Semi-metrics, closure spaces and digital topology

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Abstract

We investigate certain generalized topological structures, with the aim of showing that they can provide a suitable framework within which to compare various approaches to digital topology (including tolerance geometry) with "ordinary" topology. Within an appropriate category of these structures, ordinary spaces arise as inverse limits of digital spaces.

In the first instance, the structures can be taken to be (a slight generalization of) Čech closure spaces. The structures of this type which seem to be the most useful are those which can be realized as topological graphs (see Section 3). Domain equations for the (real) unit interval provide our main detailed application (Section 4). Another application area which seems promising is that of modal semantics.

1. Introduction

The aim of this paper is to show that certain generalized "topological" structures are worthy of study, both for their intrinsic interest and in view of the potential applications. The work presented is of a somewhat preliminary character: some of the definitions must be regarded as tentative, and the applications remain to be developed in detail.

The motivation for the work stems from a number of areas, of which digital topology is one. Standard digital topology (see [16]) is not topological in the strict sense, but graph-theoretic. Connectivity is taken in the usual graph sense, and graph analogues of the notions of arc and closed curve, and of the Jordan curve theorem, etc., are developed. More recently, there has been developed a properly topological approach, associated with the name of Khalimsky among others (see [8]). An essential feature of this approach is that the spaces directly involved (i.e. the spaces of, say, pixels along with their edges and vertices, in which the digital images sit as finite subsets) are non-Hausdorff. Despite the elegance of this topological treatment of digital topology, it is, however, by no means clear that the traditional graph-theoretic approach is about to be superseded, even for foundational studies in the area [9]. Our position is that the situation in digital topology can be clarified and explained by noting that the structures involved in the two approaches (i.e. the graphs and the spaces) are both instances of a more general structure, in terms of which the foundational aspects, including the relation between digital and "ordinary" (Euclidean)
A second area from which our work draws inspiration is that of tolerance geometry. We refer to the idea that perceptual (and, perhaps, physical) continua, as opposed to the idealized continua of classical mathematics are of finite or countable cardinality and structured by means of a binary relation of indiscernability, which is reflexive and symmetric, but not necessarily transitive. This idea was introduced by Poincaré [12] and (independently) by Zeeman [25]. By far, the most extensive treatment is the Ph.D. thesis of Poston [13]. This work poses in a rather acute form the problem of the relation between graph theory and topology. For, although a tolerance (or fuzzy) space is simply a graph of a certain sort, Poston is able to provide a quite comprehensive "topological" theory for these spaces. Poston's remarkable work seems to have been ignored by graph-theorists, topologists and digital topologists alike. An interesting recent development is Hovsepian [6] (also a Warwick Ph.D. thesis). Here the idea is to apply tolerance geometry to certain problems in Artificial Intelligence, in particular (following suggestions of Patrick Hayes) to the formulation of "naive physics".

A third area to which the material presented here appears to be relevant is that of the semantics of modal logic. This, also, has generally been presented either in terms of relational structures (Kripke frames) or by the use of topological spaces (Rasiowa and Sikorski [15], following McKinsey and Tarski), or possibly a combination of both ([18] and references there cited). But the existing techniques seem to lack sufficient generality to handle all the situations that may arise, for example with respect to intuitionistic modal logic.

It is a remarkable fact that, for at least some of the phenomena we have mentioned, a promising explanatory framework has been available for many years. We refer to the theory of closure spaces developed by Čech [2] as a generalization of ordinary topology. Tolerance spaces, in particular, are straightforward examples of closure spaces (Section 2.2).

In the work reported here, we try to show that generalized topology, somewhat in the style of Čech, has a bearing on various developments in computer science, logic, and elsewhere. At this preliminary stage, much of the effort is concerned with finding the right definitions for basic concepts. In the general setting we may find, for example, that there are several conceivable notions of compactness, all of which reduce to the usual notion for ordinary spaces. We shall find that there are grounds for adjusting some of Čech's definitions. Also of note is the fact that, for certain purposes, it seems advisable to introduce additional structure to the "generalized" spaces so that they become, in effect, topological relational structures (Section 3). What happens then is that, instead of one "generalized topology", we have a number (at least three) of significant conventional topologies associated with a given structure. The problem of generalizing notions such as compactness is to that extent replaced by the, perhaps more manageable, problem of selecting the appropriate topology with respect to which to take compactness, etc. (we might want to take compactness with respect to one topology, connectedness with respect to a second one, and $T_0$-separation with respect to yet a third).
2. Generalized spaces

2.1. Neighbourhood spaces

For our general notion of "space" we shall adopt the following, as being the best adapted to (directed) graphs and weak "metrics".

Definition 2.1. A **neighbourhood space** is a pair \((X, \mathcal{N})\), where \(\mathcal{N}: X \to \mathcal{P}(\mathcal{P}(X))\) assigns to each point \(x\) a (not necessarily proper) filter \(\mathcal{N}_x\) of subsets of \(X\) (the neighbourhoods of \(x\)).

Thus, a point need not be an element of (each of) its neighbourhoods. Moreover, a point may have the empty set as a neighbourhood, in which case every subset of the space is a neighbourhood of it.

It should be mentioned that notions of "space" as general as, or even more general than, that of the above definition are nothing new. An assignment of a system of "neighbourhoods", satisfying no particular requirement (not even that of being a filter), to each point of a set \(X\), is known as a "Fréchet \(V\)-space". A detailed treatment of \(V\)-spaces can be found in Sierpinski [19].

Definition 2.2. A **distance function** (or real-valued relation: [2]) on the set \(X\) is any map \(d: X \times X \to \mathbb{R}^0^+\) (the nonnegative reals).

Further references on weak "distance functions" (subject to few, or none, of the usual metric axioms) may be found in [19, ch. 6].

Proposition 2.3. A distance function \(d\) defined on \(X\) induces a neighbourhood space, by taking \(S \in \mathcal{N}_x\) if and only if, for some \(\varepsilon > 0\), \(\mathcal{B}_\varepsilon(x)\) (\(= \{y | d(x, y) < \varepsilon\}\)) \(\subseteq S\).

From the point of view of generating a neighbourhood space, the use of the reals is of course not essential here. What is important is that the relations \(d(x, y) < \varepsilon\) form the base of a filter (over \(X \times X\)). By extension from the ordinary topological case, we could consider any filter of binary relations on \(X\) as a "uniform neighbourhood space" on \(X\).

The case of a single relation on \(X\) – i.e. a graph – is of particular significance. Any graph gives rise to a neighbourhood space by taking as the neighbourhoods of a node \(x\) those sets (of nodes, or points) which contain the immediate successors of \(x\).

