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Gaussian density estimates for solutions to quasi-linear stochastic partial differential equations

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Abstract

In this paper we establish lower and upper Gaussian bounds for the solutions to the heat and wave equations driven by an additive Gaussian noise, using the techniques of Malliavin calculus and recent density estimates obtained by Nourdin and Viens in [17]. In particular, we deal with the one-dimensional stochastic heat equation in [0, 1] driven by the space-time white noise, and the stochastic heat and wave equations in $\mathbb{R}^d$ ($d \geq 1$ and $d \leq 3$, respectively) driven by a Gaussian noise which is white in time and has a general spatially homogeneous correlation.

Keywords: Gaussian bounds; Malliavin calculus; Spatially homogeneous Gaussian noise; Stochastic partial differential equations

1. Introduction

The Malliavin calculus, or stochastic calculus of variations, is a suitable technique for proving that a given random vector $F = (F_1, \ldots, F_m)$ on the Wiener space possesses a smooth probability density. There has been current interest in the applications of the stochastic calculus of variations to obtain lower and upper bounds of Gaussian type for the density of a given Wiener functional. The starting point of this research is the work by Kusuoka and Stroock [10], where they proved that the density of a uniformly hypoelliptic diffusion whose drift is a smooth

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combination of its diffusion coefficient has a lower bound of Gaussian type. Recently, three different approaches have been developed to derive Gaussian-type bounds for densities of general Wiener functionals using the techniques of Malliavin calculus:

(i) In the paper [9], Kohatsu-Higa proposes a general methodology for computing lower bounds for a multidimensional functional $F = (F_1, \dots, F_m)$ of a Wiener sheet in $[0, T] \times \mathbb{R}^d$. The method uses an approximation of F by means of a sequence of conditionally non-degenerate random variables adapted to the filtration generated by the white noise. This paper was inspired by the work by Kusuoka and Stroock [10], and it can be used in a nonMarkovian framework. As an application, the author obtains lower bounds for the solution to the one-dimensional heat equation driven by a space-time white noise, assuming that the diffusion coefficient σ is bounded away from zero. The ideas introduced in this paper were later developed in the work by Bally [1] to get Gaussian lower bounds for locally elliptic Itô processes.
(ii) Nourdin and Viens in [17] have proved a new formula for the density of a one-dimensional Wiener functional in terms of the Malliavin calculus (see (2.2) below). As an application, they obtain upper and lower Gaussian bounds for the density of the maximum of a general Gaussian process.
(iii) In [12] Malliavin and Nualart derived Gaussian lower bounds for multidimensional Wiener functionals under an exponential moment condition on the divergence of a covering vector field. A one-dimensional version of this result was obtained by Nualart in [19].

The purpose of this paper is to apply the results obtained by Nourdin and Viens in [17] to the solutions of different classes of stochastic partial differential equations driven by an additive Gaussian noise. Upper and lower Gaussian estimates for the density of solutions to stochastic partial differential equations are essential tools in the potential analysis for these type of equations (see the recent works [5–7]).

The paper is organized as follows. In Section 2 we recall briefly the results of [17]. Section 3 deals with the one-dimensional heat equation on the interval [0, 1] with Dirichlet boundary conditions and driven by a space-time white noise. Let us denote by $u(t, x)$ its solution evaluated at some $(t, x) \in \mathbb{R}_+ \times (0, 1)$ and set $m = E|u(t, x)|$. Then, we derive a Gaussian lower and upper bound for the density of $u(t, x)$ of the form

$$Ct^{-\frac{1}{2}} \exp\left(-\frac{(z-m)^2}{C' t^{\frac{1}{2}}}\right). \tag{1.1}$$

The lower bound differs from that obtained by Kohatsu-Higa in the paper [9], where the first factor was $Ct^{\frac{1}{4}}$.

Section 3 is devoted to the stochastic heat equation in $\mathbb{R}^d$ with an additive Gaussian noise which is white in time and it has a spatially homogeneous correlation in space. We assume that the spectral measure of the noise integrates $(1 + |\xi|^2)^{-\eta}$ for some $\eta \in (0, 1)$, and we obtain lower and upper Gaussian bounds for the density of the solution of the form (1.1), that involve the powers t and $t^{1-\eta}$ (see Theorem 4.4 for the precise statement). Let us mention at this point that it has been of much importance the fact that the Malliavin derivative of the solution defines a non-negative process, since it solves a linear parabolic equation.

Finally, in Section 4 we consider a stochastic wave equation in dimension $d = 1, 2, 3$, again driven by an additive Gaussian noise which is white in time and it has a spatially homogeneous correlation in space. In this case we obtain lower and upper bounds only for t small enough, and they involve the powers t^3 and $t^{3-2\eta}$, where again η is given by the integrability of the spectral measure. We have not been able to overcome this technical restriction on the time parameter because, in comparison with the heat equation, the Malliavin derivative of the solution in this case does not need to be a non-negative function neither a function itself.

2. Gaussian density estimates

In this section, we recall a general method set up in [17] in order to show that a smooth random variable in the sense of Malliavin calculus has a probability density admitting Gaussian lower and upper bounds.

First of all, let us briefly describe the Gaussian context in which we will be working on and the main elements of the Malliavin calculus associated to it. Namely, suppose that in a complete probability space $(\Omega, \mathcal{F}, P)$ we are given a centered Gaussian family $W = \{W(h),\ h \in H\}$ of random variables indexed by a separable Hilbert space H with covariance

$$E(W(h)W(g)) = \langle h, g\rangle_H, \quad h, g \in H.$$

The family W is usually called an isonormal Gaussian process on H. Assume that $\mathcal{F}$ is the σ-field generated by W.

Let us use standard notation for the main operators of the Malliavin calculus determined by the family W (see, for instance, [18]). More precisely, we denote by D the Malliavin derivative, defined as a closed and unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$, whose domain is denoted by $\mathbb{D}^{1,2}$. The adjoint of the operator D is denoted by δ, usually called the divergence operator. A random element $u \in L^2(\Omega; H)$ belongs to the domain of δ if and only if it satisfies

$$|E(\langle DF, u\rangle_H)| \le C\|F\|_{L^2(\Omega)},$$

for any $F \in \mathbb{D}^{1,2}$, where the constant C only depends on u. For any element u in the domain of δ, the random variable $\delta(u)$ can be characterized by the duality relationship

$$E(F\delta(u)) = E(\langle DF, u\rangle_H),$$

for every $F \in \mathbb{D}^{1,2}$.

Any random variable F in $L^2(\Omega, \mathcal{F}, P)$ can be decomposed by means of its Wiener chaos expansion (see [18, Section 1.1.1]), which is usually written as

$$F = \sum_{n=0}^{\infty} J_n F,$$

where J_n denotes the projection onto the nth Wiener chaos.

Using the chaos expansion one may define the operator L by the formula $L = \sum_{n=0}^{\infty} -nJ_n$, which is called the generator of the Ornstein–Uhlenbeck semigroup (see [18, Section 1.4]). It is related to the Malliavin derivative D and its adjoint δ through the formula

$$\delta DF = -LF,$$

in the sense that F belongs to the domain of L if and only if it belongs to the domain of δD, and in this case the above equality holds.

One can also define the inverse of L, denoted by L^{-1}, as follows: for any $F \in L^2(\Omega, \mathcal{F}, P)$, set $L^{-1}F := \sum_{n=1}^{\infty} -\frac{1}{n} J_n F$. Then it holds that $LL^{-1}F = F - E(F)$, for any $F \in L^2(\Omega, \mathcal{F}, P)$, so that L^{-1} acts as the inverse of L for centered random variables.

Let us consider $F \in \mathbb{D}^{1,2}$ with mean zero and we define the following function in $\mathbb{R}$:

$$g(z) := E[\langle DF, -DL^{-1}F\rangle_H | F = z].$$

By [16, Proposition 3.9], it holds that $g(z) \geq 0$ on the support of F. Then, Theorem 3.1 and Corollary 3.3 in [17] state that if the random variable $g(F)$ is bounded away from zero almost surely, that is

$$g(F) \geq c_1 > 0, \quad \text{a.s.},$$

for some constant c_1, then F has a density ρ whose support is $\mathbb{R}$ satisfying, almost everywhere:

$$\rho(z) = \frac{E|F|}{2g(z)} \exp\left(-\int_0^z \frac{y}{g(y)} \mathrm{d}y\right). \tag{2.2}$$

As a consequence, see [17, Corollary 3.5], if one also has that $g(F) \leq c_2$, a.s., then the density ρ satisfies, for almost all $z \in \mathbb{R}$:

$$\frac{E|F|}{2c_1} \exp\left(-\frac{z^2}{2c_2}\right) \leq \rho(z) \leq \frac{E|F|}{2c_2} \exp\left(-\frac{z^2}{2c_1}\right). \tag{2.3}$$

Let us also mention that [17, Proposition 3.7] provides a more suitable formula for $g(F)$ for computational purposes. Indeed, given a random variable $F \in \mathbb{D}^{1,2}$, one can write $DF = \Phi_F(W)$, where Φ_F is a measurable mapping from $\mathbb{R}^H$ to H, determined $P \circ W^{-1}$-almost surely (see [18, pp. 54–55]). Then

$$g(F) = \int_0^\infty \mathrm{e}^{-\theta} \mathbf{E}\left[\langle \Phi_F(W), \Phi_F(\mathrm{e}^{-\theta} W + \sqrt{1-\mathrm{e}^{-2\theta}} W')\rangle_H | F\right] \mathrm{d}\theta, \tag{2.4}$$

where W' stands for an independent copy of W such that W and W' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$. Eventually, $\mathbf{E}$ denotes the mathematical expectation with respect to $P \times P'$.

Formula (2.4) can be still rewritten in the following form:

$$g(F) = \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\langle DF, \widetilde{DF}\rangle_H\right) | F\right] \mathrm{d}\theta, \tag{2.5}$$

where, for any random variable X defined in $(\Omega, \mathcal{F}, P)$, $\widetilde{X}$ denotes the shifted random variable in $\Omega \times \Omega'$

$$\widetilde{X}(\omega, \omega') = X(\mathrm{e}^{-\theta}\omega + \sqrt{1-\mathrm{e}^{-2\theta}}\omega'), \quad \omega \in \Omega,\ \omega' \in \Omega.$$

Notice that, indeed, $\widetilde{X}$ depends on the parameter θ, but we have decided to drop its explicit dependence for the sake of simplicity.

Along the paper we denote by C a generic constant which may vary from line to line.

