Exponential Expansion with Šil’nikov’s Saddle-Focus*

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1. INTRODUCTION

Consider a system of ordinary differential equations in a neighborhood of the origin in $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^n$:

$$\begin{align*}
\dot{x} &= Ax + f(x, y) \\
\dot{y} &= By + g(x, y)
\end{align*}$$

where $(x, y) \in U$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. (1.1)

Assume it satisfies the following hypotheses:

(H1) There exist constants $M \geq 1$ and $\lambda < 0 < \mu < \mu_1$ such that $|e^{\lambda t}| \leq Me^{\mu t}$ for $t \geq 0$, $B = \text{diag}(B_0, B_1)$, where $B_0$ is either the real number $\mu$ or the elementary $2 \times 2$ matrix

$$B_0 = \begin{pmatrix}
\mu & -\omega \\
\omega & \mu
\end{pmatrix}$$

of complex eigenvalues $\mu \pm i\omega$ with $\omega$ being any real number, and $B_1$ satisfies $|e^{B_1 t}| \leq Me^{\mu t}$ for $t \leq 0$;

(H2) $f$ and $g$ are $C^{r+1}$ with $r \geq 3$ satisfying $f(0, y) = 0$, $Df(0, 0) = 0$, $g(x, 0) = 0$ and $Dg(0, 0) = 0$ for $(0, y)$ and $(x, 0) \in U$, where $D$ is the differentiation operator in $(x, y)$.

Given a triplet $(\tau, x_0, y_1)$ a solution $(x, y)(t)$ of Eq. (1.1) is a solution to the Šil’nikov problem if the Šil’nikov conditions $x(0) = x_0$ and $y(\tau) = y_1$ are satisfied. It is well known that such a solution exists and is unique with respect to the Šil’nikov conditions for $\tau \geq 0$, $|x_0| \leq \delta_0$ and $|y_1| \leq \delta_0$ with a constant $\delta_0$ being arbitrarily small but fixed. Also, as functions of $t$, $\tau$, $x_0$, and $y_1$, $x(t)$, $x(t, x_0, y_1)$ and $y(t)$, $\tau$, $x_0$, $y_1$ are also $C^{r+1}$. Furthermore, there

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exists a constant $K_0 > 0$ independent of $\tau, x_0$, and $y_1$ such that
\[ |\partial x(t)| \leq K_0 e^{\mu t} \quad \text{and} \quad |\partial y(t)| \leq K_0 e^{\mu (t - \tau)} \]
for $0 \leq t \leq \tau$ where $\partial$ is any mixed partial derivative in $\tau, x_0$, and $y_1$ of order $\leq r$. In particular, for the solutions $x(t)$ and $y(t)$ themselves, the constant $K_0$ can be replaced by $2\delta_0$. These results can be found in Deng [1]. In applications, however, what is most important is whether the solution admits an exponential expansion in the following sense.

**Definition 1.1.** The $y$-component solution $y(t)$ admits an exponential expansion of regularity $l$ if there exist $C^l$ functions $\varphi = \varphi(t, x_0, y_1)$ of $t, x_0$, and $y_1$ and $R = R(t; \tau, x_0, y_1)$ of $t, \tau, x_0$, and $y_1$ with $l \geq 1$ such that $y(t)$ can be expressed as
\[ y(t) = e^{B_\tau(t - \tau)} \varphi(t, x_0, y_1) + R(t; \tau, x_0, y_1), \]  
(1.1a)
where $B_\tau = \text{diag}(B_0, \mu I)$ with $I$ being the $(n-k) \times (n-k)$ identity matrix and $k = \text{rank } B_0$. Moreover, $\varphi$ satisfies
\[ \frac{\partial \varphi}{\partial y_1}(t, 0, 0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \varphi}{\partial x_0}(t, 0, 0) = 0 \]  
(1.1b)
with $I$ being the $k \times k$ identity matrix; and $R$ satisfies
\[ \varphi(t, 0, 0) \leq 0 \]  
(1.1c)
there exist constants $K > 0$ and $\sigma > 0$ independent of $t, \tau, x_0$, and $y_1$ such that
\[ |\partial R(t; \tau, x_0, y_1)| \leq K e^{(\mu + \sigma)(t - \tau)} \]
for $0 \leq t \leq \tau$ and for all partial derivatives $\partial$ in $t, \tau, x_0$, and $y_1$ of order $\leq l$.

Formula (1.1a) is referred to as exponential expansion, the functions $\varphi$ and $R$ are referred to as the coefficient function and the remainder term, respectively. It has been proved in Deng [1] that the solution of Eq. (1.1) does admit an exponential expansion of regularity $r - 1$ if the principal block $B_0$ is just a simple real eigenvalue $\mu$ of $B$. The proof has taken a great advantage of the fact that $e^{B_\tau t'} = e^{\mu t'}$ can be treated essentially as a real number which commutes with all matrices. In this context, Eq. (1.1) also admits an exponential expansion of regularity $r - 1$ when $\omega = 0$. The purpose of this paper is to show, in a uniform way, that an exponential expansion of regularity $r - 2$ also holds true for Eq. (1.1) if $B_0$ is the $2 \times 2$ matrix of the principal complex eigenvalues with $\omega \neq 0$. This is true only under certain admissible variable $(x, y)$ defined as follows,

**Definition 1.2.** A differentiable change of variables for Eq. (1.1) is $y$-admissible if it leaves the matrices $A$ and $B$ of the linearization unchanged, and, besides the same hypothesis (H2) is satisfied for the new system, the new higher order term $g$ is of order $O(\sum_{i=1}^{k} |y^{(i)}|^2 +$
\[ \sum_{i=k+1}^{n} |y^{(i)}| \] for each fixed \( x \) as \( |y| \to 0 \), where \( y = (y^{(1)}, \ldots, y^{(n)}) \). The new variable is called \( y \)-admissible variable.

A variable which is \( x \)-admissible is defined analogously. If it is also \( y \)-admissible then it is simply referred to as admissible.

**Theorem 1.3.** There exists a \( y \)-admissible variable \((x, y)\) for Eq. (1.1) such that the solution \( y(t) \) of the Šil’nikov problem admits an exponential expansion of regularity \( r - 2 \).

The following example shows that an exponential expansion may not hold true even for an analytic vector field if the principal unstable eigenvalues are complex conjugates and the variable is not \( y \)-admissible.