Note: Graphs, in this paper, will generally be taken to be directed and without parallel edges. Parallel edges could perhaps be accommodated by allowing neighbourhoods to be multisets, so that if there are \(n\) edges from \(x\) to \(y\), then \(y\) appears in any neighbourhood of \(x\) with at least multiplicity \(n\). But nothing of this kind will be attempted here.

We have the following easy characterization of the neighbourhood spaces which arise from graphs in this way.
Proposition 2.4. A space $X$ arises from a graph if and only if every point of $X$ has a smallest neighbourhood.

In fact, it is clear that a space in which every point has a smallest neighbourhood (an "Alexandroff space") arises from a unique graph, so that Alexandroff spaces may be identified with graphs. In particular, finite spaces are the same as finite graphs. It is also true, as we shall see in a moment, that every space gives rise to a graph in a natural way.

As with ordinary topology, neighbourhood spaces can be introduced in terms of a number of alternative primitive notions, in particular those of interior operator and closure operator.

Definition 2.5 (Rowlands-Hughes). An interior operator on a set $X$ is a map from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ which preserves finite meets. Closure operator: defined dually.

We remark in passing that these correspond to the minimal properties of $\Box$ and $\Diamond$ in modal logic.

Given a space $(X, \mathcal{N})$, we have an interior and a closure operator defined, just as usual, by

$$x \in \text{Int}(S) \iff S \in \mathcal{N}_x,$$

$$x \in \text{Cl}(S) \iff \text{every neighbourhood of } x \text{ meets } S.$$

The three notions (neighbourhood, interior, closure) are indeed inter-definable, in exactly the usual way, and any one may be taken as primitive: we omit further details.

Example. If $d$ is an arbitrary distance function on $X$, then, in the neighbourhood space induced by $d$:

$$x \in \text{Cl}(S) \iff d(x, S) = 0$$

(where $d(x, S) = \inf\{d(x, y) | y \in S\}$).

It is well-known that in the study of non-$T_1$ topological spaces the specialization (pre)order of a space (defined by: $x \leq y$ $\iff$ every open set which contains $x$ also contains $y$) assumes considerable importance. In the more general setting of neighbourhood spaces, there are two important notions of this type, both of which reduce to the specialization order for topological spaces.

Definition 2.6. Let $(X, \mathcal{N})$ be a neighbourhood space. Then the specialization preorder $\preceq_X$ is defined by

$$x \preceq_X y \iff \mathcal{N}_x \subseteq \mathcal{N}_y;$$

while the associated graph $R_x$ is defined by

$$x R_y \iff x \in \text{Cl}\{y\}$$

(equivalently: every neighbourhood of $x$ contains $y$).

In terms of the specialization order we may define $T_0$-separation of a space $X$ by

$X$ is $T_0$ $\iff \forall x, y \in X (x \equiv_X y \Rightarrow x = y)$
(i.e., distinct points cannot have exactly the same neighbourhoods). We would propose to define $T_1$-separation in the same style by the condition

$$x \leq y \Rightarrow x = y,$$

and $T_2$-separation – more tentatively – by the condition

$$x \neq y \Rightarrow \exists A \in \mathcal{N}_x \exists B \in \mathcal{N}_y. \text{Int}(A) \cap \text{Int}(B) = \emptyset$$

(we shall only have occasion to use $T_0$-separation in the sequel). With these proposals we depart significantly from Čech. In effect, he uses $R_x$ where we have $\leq_x$ in these conditions. Thus, his $T_0$ condition (in the context of closure spaces, which we consider in a moment) may be stated as

$$x R y \& y R x \Rightarrow x = y.$$

Few of the spaces which we consider later are $T_0$ in Čech’s sense, although it is in many cases important that they are $T_0$ in the weaker sense proposed here.

**Example.** For a graph $(X, R)$ considered as a space, the “associated graph” is simply the graph, or relation, $R$ itself. If $R$ is reflexive, then such a space is $T_0$ in the sense of Čech if and only if $R \cap R^{-1}$ is the identity relation.

**Definition 2.7.** A function $f : X \rightarrow Y$, where $X$, $Y$ are neighbourhood spaces, is **continuous** if any of the following equivalent conditions is satisfied:

1. $f^{-1}(\mathcal{N}_{f(x)}) \subseteq \mathcal{N}_x$ \hspace{1cm} (all $x \in X$),
2. $f^{-1}(\text{Int}(S)) \subseteq \text{Int}(f^{-1}(S))$ \hspace{1cm} (all $S \subseteq Y$),
3. $f(\text{Cl}(S)) \subseteq \text{Cl}(f(S))$ \hspace{1cm} (all $S \subseteq X$).

We denote by $\text{NSp}$ the (concrete) category of neighbourhood spaces with continuous functions as morphisms.

These morphisms reduce to the usual graph (or tolerance) morphisms in case the spaces are graphs (tolerance spaces), and (of course) to ordinary continuous functions in case the spaces are topological.

**Proposition 2.8.** Any continuous map $f : X \rightarrow Y$ is a graph morphism with respect to the associated graphs of $X$, $Y$.

The corresponding statement for specialization orders is not true, however.

**Example 2.9.** Let $X$ be the graph $\{(a, b), \emptyset\}$ and $Y$ have the same vertex set and a single edge at $b$. Then the identity map from $X$ to $Y$ is continuous, but is not monotonic with respect to the specialization orders of $X$, $Y$.

Every neighbourhood space has associated with it some significant topologies. The first of these is as follows.
Proposition 2.10. Let \((X, \mathcal{N})\) be a neighbourhood space. Then the collection of sets \(U \subseteq X\) such that \(U \subseteq \text{Int}(U)\) is a topology, denoted \(\mathcal{O}(X)\).

Proof. Almost immediate, on noting that the condition for a set \(U\) to be open may be stated as: \(U\) is a neighbourhood of each of its points. \(\square\)

Given two neighbourhood structures \(\mathcal{N}, \mathcal{N}'\) on a set \(X\), we say that \(\mathcal{N}\) is finer than \(\mathcal{N}'\) (and \(\mathcal{N}'\) coarser than \(\mathcal{N}\)) if, for each \(x \in X\), every \(\mathcal{N}'\)-neighbourhood of \(x\) is also an \(\mathcal{N}\)-neighbourhood of \(x\).

Proposition 2.11. The topology \(\mathcal{O}(X)\) associated with a neighbourhood space \((X, \mathcal{N})\) is the finest topology on \(X\) that is coarser than \(\mathcal{N}\).

