R^n, $n \geqslant 2$. Our principal result is the following

THEOREM 1. *Let M be a subspace of $C[X]$ with* $\dim(M) = N$. *Then, $2r$ points $\{p_i\}_1^r$, $\{q_i\}_1^r$ can be selected in X with the property that if $F \in C[X]$ and $f \in M$, and*

$$F(p_i) - f(p_i) = \rho \qquad i = 1, 2, 3, \ldots, r$$

$$F(q_i) - f(q_i) = -\rho \qquad i = 1, 2, 3, \ldots, r$$

where $\rho = \|F - f\|$, then f is a best uniform approximation to F on X. (Here, $r \leqslant N$.)

We shall prove this with $r = N$, and in this case also obtain the fact that f will then be the *unique* best approximation to F. Since uniqueness is an uncommon event in the approximation of functions of several variables, this is a very convincing argument that $r = N$ is much too large. This is also supported by the special cases that are examined in the present paper, and by certain observations following the proof of Theorem 1. This suggests the following conjecture for functions of n variables, $n \geqslant 2$, defined on an n-cell.

Conjecture. In general, the total number of alternation points can be reduced to N when N is even, and to $N + 1$ when N is odd. (Thus, $r = [(N + 1)/2]$.)

The division into even and odd dimension would seem, from the examples given in Section 5, to be essential. However, there may exist pathological choices for M in which many fewer alternation points are needed. Nor does it seem that such alternation conditions are necessary, or that any general statement can be made about the cardinal number or structure of the set of $p \in X$ where $|F(p) - f(p)| = \|F - f\|$, where f is an optimal approximation to F, and F and M are arbitrary.

This approach is very closely related to the H-sets studied by Collatz (see [*3*]).

2. PROOF OF THEOREM 1

The following result was proved in [*2*] (see also [*1*]). In effect, it recovers a portion of the unisolvence property for a general subspace M.

LEMMA. *Let M be a subspace of $C[X]$ of dimension N. Then, there are nonempty sets $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$, disjoint and open in X, such that*

(3) *If $f \in M$, x_i is any point in $\mathcal{O}_i$, and $f(x_i) = 0$ for $i = 1, 2, \ldots, N$, then $f \equiv 0$.*

(4) *Given points $x_i \in \mathcal{O}_i$, and real constants c_i, there is a unique $f \in M$ such that $f(x_i) = c_i$ for $i = 1, 2, \ldots, N$.*

We use this Lemma to prove Theorem 1. We may choose the sets $\mathcal{O}_i$ to be open in R^n and connected, and such that their closures are disjoint and have the property described in the Lemma. Choose two distinct points p_i, q_i in $\mathcal{O}_i$, and let β_i be an arc in $\mathcal{O}_i$ from p_i to q_i. We therefore have $2r$ points of X, with $r = N$. Suppose that $F \in C[X], f_0 \in M$, and that

$$F(p_i) - f_0(p_i) = \rho \qquad i = 1, 2, \ldots, N$$

$$F(q_i) - f_0(q_i) = -\rho \qquad i = 1, 2, 3, \ldots, N$$

where $\rho = \|F - f_0\|$. Let f^* be any optimal approximation to F on X, so that $\|F - f^*\| = \rho_M(F) = \inf\limits_{f \in M} \|F - f\| \leqslant \rho$. Set $g = f^* - f_0 = (f^* - F) + (F - f_0)$.

Then,

$$g(p_i) = (f^* - F)(p_i) + \rho \geqslant -\rho_M(F) + \rho \geqslant 0$$

$$g(q_i) = (f^* - F)(q_i) - \rho \leqslant \rho_M(F) - \rho \leqslant 0.$$

But, p_i and q_i are the ends of the arc β_i in $\mathcal{O}_i$. Either g is 0 at an end point, or g changes sign on β_i and must have a zero somewhere on β_i. In either case, g has a zero somewhere in the closure of $\mathcal{O}_i$. Since $g \in M$, $g \equiv 0$ and $f_0 = f^*$.

Note that the *uniqueness* of best approximation was obtained by the initial step of shrinking the original sets $\mathcal{O}_i$ of the Lemma, obtaining new open sets whose closures were disjoint and which had the same unisolvence property. It is very suggestive to examine the effect of taking the $\mathcal{O}_i$ as large as possible. Suppose that N is even, and that the sets $\mathcal{O}_i$ can be taken large enough so that they are mutually disjoint, but their boundaries have common points as shown in Figure 1 (for $N = 4$).