**Example 1.4.** Consider
\[
\begin{align*}
\dot{x} &= -2x, \\
\dot{y}^{(1)} &= y^{(1)} - y^{(2)}, \\
\dot{y}^{(2)} &= y^{(1)} + y^{(2)}, \\
\dot{y}^{(3)} &= 2y^{(3)} + xy^{(1)}.
\end{align*}
\]
Integrating this system with \( x_0 = y_1^{(1)} = y_1^{(3)} = 1 \) and \( y_1^{(2)} = 0 \), we have
\[
y^{(3)}(t) = e^{2(t - \tau)} + \int_{\tau}^{t} e^{2(t-s)} e^{-2s} e^{s-\tau} \cos(s - \tau) \, ds.
\]
Setting \( t = 0 \), an elementary calculation yields
\[
y^{(3)}(0) = e^{-2\tau} + \frac{1}{15} e^{-\tau} (3e^{-3\tau} - \sin \tau - 3\cos \tau).
\]
Thus \( y^{(3)}(0) \) cannot be expanded in the sense of Definition 1.1.

We remark that a slightly more complicated counter example in \( \mathbb{R}^3 \) is given as \( \dot{x} = -2x, \dot{y}^{(1)} = y^{(1)} - y^{(2)} + xy^{(1)}, \) and \( \dot{y}^{(2)} = y^{(1)} + y^{(2)} \).

In many applications, it often requires both \( x(t) \) and \( y(t) \) to admit an exponential expansion simultaneously if \(-A\) also has the form of \( B \). To be precise, we have

**Definition 1.5.** If \(-A\) has the form of \( B \), then the \( x \)-component \( x(t) \) of the solution of the Šil’nikov problem admits an exponential expansion provided that as the solution of the Šil’nikov problem for the time reversed system of Eq. (1.1) \( x(s) \) with \( s = \tau - t \) admits an exponential expansion.

**Theorem 1.6.** If, in addition to (H1), \(-A\) also has the form of \( B \), then there exists an admissible variable \((x, y)\) for Eq. (1.1) such that each of \( x(t) \) and \( y(t) \) admits an exponential expansion of regularity \( r - 2 \) simultaneously.

We remark that for the coefficient function \( \varphi \) in Theorem 1.3 all the components except for the first \( k \) ones are identically equal to zero. An analogous statement also holds true of the expansion coefficient functions in Theorem 1.6.
Note that in general the regularity for our exponential expansion is only $r - 2$ as asserted by these theorems while it is $r - 1$ for a special case where the principal eigenvalues for the linearization of Eq. (1.1) are simple and real as described above from Deng [1]. This is because the special admissible variable of Lemma 2.1 is required for the general case and any change of variables usually reduces the smoothness of the new vector field by one (at least formally).

The formulation of the exponential expansion above is inspired by L. P. Šil'nikov's mid-1960s works on the structure of flows near a homoclinic orbit of a hyperbolic equilibrium point; in particular, see Šil'nikov [2]. In that paper he considers a system of autonomous ordinary differential equations having a nondegenerate homoclinic orbit $\Gamma$ to a saddle-focus equilibrium $a$. By saddle-focus one means

(i) The principal unstable eigenvalues of the linearization of the vector field at $a$ is a pair of conjugate complexes $\mu \pm i\omega$ with $\omega \neq 0$ and in comparison with the principal stable eigenvalues $\lambda$ they are relatively contractive, i.e., $\mu < Re \lambda$.

By nondegeneracy one means

(ii) As $t \to -\infty$, $\Gamma$ approaches to $a$, being asymptotically tangent to the two dimensional linear subspace of the principal unstable eigenvectors;

(iii) The stable manifold $W^s$ and the unstable manifold $W^u$ intersect along $\Gamma$ in general position, i.e.,

$$\text{codim span } \{ T_p W^s, T_p W^u \} = 1 \quad \text{for all } p \in \Gamma,$$

where $T_p W$ means the tangent space of a given manifold $W$ at the the base point $p$;

(iv) The strong inclination property is satisfied.

One can show the following result

**Theorem 1.7.** (Šil'nikov, 1970). In an arbitrary neighborhood of a nondegenerate saddle-focus homoclinic orbit, there exists a subsystem of orbits which is in one-to-one correspondence with the set $\Omega(\rho)$ of doubly infinite sequence $(\cdots s_{-1}, s_0, s_1, \cdots)$ of the symbols 0, 1, 2, ..., satisfying $s_j + 1 < \rho s_j$ for some constant $1 < \rho < - Re \lambda / \mu$.

Here, by an orbit one means that it lies wholly within the stated neighborhood of $\Gamma$ for all the past and future time $t \in (-\infty, +\infty)$.

Note that by choosing a pair of sufficiently large integers $M$ and $N$ satisfying $M < N < \rho M$ we see that $\Omega(\rho)$ contains a "Smale horseshoe"
\( S = \{ (\cdots s_{-1}, s_0, s_1, \cdots) | s_j = M, \text{ or } N \} \). In fact, one can choose countable pairs of such integers so that \( \Omega(\rho) \) contains a countable number of Smale horseshoes. Indeed, \( \Omega(\rho) \) is in one-to-one correspondence with an invariant subset of a Poincaré map, called \( \Pi \), defined on a subset (to be defined in a moment) of a cross section \( \Sigma_0 \). Let us just sketch the idea of how to construct the Poincaré map with a countable number of Smale horseshoes as invariant sets.

Let \( \Sigma_0 \) be chosen sufficiently close to the equilibrium \( a \), intersecting transversely only that part of \( \Gamma \) which lies in the local stable manifold. Then, \( \Pi \) is the composition of a local map called \( \Pi_0 \) and a global map called \( \Pi_1 \) as follows: Take another cross section \( \Sigma_1 \) also close to \( a \) but intersecting transversely only that part of \( \Gamma \) which lies in the local unstable manifold. The domain \( \mathcal{A} \) of the local map \( \Pi_0 \) (as well as the Poincaré map \( \Pi \)) consists of those points of \( p \) in \( \Sigma_0 \) whose local orbit \( \gamma(p) \) in a small neighborhood \( U \) of \( a \) intersects \( \Sigma_1 \) at a point \( q \in \gamma(p) \cap \Sigma_1 \) for the first time \( \tau(p) \), where \( U \) contains both \( \Sigma_0 \) and \( \Sigma_1 \). \( \Pi_0 \) is defined in a natural way as \( \Pi_0(p) = q \). The global map \( \Pi_1 \) is defined pretty much in the same way as \( \Pi_0 \) except that the times that different global orbits take to travel from \( \Sigma_1 \) back to \( \Sigma_0 \) remain almost constant while the time \( \tau(p) \) for a local orbit approaches infinity as the initial point \( p \in \mathcal{A} \) tends to the local stable manifold. It is this long time behavior of local orbits that determines the dynamics of flows near the homoclinic orbit \( \Gamma \) and at the same time imposes all kinds of difficulties. A key technique in overcoming those difficulties involved is to use the solution of the Šil'nikov problem introduced in the beginning.