Proof. That a topology \(\mathcal{T}\) on \(X\) is coarser than \(\mathcal{N}\) amounts to saying that every \(\mathcal{T}\)-open set is an \(\mathcal{N}\)-neighbourhood of each of its points. Thus \(\mathcal{O}(X)\), as the collection of all sets with this property, is the finest such topology. \(\square\)

In referring to "open" ("closed") subsets of a neighbourhood space, the \(\mathcal{O}\)-topology will be understood (unless another topology is specified). Notice that the specialization preorder of a neighbourhood structure \(\mathcal{N}\) and that of its associated topology \(\mathcal{O}\) are, in general, unrelated. This contrasts with the situation in which we have two topologies \(\mathcal{T}, \mathcal{T}'\), with \(\mathcal{T}\) coarser than \(\mathcal{T}'\) (in which case \(\subseteq_{\mathcal{T}} \subseteq \subseteq_{\mathcal{T}'}\)). For example, with \((X, \mathcal{N})\) as the discrete graph of Example 2.9, \(a \leq_{\mathcal{N}} b\) but \(\neg a \leq_{\mathcal{O}} b\) since \(\mathcal{O}\) here is the discrete topology. With respect to continuity of functions, however, the expected relation does hold.

Proposition 2.12. Every \(\mathcal{N}\)-continuous function is \(\mathcal{O}\)-continuous.

Proof. Suppose that \(f:(X, \mathcal{N}) \to (Y, \mathcal{N}')\) is continuous. Let \(V \subseteq Y\) be such that \(V \subseteq \text{Int}(V)\). Then we have
\[
f^{-1}(V) \subseteq f^{-1}(\text{Int}(V)) \subseteq \text{Int}(f^{-1}(V)) \quad \text{(by continuity of } f).
\]
Thus \(f^{-1}(V)\) is open. \(\square\)

2.2. Closure spaces and semi-metrics

Definition 2.13 (Čech [2]). A (Čech) closure space is a neighbourhood space in which
\[
S \subseteq \text{Cl}(S) \quad \text{(all } S \subseteq X); \quad \text{equivalently: } \text{Int}(S) \subseteq S \quad \text{(all } S); \quad \text{equivalently: } x \text{ belongs to each of its neighbourhoods (all } x).
\]
Example. A graph, viewed as a neighbourhood space, is a closure space if and only if it is reflexive.

In considering distance functions, Čech [2] imposed (as a minimum) the axioms:

1. \( d(x, x) = 0 \),
2. \( d(x, y) = d(y, x) \)

although, evidently, only condition (1) is needed to ensure that the induced neighbourhood space is a closure space. A distance function satisfying these two conditions is called (by Čech) a semi-pseudometric.

Example. Any tolerance space \((X, R)\) is trivially semi-pseudometrizable by putting

\[
d(x, y) = \begin{cases} 0 & \text{if } x \mathcal{R} y \\ 1 & \text{otherwise} \end{cases}.
\]

Closely related to the notion of a semi-pseudometric is that of a semi-uniformity on a set \(X\): a filter of relations on \(X\) having a base of reflexive, symmetric relations. From Čech's extensive treatment of semi-uniformities, let us just cite his Theorem 23 B.3. In our terminology this may be stated as:

**Theorem 2.14.** A space is semi-uniformizable if and only if its associated graph is reflexive and symmetric.

As morphisms of semi-pseudometric spaces (indeed, of real-valued relations generally) we normally take the nonincreasing functions, i.e. functions \(f : X \to Y\) satisfying

\[
d_y(f(x), f(y)) \leq d_x(x, y).
\]

When may the "pseudo" be dropped from "semi-pseudometric"? We would suggest: at least when the space is \(T_0\). The usual practice, including that of Čech, is to impose a far stronger condition (designed to ensure at least \(T_1\) separation), namely

3. \( d(x, y) = 0 \Rightarrow x = y \).

Such a condition immediately rules out almost all the spaces that are of interest to us. A weaker, slightly more satisfactory, condition is

4. \( \forall z. d(x, z) = 0 \iff d(y, z) = 0 \Rightarrow x = y \).

**Proposition 2.15.** Any semi-pseudometric space which satisfies condition (4) is \(T_0\).

The converse is not true, although all the \(T_0\) spaces which we shall encounter below will in fact satisfy (4). The formulation of \(T_0\) separation itself in terms of the distance function, while of course possible, is rather cumbersome. A more important consideration is the following. We expect, given some notion of "\(K\)-pseudometric", to be able straightforwardly to quotient a given \(K\)-pseudometric space by the relevant equivalence relation, yielding the corresponding reduced "\(K\)-metric" space. But the possible
criteria for semi-metricity which we have looked at so far ((3), (4), and $T_0$-separation) do not seem to yield a simple and straightforward reduction of this kind. For this reason, and also because it is the weakest of the feasible criteria, we adopt the following definition.

**Definition 2.16.** A semi-metric space is a semi-pseudometric space $(X, d)$ satisfying

\[ \forall x, y, z \in X. \quad d(x, z) = d(y, z) \implies x = y. \]

Thus, points are to be identified if (and only if) they have the same distance to every point of the space. It is obvious (or it will be as soon as we have reached Proposition 2.18) that this immediately yields a semi-metric quotient for any semi-pseudometric space $X$, which we shall call the semi-metric reduction of $X$.

We conclude this subsection with two useful constructions, which are interesting in as much as they yield spaces which are (in general) at best semi-metric, even when we start with a metric space.

**Proposition 2.17.** Let $(X, d)$ be a semi-pseudometric space. Then the distance function $d_s$, defined over the nonempty subsets of $X$ by

\[ d_s(A, B) = \inf\{d(x, y) | x \in A, y \in B\} \]

is a semi-pseudometric.

We may remark that, in favourable cases (for example, when $(X, d)$ is metric, or pseudo-metric), the restriction of $d_s$ to the closed subsets is a semi-metric. We intend to make a more precise study of this "semi-metric hyperspace" construction on another occasion.

The second construction is quotienting.

**Proposition 2.18.** Let $(X, d)$ be semi-pseudometric, $E$ an equivalence relation on $X$. Then $(X/E, d_s)$, where $d_s$ is defined as in the preceding proposition, is the quotient of $X$ by $E$. Indeed, $d_s$ is the least distance function defined on $X/E$ such that the canonical map from $X$ to $X/E$ is nonincreasing.

**Example.** Let $I$ be the unit interval, with Euclidean metric. Let $I/E$ be the partitioning of $I$ into the three sets $A = \{0\}$, $B = (0, 1)$, $C = \{1\}$. Then the quotient distance is given by: $d_s(A, B) = d_s(B, C) = 0$, $d_s(A, C) = 1$. This defines a semi-metric space, which will play an important role further on.