3. Density estimates for the stochastic heat equation in [0, 1]

In this section, we will consider a stochastic heat equation in [0, 1], with Dirichlet boundary conditions, some non-linear drift b and with an additive space-time white noise perturbation. We aim to give sufficient conditions on the coefficient b ensuring Gaussian lower and upper bounds for the probability density of the solution at any point.

3.1. The stochastic heat equation in [0, 1]

We are concerned with the following one-dimensional heat equation driven by a space-time white noise:

$$\frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) = b(u(t,x)) + \sigma \dot{W}(t,x), \quad (t,x) \in [0,T] \times [0,1], \tag{3.6}$$

where $T > 0$, the initial condition is given by a continuous function $u_0 : [0,1] \to \mathbb{R}$ and we consider Dirichlet boundary conditions. That is,

$$\begin{aligned} &u(0,x) = u_0(x), \quad x \in [0,1], \\ &u(t,0) = u(t,1) = 0, \quad t \in [0,T]. \end{aligned} \tag{3.7}$$

The real-valued random field solution to Eq. (3.6) will be denoted by $\{u(t,x),\ (t,x) \in [0,T] \times [0,1]\}$. The function $b : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ having a bounded derivative and $\sigma > 0$ is a constant. We assume that $\{W(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ is a Brownian sheet on $[0,T] \times [0,1]$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$. That is, $\{W(t,x)\}$ is a centered Gaussian family with the covariance function

$$E(W(t,x)W(s,y)) = (t \wedge s)(x \wedge y).$$

For $0 \le t \le T$, let $\mathcal{F}_t$ be the σ-field generated by the random variables $\{W(s,x),\ (s,x) \in [0,t] \times [0,1]\}$ and the P-null sets.

The solution to the formal equation (3.6) is understood in the mild sense: a $\{\mathcal{F}_t\}$-adapted stochastic process $\{u(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ solves (3.6) with initial and boundary conditions (3.7) if, for any $(t,x) \in (0,T] \times (0,1)$,

$$\begin{aligned} u(t,x) = &\int_0^1 G_t(x,y) u_0(y) \mathrm{d}y + \int_0^t \int_0^1 G_{t-s}(x,y) b(u(s,y)) \mathrm{d}y \mathrm{d}s \\ &+ \sigma \int_0^t \int_0^1 G_{t-s}(x,y) W(\mathrm{d}s, \mathrm{d}y), \end{aligned} \tag{3.8}$$

where $G_t(x,y)$, $(t,x,y) \in \mathbb{R}_+ \times (0,1)^2$, denotes the Green function associated to the heat equation on [0, 1] with Dirichlet boundary conditions. We will use the following facts: for any $0 < t' < t$ and $x \in (0,1)$,

$$0 \le G_t(x,y) \le \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{4t}}, \tag{3.9}$$

$$c_x \sqrt{t-t'} \le \int_{t'}^t \int_0^1 |G_{t-s}(x,y)|^2 \mathrm{d}y \mathrm{d}s \le \frac{1}{\sqrt{2\pi}} \sqrt{t-t'}, \tag{3.10}$$

where c_x is a positive constant. We also have $\inf_{\alpha \le x \le 1-\alpha} c_x > 0$ for any $\alpha \in (0,1)$. In the case of Neumann boundary conditions, c_x does not depend on x (see, for instance, [15, Lemma A1.2]).

Let us also mention that the stochastic integral in (3.8) is understood as an integral with respect to the Brownian sheet in the sense of Walsh [25].

Existence and uniqueness of mild solution for Eq. (3.8) can be deduced from the results in [25]; for an even more general setting, see also [2]. Malliavin calculus applied to Eq. (3.8) have been dealt with in [3]. In this case, we consider the Gaussian context associated to the space-time white noise. That is, we have $H = L^2(\mathbb{R}_+ \times [0,1])$ and the Gaussian family $\{W(h),\ h \in L^2(\mathbb{R}_+ \times [0,1])\}$ is given by the Wiener integral

$$W(h) = \int_{\mathbb{R}_+} \int_0^1 h(s,y) W(\mathrm{d}s, \mathrm{d}y).$$

A consequence of Proposition 4.3 and Theorem 2.2 in [3] is that, for all $(t,x) \in (0,T] \times [0,1]$, the random variable $u(t,x)$ belongs to $\mathbb{D}^{1,2}$, the Malliavin derivative satisfies the linear parabolic equation

$$D_{r,z} u(t,x) = \sigma G_{t-r}(x,z) + \int_r^t \int_0^1 G_{t-s}(x,y) b'(u(s,y)) D_{r,z} u(s,y) \mathrm{d}y \mathrm{d}s, \tag{3.11}$$

for any $(r,z) \in [0,t] \times [0,1]$ and $(t,x) \in [0,T] \times [0,1]$, and the probability law of $u(t,x)$ has a density. The positivity of σ and G guarantees that the solution of Eq. (3.11) remains non-negative, that is $D_{r,z} u(t,x) \geq 0$, a.s. This will be a key point in proving that the density of $u(t,x)$ admits Gaussian upper and lower estimates.

3.2. Gaussian bounds for the density

We fix $T > 0$ and consider $\{u(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ the unique mild solution of Eq. (3.8). This section will be devoted to the proof of the following result.

Theorem 3.1. *Assume that the drift coefficient b is a $\mathcal{C}^1$ function with bounded derivative. Then, for all $t \in (0,T]$ and $x \in (0,1)$, the random variable $u(t,x)$ possesses a density p satisfying the following statement: for almost every $z \in \mathbb{R}$,*

$$\frac{E\,|u(t,x) - m|}{C_2 t^{\frac{1}{2}}} \exp\left\{-\frac{(z-m)^2}{C_1 t^{\frac{1}{2}}}\right\} \leq p(z) \leq \frac{E\,|u(t,x) - m|}{C_1 t^{\frac{1}{2}}} \exp\left\{-\frac{(z-m)^2}{C_2 t^{\frac{1}{2}}}\right\}, \tag{3.12}$$

where $m := E(u(t,x))$ and C_1, C_2 are positive quantities depending on σ, $\|b'\|_\infty$, T and x.

The statement of the above Theorem 3.1 will be a consequence of formula (2.2) (see [17, Theorem 3.1]) and the following proposition. In order not to overload notations, set $F := u(t,x) - E(u(t,x))$, and recall that, as it has been specified in Section 2 (see (2.5) therein),

$$\begin{aligned} g(F) &= \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\int_0^t \int_0^1 D_{r,z} F \left(\widetilde{D_{r,z} F}\right) \mathrm{d}z \mathrm{d}r \right) \Big| F \right] \mathrm{d}\theta \\ &= \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\int_0^t \int_0^1 D_{r,z} u(t,x) \left(\widetilde{D_{r,z} u(t,x)}\right) \mathrm{d}z \mathrm{d}r \right) \Big| F \right] \mathrm{d}\theta, \end{aligned} \tag{3.13}$$

where $\widetilde{DF} = (DF)(\mathrm{e}^{-\theta}\omega + \sqrt{1 - \mathrm{e}^{-2\theta}}\omega')$.

Proposition 3.2. *Fix $T > 0$ and assume that the drift coefficient b is of class $\mathcal{C}^1$ and has a bounded derivative. There exist positive constants C_1, C_2 such that*

$$C_1 t^{\frac{1}{2}} \leq g(F) \leq C_2 t^{\frac{1}{2}}, \tag{3.14}$$

for all $t \in (0,T]$.

In order to prove Proposition 3.2, we will need the following technical lemma:

Lemma 3.3. *Let $t > 0$. Assume that $b \in \mathcal{C}^1$ and it has a bounded derivative. There exists a positive constant K depending on σ and $\|b'\|_\infty$, such that, for any $\delta \in (0, 1]$:*

$$\sup_{\substack{(1-\delta)t \le \nu \le t \\ 0 \le y \le 1}} \int_{(1-\delta)t}^{t} \int_0^1 E\left[\left| D_{r,z} u(\nu, y) \right|^2 \middle| F \right] \mathrm{d}z \mathrm{d}r \le K(\delta t)^{\frac{1}{2}} \tag{3.15}$$

and

$$\sup_{\theta \in \mathbb{R}} \sup_{\substack{(1-\delta)t \le \nu \le t \\ 0 \le y \le 1}} \int_{(1-\delta)t}^{t} \int_0^1 E\left[E'\left(\left| \widetilde{D_{r,z} u(\nu, y)} \right|^2 \right) \middle| F \right] \mathrm{d}z \mathrm{d}r \le K(\delta t)^{\frac{1}{2}}, \tag{3.16}$$

P-almost surely.

Proof. We will only deal with the proof of (3.15), since (3.16) may be checked using exactly the same arguments.

Let us first invoke the linear equation (3.11) satisfied by the Malliavin derivative $Du(\nu, v)$, for $(\nu, v) \in [(1-\delta)t, t] \times (0, 1)$, and then take the square L^2-norm on $[(1-\delta)t, t] \times [0, 1]$:

$$\begin{aligned} \int_{(1-\delta)t}^{t} \int_0^1 |D_{r,z} u(\nu, v)|^2 \mathrm{d}z \mathrm{d}r &\le 2\sigma^2 \int_{(1-\delta)t}^{t} \int_0^1 |G_{\nu-r}(v, z)|^2 \mathrm{d}z \mathrm{d}r \\ &\quad + 2 \int_{(1-\delta)t}^{t} \int_0^1 \left(\int_r^\nu \int_0^1 G_{\nu-s}(v, y) b'(u(s, y)) D_{r,z} u(s, y) \mathrm{d}y \mathrm{d}s \right)^2 \mathrm{d}z \mathrm{d}r. \end{aligned} \tag{3.17}$$

By (3.10), the first term in the right-hand side of (3.17) can be bounded by $\sqrt{\frac{2}{\pi}} \sigma^2 (\delta t)^{\frac{1}{2}}$. For the second one, we apply Hölder's inequality, the fact that b' is bounded and Fubini's theorem, so that we end up with

$$\begin{aligned} \int_{(1-\delta)t}^{t} \int_0^1 |D_{r,z} u(\nu, v)|^2 \mathrm{d}z \mathrm{d}r &\le \sqrt{\frac{2}{\pi}} \sigma^2 (\delta t)^{\frac{1}{2}} + 2\|b'\|_\infty^2 \delta t \int_{(1-\delta)t}^{\nu} \int_0^1 |G_{\nu-s}(v, y)|^2 \\ &\quad \times \left(\int_{(1-\delta)t}^{t} \int_0^1 |D_{r,z} u(s, y)|^2 \mathrm{d}z \mathrm{d}r \right) \mathrm{d}y \mathrm{d}s. \end{aligned}$$