Let the local system in the neighborhood \( U \) be the same as Eq. (1.1). Let the initial point \( p \) be as \((x_0, y_0)\), the end point \( q \) as \((x_1, y_1)\) and the time \( \tau(p) \) as \( \tau \). Then the initial and end points can be expressed as \((x_0, y(0; \tau, x_0, y_1)) \) and \((x(\tau; \tau, x_0, y_1), y_1) \), respectively, by the solution of the Šil'nikov problem. This correspondence between the triplet \((\tau, x_0, y_1)\) as independent variable and the initial and end points as dependent variables would remain mysterious unless we regard the former correspondence as a change of variables on the domain \( \mathcal{A} \) and the latter as the local map \( \Pi_0 \) under the new variable \((\tau, x_0, y_1)\). In fact, the diffeomorphic property for this change of variables can be easily justified since it has a differentiable inverse map \((x_0, y_0) \rightarrow (\tau, x_0, y_1)\). Note that with the constraints of \((x_0, y_0) \in \Sigma_0 \) and \((x_1, y_1) \in \Sigma_1 \) the triplet \((\tau, x_0, y_1)\) is actually inbedded in \( \mathbb{R}^{d-1} \). Using this new variable the image or preimage of a given set in the domain or range of the Poincaré map can be easily described as demonstrated by the following case in \( \mathbb{R}^3 \).

Let the local coordinate \((x, r, \theta)\) with \((r, \theta)\) being the polar coordinate for the local unstable manifold \( \{x = 0\} \cap U \). Let \( \Sigma_0 = \{(x, r, \theta) | x = \delta_0, r < 1 \} \cap U \) and \( \Sigma_1 = \{(x, r, \theta) | r = \delta_0, |\theta| < \theta^* \} \cap U \) with the point
(0, δ₀, 0) as the intersection of Γ with Σ₁. For points in Σ₀ or Σ₁ they are parametrized by (r, θ) or (x, θ), respectively. Also the corresponding Sil’nikov variable is (τ, θ) and the domain and range of the local map are given by \( \mathcal{A} = \{(φ₁(θ)e^{-μτ}, -ωτ + φ₂(θ)) + \text{h.o.t.} | τ \geq 1, |θ| ≤ θ^*\} \) and \( \Pi_0(\mathcal{A}) = \{(ψ(θ)e^{iτ}, θ) + \text{h.o.t.} | τ \geq 1, |θ| \leq θ^*\} \), respectively, where h.o.t. means terms of order \( e^{-(μ + α)τ} \). Here, \( φ₁, φ₂, \) and \( ψ \) are the corresponding expansion coefficient functions for the Sil’nikov solution and satisfy \( φ₁(θ) ≈ δ₀, φ₂(θ) ≈ θ, \) and \( ψ(θ) ≈ δ₀ \) for all \(|θ| ≤ θ^*\). We emphasize the preciseness of \( λ \) and \( μ \) here as the real parts of eigenvalues in these expressions. For simplicity we will assume \( ω = 1 \) in this paragraph. Let \( Q \) be a parallelogram with boundaries \( a, b, c, \) and \( d \) in the Sil’nikov variable space, where \( a = \{(τ, θ) | θ = θ^*, τ₀ - 2π ≤ τ ≤ τ₀ \}, b = \{(τ, θ) | θ = θ^*, τ₀ - 2π + φ₂(θ^*) - φ₂(θ^*) ≤ τ ≤ τ₀ + φ₂(θ^*) - φ₂(θ^*) \}, \) and \( c \) and \( d \) are two parallel lines connecting the end points of \( a \) and \( b \), respectively (see Fig. 1). Let \( Q₀ \) and \( Q₁ \) be the corresponding preimage and image of the
local map $\Pi_0$, respectively; and let $a_0, b_0, \cdots$ etc. be the corresponding boundaries. Then it is easy to check by using the representation of the Šil'nikov variable that the preimages $a_0$ and $b_0$ are two spiral arcs with their central angles approximately equal to $2\pi$ and $c_0$ and $d_0$ approximately lie on a same radial line of angle $\tau_0 - 2\pi + \varphi_2(-\vartheta^*)$. Note that we can start the two spiral curves at any angle we want by varying $\tau_0$. In contrast to the spiral strip $Q_0$, the image $Q_1$ looks like another parallelogram and so does its image $Q_0$ under the global return map $\Pi$. Note that both $Q_1$ and $Q_0$ are within distance of order at most $e^{\lambda t_0}$ to the local unstable manifold $\{x = 0, n \in C\}$, and the center point $x = 0$ and $r = 0$ in $\Sigma_0$, respectively, while the spiral strip $Q_0$ lies at least outside a circle of the center in $\Sigma_0$ with the radius of order $e^{\mu t_0}$. Thus, because of the relative contraction assumption $\mu < -\lambda$, $Q_0$ must superimpose $Q_0$ and, therefore, a "horseshoe" is created. To verify this is indeed a Smale horseshoe one needs to show that the invariant tangent bundle cone condition from Moser [3], for instance, is also satisfied. In fact, by using the Šil'nikov change of variables, the image and the preimage of the linearization of the Poincare map on the tangent bundle space are as easily manageable as the map itself above. It is not difficult to see now that a countable number of horseshoes can be constructed in the same way for an ever increasing sequence of $\tau_0$.

Let's take another look at the role played by the Šil'nikov solution in the discussion above. Since the $y$ component of a solution of the initial value problem is stretching we will lose control on the magnitude of the image of a given set in $\Sigma_0$ if there are some additional nonprincipal unstable directions involved. Even if an image is given first, the same problem still persists if there are some nonprincipal stable directions involved. Indeed, these difficulties are identical up to the time reverse for the flow. In this context, the Šil'nikov solution comes in very naturally, treating the troublesome initial $x$ and end $y$ components as independent variables. One of the advantages of doing so is that independent variables can be confined within a bounded domain a priori. Also note that the Šil'nikov variable $(\tau, x_0, y_1)$ is invariant under the time reverse. Therefore it becomes important to understand the structure of the Šil'nikov solution, in particular the exponential expansion.