### 2.3. Topological topics

In this subsection we look at some of the standard basic concepts of topology (such as constructions of spaces, bases, connectedness, and compactness) in the setting of generalized spaces. Inevitably, the discussion will be brief and sketchy.
As regards constructions, we note that the concrete category $\mathbf{NSp}$ admits initial structures for arbitrary sources. That is, given a set $S$ and a family of maps $f_t: S \to X_t (t \in I)$, where each $X_t$ is a neighbourhood space, there is a coarsest neighbourhood structure on $S$ which makes all the $f_t$ continuous. Namely, for each $x \in S$, we must take $\mathcal{N}_x$ as the least filter which contains all the filters $\mathcal{F}_x = f_t^{-1}(\mathcal{N}_{f_t(x)})$ (cf. Definition 2.7 (1)). Thus, $\mathcal{N}_x$ is the filter obtained by taking all possible finite intersections of members of $\bigcup_{t \in I} \mathcal{F}_i$.

Likewise, final structures exist for arbitrary sinks. By standard arguments (e.g. [14]) it follows that $\mathbf{NSp}$ is complete and cocomplete, as well as having well-defined notions of subspace, quotient, etc. Note that $\mathbf{NSp}$ is not quite a topological category, since there are two neighbourhood structures on the singleton set $1$: that in which $\text{Int}(\emptyset) = 1$, and that in which $\text{Int}(\emptyset) = \emptyset$. This means that constant maps defined on a neighbourhood space are not necessarily continuous. On the other hand, $\mathbf{CSp}$ (the category of closure spaces and continuous maps) is clearly a topological category, and is thus slightly better behaved than $\mathbf{NSp}$.

For the notion of a base we propose:

**Definition 2.19.** A base of the neighbourhood space $(X, \mathcal{N})$ is a collection $\mathcal{B}$ of subsets of $X$ satisfying:

$$S \in \mathcal{N}_x \Rightarrow \exists B, B' \in \mathcal{B}. x \in B \subseteq \text{Int}(B') \& B' \subseteq S.$$ 

A slightly less stringent definition would simply require that every neighbourhood of a point $x$ contains a basic neighbourhood of $x$ (i.e. a neighbourhood of $x$ which is a member of $\mathcal{B}$). There is little difference in effect between the two versions. A base in the weaker sense evidently gives rise to a base as in Definition 2.19 simply by incorporating the interiors of the given basic sets; i.e. by taking

$$\mathcal{B}^+ = \mathcal{B} \cup \{ \text{Int}(B) \mid B \in \mathcal{B} \}.$$ 

There is no requirement that basic sets be open, and the notion does not reduce exactly to what is standard in topology. But the connection is sufficiently close.

**Proposition 2.20.** Suppose that $(X, \mathcal{N})$ is topological. Then a collection $\mathcal{B}$ is a base of $X$ (as a neighbourhood space) if and only if $\{ \text{Int}(B) \mid B \in \mathcal{B} \}$ is a base of open sets of $X$.

The definition of connectedness reads exactly as usual.

**Definition 2.21.** A space $X$ is connected if it cannot be partitioned into two nonempty open subsets.

Thus connectedness is in effect taken with respect to the $\emptyset$-topology. Nevertheless, the standard graph-theoretic notion is captured as well, as the reader may check.

As a check on whether the notion is satisfactory in terms of categorical topology, we have the following proposition.
Proposition 2.22. A neighbourhood space $X$ is connected if and only if there is no $(\mathcal{N}^\sim)$-continuous surjection from $X$ to the discrete two-point space.

We turn now to compactness. In order to discuss this in terms of covers, we need to generalize open covers in the following manner [2].

Definition 2.23. Let $S$ be a subset of the neighbourhood space $X$. A collection $\mathcal{C}$ of subsets of $X$ is an interior cover of $S$ if each point of $S$ is in the interior of some member of $\mathcal{C}$.

There seem now to be two possibilities for defining compactness:

(I) $S$ is compact if every interior cover of $S$ contains a finite cover of $S$;

(II) $S$ is compact if every interior cover of $S$ contains a finite interior cover of $S$.

Čech adopts (I); and this choice agrees, in the main, with Poston’s practice in [13] (Poston works very little with compactness, and admits to some uncertainty as to the best definition). However, (I) has some disadvantages:

- it makes little sense for general neighbourhood spaces (for example, the one-point graph having no edges is not compact in the sense of (I))
- there seems no possibility of a Hofmann-Mislove theorem for it (we refer to the type of theorem, extremely important in the study of spaces as domains of computation, which gives a “logical” characterization of compact saturated sets as the meets of suitable filters of neighbourhoods).

By contrast, compactness as defined in (II) makes sense in all spaces, and permits a Hofmann-Mislove theorem for it (this will be explained in a moment); but there is the serious objection that compactness in the sense of (II) is not (in general) preserved by continuous functions.

In the next section we shall consider spaces with added structure which will enable us, among other things, to avoid the difficulties just noted. For now, let us observe that compactness in the sense (II) (which we are inclined to prefer to (I)) is equivalent to compactness, in the ordinary sense, with respect to the topology of interior sets for $X$: i.e. the topology which has as base (in the ordinary topological sense) the collection \{\text{Int}(A) | A \subseteq X\}. By requiring sobriety with respect to this topology on $X$, we can thus, in a rather trivial way, obtain a Hofmann-Mislove theorem for $X$. By requiring that functions be continuous with respect to the interior sets topology we could, even more trivially, ensure that morphisms preserve compactness. The added structure to be introduced in the next section will, however, enable us to proceed in a somewhat less strained manner.

We conclude this section with a remark on duality. In trying to dualize neighbourhood spaces, an obvious suggestion is to consider frames (in the sense of complete Heyting algebras, not “Kripke frames”) with an additional “interior operator”. The difficulty with this is that we have no canonical choice of subsets of a neighbourhood space $X$ to serve as the elements of the “dual” $\mathcal{L}(X)$. True, we have the $\Theta$-topology
and the interior sets topology; but it does not seem possible to base a general duality
type on either of these. Our solution, roughly, will be to impose additional structure
on spaces, in such a way that a space is taken together with a specified base.