Taking the conditional expectation $E[\cdot|F]$ and using again (3.10) we obtain

$$\begin{aligned} \sup_{\substack{(1-\delta)t \le \rho \le \nu \\ 0 \le v \le 1}} \int_{(1-\delta)t}^{t} \int_0^1 E\left[|D_{r,z} u(\rho, v)|^2 | F \right] \mathrm{d}z \mathrm{d}r &\le \sqrt{\frac{2}{\pi}} \sigma^2 (\delta t)^{\frac{1}{2}} \\ &\quad + 2\|b'\|_\infty^2 \delta t \int_{(1-\delta)t}^{\nu} \left(\sup_{\substack{(1-\delta)t \le \tau \le s \\ 0 \le y \le 1}} \int_{(1-\delta)t}^{t} \int_0^1 E\left[|D_{r,z} u(\tau, y)|^2 | F \right] \mathrm{d}z \mathrm{d}r \right) \frac{1}{\sqrt{\nu - s}} \mathrm{d}s. \end{aligned}$$

If we set

$$\Psi_{\delta,t}(\nu) := \sup_{\substack{(1-\delta)t \le \rho \le \nu \\ 0 \le v \le 1}} \int_{(1-\delta)t}^{t} \int_0^1 E\left[|D_{r,z} u(\rho, v)|^2 | F \right] \mathrm{d}z \mathrm{d}r, \quad \nu \in [(1-\delta)t, t],$$

then we have seen that

$$\Psi_{\delta,t}(\nu) \leq K(\delta t)^{\frac{1}{2}} + K\delta t \int_{(1-\delta)t}^{\nu} \Psi_{\delta,t}(s)\frac{1}{\sqrt{\nu - s}}\mathrm{d}s, \quad \nu \in [(1-\delta)t, t], \quad \text{a.s.}$$

Now we can conclude by applying Gronwall's lemma [4, Lemma 15]. □

Proof of Proposition 3.2. We first recall that the Malliavin derivative of $u(\nu, v)$, $(\nu, v) \in [0, T] \times [0, 1]$, satisfies that $D_{r,z}u(\nu, v) \geq 0$, for all $(r, z) \in [0, T] \times [0, 1]$, a.s. This is because the Malliavin derivative solves the linear parabolic equation (3.11). Let us deal with the proof of (3.14) in two steps:

Step 1: *The lower bound*. Fix $\delta \in (0, 1]$ and let us first derive the lower bound in (3.14). Since the Malliavin derivative of $u(t, x)$ is non-negative, formula (3.13) yields

$$g(F) \geq \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\int_{(1-\delta)t}^t \int_0^1 D_{r,z}u(t,x)\left(\widetilde{D_{r,z}u(t,x)}\right)\mathrm{d}z\mathrm{d}r \right)\Bigg| F\right]\mathrm{d}\theta.$$

By Eq. (3.11), we can decompose the right-hand side of the above inequality in a sum of four terms:

$$A_0(t,x;\delta) = \sigma^2 \int_{(1-\delta)t}^t \int_0^1 |G_{t-r}(x,z)|^2 \mathrm{d}z\mathrm{d}r,$$

$$\begin{aligned} A_1(t,x;\delta) &= \sigma \int_{(1-\delta)t}^t \int_0^1 G_{t-r}(x,z) \\ &\quad \times E\left[\int_r^t \int_0^1 G_{t-s}(x,y) b'(u(s,y)) D_{r,z}u(s,y)\mathrm{d}y\mathrm{d}s \Bigg| F\right]\mathrm{d}z\mathrm{d}r, \\ A_2(t,x;\delta) &= \sigma \int_0^\infty \mathrm{e}^{-\theta} \int_{(1-\delta)t}^t \int_0^1 G_{t-r}(x,z) \\ &\quad \times E\left[E'\left(\int_r^t \int_0^1 G_{t-s}(x,y) b'(\widetilde{u(s,y)})\left(\widetilde{D_{r,z}u(s,y)}\right)\mathrm{d}y\mathrm{d}s \right)\Bigg| F\right]\mathrm{d}z\mathrm{d}r\mathrm{d}\theta, \\ A_3(t,x;\delta) &= \int_0^\infty \mathrm{e}^{-\theta} \int_{(1-\delta)t}^t \int_0^1 E\left[\left(\int_r^t \int_0^1 G_{t-s}(x,y) b'(u(s,y)) D_{r,z}u(s,y)\mathrm{d}y\mathrm{d}s\right)\right. \\ &\quad \times \left. E'\left(\int_r^t \int_0^1 G_{t-s}(x,y) b'(\widetilde{u(s,y)}) \left(\widetilde{D_{r,z}u(s,y)}\right)\mathrm{d}y\mathrm{d}s\right)\Bigg| F\right]\mathrm{d}z\mathrm{d}r\mathrm{d}\theta. \end{aligned}$$

First we notice that, by (3.10):

$$A_0(t,x;\delta) \geq \sigma^2 c_x (\delta t)^{\frac{1}{2}}. \tag{3.18}$$

Thus we can write

$$g(F) \geq \sigma^2 c_x (\delta t)^{\frac{1}{2}} - |A_1(t,x;\delta) + A_2(t,x;\delta) + A_3(t,x;\delta)|, \tag{3.19}$$

so that we will need to obtain upper bounds for the terms $|A_i(t,x;\delta)|$, $i = 1, 2, 3$.

We apply Fubini's theorem for the conditional expectation, the boundedness of b', the Cauchy–Schwarz inequality and the bound (3.10), so we have the following estimates:

$$
\begin{aligned}
|A_1(t,x;\delta)| &\le C\left(\int_{(1-\delta)t}^{t}\int_0^1 |G_{t-r}(x,z)|^2 \mathrm{d}z\mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{(1-\delta)t}^{t}\int_0^1 \left|\int_r^t\int_0^1 G_{t-s}(x,y)E\left[|D_{r,z}u(s,u)|\;|F\right]\mathrm{d}y\mathrm{d}s\right|^2 \mathrm{d}z\mathrm{d}r\right)^{\frac{1}{2}} \\
&\le C(\delta t)^{\frac{3}{4}}\left(\int_{(1-\delta)t}^{t}\int_0^1\left(\int_r^t\int_0^1 |G_{t-s}(x,y)|^2 E\left[|D_{r,z}u(s,y)|^2|F\right]\mathrm{d}y\mathrm{d}s\right)\mathrm{d}z\mathrm{d}r\right)^{\frac{1}{2}} \\
&\le C(\delta t)^{\frac{3}{4}}\left(\int_{(1-\delta)t}^{t}\int_0^1 |G_{t-s}(x,y)|^2\left(\int_{(1-\delta)t}^{t}\int_0^1 E\left[|D_{r,z}u(s,y)|^2|F\right]\mathrm{d}z\mathrm{d}r\right)\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C(\delta t)\left(\sup_{\substack{(1-\delta)t\le s\le t\\ 0\le y\le 1}}\int_{(1-\delta)t}^{t}\int_0^1 E\left[|D_{r,z}u(s,y)|^2|F\right]\mathrm{d}z\mathrm{d}r\right)^{\frac{1}{2}}.
\end{aligned}
$$

At this point we are in position to apply (3.15) in Lemma 3.3. Therefore

$$
|A_1(t,x;\delta)| \le C(\delta t)^{\frac{5}{4}}, \quad \text{a.s.}, \tag{3.20}
$$

where $C = \frac{1}{\sqrt{\pi}(2\pi)^{\frac{1}{4}}}\sigma^2\|b'\|_\infty$.

In order to get a bound for $|A_2(t,x;\delta)|$, one can use analogous arguments as for $|A_1(t,x;\delta)|$ but applying (3.16) instead of (3.15) in Lemma 3.3. Hence, one obtains

$$
|A_2(t,x;\delta)| \le C(\delta t)^{\frac{5}{4}}, \quad \text{a.s.} \tag{3.21}
$$

Let us finally estimate $|A_3(t,x;\delta)|$. For this, we apply Fubini's theorem, the fact that b' is bounded, the Cauchy–Schwarz inequality with respect to $\mathrm{d}z\mathrm{d}r\mathrm{d}P_{|F}\mathrm{d}P'$ and we finally invoke Lemma 3.3:

$$
\begin{aligned}
|A_3(t,x;\delta)| &\le C\int_0^\infty \mathrm{e}^{-\theta}\int_{(1-\delta)t}^{t}\int_0^1\int_{(1-\delta)t}^{t}\int_0^1 G_{t-s}(x,y)G_{t-\bar{s}}(x,\bar{y}) \\
&\quad\times\left(\int_{(1-\delta)t}^{t}\int_0^1 E\left[E'\left(|D_{r,z}u(s,y)\widetilde{D_{r,z}u(\bar{s},\bar{y})}|\right)\Big| F\right]\mathrm{d}z\mathrm{d}r\right)\mathrm{d}y\mathrm{d}s\mathrm{d}\bar{y}\mathrm{d}\bar{s}\mathrm{d}\theta \\
&\le C\int_0^\infty \mathrm{e}^{-\theta}\int_{(1-\delta)t}^{t}\int_0^1\int_{(1-\delta)t}^{t}\int_0^1 G_{t-s}(x,y)G_{t-\bar{s}}(x,\bar{y}) \\
&\quad\times\left(\int_{(1-\delta)t}^{t}\int_0^1 E\left[|D_{r,z}u(s,y)|^2|F\right]\mathrm{d}z\mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{(1-\delta)t}^{t}\int_0^1 E\left[E'\left(|\widetilde{D_{r,z}u(\bar{s},\bar{y})}|^2\right)\Big| F\right]\right)^{\frac{1}{2}}\mathrm{d}y\mathrm{d}s\mathrm{d}\bar{y}\mathrm{d}\bar{s}\mathrm{d}\theta
\end{aligned}
$$

$$\leq C(\delta t)^{\frac{1}{2}} \left(\int_{(1-\delta)t}^{t} \int_0^1 G_{t-s}(x, y) \mathrm{d}y \mathrm{d}s \right)^2$$
$$\leq C(\delta t)^2, \tag{3.22}$$

where $C = \frac{1}{\sqrt{2}\pi^{\frac{3}{2}}} \sigma^2 \|b'\|_\infty^2$. The very last estimate in (3.22) has been obtained after applying the Cauchy–Schwarz inequality and the bound (3.10). Eventually, plugging the bounds (3.20)–(3.22) in (3.19) we have

$$g(F) \geq \sigma^2 c_x (\delta t)^{\frac{1}{2}} - c_1 \left((\delta t)^{\frac{5}{4}} + (\delta t)^2 \right),$$

where c_1 is a positive constant depending on σ and $\|b'\|_\infty$. Hence, if we assume that $\delta < 1 \wedge \frac{1}{T}$, then we can write

$$g(F) \geq \sigma^2 c_x (\delta t)^{\frac{1}{2}} - 2c_1 (\delta t)^{\frac{5}{4}} \geq \left(\sigma^2 c_x \delta^{\frac{1}{2}} - 2c_1 \delta^{\frac{5}{4}} T^{\frac{3}{4}} \right) t^{\frac{1}{2}}.$$

It only remains to observe that the quantity $\sigma^2 c_x \delta^{\frac{1}{2}} - 2c_1 \delta^{\frac{5}{4}} T^{\frac{3}{4}}$ is strictly positive whenever δ is sufficiently small, namely $\delta \in (0, \delta_0)$, with

$$\delta_0 = 1 \wedge \frac{1}{T} \wedge \frac{1}{T} \left(\frac{\sigma^2 c_x}{2c_1} \right)^{\frac{4}{3}}.$$

Thus, the lower bound in (3.14) has been then proved.