In [2] and [4] Šil'nikov assumes the vector field considered to be analytic and uses a form of exponential expansion. Unfortunately, that is not quite clear as demonstrated by Example 1.4. It can be achieved under admissible variables and Theorem 1.3. Probably out of this concern in his recent joint work with I. M. Ovsyannikov [5] in their studying the same chaotic system with saddle-focus homoclinic orbit as in Theorem 1.7, he takes a conventional approach instead via the solution of the initial value problem. However, a proof using this approach must restrict either the stable or the unstable manifold to be principal ones as we have pointed out.
in the previous paragraph. In that paper they prove that structurally unstable systems are dense in the set of systems with saddle-focus homoclinic orbit. To be more precise, let $X$ be the subset of vector fields satisfying the conditions (i), (ii), (iii), and (iv) above but with only one dimensional stable manifold. Let $X$ be equipped with the $C^5$-topology. They prove the following main result among other things.

**Theorem 1.8.** (Ovsyannikov and Šil‘nikov, 1986). Each of the following sets of vector fields are dense in $X$:

1. those possessing a structurally unstable periodic orbit; and
2. those possessing a hyperbolic periodic orbit whose stable and unstable manifolds intersect nontransversely.

Another approach that should be mentioned is through a $C^1$-linearization change of variables, transforming the nonlinear local system in $U$ into a linear one. However, additional assumptions must be included for all kinds of existing $C^1$-linearization theorems with only a few exceptions. One of the exceptional cases is in $R^3$ due to a $C^1$-linearization theorem by Belickii [6] and Theorem 1.7 even holds true for a $C^{1,1}$ vector field (see Tresser [7]). We remark that $R^4$ is another exceptional case when the equilibrium point is bifocus and the same statement also holds true. No matter which method we use, the key step involved is to find a tractable coordinate for the local map $\Pi_0$, but with the $C^1$-linearization approach more conditions than necessary are required in general.

In summary, any restriction on the dimension of the stable or the unstable manifold (as far as systems of finite dimensionality are concerned) or any additional $C^1$-linearization assumption is just superfluous and technical for Theorems 1.7 and 1.8. These theorems are true for $C^4$ vector fields and for all finite dimensional stable and unstable manifolds. Their proofs basically follow the same way as their original counterparts, using the exponential expansion of our kind as a basis. We remark that an additional condition called (D) in [2] and (3) in [5], respectively, included in the sufficient conditions for both Theorems 1.7 and 1.8 of the original works. This condition is automatically satisfied in $R^3$ and is equivalent to our so-called strong inclination property (iv). In addition, the exponential expansion theory also has many important applications in the modern theory of dynamical systems. For example, as demonstrated by Deng [1] for the case where $B_0 = \mu$, it implies the strong $\lambda$-lemma which plays an important role in classifying some fundamental problems of codimension two homoclinic and heteroclinic bifurcations (see Chow, et al. [8], [9], and Deng [10]). It also implies the $C^1$-linearization theorem in $R^2$. 

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Analogous results can also be obtained for \( B_0 \) which is a \( 2 \times 2 \) matrix of complex eigenvalues once the exponential expansion becomes available. In particular, the generalized strong \( \lambda \)-lemma will imply that the strong inclination property is actually generic. However, we will not pursue any one of these problems nor intend to give a rigorous proof to Theorem 1.7 or 1.8 here since each of them deserves an independent treatment elsewhere.

2. Admissible Variables

In this section, we replace \( y \) by \((y, z)\), \( B_0 \) by \( B \), and \( B_1 \) by \( C \) in Eq. (1.1) for simplicity of notation and rewrite the equation as

\[
\begin{align*}
\dot{x} &= Ax + f(x, y, z) \\
\dot{y} &= By + g_{11}(x, y, z) y + g_{12}(x, y, z) z, \\
\dot{z} &= Cz + g_{21}(x, y, z) y + g_{22}(x, y, z) z,
\end{align*}
\]

where \( f \) is \( C^{r+1} \), \( g_{ij} \) is \( C^r \), \( f = O(|x| + |y| + |z|) |x| \), and \( g_{ij} = O(|x| + |y| + |z|) \) for \( i, j = 1 \) and \( 2 \).

**Lemma 2.1.** There exists a \( C^r \) \( y \)-admissible change of variables such that \( g_{ii}(x, 0, 0) = 0 \) for \( i = 1 \) and \( 2 \), and the resulting new vector field is \( C^{r-1} \).

**Proof.** The proof follows an idea of [5]. We perform a \( y \)-admissible change of variables

\[
\begin{align*}
\xi &= x, \\
\eta &= y - p(x) y, \\
\zeta &= z - q(x) y,
\end{align*}
\]

where the \( k \times k \) matrix function \( p \) and \((n-k) \times k \) matrix function \( q \) of \( x \) are to be chosen so that \( p(0) = 0 \), \( q(0) = 0 \) and the resulting \( g_{ii}(x, 0, 0) \) are vanishing for \( i = 1 \) and \( 2 \). Here \( k = 1 \) if \( B = \mu \) and \( k = 2 \) otherwise. Writing out the derivatives of \( \eta \) and \( \zeta \) with respect to the time \( t \), we have

\[
\begin{align*}
\dot{\eta} &= \dot{y} - \dot{p} y - p \dot{y} \\
&= By + g_{11}(x, y, z) y + g_{12}(x, y, z) z - \dot{p} y - p \dot{y} \\
&= B\eta + (\dot{p} B + p B + g_{11} - p g_{11} - p g_{12} q)(I - p)^{-1} \eta \\
&\quad + (g_{12} - p g_{12}) \zeta \\
&\equiv B\eta + G_{11}(\xi, \eta, \zeta) \eta + G_{12}(\xi, \eta, \zeta) \zeta
\end{align*}
\]
and
\[ \xi = \dot{z} - \dot{q}v - \dot{q}y \]
\[ = Cz + g_{21}y + g_{22}z - \dot{q}v - qBy - qg_{11}y - qg_{12}z \]
\[ = C\xi + (\dot{q}C - qB + g_{21} + g_{22}q - qg_{11} - qg_{12}q)(I - p)^{-1} \eta \]
\[ + (g_{22} - qg_{12})\xi \]
\[ \overset{\text{def}}{=} C\xi \overset{\text{def}}{=} G_{21}(\xi, \eta, \zeta) \eta \overset{\text{def}}{=} G_{22}(\xi, \eta, \zeta) \zeta, \]