3. Topological graphs

We consider structures \((X, \mathcal{F}, R)\), where \(\mathcal{F}\) is a topology, and \(R\) a binary relation,
on the set \(X\). Given \((X, \mathcal{F}, R)\), we have an interior (finite-meet preserving) operation
on \(\mathcal{P}(X)\), defined by

\[ x \in \text{Int}(S) \iff \exists U, V \in \mathcal{F}, x \in U \land R(U) \subseteq V \land V \subseteq S. \]

Thus we have a neighbourhood space, which will be denoted \((X, \mathcal{F}^* R)\). This neigh-
bourhood space has \(\mathcal{F}\) as a base, and Int restricts to an operation on \(\mathcal{F}\) (explicitly:
\(\text{Int}(V) = \bigcup \{ U \in \mathcal{F} | R(U) \subseteq V \}\)). The space \((X, \mathcal{F}^* R)\) has its associated graph, say
\(R'\). How are \(R, R'\) related? Clearly, we have \(R \subseteq R'\);
but the reverse inclusion does not hold in general. We are particularly interested in the
case that \(R, R'\) coincide, and we enshrine this in a definition.

**Definition 3.1.** A structure \((X, \mathcal{F}, R)\) is a topological graph provided that \(R\) coincides
with the associated graph of \((X, \mathcal{F}^* R)\).

The following proposition provides two straightforward reformulations of the
definition. The first of these (i.e. condition (2)) is intended to be helpful in showing how
to think about topological graphs. The second reformulation is frequently advantage-
ous in checking the defining condition in concrete cases.

**Proposition 3.2.** Given a structure \((X, \mathcal{F}, R)\), the following are equivalent:
(1) \((X, \mathcal{F}, R)\) is a topological graph
(2) for all \(x, y \in X\),
\[ xRy \iff [\forall U \in \mathcal{F}, x \in \text{Int}(U) \Rightarrow y \in U] \]
(3) for every pair of points \(x, y\) such that \(\neg xRy\), there is a pair \(U, V\) of \((\mathcal{F}\)-open sets
which witnesses to this, i.e.
\[ x \in U \land R(U) \subseteq V \land y \notin V. \]

**Example A.** Finite topological graphs. If \(X\) is finite, then the condition for \((X, \mathcal{F}, R)\)
to be a topological graph amounts to saying that \(R\) is compatible with the special-
ization order \(\leq\) of \(X\), in the sense that
\[ x \leq yRz \leq w \Rightarrow xRw \]
Example B. Linear (topological) graphs. Given an arbitrary linear order \((L, \leq)\), let \(L\) be the usual interval topology of \(L\), with base the open intervals \(\{x | a < x < b\}\), and let \(R\) be the adjacency relation of \(L\) \((xRy \iff \forall z [x < z < y \Rightarrow z = x \text{ or } z = y])\). Then \((X, \mathcal{T}, R)\) is a topological graph. Structures of this kind will sometimes simply be called linear graphs.

With the obvious choice of morphisms (\(\mathcal{T}\)-continuous, \(R\)-preserving), we have a category \(\text{TopGr}\). The situation with regard to \(\text{TopGr}\) is much as with \(\text{NSp}\) and \(\text{CSp}\) (Section 3.2). That is, \(\text{TopGr}\) is "not quite" a topological category, since there are two possible structures on the singleton set (and constant functions are not necessarily morphisms); a topological category is obtained on restricting to the reflexive (topological) graphs.

We will in due course examine a number of further conditions on structures of the type \((X, \mathcal{T}, R)\), including that which one would most often expect to see in mathematical contexts (namely, that \(R\) is closed as a subset of \(X \times X\)).

From the existing literature, Definition 3.1 is most closely related to what one finds in discussions of the semantics of modal logic; see especially Sambin and Vaccaro [18]. The significance of the condition expressed in Definition 3.1 (as reformulated in Proposition 3.2(2)) is that the relation \(R\) between points is determined by the ("modal") operator \(\text{Int}\) on the open sets, i.e. by the "logic" of the structure. Indeed, it is easy to see that, given any finite-meet preserving operator \(\Box\) on a topology \(\mathcal{T}\) over a set \(X\), the associated graph \(R\) given by

\[ xRy \iff \forall U \in \mathcal{T}, x \in \Box U \Rightarrow y \in U \]

is a topological graph with respect to \(\mathcal{T}\). What we have here is in effect the object part of an adjunction between structures of the type \((X, \mathcal{T}, R)\) and what we might call "modal locales": structures \((L, \Box)\) where \(L\) is a locale and \(\Box\) a (finite-)meet preserving operation on \(L\). (Unfortunately, the term "modal frame" is currently in use to refer to certain structures of the type \((X, \mathcal{T}, R)\), i.e. to structures on the spatial rather the logical side of the adjunction.) It is easy to introduce suitable morphisms for these structures, so as to obtain an adjunction of categories. A structure \((X, \mathcal{T}, R)\) is isomorphic to its "bidual" under this adjunction if and only if it is a topological graph and \(\mathcal{T}\) is sober.

At this point we prefer not to go further into the details of adjunctions/dualities for topological graphs, or into the connections with modal logic. The conditions studied below are related to those considered in modal semantics, but we shall not draw the connections explicitly: see the concluding remarks to this section.

Proposition 3.3. Let \((X, \mathcal{T}, R)\) be a topological graph. Then

1. for any \(y \in X\), \(\{x | xRy\}\) is \((\mathcal{T})\)-closed.
2. for any \(x \in X\), \(\{y | xRy\}\) is saturated (i.e. a meet of open sets, or equivalently, an upper set with respect to the specialization order \(\preceq\) of the space \((X, \mathcal{T})\)).

Proof. (1) Let \(y \in X\) be given. If \(\neg xRy\), then (Proposition 3.2) we have an open set \(U\) such that \(x \in U\) and \(R(U) \subseteq X - \{y\}\). Thus, \(\{x | \neg xRy\}\) is open.
(2) Let \( x \in X \) be given. Suppose that \( xRy \), and \( y \leq z \). That \( xRy \) means that, for every open set \( U \) such that \( x \in \text{Int}(U) \), \( y \in U \). Since every open set containing \( y \) also contains \( z \), we have \( xRz \) a fortiori.  

In the case that \( R \) is a partial order, the property (1) of Proposition 3.3 has been called lower semi-continuity of \( R \) [4], with upper semi-continuity of \( R \) defined similarly as the property that \( R[x] \) is closed for all \( x \). However, we shall not use these terms in this way, as this usage is in conflict with the view of relations as many-valued maps which we shall (in effect) adopt.

**Definition 3.4.** A relation \( R \) on a topological space \((X, \mathcal{T})\) is continuous [18] if, for each open set \( U \), the set \( \{x | R[x] \subseteq U\} \) is open. A continuous graph is a topological graph \((X, \mathcal{T}, R)\) such that \( R \) is continuous.

Continuity of \( R \) is the condition that \( R \), regarded as a multifunction, is upper semi-continuous [11, 12].