Step 2: *The upper bound*. The upper estimation in (3.14) is almost an immediate consequence of the computations which we have just performed for the lower bound. More precisely, according to (3.13) and the considerations in the first part of the proof, we have the following:

$$g(F) \leq \sum_{i=0}^{3} |A_i(t, x; 1)|, \tag{3.23}$$

where we notice that we have substituted δ by 1 in $A_i(t, x; \delta)$, $i = 0, 1, 2, 3$. We have already seen that $|A_i(t, x; 1)| \leq C t^{\frac{5}{4}}$, for $i = 1, 2$, and $|A_3(t, x; \delta)| \leq C t^2$, so we just need to bound $|A_0(t, x; 1)|$, which follows directly from (3.10). Thus

$$g(F) \leq C \left(t^{\frac{1}{2}} + t^{\frac{5}{4}} + t^2 \right), \quad \text{a.s.},$$

for a constant $C > 0$. Therefore $g(F) \leq C_2 t^{\frac{1}{2}}$, where the constant $C_2 > 0$ depends on σ, $\|b'\|_\infty$ and T. □

We are now in position to prove the main result of this section:

Proof of Theorem 3.1. The random variable $F = u(t, x) - E(u(t, x))$ is centered, belongs to $\mathbb{D}^{1,2}$ and, by Proposition 3.2, it holds that $0 < C_1 t^{\frac{1}{2}} \leq g(F)$, for all $t \in (0, T]$. We apply then [17], Theorem 3.1 and Corollary 3.3, and we obtain that the probability density $\rho : \mathbb{R} \to \mathbb{R}$ of F is given by

$$\rho(z) = \frac{E|u(t, x) - E(u(t, x))|}{2g(z)} \exp\left(-\int_0^z \frac{y}{g(y)} \mathrm{d}y \right),$$

for almost every $z \in \mathbb{R}$. Thus, the density p of the random variable $u(t,x)$ satisfies

$$p(z) = \frac{E|u(t,x) - E(u(t,x))|}{2g(z - E(u(t,x)))} \exp\left(-\int_0^{z-E(u(t,x))} \frac{y}{g(y)} \mathrm{d}y\right). \tag{3.24}$$

In order to conclude the proof, we only need to use the bounds (3.14) in the above expression (3.24). □

4. The stochastic heat equation in $\mathbb{R}^d$

In this section we are interested in the stochastic heat equation in $\mathbb{R}^d$ with an additive Gaussian noise which is white in time and it has a spatially homogeneous correlation in space. We aim to find out sufficient conditions on the drift term and the noise's spatial correlation ensuring that the density of the solution admits Gaussian-type estimations. The fact that we deal with a SPDE in $\mathbb{R}^d$ with a non-trivial spatial correlation makes the analysis in this case much more involved in comparison to the one-dimensional setting in Section 3.

4.1. SPDEs with spatially homogeneous noise

Let us consider the following stochastic parabolic Cauchy problem in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = b(u(t,x)) + \sigma \dot{W}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.25}$$

where $T > 0$, Δ denotes the Laplacian operator in $\mathbb{R}^d$, $b : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ function with bounded derivative, and suppose that we are given an initial condition of the form

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d,$$

with $u_0 : \mathbb{R}^d \to \mathbb{R}$ measurable and bounded.

The random perturbation $\dot{W}$ (formally) stands for a Gaussian noise which is white in time and with some spatially homogeneous correlation. More precisely, on a complete probability space $(\Omega, \mathcal{F}, P)$, we consider a family of mean zero Gaussian random variables $W = \{W(\varphi), \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})\}$, where $\mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ denotes the space of infinitely differentiable functions with compact support, with covariance functional

$$E(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} (\varphi(t) * \psi_s(t))(x) \Lambda(\mathrm{d}x)\mathrm{d}t, \quad \varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1}), \tag{4.26}$$

where $\psi_s(t,x) := \psi(t,-x)$ and Λ is a non-negative and non-negative definite tempered measure. By [24, Chapter VII, Théorème XVII], Λ is symmetric and there exists a non-negative tempered measure μ whose Fourier transform is Λ. That is, by definition of the Fourier transform on the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions, for all ϕ belonging to the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing $\mathcal{C}^\infty$ functions,

$$\int_{\mathbb{R}^d} \phi(x)\Lambda(\mathrm{d}x) = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(\mathrm{d}\xi),$$

and there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-m} \mu(\mathrm{d}\xi) < \infty.$$

The measure μ is usually called the spectral measure of the noise W. In particular, we have

$$E(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t)(\xi)\overline{\mathcal{F}\psi(t)(\xi)}\mu(\mathrm{d}\xi)\mathrm{d}t, \quad \varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1}).$$

Notice that we have used the symbol "$*$" for the standard convolution in $\mathbb{R}^d$.

Example 4.1. Usual examples of spatial correlations are given by $\Lambda(\mathrm{d}x) = f(x)\mathrm{d}x$, where f is a non-negative continuous function on $\mathbb{R}^d \setminus \{0\}$ which is integrable in a neighborhood of 0; for instance, one can take f to be a Riesz kernel $f_\epsilon(x) = |x|^{-\epsilon}$, for $0 < \epsilon < d$. The space-time white noise would correspond to consider f equal to the Dirac delta on the origin. In this latter case, the spectral measure μ is the Lebesgue measure on $\mathbb{R}^d$.

We denote by $\mathcal{H}$ the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ endowed with the semi-inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} (\phi_1 * (\phi_2)_s)(x)\Lambda(\mathrm{d}x) = \int_{\mathbb{R}^d} \mathcal{F}\phi_1(\xi)\overline{\mathcal{F}\phi_2(\xi)}\mu(\mathrm{d}\xi),$$

$\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$, and associated semi-norm $\|\cdot\|_{\mathcal{H}}$. Notice that $\mathcal{H}$ is a Hilbert space that may contain distributions (see [4, Example 6]). Set $\mathcal{H}_T := L^2([0,T];\mathcal{H})$.

Then, it turns out that the Gaussian noise W can be naturally extended to $\mathcal{H}_T$, so that we obtain a family $\{W(h),\ h \in \mathcal{H}_T\}$ of centered Gaussian random variables such that

$$E(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathcal{H}_T} = \int_0^T \langle h_1(t), h_2(t) \rangle_{\mathcal{H}} \mathrm{d}t, \quad h_1, h_2 \in \mathcal{H}_T.$$

The details of that extension can be found, for instance, in [14, p. 805] or [22, p. 3]. Then, if we set

$$W_t(g) := W(\mathbf{1}_{[0,t]}g), \quad t \in [0,T],\ g \in \mathcal{H},$$

we obtain a cylindrical Wiener process $\{W_t(g),\ t \in [0,T], g \in \mathcal{H}\}$ on the Hilbert space $\mathcal{H}$. That is, for any $t \in [0,T]$ and $g \in \mathcal{H}$, $W_t(g)$ is a mean zero Gaussian random variable and

$$E(W_t(g_1)W_s(g_2)) = (t \wedge s)\langle g_1, g_2 \rangle_{\mathcal{H}}, \quad s, t \in [0,T],\ g_1, g_2 \in \mathcal{H}.$$

We denote by $\mathcal{F}_t$ the σ-algebra generated by $\{W_s(g),\ s \in [0,t], g \in \mathcal{H}\}$ and the P-null sets. As it has been explained in [20, Section 3], one can construct (real-valued) stochastic integrals of predictable processes in $L^2(\Omega \times [0,T];\mathcal{H})$ with respect to the cylindrical Wiener process W; notice that we are making an abuse of notation since W denoted the Gaussian family given at the beginning. The resulting stochastic integral turns out to extend Walsh's integration theory [25] and it is equivalent, in some particular situations, to Dalang's stochastic integral set up in [4]. The stochastic integrals appearing throughout this section will be understood as integrals with respect to the cylindrical Wiener process W.

We are now in position to rigorously define the mild solution of Eq. (4.25): a square integrable $\{\mathcal{F}_t\}$-adapted random field $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ solves Eq. (4.25) if, for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\begin{aligned} u(t,x) = {} & \int_{\mathbb{R}^d} G_t(x-y)u_0(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b(u(s,y))\mathrm{d}y\mathrm{d}s \\ & + \sigma \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)W(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{4.27}$$

Here, the function $G_t(x)$, $t > 0$, $x \in \mathbb{R}^d$, denotes the fundamental solution associated to the heat equation in $\mathbb{R}^d$, that is the centered Gaussian kernel of variance $2t$:

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

General existence and uniqueness results for Eq. (4.27) may be found in [4]. More precisely, it turns out that sufficient conditions for existence and uniqueness of solution to Eq. (4.27) are the following: b is Lipschitz, u_0 is measurable and bounded, and the noise's spatial correlation is related with the fundamental solution G through the condition

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}G_t(\xi)|^2 \mu(d\xi) dt < +\infty. \tag{4.28}$$

As it has been shown in [4, Example 8], condition (4.28) holds if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < +\infty, \tag{4.29}$$

and this integrability condition will be assumed to be satisfied in the remainder of this section.