where \( g_{ij} \) are understood to be composed with the new variables; and \( \dot{p} \) and \( \dot{q} \) mean derivatives of \( p \) and \( q \) in time \( t \) along the solution of Eq. (2.1). We require \( G_{ii}(\xi, 0, 0) = 0 \) for \( i = 1 \) and 2. It suffices to solve the following coupled equations for \( p, q, \) and \( x \) on the stable manifold \( W^s = \{ y = 0, z = 0 \} \) of the old Eq. (2.1):
\[ \dot{p} = Bp - pB + g_{11}(x, 0, 0) + g_{12}(x, 0, 0)q - pg_{11}(x, 0, 0) - pg_{12}(x, 0, 0)q \]
\[ \dot{\eta} = Cq - qB + g_{21}(x, 0, 0) + g_{22}(x, 0, 0)q - qg_{21}(x, 0, 0) - qg_{22}(x, 0, 0)q \]
\[ \dot{x} = Ax + f(x, 0, 0). \]

Write the matrix \( p \) in the form of a column vector by taking the matrix's first row as the vector's first \( k \) components, the second row as the next \( k \) components, and so on. Do the same thing to the matrix \( q \). Putting these two vectors and \( x \) together, we obtain a large system of \( (n - k)k + k^2 + m \) equations. The origin is a trivial equilibrium point for this system. By P. Lancaster [11], the linearization of the resulting vector field at this equilibrium point has the same stable eigenvalues as \( A \)'s and the center unstable eigenvalues of the form \( 0, 0, i2\omega, -i2\omega \), and \( v - \mu \pm iw \) if \( k = 2 \) or \( 0 \) and \( v - \mu \) if \( k = 1 \), where \( v \) is any one of the eigenvalues for the nonprincipal unstable block \( C \) of Eq. (2.1). Since the projection from the linear subspace \( p = 0, q = 0 \) (i.e., the \( x \)-axis) to the stable eigensubspace of this linearization is one-to-one and onto, we can choose matrix functions \( p = p(x) \) and \( q = q(x) \) which give rise to the local stable manifold of this \( (p, q, x) \) system. By the theory of stable manifolds \( p \) and \( q \) are \( C^r \) functions satisfying \( p(0) = 0 \) and \( q(0) = 0 \). Hence, the vector field of Eq. (2.1) under the new variables is \( C^{r - 1} \).

Remark 2.2. (a) Note that the change of variables (2.2) as above leaves the \( x \)-coordinate fixed. Thus, if \( -A \) has the same form as \( \text{diag}(B, C) \), then we can keep performing another \( x \)-admissible change of variables again to "kill" the corresponding terms \( f_{ii}(0, y, z) \) for \( i = 1 \) and 2 while the already obtained property \( g_{ii}(x, 0, 0) = 0 \) is still preserved. However, this procedure will at least formally further reduce the smoothness of the final resulting vector field to \( C^{r - 2} \). Fortunately, this artificial
loss of regularity can be remedied by taking these \((x, y)\) admissible changes of variables simultaneously. This observation together with Theorem 1.3 implies Theorem 1.6. Hence, we only need to prove the first theorem.

(b) Following the procedure in the proof above we find that a \(y\)-admissible change of variables for Example 1.4 is as follows, \(p = 0, q_1 = -\frac{1}{10}x, q_2 = \frac{1}{10}x\). Incidentally, the system under thus obtained \(y\)-admissible variable is completely linearized.

3. PROOF OF THEOREM 1.3

The proof follows an idea from Deng [1]. We break it up into three steps.

(I) Let \((\tilde{x}, \tilde{y}, \tilde{z})(t)\) denote the solution of the Šil'nikov problem for the time reversed system of Eq. (2.1) satisfying the corresponding Šil'nikov conditions \(\tilde{x}(\tau) = x_0\) and \((\tilde{y}, \tilde{z})(0) = (y_1, z_1)\). Then by the uniqueness of solutions we have \((\tilde{x}, \tilde{y}, \tilde{z})(t; \tau, x_0, y_1, z_1) = (x, y, z)(\tau - t; \tau, x_0, y_1, z_1)\) or \((\tilde{x}, \tilde{y}, \tilde{z})(t) = (x, y, z)(\tau - t)\) for short. By Lemma A of Appendix, there exist constants \(\delta_0, K_1\), and \(0 < \sigma < \min\{\lambda, -\mu, \mu - \mu\}\) independent of \(t, \tau, x_0, y_1,\) and \(z_1\) such that \(|\partial \tilde{x}(t)| \leq K_1 e^{-\lambda(t - \tau)}\), \(|\partial \tilde{y}(t)| \leq K_1 e^{-\mu t}\) and \(|\partial \tilde{z}(t)| \leq K_1 e^{(\mu - \sigma)t}\) for \(0 \leq t \leq \tau, |x_0|, |y_1|, |z_1| \leq \delta_0/M\) and for all partial derivatives \(\partial\) in \(\tau, x_0, y_1,\) and \(z_1\) of order \(\leq \rho - 1\); in particular, for the solutions themselves the constant \(K_1\) can be replaced by \(2\delta_0\). Moreover, when \(\partial\) contains at least one partial derivative in \(\tau\), smaller bounds like \(|\partial \tilde{y}(t)| \leq K_1 e^{-\lambda(t - \tau) - \mu t}\) and \(|\partial \tilde{z}(t)| \leq K_1 e^{-\lambda(t - \tau) + (-\mu - \sigma)t}\) hold true; in particular, when \(\partial = \partial / \partial \tau\) the constant \(K_1\) can be replaced by an order of \(\delta_0, \) i.e., \(K_1 \sim O(\delta_0)\). Since these results are simply a modification of Theorem 3.1 from Deng [1\], the proof will be given in Appendix. Now, we can conclude that when \(\partial\) contains at least one partial derivative in \(\tau\) these exponential bounds together with the special \(y\)-admissible variables \(x, y,\) and \(z\) from Lemma 2.1 imply

\[
|\partial [g_{11} (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)), \tilde{y}(t) + g_{12} (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)), \tilde{z}(t)]| 
\leq K_2 e^{-\lambda(t - \tau)} + (\mu - \sigma)t \quad \text{for} \quad i = 1 \text{ and } 2,
\]

where \(K_2\) is some constant depending on the usual \(C^{r+1}\) norms for \(f\) and \(g\) in the neighborhood \(U\) among other things but independent of \(t, \tau, x_0, y_1,\) and \(z_1\). Indeed, this is due to the fact that under the special \(y\)-admissible variables we have chosen \(g_{11} y + g_{12} z\) has the order of \(|y|^2 + |z|\)
as $|y| + |z| \to 0$ for $i = 1$ and 2. The last exponential bounds (3.1) are very crucial in what follows.