In the case of a \( T_1 \)-space, a continuous relation automatically defines a topological graph:

**Proposition 3.5.** If \((X, \mathcal{T})\) is \( T_1 \) and \( R \) is a continuous relation on \( X \), then \((X, \mathcal{T}, R)\) is a topological graph.

**Proof.** Assume that \( X \) is \( T_1 \), and \( R \) continuous. Suppose that \( x, y \in X \) are such that \( \neg xRy \). Then \( V = X - \{y\} \) is open, and \( x \in U = \{x | R[x] \subseteq V\} \). But \( U \) is open by continuity of \( R \), and so the result is proved (by Proposition 3.2).  

Finally (in this series of definitions concerning structures \((X, \mathcal{T}, R)\)), a condition which is sometimes (e.g. [1]) taken as part of the definition of upper semi-continuity.

**Definition 3.6.** A continuous relation \( R \) on \((X, \mathcal{T})\) is finitary if \( R[x] \) is compact for every \( x \in X \). A finitary graph is a topological graph \((X, \mathcal{T}, R)\) such that \( R \) is finitary.

When topological spaces with an associated relation are studied (in the mathematical literature), the condition most usually imposed on the relation is that it be closed as a subset of the product space. It tends to be assumed also that the space is Hausdorff (an exception is provided by [4], and works on pospaces cited there). Generally, only the case that the relation is a partial order is considered. The very important compact ordered spaces of course exemplify all of this. The case in which the relation is not required to be a partial order seems to be only rarely discussed (we have found little besides [10], where graphs which are Hausdorff spaces with a closed
The following proposition shows how our notion is related to some of those studied previously, in the case that the space is Hausdorff:

**Proposition 3.7.** Let \((X, \mathcal{T})\) be a Hausdorff space. If \((X, \mathcal{T}, R)\) is a finitary graph, then \(R\) is a closed relation. The converse holds in case \(X\) is compact.

**Proof.** Assume that \((X, \mathcal{T}, R)\) is finitary. Let \(x, y \in X\) be such that \(\neg xRy\). Since \(R[x]\) is compact, there are disjoint open sets \(A, B\) such that \(R[x] \subseteq A, y \in B\). Since \(R\) is upper semi-continuous, there is an open neighbourhood \(O\) of \(x\) (viz., \(O = \{x|R[x] \subseteq A\}\)) such that \(R(O) \subseteq A\). Thus we have a neighbourhood \(O \times B\) of \((x, y)\) which does not meet \(R\), showing that \(R\) is closed.

For a converse, assume that \((X, \mathcal{T})\) is compact, and that \(R\) is closed. For each \(x \in X\), \(R[x]\) is closed, hence compact. For continuity, let \(x \in X\) be given, and let \(A\) be an open neighbourhood of \(R[x]\). Since \(R\) is closed, we have for each \(y \in X - A\) open neighbourhoods \(O_y, B_y\) of \(x, y\), respectively, such that \(O_y \times B_y\) is disjoint from \(R\). By compactness, finitely many of the \(B_y\) cover \(X - A\). The meet of the corresponding \(O_y\) is then an open neighbourhood \(O\) of \(x\) such that \(R(O)\) is disjoint from \(X - A\), i.e. \(R(O) \subseteq A\). Thus \(R\) is continuous, and so (Proposition 3.5) \((X, \mathcal{T}, R)\) is finitary.  

With regard to morphisms for topological graphs \((X, \mathcal{T}, R)\), we have the obvious choice of the \(\mathcal{T}\)-continuous, \(R\)-preserving maps. Likewise, compactness is unproblematic (cf. the discussion in the preceding Section) if taken with respect to \(\mathcal{T}\). But these are not the only possible choices. We have also the \(O\)-topology and the interior sets topology of the graph (viewed as a neighbourhood space). It may seem that we have an embarrassment of riches here. But in favourable cases, at least, some of these topologies coincide. In particular, we have the following proposition.

**Proposition 3.8.** Let \((X, \mathcal{T}, R)\) be a continuous graph, with \(R\) a preorder. Then the topology \(\mathcal{O}(X)\) (Proposition 2.10) of \(X\) with respect to its neighbourhood structure \(\mathcal{T}^*R\) coincides with the topology of interior sets (discussion following Definition 2.23).

**Proof.** Since \(R\) is reflexive, each \(\mathcal{T}^*R\)-open set \(O\) satisfies \(O = \text{Int}(O)\) and is thus an interior set. On the other hand, suppose that \(U\) is an interior set. By continuity of \(R\), \(U - R^{-1}(V)\) for some \(\mathcal{T}\)-open set \(V\) and, since \(R\) is a preorder, \(R^2(U) \subseteq V\), so that \(R(U) \subseteq R^{-1}(V)\). Thus we have

\[
U \subseteq R(U) \subseteq R^{-1}(V),
\]

and so \(U = R(U)\). Thus \(U = \text{Int}(U)\); \(U\) is \(\mathcal{T}^*R\)-open.  

**Example.** We recall that Scott domains of all the usual varieties can be viewed as compact ordered spaces in their Lawson topology and information order (see, e.g., \([20, 22, 4, 7]\)). When a domain \(D\) is thus viewed as \((D, \text{Law}(D), \subseteq)\), the Scott topology is recovered as the collection of open upper sets of the compact ordered space. Now,
viewing the structure as a topological graph, and hence as a neighbourhood space, it is easy to see that the open upper sets coincide with the $\mathcal{O}$-open sets; thus the $\mathcal{O}$-topology and (in view of Proposition 3.8) the topology of interior sets both reduce to the Scott topology.

As the last main topic of the section, we would like to characterize the inverse limits of finite topological graphs. Thus we consider $(X, \mathcal{I}, R) = \text{Lim}_{\mathcal{I}} (X_i, \mathcal{I}_i, R_i)$, where we assume the $(X_i, \mathcal{I}_i)$ to be $T_0$ (and finite), so that they may be taken to be in effect posets $(X_i, \leq_i)$. It is well-known [7, 22, 3] that $(X, \mathcal{I})$ may be characterized as a spectral ($= \text{coherent}$: Johnstone [7]) space, or; alternatively, as an ordered Stone space ($= \text{Priestley space}$) $(X, \mathcal{I}, <)$ where $(X, \mathcal{I})$ is the inverse limit of the $X_i$ taken as discrete, while $\leq$ is the inverse limit of the $\leq_i$. The relation between $\mathcal{I}$ and $(X, \mathcal{I})$ is as follows. The $\mathcal{I}$-open sets are the $\mathcal{I}$-open upper sets, while, starting from $\mathcal{I}$, we obtain $\mathcal{I}$ as its patch (i.e. $\mathcal{I}$ is the least topology which includes $\mathcal{I}$ and contains also the complements of the compact saturated sets of $(X, \mathcal{I})$ [4, 18]) and $\leq$ as its specialization order. With regard to $R$, we note that if $x, y \in X$, where $x = (x_i), y = (y_i)$, then

$$xRy \iff \forall i. x_iRy_i,$$
$$\neg xRy \iff \exists k. \forall i \geq k. \neg x_iRy_i.$$