Let us turn now to the question whether the probability law of the solution at any point has a density. The Gaussian setting in which we apply the Malliavin calculus machinery is determined by the Gaussian family $\{W(h),\ h \in \mathcal{H}_T\}$ given before.

Using this framework, it is a consequence of [20, Theorem 5.2] that, if the drift coefficient $b \in \mathcal{C}^1$ has a bounded and Lipschitz continuous derivative and condition (4.29) is fulfilled, then the solution $u(t,x)$ to Eq. (4.27), at any point $(t,x) \in (0,T] \times \mathbb{R}^d$, is differentiable in the Malliavin sense, that is $u(t,x) \in \mathbb{D}^{1,2}$, and its law has a density with respect to the Lebesgue measure. At this point, we should mention that in [20] all the results are proved in the case where the noise's correlation is given by a function f, that is $\Lambda(dx) = f(x)dx$. However, the extension of those results to a general tempered measure Λ is straightforward. See also the works [13,22,23] for related results with slightly stronger conditions on the spectral measure.

It will be of much importance for us the equation satisfied be the Malliavin derivative of $u(t,x)$. Indeed, see either [13,22] or [20], the Malliavin derivative $Du(t,x)$ takes values in the Hilbert space $\mathcal{H}_T$ and satisfies the following linear parabolic equation:

$$Du(t,x) = \sigma G_{t-\cdot}(x-\star) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) Du(s,y) dy ds, \tag{4.30}$$

where "$\star$" stands for the $\mathcal{H}$-variable. Moreover, one proves that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E(\|Du(t,x)\|^2_{\mathcal{H}_T}) < +\infty. \tag{4.31}$$

Eq. (4.30) may be interpreted in the following sense: for any $r \in [0,t)$, $D_r u(t,x)$ satisfies the equation in $\mathcal{H}$

$$D_r u(t,x) = \sigma G_{t-r}(x-\star) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) D_r u(s,y) dy ds. \tag{4.32}$$

The integral on the right-hand side of (4.32) is understood as a $\mathcal{H}$-valued pathwise integral. Before going on with our analysis, let us briefly describe how this integral is rigorously defined. Let $\{e_j,\ j \geq 1\}$ be a complete orthonormal system of $\mathcal{H}$. Then, using the properties of G, the boundedness of b' and (4.31), the $\mathcal{H}$-valued integral

$$\mathcal{I}_t = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) D_r u(s,y) \mathrm{d}y\mathrm{d}s$$

can be defined through its components

$$\left\{ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) \langle D_r u(s,y), e_j \rangle_{\mathcal{H}} \mathrm{d}y\mathrm{d}s, \ j \geq 1 \right\}$$

with respect to the basis $\{e_j, \ j \geq 1\}$, and those latter integrals takes values in $\mathbb{R}$. Moreover, one can obtain an upper bound for the square moment of $\mathcal{I}_t$ (for the general setting see, for instance, [21, p. 24]):

$$E(|\mathcal{I}_t|^2) \leq C \int_0^t \int_{\mathbb{R}^d} |G_{t-s}(x-y)|^2 |b'(u(s,y))|^2 E(\|D_r u(s,y)\|_{\mathcal{H}}^2) \mathrm{d}y\mathrm{d}s. \tag{4.33}$$

Let us go back now to Eq. (4.32). Since $G_{t-r}(x-\star) : \mathbb{R}^d \to \mathbb{R}$ defines a function (indeed in $\mathcal{S}(\mathbb{R}^d)$), this implies that either $D_r u(t,x)$ and the integral in the right-hand side of (4.32) define elements of $\mathcal{H}$ which are functions in z. Therefore, for any fixed $(t,x) \in (0,T] \times \mathbb{R}^d$, we can state that the Malliavin derivative satisfies, for all $(r,z) \in [0,t) \times \mathbb{R}^d$:

$$D_{r,z} u(t,x) = \sigma G_{t-r}(x-z) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) D_{r,z} u(s,y) \mathrm{d}y\mathrm{d}s. \tag{4.34}$$

A crucial consequence of this fact is that, as in the case of the one-dimensional stochastic heat equation with boundary conditions, the Malliavin derivative $D_{r,z}u(t,x)$ is non-negative, for all $(r,z) \in [0,t) \times \mathbb{R}^d$, a.s.

We will need a slightly stronger condition on the spectral measure μ than (4.29). Namely, consider the following hypothesis:

Hypothesis H_η**.** There exists $\eta \in (0,1)$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^\eta} \mu(\mathrm{d}\xi) < +\infty. \tag{4.35}$$

Then, one can prove the following estimates (see [13, Lemma 3.1]):

Lemma 4.2. *Assume that the spectral measure μ satisfies $\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(\mathrm{d}\xi) < +\infty$.*

1. *Let $T > 0$. Then, there exists a constant $k_1 > 0$ such that, for all $t \in [0,T]$,*

$$k_1 t \leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}G_s(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}s.$$

The constant k_1 depends on T and, indeed, it converges to zero as T tends to infinity.

2. *Suppose that Hypothesis H_η holds. Then, there exists a constant $k_2 > 0$ such that, for all $t \geq 0$,*

$$\int_0^t \int_{\mathbb{R}^d} |\mathcal{F}G_s(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}s \leq k_2 t^\beta, \tag{4.36}$$

for all $\beta \in (0, 1-\eta]$.

Remark 4.3. It is worth mentioning that the integrability condition (4.28) was sufficient for us to prove the existence of density for the solution $u(t,x)$, at any point $(t,x) \in (0,T] \times \mathbb{R}^d$ (see [20, Theorem 5.2]). However, as it will be made clearer in Section 4.2, we will really need upper bounds of the form (4.36) in order to obtain Gaussian estimates for the density of $u(t,x)$.

4.2. Gaussian estimates for the density of the solution

Let us consider $T > 0$ and $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ the unique mild solution to Eq. (4.27). This section is devoted to proof the following result:

Theorem 4.4. *Fix $t \in (0,T]$ and $x \in \mathbb{R}^d$. Suppose that Hypothesis H_η is satisfied for some $\eta \in (0, \frac{3}{4})$ and that the coefficient b is of class $\mathcal{C}^1$ and has a bounded Lipschitz continuous derivative. Then, the random variable $u(t,x)$ has a density p with respect to Lebesgue measure which satisfies the following: for almost every $z \in \mathbb{R}$,*

$$\frac{E\,|u(t,x)-m|}{C_2 t^{1-\eta}} \exp\left\{-\frac{(z-m)^2}{C_1 t}\right\} \le p(z) \le \frac{E\,|u(t,x)-m|}{C_1 t} \exp\left\{-\frac{(z-m)^2}{C_2 t^{1-\eta}}\right\},$$

where $m = E(u(t,x))$ and C_1, C_2 are positive constants depending on σ, $\|b'\|_\infty$, η and T.

Theorem 4.4 will be a consequence of [17, Theorem 3.1] and Proposition 4.5. As we have done in Section 3.2, we use the notation $F = u(t,x) - E(u(t,x))$ and we remind that we will need to find almost sure lower and upper bounds for the random variable $g(F)$, where

$$g(F) = \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\langle Du(t,x), \widetilde{Du(t,x)}\rangle_{\mathcal{H}_T}\right)\Big|\, F\right] \mathrm{d}\theta.$$

Proposition 4.5. *Fix $T > 0$. Assume that Hypothesis H_η holds for some $\eta \in (0, \frac{3}{4})$ and the coefficient b is of class $\mathcal{C}^1$ and has a bounded Lipschitz continuous derivative. There exist positive constants C_1, C_2 such that,*

$$C_1 t \le g(F) \le C_2 t^{1-\eta}, \quad a.s., \tag{4.37}$$

for all $t \in (0,T]$.

In order to prove Proposition 4.5, we will need the following technical lemma, which plays the role of Lemma 3.3 in our standing setting.

Lemma 4.6. *Let $t > 0$ and assume that Hypothesis H_η holds. Then, there exists a positive constant C depending on σ, $\|b'\|_\infty$ and the constant k_2 in (4.36), such that, for all $\delta \in (0,1]$,*

$$\sup_{\substack{(1-\delta)t \le v \le t \\ y \in \mathbb{R}^d}} E\left[\int_{(1-\delta)t}^t \|D_r u(v,y)\|_{\mathcal{H}}^2 \mathrm{d}r \,\Big|\, F\right] \le C(\delta t)^\beta, \quad a.s. \tag{4.38}$$

and

$$\sup_{\theta \ge 1} \sup_{\substack{(1-\delta)t \le v \le t \\ y \in \mathbb{R}^d}} E\left[E'\left(\int_{(1-\delta)t}^t \|\widetilde{D_r u(v,y)}\|_{\mathcal{H}}^2 \mathrm{d}r\right)\Big|\, F\right] \le C(\delta t)^\beta, \quad a.s., \tag{4.39}$$

for any $\beta \in (0, 1-\eta]$.

Proof. It is very similar to that of Lemma 3.3. In fact, owing to Eq. (4.32), we have, for any $(\nu, v) \in [(1-\delta)t, t] \times \mathbb{R}^d$,

$$\int_{(1-\delta)t}^{t} \|D_r u(\nu, v)\|_{\mathcal{H}}^2 \mathrm{d}r \leq 2\sigma^2 \int_{(1-\delta)t}^{t} \|G_{\nu-r}(v-\star)\|_{\mathcal{H}}^2 \mathrm{d}r$$

$$+ 2\|b'\|_\infty^2 (\delta t) \int_{(1-\delta)t}^{\nu} \int_{\mathbb{R}^d} |G_{\nu-s}(x-y)|^2 \left(\int_{(1-\delta)t}^{t} \|D_r u(s, y)\|_{\mathcal{H}}^2 \mathrm{d}r \right) \mathrm{d}y \mathrm{d}s, \quad (4.40)$$

where we have applied Minkowski's and the Cauchy–Schwarz inequalities. By (4.36) in Lemma 4.2,

$$\int_{(1-\delta)t}^{t} \|G_{\nu-r}(v-\star)\|_{\mathcal{H}}^2 \mathrm{d}r = \int_0^{\delta t} \int_{\mathbb{R}^d} |\mathcal{F}G_r(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}r \leq k_2(\delta t)^\beta,$$

for all $\beta \in (0, 1-\eta]$. Therefore, plugging this bound in (4.40) and taking conditional expectation, we obtain:

$$E\left[\int_{(1-\delta)t}^{t} \|D_r u(\nu, v)\|_{\mathcal{H}}^2 \mathrm{d}r \,\middle|\, F \right] \leq 2\sigma^2 k_2 (\delta t)^\beta + \frac{1}{2\sqrt{2\pi}} \|b'\|_\infty^2 (\delta t)$$

$$\times \int_{(1-\delta)t}^{\nu} \left(\sup_{\substack{(1-\delta)t \leq \tau \leq s \\ y \in \mathbb{R}^d}} E\left[\int_{(1-\delta)t}^{t} \|D_r u(\tau, y)\|_{\mathcal{H}}^2 \mathrm{d}r \,\middle|\, F \right] \right) \frac{1}{\sqrt{\nu - s}} \mathrm{d}s, \quad \text{a.s.}$$