(II) For simplicity of notation, let us switch back to our old notation: $y$ for $(y, z)$ and $g(x, y)$ for the nonlinear term of the $\dot{y}$ equation as a whole. Then what has just been described above in the first step can be reinterpreted as $|\partial g(\bar{x}(t), \bar{y}(t))| \leq K_2 e^{-\alpha(t - \tau) + (-\mu - \sigma)t}$ provided $\partial$ contains at least one partial derivative in $\tau$. Let

$$Y(t; \tau, x_0, y_1) \overset{\text{def}}{=} e^{-B_*(t - \tau)} y(t; \tau, x_0, y_1),$$

where $B_* = \text{diag}(B_0, \mu I)$ as in the theorem. We need to show for $t, x_0,$ and $y_1$ from an arbitrarily given compact set the function $Y$ together with its derivatives converges to some $C^{r-2}$ function $\varphi - \varphi(t, x_0, y_1)$ uniformly at a rate of $e^{-\sigma t}$ as $t \to +\infty$. Before doing so in the next step, we prove a claim: There exists a constant $K_3$ independent of $t, \tau, x_0,$ and $y_1$ such that for all partial derivatives $\partial$ in $t, \tau, x_0,$ and $y_1$ of order $\leq r - 2$ it satisfies

$$\left| \frac{\partial}{\partial \tau} (\partial Y(t; \tau, x_0, y_1)) \right| \leq K_3 e^{\alpha(t - \tau)}. \quad (3.3)$$

Since the solution of the Šil'nikov problem for the differential equations also satisfies the following equivalent integral equations

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} f(x(s), y(s)) \, ds$$

$$y(t) = e^{B(t-\tau)} y_1 + \int_\tau^t e^{B(t-s)} g(x(s), y(s)) \, ds,$$

$Y(t; \tau, x_0, y_1)$ must satisfy

$$Y(t; \tau, x_0, y_1) = e^{(B - B_*)(t - \tau)} y_1 + \int_\tau^t e^{(B - B_*)(t-s)} e^{B_*(t-s)} g(x(s), y(s)) \, ds$$

$$= e^{(B - B_*)(t - \tau)} y_1 + \int_\tau^t e^{(B - B_*)(t-s)} e^{B_*(t-s)}$$

$$\times g(\bar{x}(\tau - s), \bar{y}(\tau - s)) \, ds$$

$$= e^{(B - B_*)(t - \tau)} y_1 - \int_0^t e^{(B - B_*)(t - \tau + s)} e^{B_*(t-s)} g(\bar{x}(\tau - s), \bar{y}(\tau - s)) \, ds.$$
Thus
\[
\frac{\partial Y}{\partial \tau} (t, \tau, x_0, y_1) = - (B - B^*) e^{(B - B^*) (t - \tau)} y_1
\]
\[
+ \int_0^{t - \tau} (B - B^*) e^{(B - B^*) (t - \tau + z)} e^{B x^2} g(\tilde{x}(\alpha), \tilde{y}(\alpha)) d\alpha
\]
\[
- \int_0^{t - \tau} e^{(B - B^*) (t - \tau + z)} e^{B x^2} \frac{\partial}{\partial \tau} \left[ g(\tilde{x}(\alpha), \tilde{y}(\alpha)) \right] d\alpha
\]
def \[= I_1 + I_2 + I_3 + I_4.\]

Before we start to estimate the exponential bounds to these terms \(I_i\), notice the following trivial inequalities: \(|(B - B^*) e^{(B - B^*) t}| \leq M_0 e^{(\mu_1 - \mu) t}\) for \(t \leq 0\) and some constant \(M_0 \geq M\) since the first \(k \times k\) principal block of the block diagonal matrix is identically zero; \(|e^{B x^2}| \leq M_0 e^{\mu t}\) for all \(- \infty < t < + \infty\) since the amplitude is exactly \(e^{\mu t}\); and last we only have \(|e^{(B - B^*) t}| \leq M_0\) for \(t \leq 0\) since the principal diagonal block is the \(k \times k\) identity matrix. Let \(G = \|g\|_{C^r - 1}\), the usual \(C^r - 1\) topology for \(C^r - 1\) functions in \(U\). Then, for \(0 \leq t \leq \tau\) we have

\[|I_1| \leq M_0 \delta_0 e^{(\mu_1 - \mu) (t - \tau)} \leq M_0 \delta_0 e^{\sigma (t - \tau)},\]

\[|I_2| \leq M_0 e^{\mu (t - \tau)} G 4 \delta_0^2 (e^{-2 \mu (t - \tau)} + e^t (\mu^2 \sigma (t - \tau))) / 2 \]

\[\leq 4 M_0 G \delta_0^2 \sigma (t - \tau),\]

\[|I_3| \leq \int_0^{t - \tau} M_0^2 e^{(\mu_1 - \mu) (t - \tau + \alpha)} e^{\mu_2 G 4 \delta_0^2 (e^{-2 \mu x} + e^{(-\mu - \sigma) x}) / 2} d\alpha \]

\[\leq 4 M_0^2 G \delta_0^2 e^{(\mu_1 - \mu - \sigma) \alpha} \int_0^{t - \tau} e^{(\mu_1 - \mu - \sigma) \alpha} d\alpha \]

\[\leq \frac{4 M_0^2 G \delta_0^2}{\mu_1 - \mu - \sigma} e^{\sigma (t - \tau)},\]

\[|I_4| \leq \int_0^{t - \tau} M_0^2 e^{\mu_2 K_2} e^{-\lambda (t - \tau)} \left[ e^{(\mu - \sigma) x} + (-\mu - \sigma) x \right] d\alpha \]

\[\leq \frac{M_0^2 K_2}{-\lambda - \sigma} e^{2 \tau + \sigma (t - \tau)} \]

\[\leq \frac{M_0^2 K_2}{-\lambda - \sigma} e^{\sigma (t - \tau)}.\]
This proves the claim for $\partial$ to be the order of zero. It is not difficult to see that the same fashion works for any $\partial$ of order $\leq r - 2$.