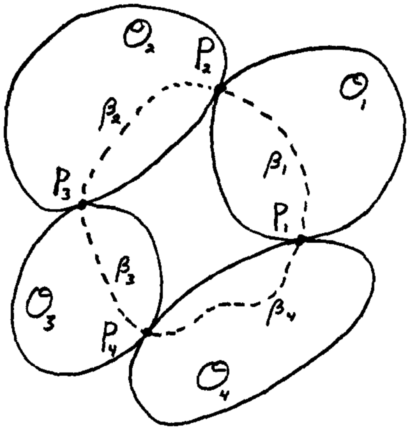

FIG. 1

It is then clear that we can choose these common points as the alternation points, in effect coalescing pairs of p_i and pairs of q_i, and reduce their total number from $2N$ to N. In all the cases I have studied, this simplification is possible by a proper choice of the sets $\mathcal{O}_i$. Since their closures will *not* be unisolvence sets, we must carry through the argument of the proof using strict inequality, thereby proving only that $\rho = \rho_M(F)$, so that f_0 is *a* best approximation, but it is not necessarily true that $f = f^*$.

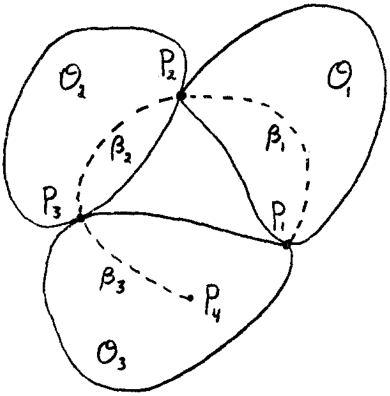

FIG. 2

Likewise, when N is odd, and the sets $\mathcal{O}_i$ can be chosen so that they have touching boundaries as shown in Figure 2, then in addition to the common boundary points, one extra point of $\mathcal{O}_N$ may be selected in order to have the desired behavior on the arcs β_i; in effect, we have merged the original points in pairs, but have one point left over. Thus, in this case, we have been able to reduce the total number of alternation points required to $N + 1$.

In the next sections, we examine in detail certain very simple cases where such sets $\mathcal{O}_i$ can be found explicitly.

3. SPECIAL CASES: $N = 1$

As yet, special methods must be used to determine optimal unisolvence sets $\mathcal{O}_i$ for a specific function subspace M, so that Theorem 1 can be obtained in its sharper form, with $r = [(N+1)/2]$. In general, the starting point is to look at the components of the complement of the zero set of functions $f \in M$, and then construct the sets $\mathcal{O}_i$ as the intersection of certain of these. The unisolvence property can be described by saying that the sets $\mathcal{O}_i$ are such that no zero set of any function $f \in M, f \not\equiv 0$, can touch all of the sets $\mathcal{O}_i$.

Since the points p_i, q_i of Theorem 1 will often be boundary points of the $\mathcal{O}_i$, especially when we are seeking to coalesce them, it is reasonable to select them on the zero set of some function in M. From this viewpoint, the property that lies behind Theorem 1 would be the requirement that if a finite set of points $P_1, P_2, \ldots, P_{2r}$ (properly labeled) lies on the zero set Γ of a function $g_0 \in M$, then no function $g \in M$ can obey $g(P_j)(-1)^j > 0$. Note that if there is an arc γ contained in Γ which passes through the P_j, then one can infer that g must have certain zeros on γ, in number $2r$ if γ is closed, $2r-1$ if γ is not closed. The desired property would then follow if we could argue that any $g \in M$ with this number of zeros in common with g_0, must in fact vanish on Γ.

When $n=2$, zero sets tend normally to be curves, and the intersection of two zero sets is apt to be a finite set, so such an argument is apt to be possible in dealing with functions of only two variables. However, with $n \geqslant 3$, this is no longer the case, and one should expect additional restrictions on the choice of alternation points P_j.