As we have done in the proof of Lemma 3.3, we are now in position to apply Gronwall's lemma [4, Lemma 15]. Hence (4.38) is proved. The estimation (4.39) can be checked using exactly the same arguments. □

Proof of Proposition 4.5. The framework of the proof is similar to that of Proposition 3.2 in Section 3.2. However, computations here will be slightly more involved since we are working in a Hilbert-space-valued setting determined by $\mathcal{H}_T = L^2([0, T]; \mathcal{H})$. Let us first deal with the lower bound in (4.37):

Step 1: *The lower bound*. Recall that $F = u(t, x) - E(u(t, x))$ and the random variable $g(F)$ can be written as

$$g(F) = \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\langle Du(t, x), \widetilde{Du(t, x)} \rangle_{\mathcal{H}_T} \right) \middle| F \right] \mathrm{d}\theta$$

$$= \int_0^\infty \mathrm{e}^{-\theta} E\left[E'\left(\int_0^t \langle D_r u(t, x), \widetilde{D_r u(t, x)} \rangle_{\mathcal{H}} \mathrm{d}r \right) \middle| F \right] \mathrm{d}\theta,$$

where $\widetilde{Du(t, x)}$ denotes the shifted random variable $(Du(t, x))(\mathrm{e}^{-\theta}\omega + \sqrt{1-\mathrm{e}^{-2\theta}}\omega')$.

According to Eq. (4.30), for any $\delta \in (0, 1]$ we have the decomposition

$$g(F) \geq \sigma^2 B_0(t, x; \delta) - |B_1(t, x; \delta) + B_2(t, x; \delta) + B_3(t, x; \delta)|, \quad (4.41)$$

where

$$B_0(t, x; \delta) = \int_{(1-\delta)t}^{t} \|G_{t-r}(x-\star)\|_{\mathcal{H}}^2 \mathrm{d}r,$$

$$B_1(t,x;\delta) = \sigma E\Bigg[\int_{(1-\delta)t}^t \Big\langle G_{t-r}(x-\star), \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b'(u(s,y))D_r u(s,y)\mathrm{d}y\mathrm{d}s\Big\rangle_{\mathcal{H}} \mathrm{d}r \Bigg| F\Bigg],$$

$$B_2(t,x;\delta) = \sigma \int_0^\infty \mathrm{e}^{-\theta} E\Bigg[E'\Bigg(\int_{(1-\delta)t}^t \Big\langle G_{t-r}(x-\star), \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b'(\widetilde{u(s,y)})\widetilde{D_r u(s,y)}\mathrm{d}y\mathrm{d}s\Big\rangle_{\mathcal{H}} \mathrm{d}r\Bigg)\Bigg| F\Bigg]\mathrm{d}\theta,$$

$$B_3(t,x;\delta) = \int_0^\infty \mathrm{e}^{-\theta} E\Bigg[E'\Bigg(\int_{(1-\delta)t}^t \Big\langle \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b'(u(s,y))D_r u(s,y)\mathrm{d}y\mathrm{d}s, \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b'(\widetilde{u(s,y)})\widetilde{D_r u(s,y)}\mathrm{d}y\mathrm{d}s\Big\rangle_{\mathcal{H}} \mathrm{d}r\Bigg)\Bigg| F\Bigg]\mathrm{d}\theta.$$

By part 1 in Lemma 4.2, notice first that

$$B_0(t,x;\delta) = \int_0^{\delta t} \int_{\mathbb{R}^d} |\mathcal{F}G_s(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}s \geq k_1 \delta t. \tag{4.42}$$

Concerning the second term $B_1(t,x;\delta)$, we can apply the Cauchy–Schwarz and Minkowski's inequalities, so that we obtain

$$\begin{aligned}
|B_1(t,x;\delta)| &\leq C(\delta t)^{\frac{1}{2}} \left(\int_{(1-\delta)t}^t \|G_{t-r}(x-\star)\|_{\mathcal{H}}^2 \mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad \times \left(E\left[\int_{(1-\delta)t}^t \left\|\int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b'(u(s,y))D_r u(s,y)\mathrm{d}y\mathrm{d}s\right\|_{\mathcal{H}}^2 \mathrm{d}r \,\middle|\, F\right]\right)^{\frac{1}{2}} \\
&\leq C(\delta t)^{\frac{1}{2}} \left(\int_0^{\delta t} \int_{\mathbb{R}^d} |\mathcal{F}G_r(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{(1-\delta)t}^t \int_{\mathbb{R}^d} |G_{t-s}(x-y)|^2 E\left[\int_{(1-\delta)t}^t \|D_r u(s,y)\|_{\mathcal{H}}^2 \mathrm{d}r \,\middle|\, F\right]\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, by (4.36), Lemma 4.6 and the fact that

$$\int_{(1-\delta)t}^t \int_{\mathbb{R}^d} |G_{t-s}(x-y)|^2 \mathrm{d}y\mathrm{d}s \leq C(\delta t)^{\frac{1}{2}}, \tag{4.43}$$

we have:

$$|B_1(t,x;\delta)| \leq C(\delta t)^{\beta+\frac{3}{4}}, \tag{4.44}$$

for all $\beta \in (0, 1-\eta]$. The term $|B_2(t,x;\delta)|$ can be treated in the same way as we have just done for $|B_1(t,x;\delta)|$. Namely, one proves that

$$|B_2(t,x;\delta)| \le C(\delta t)^{\frac{1}{2}} \left(\int_{(1-\delta)t}^t \|G_{t-r}(x-\star)\|_{\mathcal{H}}^2 \mathrm{d}r \right)^{\frac{1}{2}} \times \int_0^\infty \mathrm{e}^{-\theta} \left(E\left[E'\left(\int_{(1-\delta)t}^t \left\| \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) \times D_r u(s,y) \mathrm{d}y \mathrm{d}s \right\|_{\mathcal{H}}^2 \mathrm{d}r \right) \middle| F \right] \right)^{\frac{1}{2}} \mathrm{d}\theta$$

$$\le C(\delta t)^{\frac{1}{2}} \left(\int_0^{\delta t} \int_{\mathbb{R}^d} |\mathcal{F}G_r(\xi)|^2 \mu(\mathrm{d}\xi) \mathrm{d}r \right)^{\frac{1}{2}} \int_0^\infty \mathrm{e}^{-\theta} \left(\int_{(1-\delta)t}^t \int_{\mathbb{R}^d} |G_{t-s}(x-y)|^2 E \times \left[E'\left(\int_{(1-\delta)t}^t \|D_r u(s,y)\|_{\mathcal{H}}^2 \mathrm{d}r \right) \middle| F \right] \mathrm{d}y \mathrm{d}s \right)^{\frac{1}{2}} \mathrm{d}\theta.$$

Taking into account (4.36), Lemma 4.6 and (4.43), we also get

$$|B_2(t,x;\delta)| \le C(\delta t)^{\beta+\frac{3}{4}}, \tag{4.45}$$

for all $\beta \in (0, 1-\eta]$.

Eventually, in order to deal with the term $|B_3(t,x;\delta)|$, we mainly apply the Cauchy–Schwarz inequality with respect to $E\left[E'\left(\int_{(1-\delta)t}^t \| \bullet \|_{\mathcal{H}}^2 \mathrm{d}r\right) | F\right]$, so that we will take advantage of the computations which we have performed so far. More precisely, we have

$$B_3(t,x;\delta) \le \left(E\left[\int_{(1-\delta)t}^t \left\| \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) D_r u(s,y) \mathrm{d}y \mathrm{d}s \right\|_{\mathcal{H}}^2 \mathrm{d}r \middle| F \right] \right)^{\frac{1}{2}} \times \int_0^\infty \mathrm{e}^{-\theta} \left(E\left[E'\left(\int_{(1-\delta)t}^t \left\| \int_{(1-\delta)t}^t \int_{\mathbb{R}^d} G_{t-s}(x-y) b'(u(s,y)) \times D_r u(s,y) \mathrm{d}y \mathrm{d}s \right\|_{\mathcal{H}}^2 \mathrm{d}r \right) \middle| F \right] \right)^{\frac{1}{2}} \mathrm{d}\theta.$$

The two terms in the right-hand side of the above inequality already appeared in the analysis of $B_1(t,x;\delta)$ and $B_2(t,x;\delta)$, respectively, and each of them may be bounded, up to some constant, by $(\delta t)^{\frac{\beta}{2}+\frac{3}{4}}$. Therefore

$$|B_3(t,x;\delta)| \le C(\delta t)^{\beta+\frac{3}{2}}, \tag{4.46}$$

for any $\beta \in (0, 1-\eta]$. Estimations (4.41), (4.42) and (4.44)–(4.46) yield

$$g(F) \ge \sigma^2 k_1 \delta t - c_1 \left((\delta t)^{\beta+\frac{3}{4}} + (\delta t)^{\beta+\frac{3}{2}} \right),$$

where c_1 depends on σ, $\|b'\|_\infty$ and k_2. Hence, if $\delta < 1 \wedge \frac{1}{T}$ and $\beta \in \left(\frac{1}{4}, 1-\eta\right]$,

$$g(F) \ge t\left(\sigma^2 k_1 \delta - 2c_1 \delta^{\beta+\frac{3}{4}} t^{\beta-\frac{1}{4}}\right) \ge t\left(\sigma^2 k_1 \delta - 2c_1 \delta^{\beta+\frac{3}{4}} T^{\beta-\frac{1}{4}}\right).$$

Observe that the quantity

$$C_1 := \sigma^2 k_1 \delta - 2c_1 \delta^{\beta+\frac{3}{4}} T^{\beta-\frac{1}{4}}$$

defines a positive constant whenever $\delta \in (0, \delta_0)$, with

$$\delta_0 = 1 \wedge \frac{1}{T} \wedge \frac{1}{T} \left(\frac{\sigma^2 k_1}{2c_1} \right)^{\frac{1}{\beta - \frac{1}{4}}}.$$

Therefore, we obtain the desired lower bound in (4.37).