(III) The claim (3.3) above implies $\partial Y$ is Cauchy in $\tau$ uniformly for $t, x_0,$ and $y_1$ from a given compact set since $\partial(\partial Y)/\partial \tau$ is integrable over $\tau \in [0, +\infty)$. Indeed, for $0 < \tau' < \tau''$ we have from (3.3)

$$|\partial Y(t; \tau'', x_0, y_1) - \partial Y(t; \tau', x_0, y_1)| \leq \int_{\tau'}^{\tau''} \left| \frac{\partial}{\partial \tau} \partial Y(t; \tau, x_0, y_1) \right| d\tau \leq \frac{K_3}{\sigma} e^{\sigma(t-\tau')} \to 0 \quad \text{as} \quad \tau' \to +\infty.$$

Therefore, let

$$\varphi = \varphi(t, x_0, y_1) = \lim_{\tau \to +\infty} Y(t; \tau, x_0, y_1)$$

be the expansion coefficient function and

$$R(t; \tau, x_0, y_1) = e^{B_*^{(t-\tau)}} (Y(t; \tau, x_0, y_1) - \varphi(t, x_0, y_1))$$

be the natural remainder term, the desired exponential expansion is obtained. In fact, the property (1.1b) follows

$$\frac{\partial Y}{\partial y_1} (t; \tau, 0, 0) = e^{B_*^{(t-\tau)}} \cdot \text{diag}(I, 0) \quad \text{as} \quad \tau \to +\infty,$$

which is obtained by differentiating Eq. (3.1) and setting $(x, y)(t; \tau, 0, 0) = 0$ in $g(x, y)$. Property (1.1c) follows exactly in the same way.}

Remark 3.1. (a) In applications we often require that the exponential expansion for the Šil'nikov solution varies smoothly with a parameter if the vector field also smoothly depends on the parameter. The method presented above can be immediately extended to achieve this goal. The key point behind this is that for the principal block $B_0$ the amplitude of $|e^{B_*^t}|$ for $-\infty < t < +\infty$ has precisely the order of $e^{\mu t}$ Here, $\mu$ and $B_*$ are implicitly depending on the parameter. Hence, the exponential expansion holds true for those nearby parameters which give rise to the same symmetric or antisymmetric form $B_0$ as in (H1). In particular, if the principal eigenvalues are structurally stable, i.e., $\omega \neq 0$, then the exponential expansion sustains all small perturbations.

(b) Using the time reversed flow in our approach imposes a great difficulty to extend our result to infinite dimensional problems. In addition, what is an admissible variable for a given infinite dimensional system is unclear at this moment. Because of this, little progress has been achieved in this direction.
(c) The formulation of the Šil'nikov problem and the results for flows as above can also be extended for diffeomorphisms (see Deng [1]). It is interesting to point out a difference in the smoothness of exponential expansions between vector fields and diffeomorphisms: It is $C^{r-1}$ with $r \geq 2$ for diffeomorphisms while $C^{r-2}$ with $r \geq 3$ for vector fields under the same $C^{r+1}$ nonlinearity assumption (H2). This is because any $C^{r+1}$ change of variables for a given diffeomorphism does not reduce the smoothness of the new diffeomorphism.

APPENDIX

**Lemma A.** For every sufficiently small $\delta_0$ and $0 < \sigma < \min\{ -\lambda, \mu, \mu_1 - \mu \}$, there exists a constant $K_1 = K_1(\delta_0, \sigma)$ independent of $t, \tau, x_0, y_1, z_1$, such that $|\partial \bar{x}(t)| \leq K_1 e^{-\lambda(t-\tau)}$, $|\partial \bar{y}(t)| \leq K_1 e^{-\mu t}$ and $|\partial \bar{z}(t)| \leq K_1 e^{(-\mu - \sigma)t}$ for $0 \leq t \leq \tau$, $|x_0|, |y_1|, |z_1| \leq \delta_0/M$ and all the partial derivatives $\partial$ in $\tau, x_0, y_1, z_1$, and $z_1$ of order $\leq r-1$; in particular, when there is no derivative taken $K_1$ can be replaced by $2\delta_0$. Moreover, when $\partial$ contains at least one partial derivative in $\tau$, the exponential bounds for $\partial \bar{y}$ and $\partial \bar{z}$ can be replaced by $K_1 e^{-\lambda(t-\tau)-\mu t}$ and $K_1 e^{-\lambda t + (\mu - \sigma)t}$, respectively; in particular, when $\partial = \partial/\partial \tau$, the constant $K_1$ can be replaced by an order of $O(\delta_0)$.

**Proof:** If the exponent $(-\mu - \sigma)t$ in all the bounds for $\partial \bar{z}$ is replaced by $-\mu t$ instead, this lemma is the same as Theorem 3.1 of Deng [1] which only requires $g_0(x, y, z)$ to be the order of $|x| + |y| + |z|$ as $|x| + |y| + |z| \to 0$. We will assume Theorem 3.1 and use the additional information $g_0 = O(|y| + |z|)$ to derive the sharper bounds. For simplicity of notations, we suppress all the “bars” from the solution $(\bar{x}, \bar{y}, \bar{z})$ of the time reversed system of Eq. (2.1). We only give the necessary modification in details for $\partial$ with zero order and for $\partial$ to be $\partial/\partial \tau$. The other cases are identical.

The following elementary inequality is essential in what follows,

$$\int_{0}^{t} e^{-\mu_1(s-s') e^{\kappa(s-s') + (-\mu - \sigma)s}} ds \leq \frac{1}{\mu_1 - \mu - \sigma} e^{\kappa(t-s) + (-\mu - \sigma)s},$$

where $\kappa = 0$ or $-\lambda$ and $\tau = 0$ or $\sigma$.