The case $\dim(M)=1$, while very special, casts some light on the general theory. The single unisolvence set $\mathcal{O}$ can be chosen as any component of the set of points $p \in X$ where $\phi(p) \neq 0$, with $M = \{\text{all } f = c\phi,\ c \text{ real}\}$. Condition (3) is clearly obeyed, and the corresponding form of Theorem 1 becomes:

THEOREM 2. *If p, q are in the closure of $\mathcal{O}$, and $f_0 \in M$ satisfies*

$$\begin{aligned} F(p)-f_0(p) &= \|F-f_0\| \\ F(q)-f_0(q) &= -\|F-f_0\| \end{aligned} \tag{5}$$

then f_0 is a best uniform approximation to F among the multiples of ϕ. If both points are in $\mathcal{O}$, then f_0 is the unique best approximation to F.

In this special case, the fact that all functions in M also vanish on the zero set Γ of ϕ, permits Theorem 2 to be strengthened by adding the following statement:

If $p \in \Gamma$ and $|F(p)-f_0(p)| = \|F-f_0\|$, then f_0 is an optimal approximation to F in M.

If both p and q are in the closure of $\mathcal{O}$, but not in Γ, and conditions (5) *hold, then f_0 is the unique best approximation to F.*

Several simple examples will illustrate this. Take X as the unit square $[0,1] \times [0,1]$, and let $\phi(x,y) = x+y$. Here, $\Gamma = \{(0,0)\}$, and $\mathcal{O}$ can be taken as the interior of X. If $F(x,y) = x^2+y^2-\frac{1}{2}$, then we may take $p=(1,1)$ and $q = (\frac{1}{2}\sqrt{6}-1, \frac{1}{2}\sqrt{6}-1)$ as alternation points, with $f(x,y) = (\sqrt{6}-2)(x+y)$ and $\rho = \|F-f\| = (\frac{11}{2}) - 2\sqrt{6}$. Thus, we conclude that f is *the* best approximation to F in M.

With the same choice of X, let $\phi(x,y) = x^2-y$ and let $\mathcal{O}$ be the open subset of X lying above $y=x^2$. With $F(x,y) = x+y$, we find that the optimal approximations are $f = c\phi$, for $-1 \leqslant c \leqslant \frac{1}{2}$. This is true because in each case, $|F(1,1)-f(1,1)| = \|F-f\| = 2$, noting that the point $(1,1)$ lies in the zero set of ϕ. (Note that this shows that there may be only a *single* point in X where $|F-f|$ peaks.)

4. SPECIAL CASES: N EVEN

Let us choose M_1 as the space of all functions

$$f(x, y) = A(x^2 + y^2) + Bx + Cy + D.$$

Take X as any convex set. The zero sets of $f \in M_1$ will be a line, a circle, or all of X. A class of unisolvent sets for M_1 is given by the following:

LEMMA. *Let D_1, D_2, D_3 be open discs such that each of the following sets contains points of X:*

$$\mathcal{O}_1 = D_1 - (D_2 \cup D_3)$$
$$\mathcal{O}_2 = D_2 - (D_3 \cup D_1)$$
$$\mathcal{O}_3 = D_3 - (D_1 \cup D_2)$$
$$\mathcal{O}_4 = D_1 \cap D_2 \cap D_3.$$

Then, any function in M_1 which is zero at some point in each set $\mathcal{O}_i$ is identically zero.

Proof. What must be shown is that no line or circle can pass through a point in each of the sets $\mathcal{O}_i$. Suppose that $P_i \in \mathcal{O}_i$ and that these lie on a circle Γ in the order P_1, P_2, P_3, P_4. Then, the open disc D_j must contain the line segment from P_4 to P_j and it is evident that the disc D_2 must contain either P_1 or P_3, contradicting the fact that the sets $\mathcal{O}_i$ are disjoint.