Step 2: *The upper bound*. The upper in (4.37) is an almost immediate consequence of the computations in the Step 1 and the decomposition

$$g(F) \le \sum_{i=0}^{3} |B_i(t, x; 1)|. \tag{4.47}$$

Indeed, we have already found upper bounds for the last three terms on the right-hand side of (4.47). On the other hand, observe that (4.36) yields

$$B_0(t, x; 1) \le C t^{\beta},$$

for all $\beta \in (0, 1 - \eta]$. This bound, together with (4.44)–(4.46) in the case $\delta = 1$, implies

$$g(F) \le C \left(t^{\beta} + t^{\beta + \frac{3}{4}} + t^{\beta + \frac{3}{2}} \right) \le C_2 t^{\beta},$$

for all $\beta \in (0, 1-\eta]$, where the constant C_2 depends on T. Therefore we conclude the proof. □

The proof of Theorem 4.4 can be finished as in the case of Theorem 3.1. □

5. Gaussian bounds for the stochastic wave equation at small time

The main objective here is to extend the results in Section 4 to a stochastic wave equation in space dimension $d \le 3$ and controlled by the spatially homogeneous Gaussian noise considered there. The intrinsic properties of the differential operator driving the equation will not allow us to obtain optimal results for all time parameter $T > 0$, even if we assume that the noise's space correlation satisfies Hypothesis H_η, an assumption which is slightly stronger than the one needed to have existence of density for the corresponding mild solution (see [20]).

5.1. *The stochastic wave equation in dimension $d = 1, 2, 3$*

We consider here the same setting as in Section 4 but for the stochastic wave equation in spatial dimension $d \le 3$:

$$\frac{\partial^2 u}{\partial t^2}(t, x) - \Delta u(t, x) = b(u(t, x)) + \sigma \dot{W}(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{5.48}$$

where $T > 0$, $b : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ function with bounded derivative, and suppose that we are given initial conditions of the form

$$u(0, x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0, x) = v_0(x), \quad x \in \mathbb{R}^d,$$

with $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$ measurable and bounded functions such that u_0 is of class $\mathcal{C}^1(\mathbb{R}^d)$ and has a bounded derivative ∇u_0. The random perturbation $\dot{W}$ corresponds to the spatially homogeneous Gaussian noise described in the previous Section 4.1. We recall that μ denotes the corresponding spectral measure and $\{\mathcal{F}_t,\ t \geq 0\}$ the filtration defined by the cylindrical Wiener process associated to the noise W.

The mild solution of Eq. (5.48) is given by a $\{\mathcal{F}_t\}$-adapted process $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that, for all $(t,x) \in (0,T] \times \mathbb{R}^d$,

$$u(t,x) = \int_{\mathbb{R}^d} v_0(x-y)\Gamma_t^d(\mathrm{d}y) + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^d} u_0(x-y)\Gamma_t^d(\mathrm{d}y)\right) + \int_0^t \int_{\mathbb{R}^d} b(u(s,x-y))\Gamma_{t-s}^d(\mathrm{d}y)\mathrm{d}s + \sigma \int_0^t \int_{\mathbb{R}^d} \Gamma_{t-s}^d(x-y) W(\mathrm{d}s,\mathrm{d}y), \tag{5.49}$$

where Γ_t^d, $t > 0$, denotes the fundamental solution of the wave equation in dimension $d = 1, 2, 3$:

$$\Gamma_t^1(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}},$$
$$\Gamma_t^2(x) = \frac{1}{2\pi}(t^2 - |x|^2)_+^{-1/2},$$
$$\Gamma_t^3 = \frac{1}{4\pi t}\sigma_t.$$

The element σ_t stands for the surface measure on the three-dimensional sphere of radius t. In particular, for each t, Γ_t^d has compact support and, in the case $d = 3$, Γ_t^3 is no more a function but measure on $\mathbb{R}^3$. It is important to remark that only in these cases, Γ^d, $d = 1, 2, 3$, defines a non-negative measure. Existence and uniqueness of mild solution to Eq. (5.49) is a consequence of the results in [8], whenever the space correlation satisfies

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(\mathrm{d}\xi) < +\infty. \tag{5.50}$$

We also point out that the stochastic integral in the right-hand side of (5.49) is a well-defined integral of a deterministic element in $\mathcal{H}_T$ with respect to the cylindrical Wiener process associated to the noise (see Lemma 3.2 and Example 4.2 in [20]).

For all dimensions $d \geq 1$, we have a unified expression for the Fourier transform of Γ_t^d:

$$\mathcal{F}\Gamma_t^d(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Using this fact and assuming that (5.50) holds, one proves the following lemma (see [11], Lemmas 5.4.1 and 5.4.3):

Lemma 5.1. *For any $t \geq 0$ it holds that*

$$c_1(t \wedge t^3)\frac{1}{1+|\xi|^2} \leq \int_0^t |\mathcal{F}\Gamma_s^d(\xi)|^2 \mathrm{d}s \leq c_2(t + t^3)\frac{1}{1+|\xi|^2}, \tag{5.51}$$

with some positive constants $c_1, c_2 > 0$.

Thus, if we assume that $\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^2} < \infty$, (5.51) yields

$$d_1(t \wedge t^3) \leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma_s^d(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}s \leq d_2(t + t^3),$$

for all $t \geq 0$, with some positive constants d_1, d_2. In particular, for $t \in [0, 1)$ we have

$$d_1 t^3 \leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma_s^d(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}s \leq d_2 t. \tag{5.52}$$

However, under Hypothesis H_η (see (4.35)) one can get a slightly sharper upper estimation (see [22, Lemma 3]):

Lemma 5.2. *Let $T > 0$ and assume that Hypothesis H_η holds. Then*

$$\int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma_s^d(\xi)|^2 \mu(\mathrm{d}\xi)\mathrm{d}s \leq d_3 t^{3-2\eta}, \tag{5.53}$$

for all $t \in [0, T]$.

Eventually, if $d = 1, 2, 3$, explicit computations yield that, for any $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}^d} \Gamma_s^d(\mathrm{d}y)\mathrm{d}s \leq C t^2, \tag{5.54}$$

where C is a positive constant that only depends on d.

If b belongs to $\mathcal{C}^1$ and it has a Lipschitz continuous bounded derivative, then the solution $u(t, x)$, at any $(t, x) \in (0, T] \times \mathbb{R}^d$, belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative, as a random variable taking values in $\mathcal{H}_T = L^2([0, T]; \mathcal{H})$, satisfies

$$Du(t, x) = \sigma \Gamma_{t-\cdot}^d(x - \star) + \int_0^t \int_{\mathbb{R}^d} b'(u(s, x - y)) Du(s, x - y) \Gamma_{t-s}^d(\mathrm{d}y)\mathrm{d}s, \tag{5.55}$$

where "$\star$" stands for the $\mathcal{H}$-variable (see [20, Proposition 5.1]). This linear equation is understood in $L^2(\Omega \times [0, T]; \mathcal{H})$ and let us remark that $\sigma \Gamma_{t-\cdot}^d(x - \star)$ is a well-defined element in $\mathcal{H}_T$ (see [20, Lemma 3.2]). Moreover, under the standing hypothesis, the random variable $u(t, x)$, for $(t, x) \in (0, T] \times \mathbb{R}^d$, has a density with respect to the Lebesgue measure (see [20, Theorem 5.2]). Of course, the Gaussian setting here is the same as the one that has been considered in Section 4.

5.2. Gaussian estimates of the density at small time

For $T > 0$, consider the unique mild solution $\{u(t, x),\ (t, x) \in [0, T] \times \mathbb{R}^d\}$ to Eq. (5.48). In this section, we will prove that the density $p : \mathbb{R}^d \to \mathbb{R}$ of $u(t, x)$ has lower and upper Gaussian bounds whenever T is *small*, where this essentially means that $T < 1$. The main result is the following:

Theorem 5.3. *Suppose that Hypothesis H_η is satisfied and that the coefficient b is of class $\mathcal{C}^1$ and has a bounded Lipschitz continuous derivative. Then, there exists $T_0 \in (0, 1]$ such that the following statement is satisfied: for any $T \in (0, T_0)$ and $(t, x) \in (0, T] \times \mathbb{R}^d$, the random variable $u(t, x)$ has a density p with respect to Lebesgue measure such that, for almost every $z \in \mathbb{R}$,*

$$\frac{E\,|u(t,x) - m|}{C_2 t^{3-2\eta}} \exp\left\{-\frac{(z-m)^2}{C_1 t^3}\right\} \leq p(z) \leq \frac{E\,|u(t,x) - m|}{C_1 t^3} \exp\left\{-\frac{(z-m)^2}{C_2 t^{3-2\eta}}\right\},$$

where $m = E(u(t, x))$ and C_1, C_2 are positive constants depending on σ, $\|b'\|_\infty$, T_0 and η.

Remark 5.4. In the case of the stochastic heat equation presented in Section 4, we have been able to obtain Gaussian upper and lower bounds for any $T > 0$, while here we restrict our analysis to *small* T. As we will precisely point out in the next Proposition 5.5, that difference is due to the fact that the Malliavin derivative of the solution to the stochastic wave equation does not need to be a non-negative function.

The statement of Theorem 5.3 is an immediate consequence of Theorem 3.1 and Corollary 3.3 in [17] and the following proposition. For $t > 0$ and $x \in \mathbb{R}^d$, set $F = u(t,x) - E(u(t,x))$ and we remind that we will need to find almost sure lower and upper bounds for the random variable $g(F)$, where

$$g(F) = \int_0^\infty e^{-\theta} E\left[E'\left(\langle Du(t,x), \widetilde{Du(t,x)} \rangle_{\mathcal{H}_T} \right) \Big| F \right] d\theta. \tag{5.56}$$

Proposition 5.5. *Assume that Hypothesis* H_η *holds. There exist* $T_0 \in (0,1]$ *and positive constants* C_1, C_2 *such that, for any* $T \in (0, T_0)$,

$$C_1 t^3 \le g(F) \le C_2 t^{3-2\eta}, \quad a.s. \tag{5.57}$$

for all $t \in [0, T]$.