From this we see that $\neg xRy$ holds iff, for some $\mathcal{I}$-open neighbourhoods $U, V$ of $x, y$ we have: $\forall w \in U \forall z \in V. \neg wRz$. This amounts to saying that $R$ is a closed relation on $X$ (with respect to its Stone topology). We have not yet taken into account that the $(X_i, \mathcal{I}_i, R_i)$ are topological graphs, in other words (Example A above) that $R_i$ is compatible with $\leq_i$ for each $i$. If we assume, as we may without loss of generality, that each $R_i$ is a quotient of $R$ (i.e. $p_i$ is surjective, and $x_iRy_i$ holds only if $xRy$) we find that $R$ is compatible with $\leq$ iff, for all $i$, $R_i$ is compatible with $\leq_i$. Thus we have as a characterization that (1) $\mathcal{I}$ is spectral and (2) $R$ is closed with respect to $\mathcal{I}$ and compatible with $\leq$. This leads us to:

**Theorem 3.9.** A topological graph $G = (X, \mathcal{I}, R)$ is expressible as the inverse limit of finite graphs if and only if $\mathcal{I}$ is spectral and $G$ is finitary.

**Proof.** We adopt the notation and results of the preceding discussion. Suppose that $G$ is (isomorphic with) the inverse limit of finite graphs. Then we know that $R$ is closed with respect to the patch $\mathcal{I}$ of the spectral space $(X, \mathcal{I})$ and compatible with the specialization order $\leq$ of $\mathcal{I}$. Let $V$ be a $\mathcal{I}$-open set, i.e. an open upper set of $(X, \mathcal{I}, \leq)$. By Proposition 3.7, $\{x | R[x] \subseteq V\}$ is $\mathcal{I}$-open, and is an upper set since $R$ is compatible with $\leq$. This means that $R$ is $(\mathcal{I})$-continuous. Trivially, each $R[x]$ is $\mathcal{I}$-compact since it is $(\mathcal{I}\text{-closed hence}) \mathcal{I}$-compact. We have shown that $G$ is finitary.

Suppose conversely that $G$ is finitary. Then $R$ is compatible with $\leq$ by Proposition 3.3. Next, let $x, y$ be such that $\neg xRy$. Since $(X, \mathcal{I}, \leq)$ is an ordered Stone space, there is an upper ($\mathcal{I}$-)clopen neighbourhood $V$ of the compact set $R[x]$ such that $y \notin V$. 


Now $V$ is $\mathcal{T}$-open, so we have an open neighbourhood $U$ of $x$ such that $R(U) \subseteq V$. Thus we have a neighbourhood of $(x, y)$, namely $U \times (X - V)$, which is disjoint from $R$. This establishes that $R$ is closed; we conclude that $G$ is an inverse limit of finite graphs.

**Discussion.** We concluded the previous section by pointing out that it may be advisable to consider neighbourhood spaces with a specified base. In a neighbourhood space of type $(X, \mathcal{T}^* R)$, $\mathcal{T}$ is of course a base. But we have not given any justification for requiring that the base be a topology, rather than, say, just a lattice. In studying “modal frames” $(X, \mathcal{T}, R)$, a typical requirement is that $\mathcal{T}$ be a field of subsets of $X$, although the topology generated by $\mathcal{T}$ is also of considerable importance [18]. This requirement is suited to the case that we are studying classical (modal) logic. The requirement that $\mathcal{T}$ be a topology, which we have adopted here, is closer to the mathematical literature. We leave for another occasion the study of more general structures involving weaker conditions on $\mathcal{T}$.

4. Domain equations for the unit interval

Our general aim in considering inverse limits of finite spaces is to bring the structures useful in digital topology (and other areas of computation) into a common framework with “standard” topological spaces. The approximation of (Euclidean) images by finite structures is one aspect of this. But, as regards the computational aspect of the framework, we seek also to construe basic standard spaces such as the unit interval as first-class “data types”, and indeed to characterize them by suitable domain equations. We have on previous occasions (for example [21], unpublished) attempted this latter task in terms of quasi-metrics and non-Hausdorff topologies. We should like to propose here (in outline) what seems to be a simpler approach using semi-metrics and closure spaces/topological graphs.

For a first version of this we can look at linear topological graphs (Example B, Section 3). Let $G_n$ be the set of binary sequences ($\in \{0, 1\}^n$) of length $n$, ordered lexicographically. With truncation providing the connecting maps $(f_{n+1,n}(\sigma_n e) = \sigma_n$, where $\sigma_n \in G_n$, $e \in \{0, 1\}$), we have an inverse sequence. If the $G_n$ are taken with the discrete topology, the limit of this sequence is of course simply Cantor space. However, the $G_n$ are connected as topological graphs (or closure spaces), and so is their inverse limit $G$. The limit space $G$ differs from the unit interval $I$ in as much as it has two versions of each dyadic rational belonging to $I$. The two versions of a given dyadic are indistinguishable by neighbourhoods, and indeed the $T_0$ification of $G$ (as a closure space) coincides with $I$.

To get a more concrete view of this, we can regard the $G_n$ as being obtained by successive subdivision of the interval $I$. Viewed this way, the elements of $G_n$ are (closed) intervals $[k \cdot 2^{-n}, (k + 1) \cdot 2^{-n}]$, $k = 0, \ldots, 2^n - 1$. Endowing the $G_n$ with the semi-metric of closed subsets (Proposition 2.17), the connecting maps are now nonincreasing (notice that this does not happen if the $G_n$ are taken with Hausdorff metric).
We thus obtain $G$ as a semi-pseudometric space. It should be evident that the two copies of each dyadic have the same distance to every point of $G$, and that the (semi-)metric reduction of $G$ is exactly $I$ with its Euclidean metric (a general result of this type will be proved in a moment).