Using this as suggested in Figure 3 and in the discussion of Theorem 1, we arrive at the following:

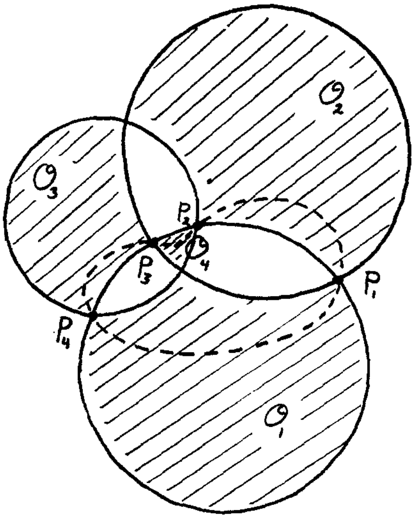

FIG. 3

THEOREM 3. *Let P_1, P_2, P_3, P_4 be concyclic points in X, in this order. Let $F \in C[X]$ and $f \in M_1$ satisfy*

$$F(P_i) - f(P_i) = (-1)^i \|F - f\|.$$

Then, f is an optimal uniform approximation to F in M_1.

A similar theorem can be obtained for the space

$$M_2 = \{\text{all } Axy + Bx + Cy + D\}.$$

Here, the zero sets are a special class of hyperbolas, and the corresponding unisolvence sets are those illustrated in Figure 4.

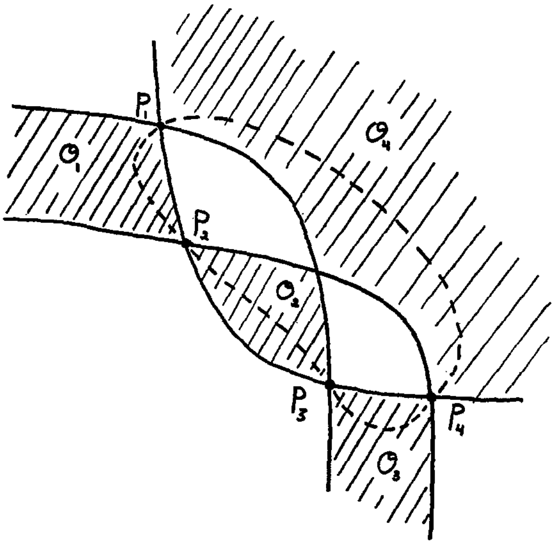

FIG. 4

THEOREM 4. *Let P_1, P_2, P_3, P_4 lie on a branch of a hyperbola with the equation $g_0(x, y) = 0$, $g_0 \in M_2$, $g_0 \not\equiv 0$ the points being labeled in their natural order on this curve. Then, if $F \in C[X]$ and $f \in M_2$ satisfy $F(P_j) - f(P_j) = (-1)^j \|F - f\|$, f is an optimal uniform approximation to F in M_2.*

These examples confirm the conjecture that one needs only N alternation points when N is even, in order to obtain a sufficient characterization of an optimal approximation. However, as observed earlier, alternation is not necessary. This is also shown by the following observation. With the space M_2 above, choose $F(x, y) = x^2 + y^2$ and X as the unit disc. Then, the best approximation to F is easily seen to be the constant function with value $\frac{1}{2}$, and it is not possible to find four distinct points P_i in X which have the alternation property.

In the last two examples, with $n = 2$, the zero sets of functions in M have been curves; it was therefore to be expected that order conditions similar to those of the classical one-variable theory should apply to the choice of the alternation points and the signs of the difference $F(p) - f(p)$. When $n \geqslant 3$, zero sets will in general no longer be curves, and the criteria imposed on the P_j will be more complicated.

Let $M_3 = \{\text{all } A(x^2 + y^2 + z^2) + Bx + Cy + Dz\}$, again with $\dim(M_3) = 4$, and choose X to be a ball of radius R, containing the origin in its interior.

THEOREM 5. *Let* $P_i = (x_i, y_i, z_i)$, $i = 1, 2, 3, 4$, *obey the conditions*:

$$\det \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \neq 0, \tag{6}$$

$$\det \begin{vmatrix} x_1 & y_1 & z_1 & x_1^2 + y_1^2 + z_1^2 \\ x_2 & y_2 & z_2 & x_2^2 + y_2^2 + z_2^2 \\ x_3 & y_3 & z_3 & x_3^2 + y_3^2 + z_3^2 \\ x_4 & y_4 & z_4 & x_4^2 + y_4^2 + z_4^2 \end{vmatrix} = 0, \tag{7}$$

(8) *there are constants* $\alpha_i > 0$ *such that*

$$P_4 = \alpha_1 P_1 - \alpha_2 P_2 + \alpha_3 P_3.$$

Then, if $F \in C[X]$ *and* $f_0 \in M_3$ *satisfy*

$$F(P_i) - f_0(P_i) = (-1)^i \|F - f_0\|,$$

the function f_0 *is an optimal uniform approximation to* F *in* M_3.