When the derivative is not taken, $z(t)$ satisfies

$$z(t) = e^{-\mu t} z_1 - \int_{0}^{t} e^{-\mu(s-s')} (g_{21} y(s) + g_{22} z(s)) ds$$

(A.1)

and $|z(t)| \leq 2\delta_0 e^{-\mu t}$ for $0 \leq t \leq \tau$. Note that we are using notations from Section 2 here. For $\tau = 0$ or $\sigma$ we define a subset $\Gamma_i$ of all functions $p$
of interval $[0, \tau]$ satisfying $|p(t)| \leq 2\delta_0 e^{(-\mu - \sigma) t}$. \( \Gamma_i \) becomes a complete metric space under the weighted norm

$$
\|p\|_i = \sup_{0 \leq t \leq \tau} |p(t)| e^{(\mu + \sigma) t}.
$$

Define an operator, called \( T_i \), by the right hand side of Eq. (A.1) as follows

$$
T_i(p)(t) = e^{-\mu t} z_1 - \int_0^t e^{-\mu (t-s)} \left[ g_{21}(x(s), y(s), p(s)) y(s) + g_{22}(x(s), y(s), p(s)) p(s) \right] ds.
$$

We claim that for an arbitrarily small but fixed \( \delta_0 \) the operator \( T_i \) maps \( \Gamma_i \) into itself and it is contractive. Indeed, we have

$$
\|T_i(p)(t)\| \leq \delta_0 e^{-\mu t} + \int_0^t Me^{-\mu (t-s)} 4G \delta_0^2 (e^{-2\mu s} + e^{(-\mu - \sigma) s})/2 ds
$$

$$
= \delta_0 e^{-\mu t} \left[ 4MG \delta_0^2 \int_0^t e^{-\mu (t-s)} e^{(-\mu - \sigma) s} ds \right]
$$

$$
\leq \left( \delta_0 + \frac{4MG \delta_0^2}{\mu_1 - \mu - \sigma} \right) e^{(-\mu - \sigma) t},
$$

where \( \kappa = 0, |z_1| \leq \delta_0/M, G = \|g\|_{C^{r-1}}, \) and \( M \) is as in the hypothesis (H1). Thus for small \( \delta_0 \) we have \( |T_i(p)(t)| \leq 2\delta_0 e^{(-\mu - \sigma) t} \), implying \( T_i: \Gamma_i \to \Gamma_i \). Similarly, it is easy to show

$$
|T_i(p)(t) - T_i(q)(t)| \leq \int_0^t MG \delta_0 e^{-\mu (t-s)} e^{(-\mu - \sigma) s} ds \cdot \|p - q\|,
$$

$$
\leq \frac{MG \delta_0}{\mu_1 - \mu - \sigma} e^{(-\mu - \sigma) t} \|p - q\|.
$$

Hence,

$$
\|T_i(p) - T_i(q)\| \leq \frac{MG \delta_0}{\mu_1 - \mu - \sigma} \|p - q\|,
$$

showing that \( T_i \) is contractive provided \( \delta_0 \) is small. Since \( \Gamma_\sigma \) is a subset of \( \Gamma_0 \), the unique fixed point of \( T_\sigma \) must be an element of \( \Gamma_0 \) and vice versa. Therefore, being a fixed point of \( T_\sigma \), the solution \( z(t) \) must be in \( \Gamma_\sigma \). This proves our claim for \( \sigma \) of order zero.
To show the other case when \( \partial = \partial / \partial \tau \), we differentiate Eq. (A.1) with respect to \( \tau \) and obtain

\[
\frac{\partial z}{\partial \tau} (t) = - \int_0^t e^{-c(t-s)} \left[ \frac{\partial g}{\partial x} \frac{\partial x}{\partial \tau} (s) + \frac{\partial g}{\partial y} \frac{\partial y}{\partial \tau} (s) \right] ds - \int_0^t e^{-c(t-s)} \frac{\partial g}{\partial z} \frac{\partial z}{\partial \tau} (s) ds
\]

where \( g(x, y, z) \) denotes the whole nonlinear term \( g_{21}y + g_{22}z \). Note that the original "constant" term \( e^{-ct_1} \) drops out from this variational equation. Because of the property \( g = O(|y|^2 + |z|) \), we have

\[
\left| \frac{\partial g}{\partial x} \right| \leq 2G \delta_0 e^{-(\mu - \sigma) t} K_1 e^{-\lambda(t - \tau)},
\]

and

\[
\left| \frac{\partial g}{\partial y} \right| \leq 2G \delta_0 e^{-\mu t} K_1 e^{-\lambda(t - \tau) - \mu t}.
\]

Hence, the "constant" element \( p_0 \) in this equation satisfies

\[
|p_0(t)| \leq 4MGK_1 \delta_0 \int_0^t e^{-\mu(t-s)} \cdot e^{-\lambda(t-s) + (-\mu - \sigma)s} ds
\]

\[
\leq \frac{4MGK_1 \delta_0}{\mu_1 - \mu - \sigma} e^{-\lambda(t - \tau) + (-\mu - \sigma)t}
\]

\[
\leq \frac{K_1}{2} e^{-\lambda(t - \tau) + (-\mu - \sigma)t}
\]

provided \( \delta_0 \) is small but fixed. Note that \( \delta_0 \) can be made independent of \( K_1 \). Define a subset \( \Sigma_{\lambda, t} \) to be all functions \( p \) of the interval \( 0 \leq t \leq \tau \) satisfying \( |p(t)| \leq K_1 e^{-\lambda(t - \tau) + (-\mu - \sigma)t} \). Equip it with the weighted norm

\[
\| p \|_{\Sigma_{\lambda, t}} = \sup_{0 \leq t \leq \tau} |p(t)| e^{\lambda(t - \tau) + (\mu + t)t},
\]

where \( t = 0 \) or \( \sigma \) as above. Analogously, \( \Sigma_{\lambda, t} \) is complete. Furthermore, \( p_0 \in \Sigma_{\lambda, \sigma} \subset \Sigma_{\lambda, 0} \). Define an operator, called \( S_{\lambda, t} \), as the right hand side of the equation for \( \partial z / \partial \tau \) as above

\[
S_{\lambda, t}(p)(t) = p_0(t) - \int_0^t e^{-c(t-s)} \frac{\partial g}{\partial z} p(s) ds.
\]
Then one can show that $S_{\lambda,i}$ maps $\Sigma_{\lambda,i}$ into itself and it is contractive, in a similar way to the operator $T_i$ and $\Gamma_i$. Therefore, the unique fixed point $\frac{\partial z}{\partial \tau}$ of $S_{\lambda,0}$ must be in $\Sigma_{\lambda,\sigma}$ for the same reason as above. 

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