For a "domain equation" generating $G$ (as its canonical solution), we can consider

$$D \cong D + D.$$ (1)

Here we suppose that we are working in the category of linearly ordered sets, viewed as graphs as in Example B, and with morphisms the surjective monotonic graph morphisms (less restrictive choices of morphisms are also possible). The operator $+$ is the usual sum (concatenation) of linear orders; this is clearly functorial for the chosen morphisms, and preserves inverse limits. The category has the one-point structure as terminal object. By standard arguments, we obtain $G$ as the canonical solution of (1) (technically, we get the isomorphism $G \to G + G$ as terminal coalgebra for $\lambda D.D + D$). This construction, however, does not take account of the semi-metric structure of the spaces. Indeed, if the spaces are augmented with their (intended) semi-metrics, it is difficult to see how the operator $+$ can be defined in such a way that the successive approximations $G_0, G_1 = G_0 + G_0, \ldots$ receive the "correct" semi-metric. We will shortly consider a slightly different approach, by which this difficulty can be overcome.

The approximation of $G$ (or $I$) by the $G_i$ may be regarded as an instance of the following general result.

**Theorem 4.1.** Let $(X,d)$ be a compact metric space. Then $X$ can be expressed as the (semi-metric reduction of) of the inverse limit of finite semi-metric spaces (via nonincreasing maps).

**Proof.** Let $\mathcal{B}_1, \mathcal{B}_2, \ldots$ be a sequence of finite covers of $X$, chosen in such a way that

1. Every member of $\mathcal{B}_n$ is a closed ball of radius $2^{-n}$
2. $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$: $\forall B \in \mathcal{B}_{n+1} \exists A \in \mathcal{B}_n. B \subseteq A$.

Each $\mathcal{B}_n$ is endowed with its "hyperspace" semi-metric (Proposition 2.17). Next, maps $f^n: \mathcal{B}_{n+1} \to \mathcal{B}_n$ are chosen such that

$$\forall B \in \mathcal{B}_{n+1}. B \subseteq f_n(B).$$

Clearly, the maps $f_n$ are nonincreasing. Let $X^* = \lim_{\leftarrow} (\mathcal{B}_n, f_n)$. If $\xi = (B_1, B_2, \ldots) \in X^*$ then, since $X$ is complete, $\cap \xi$ is a singleton (point) of $X$. Moreover, it is easy to see that, for $\xi, \eta \in X^*$,

$$d_{X^*}(\xi, \eta) = d_X(\cap \xi, \cap \eta).$$

From this it follows that $X$ is the (semi-metric) reduction of $X^*$. ♦

For our next version of the domain equation for $I$, we consider the category $K$ whose objects are semi-(pseudo)metric spaces having two distinguished points, labelled $0, 1$, subject to the axiom

$$d(0, 1) = 1.$$
As the morphisms of $K$ we take the nonincreasing maps which both preserve and reflect distinguished points, i.e. which satisfy

$$f(x) = 0(1) \text{ iff } x = 0(1).$$

The terminal object of $K$ is the semi-metric space $T = \{0, a, 1\}$, where

$$d(0, a) = d(a, 1) = 0.$$ 

Given two objects $X, Y$ of $K$, we define $X +_1 Y$ as the result of identifying $1_x$ with $0_Y$ in the disjoint union of $X, Y$, with distance defined by

$$d(x, y) = \begin{cases} 
2^{-1} \cdot d_X(x, y) & \text{ if } x, y \in X, \\
2^{-1} \cdot d_Y(x, y) & \text{ if } x, y \in Y, \\
2^{-1} \cdot (d_X(x, 1_X) + d_Y(0_Y, y)) & \text{ if } x \in X, y \in Y.
\end{cases}$$

Since morphisms are required to preserve distinguished points, this operator is functorial; almost as evident is that it commutes with inverse limits. Thus we may consider the canonical solution $H$ of

$$D \cong D +_1 D.$$ 

For a concrete version of this, we may note that the successive approximations $H_0 = T, H_1 = H_0 +_1 H_0, \ldots$ in the iterative solution of (2) can be seen as quotients of $I$ corresponding to finer and finer partitions of $I$. Thus, $H_0$ is the semi-metric quotient (Proposition 2.18) of $I$ under the trivial partition $\{\{0\}, \{0, 1\}, \{1\}\}$. Making a binary subdivision of the open interval of this partition we obtain the finer partition $\{\{0\}, \{0, 2^{-1}\}, \{2^{-1}\}, \{2^{-1}, 1\}, \{1\}\}$, for which the corresponding quotient is $H_1$; and so on. This construction of $H$ is in effect very similar to the previous construction of $G$. The only difference between $H$ and $G$ is that in $H$ we get three copies of each dyadic number in $\{0, 1\}$. For example, for the dyadic number $2^{-1}$ we have the sequence $(x_i) \in H$, where $x_i = \{2^{-1}\}$ for all $i \geq 1$; but we also have the reducing sequences of open (dyadic) intervals adjacent to $2^{-1}$ (on the left and on the right, respectively). The reduction of $H$ is, again, $I$.

5. Concluding remarks

We list here some of the further developments which we envisage (besides those emphasized already in the text, such as modal semantics).

First, we need to develop a theory of many-valued functions and power spaces. An apparatus of functions on graph-like structures that are required to preserve adjacency of nodes (or pixels) is rather restrictive. This may be one reason for the lack of popularity of Poston's theory. (Poston himself [13] characterized the technique as "chain-mail" geometry, in contradistinction to the rubber-sheet geometry of ordinary topology.) This rigidity can be overcome by allowing many-valued functions (for approximation of ordinary functions, and for building function spaces). A technical
point relevant to the results above: by permitting many-valued functions we can obtain spaces such as \( I \) as exact inverse limits of finite spaces, rather than only as reductions of such. We have already noted (Proposition 2.17) that the semi-metric setting allows for new, very simple power space constructions.

Next, we would mention continuum theory. We expect the technique of representation via inverse limits of finite connected (generalized) spaces to cast some light on this traditional topic. Beyond this, we envisage generalizing continuum theory to non-Hausdorff (and even nontopological) spaces.

The framework presented in this paper is by no means the most general that may usefully be considered. It would for a number of reasons be desirable to allow structures that have a family of interior operators (modalities) rather than just one. (Conceivably we should have explicit paired closure operators as well, since in a constructive setting these might not be interdefinable with the interior operators.) This may be relevant to the handling of digital topology in our setting, since some systematizations of this field (for example [9]) work with "spaces" which have two or more adjacency relations simultaneously on the same set of vertices. On the wider aspect we should mention that, for a considerable time, Rowlands-Hughes (unpublished, but see [17]) has been developing a very general "topological" framework in terms of a notion of covering degree. This is certainly broad enough to encompass structures with families of interior operators (an aspect which he has emphasized in his work), and may provide what we are looking for in this direction.

Finally, completeness. This important topic (in relation to semi-metrics and related structures) is conspicuous by its absence from Čech's text [2]. It requires substantial work. At present, we envisage handling it by an adaptation of the theory developed in Smyth [23] (see also [24]).

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