Proof. It follows the same lines as the proof of Proposition 4.5, so that we will only point out the main steps. More precisely, we observe first that the Malliavin derivative $Du(t,x)$ solves the linear equation (5.55) with a non-negative initial condition but driven by a hyperbolic operator. Thus, in comparison with the stochastic heat equation, $D_{r,z}u(t,x)$ does not need to be non-negative as a function of (r,z); indeed, in the case $d = 3$, even the Malliavin derivative does not need to be a function. Hence, in order to deal with the lower bound of $g(F)$ (see (5.56)), we will not be able to restrict the integral with respect to dr on a small time interval as we have done in the proof of Proposition 4.5. This is the reason why we will be forced to consider $T < 1$.

In fact, by (5.55), we are only able to consider the decomposition

$$g(F) \ge D_0(t) - (|D_1(t)| + |D_2(t)| + |D_3(t)|), \tag{5.58}$$

where

$$D_0(t) = \sigma^2 \int_0^t \| \Gamma^d_{t-r}(x - \star) \|^2_{\mathcal{H}} dr,$$

$$D_1(t) = \sigma E\left[\int_0^t \left\langle \Gamma^d_{t-r}(x-\star), \int_0^t \int_{\mathbb{R}^d} b'(u(s,x-y)) D_r u(s,x-y) \Gamma^d_{t-s}(dy) ds \right\rangle_{\mathcal{H}} dr \Bigg| F \right],$$

$$\begin{aligned} D_2(t) \;=\; & \int_0^\infty e^{-\theta} \sigma E\Bigg[E'\Bigg(\int_0^t \Big\langle \Gamma^d_{t-r}(x-\star), \\ & \int_0^t \int_{\mathbb{R}^d} b'(u(\widetilde{s,x-}y)) \left(D_r u(\widetilde{s,x-}y) \right) \Gamma^d_{t-s}(dy) ds \Big\rangle_{\mathcal{H}} dr \Bigg) \Bigg| F \Bigg] d\theta, \end{aligned}$$

$$\begin{aligned} D_3(t) \;=\; & \int_0^\infty e^{-\theta} E\Bigg[E'\Bigg(\int_0^t \Big\langle \int_0^t \int_{\mathbb{R}^d} b'(u(s,x-y)) D_r u(s,x-y) \Gamma^d_{t-s}(dy) ds, \\ & \int_0^t \int_{\mathbb{R}^d} b'(u(\widetilde{s,x-}y)) \left(D_r u(\widetilde{s,x-}y) \right) \Gamma^d_{t-s}(dy) ds \Big\rangle_{\mathcal{H}} dr \Bigg) \Bigg| F \Bigg] d\theta. \end{aligned}$$

By the lower bound in (5.52), we have

$$D_0(t) \ge d_1 t^3. \tag{5.59}$$

Concerning the term D_1, we can argue as follows:

$$
\begin{aligned}
|D_1(t)| &\le C\left(\int_0^t \|\Gamma^d_{t-r}(x-\star)\|^2_{\mathcal{H}}\mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad\times\left(E\left[\int_0^t \left\|\int_0^t\int_{\mathbb{R}^d} b'(u(s,x-y))D_r u(s,x-y)\Gamma^d_{t-s}(\mathrm{d}y)\mathrm{d}s\right\|^2_{\mathcal{H}}\mathrm{d}r\,\middle|\, F\right]\right)^{\frac{1}{2}} \\
&\le Ct^{\frac{3-2\eta}{2}}\left(E\left[\int_0^t\left(\int_0^t\int_{\mathbb{R}^d}\|D_r u(s,x-y)\|_{\mathcal{H}}\Gamma^d_{t-s}(\mathrm{d}y)\mathrm{d}s\right)^2\mathrm{d}r\,\middle|\, F\right]\right)^{\frac{1}{2}} \\
&\le Ct^{\frac{5-2\eta}{2}}\left(\int_0^t\int_{\mathbb{R}^d} E\left[\int_0^t \|D_r u(s,x-y)\|^2_{\mathcal{H}}\mathrm{d}r|F\right]\Gamma^d_{t-s}(\mathrm{d}y)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le Ct^{5-2\eta},
\end{aligned}
\tag{5.60}
$$

where we have used (5.53), (5.54) and (5.63) in Lemma 5.6 below.

Using similar arguments one proves that

$$
|D_2(t)| \le Ct^{5-2\eta}. \tag{5.61}
$$

The analysis of $|D_3(T)|$ can also be performed by following the calculations above to obtain (5.60), so that we end up with

$$
|D_3(t)| \le Ct^{7-2\eta}. \tag{5.62}
$$

Plugging the estimates (5.59)–(5.62) in (5.58) yields

$$
g(F) \ge d_1 t^3 - c_3\left(t^{5-2\eta} + t^{7-2\eta}\right),
$$

for all $t \in [0,T]$, where c_3 is a positive constant depending on σ, $\|b'\|_\infty$ and η. Hence, if $T < 1$ we have

$$
g(F) \ge t^3\left(d_1 - 2c_3T^{2-2\eta}\right),
$$

and the quantity $C_1 := d_1 - 2c_3T^{2-2\eta}$ is strictly positive whenever $T < T_0$, where

$$
T_0 = 1 \wedge \left(\frac{d_1}{2c_3}\right)^{\frac{1}{2-2\eta}}.
$$

Therefore, we have proved the lower bound in (5.57).

The upper bound in (5.57) is an immediate consequence of what we have done so far and (5.53), because

$$
g(F) \le \sum_{i=0}^3 |D_i(t)| \le C_2 t^{3-2\eta}. \quad \square
$$

In the proof of Proposition 5.5, we have applied the following technical lemma, whose proof is very similar to that of Lemma 4.6:

Lemma 5.6. *Let $t > 0$ and assume that Hypothesis H_η holds. Then, there exists a positive constant K depending on σ, $\|b'\|_\infty$ and the constant d_3 in Lemma 5.2, such that*

$$\sup_{\substack{0\le s\le t\\ y\in\mathbb{R}^d}} E\left[\int_0^t \|D_r u(s,y)\|_{\mathcal{H}}^2 \mathrm{d}r \,\middle|\, F\right] \le K t^{3-2\eta} \tag{5.63}$$

and

$$\sup_{\theta\ge 1}\sup_{\substack{0\le s\le t\\ y\in\mathbb{R}^d}} E\left[E'\left(\int_0^t \|D_r\widetilde{u(s,y)}\|_{\mathcal{H}}^2 \mathrm{d}r\right) \,\middle|\, F\right] \le K t^{3-2\eta}. \tag{5.64}$$

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References

[1] V. Bally, Lower bounds for the density of locally elliptic Itô processes, Ann. Probab. 34 (6) (2006) 2406–2440.
[2] V. Bally, I. Gyöngy, É. Pardoux, White noise driven parabolic SPDEs with measurable drift, J. Funct. Anal. 120 (2) (1994) 484–510.
[3] V. Bally, E. Pardoux, Malliavin calculus for white noise driven parabolic SPDEs, Potential Anal. 9 (1) (1998) 27–64.
[4] R.C. Dalang, Extending martingale measure stochastic integral with applications to spatially homogeneous SPDE's, Electron. J. Probab. 4 (6) (1999) 29.
[5] R.C. Dalang, D. Khoshnevisan, E. Nualart, Hitting probabilities for systems of non-linear stochastic heat equations with additive noise, ALEA Lat. Am. J. Probab. Math. Stat. 3 (2007) 231–271.
[6] R.C. Dalang, D. Khoshnevisan, E. Nualart, Hitting probabilities for systems of non-linear stochastic heat equations with multiplicative noise, Probab. Theory Related Fields 144 (3–4) (2009) 371–427.
[7] R.C. Dalang, E. Nualart, Potential theory for hyperbolic SPDEs, Ann. Probab. 32 (3A) (2004) 2099–2148.
[8] R.C. Dalang, L. Quer-Sardanyons (in preparation).
[9] A. Kohatsu-Higa, Lower bounds for densities of uniformly elliptic random variables on Wiener space, Probab. Theory Related Fields 126 (3) (2003) 421–457.
[10] D. Kusuoka, D. Stroock, Applications of the Malliavin calculus, Part III, J. Fac. Sci. Univ. Tokyo IA 34 (1987) 391–442.
[11] O. Lévêque, Hyperbolic stochastic partial differential equations driven by boundary noises, Ph.D. Thesis, EPFL, 2001.
[12] P. Malliavin, E. Nualart, Density minoration of a strongly non degenerated random variable, J. Funct. Anal. 256 (12) (2009) 4197–4214.
[13] D. Márquez-Carreras, M. Mellouk, M. Sarrà, On stochastic partial differential equations with spatially correlated noise: Smoothness of the law, Stochastic Process. Appl. 93 (2001) 269–284.
[14] A. Millet, M. Sanz-Solé, A stochastic wave equation in two space dimensions: Smoothness of the law, Ann. Probab. 27 (2) (1999) 803–844.
[15] P.L. Morien, The Hölder and the Besov regularity of the density for the solution of a parabolic stochastic partial differential equation, Bernoulli 5 (2) (1999) 275–298.

[16] I. Nourdin, G. Peccati, Stein's method on Wiener chaos, Probab. Theory Related Fields 145 (1) (2009) 75–118.
[17] I. Nourdin, F. Viens, Density estimates and concentration inequalities with Malliavin calculus, Electron. J. Probab. (in press).
[18] D. Nualart, The Malliavin Calculus and Related Topics, second ed., Springer-Verlag, Berlin, 2006.
[19] E. Nualart, Exponential divergence estimates and heat kernel tail, C. R. Math. Acad. Sci. Paris 338 (1) (2004) 77–80.
[20] D. Nualart, L. Quer-Sardanyons, Existence and smoothness of the density for spatially homogeneous SPDEs, Potential Anal. 27 (3) (2007) 281–299.
[21] L. Quer-Sardanyons, The stochastic wave equation: Study of the law and approximations, Ph.D. Thesis, Universitat de Barcelona, 2005.
[22] L. Quer-Sardanyons, M. Sanz-Solé, Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation, J. Funct. Anal. 206 (1) (2004) 1–32.
[23] L. Quer-Sardanyons, M. Sanz-Solé, A stochastic wave equation in dimension 3: Smoothness of the law, Bernoulli 10 (1) (2004) 165–186.
[24] D. Schwartz, Théorie des Distributions, Hermann, Paris, 1966.
[25] J.B. Walsh, An introduction to stochastic partial differential equations, in: P.L. Hennequin (Ed.), École d'été de probabilités de Saint-Flour XIV-1984, in: Lect. Notes Math., vol. 1180, Springer, Berlin, 1986, pp. 265–437.