Conditions (6) and (7) imply that the points P_i lie in the zero set Γ of a function $g_0 \in M_3$ of the form

$$g_0(x, y, z) = x^2 + y^2 + z^2 - B_0 x - C_0 y - D_0 z.$$

Since this set Γ is a sphere passing through the origin, it is possible to construct a closed path γ on Γ which passes through the points P_i in the order of the subscripts. Condition (8) ensures that no zero set of any $g \in M_3$ can separate $\{P_1, P_3\}$ from $\{P_2, P_4\}$; thus, no $g \in M$ can obey $g(P_i)(-1)^i > 0$. Geometrically, condition (8) means that P_4 lies in the interior of the convex cone generated by P_1, $-P_2$ and P_3, or that no P_i lies in the spherical convex hull of the others.

In the present case, it may be instructive to give a simple algebraic proof of the crucial step. If $g \in M_3$, there are constants a, b, c so that for any point (x, y, z) on the zero set of g_0, $g(x, y, z) = ax + by + cz$. The alternation hypothesis on g is equivalent to $(u \cdot P_i)(-1)^i > 0$, for $i = 1, 2, 3, 4$, where $u = (a, b, c)$. Using (8), we have

$$\begin{aligned} u \cdot P_4 &= \alpha_1(u \cdot P_1) - \alpha_2(u \cdot P_2) + \alpha_3(u \cdot P_3), \\ &< 0 \end{aligned}$$

contradicting $u \cdot P_4 > 0$.

5. Special Cases: N Odd

In this section, we shall illustrate the remarks following the proof of Theorem 1, showing that one should expect $N+1$ alternation points, instead of N, when N is odd. The simplest representative case to study is

$$M_4 = \{\text{all } Ax + By + C\}.$$

The zero sets are lines, and the typical collection of unisolvence sets is that shown in Figure 5. This yields the following:

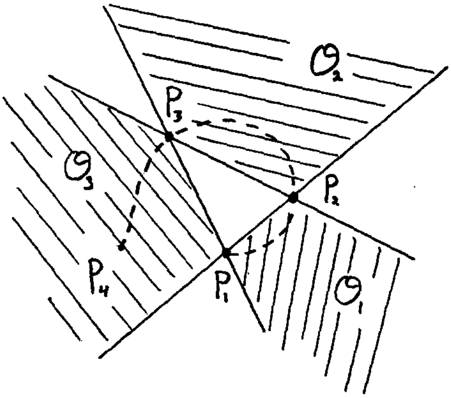

Fig. 5

Theorem 6. *Let X be a convex region in the plane, and let P_i be points interior to X such that*

$$P_4 = \alpha_1 P_1 - \alpha_2 P_2 + \alpha_3 P_3 \tag{9}$$

where $\alpha_i > 0$ and $\alpha_1 - \alpha_2 + \alpha_3 = 1$. Then, if $F \in C[X]$ and $f \in M_4$ satisfy

$$F(P_i) - f(P_i) = (-1)^i \|F - f\|,$$

f is a best uniform approximation to F from M_4 on X.

The special condition (9) on the P_i is equivalent to the geometric condition that no point P_i lies in the triangle determined by the others, together with the correct ordering of subscripts to fit with the signs; what is needed is merely that no line will separate the positive alternation points from the negative points.

As a familiar application, the function $f(x, y) = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{4}$ is a best approximation to $F(x, y) = xy$, with $\|F - f\| = \frac{1}{4}$ and having the points (0,0), (1,0), (1,1), (0,1) of the unit square X as alternation points.

It is not sufficient to have only three alternation points, even though the dimension of M is 3. This can be seen from the example $f(x, y) = \frac{1}{3}x + \frac{2}{3}y - \frac{1}{3}$, where $\|F - f\| = \frac{1}{3}$ and (0,0), (0,1), and (1,1) are alternation points, but f is not an optimal approximation to $F(x, y) = xy$.

However, this leaves many questions about alternation points and unisolvence sets still open. In special cases, the existence of a very small number of alternation points may guarantee that a function f in a space M is in fact a best approximation to a specific function F.

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