JOURNAL OF COMBINATORIAL THEORY, Series B 34, 177–186 (1983)

Equitable and Proportional Coloring of Trees

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Communicated by the Editors

Received July 27, 1977

We show that almost all trees can be equitably 3-colored, that is, with three color classes of cardinalities differing by at most one. Also, except in some extreme cases, they can be 3-colored with color classes of sizes in given proportions.

An *equitable k-coloring* of a graph on n vertices is a partition of the vertices into k sets or color classes of as near equal sizes $\lfloor (n+i)/k \rfloor$, $0 \leqslant i \leqslant k-1$, as possible, with no two adjacent vertices in the same set. A deep result of Hajnal and Szemerédi [5] states that a graph of maximum degree Δ is equitably k-colorable if $k > \Delta + 1$. A special case had been proved earlier by Corrádi and Hajnal [3]. For an exposition of these and related results, see [2, Chaps. III and VI], from which we take our notation. The equitable coloring of general graphs and hypergraphs has also been considered by Berge and Sterboul [1]. For special classes of graphs the general theorem of Hajnal and Szemerédi can be improved considerably. Here we restrict our attention to trees. Meyer [6] (and see also Eggleton in [4]) proved that trees are equitably k-colorable if $k \geqslant \lceil \Delta/2 \rceil + 1$, where Δ is the maximum degree of the tree. We improve this to $k \geqslant 3$ by proving

THEOREM 1. *A tree on n vertices with maximum degree Δ is equitably* 3-*colorable if $n \geqslant 3\Delta - 8$ or if $n = 3\Delta - 10$.*

A tree is bipartite and hence 2-colorable, but if $\Delta \geqslant 3$, not necessarily equitably so. We call the two natural color classes *amber* and *blue*, and suppose that their cardinalities are a and b with $a \geqslant b$ and $a + b = n$, the number of vertices in the tree, so that $b \leqslant \lfloor n/2 \rfloor$. The final allocation of colors to vertices will be indicated by capitalizing the names of the colors, Amber, Blue and Crimson. The method of proof of Theorem 1 is algorithmic, and we also show how to equitably 3-color those few trees which can be so colored, even though they do not satisfy the condition of Theorem 1. We first prove

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LEMMA 2. *If G is a bipartite graph with n vertices and m edges, and n_1, n_2 are integers such that $n_1 \geqslant n_2 \geqslant m$ and $n_1 + n_2 = n$, then G is 2-colorable with n_1 Amber and n_2 Blue vertices.*

Proof. Let S be the set of isolated (degree 0) vertices of G. Then $G - S$ has a 2-coloring, say with m_1 Amber and m_2 Blue vertices, where $m_1 \leqslant m_2 \leqslant m$. Since $m_1 \leqslant m \leqslant n_1$ and $m_2 \leqslant m \leqslant n_2$, we can color $n_1 - m_1$ of the vertices of S with Amber and $n_2 - m_2$ with Blue.

A graph with a vertex of degree Δ has at least $\Delta + 1$ vertices, so that Theorem 1 implies that all trees with $\Delta \leqslant 4$ are equitably 3-colorable. Of course, for $\Delta = 0$, the tree is equitably k-colorable for $k \geqslant 1$, and for $\Delta = 1$ or 2, the trees are paths which are equitably k-colorable for $k \geqslant 2$. The theorem is best possible in the sense that for $\Delta \geqslant 5$ there are trees with $3\Delta - 9$ vertices, and for $\Delta \geqslant 6$ trees with n vertices, $\Delta + 1 \leqslant n \leqslant 3\Delta - 11$, which are *not* equitably 3-colorable. Such trees are exemplified in Fig. 1 (where $\Delta = 10$) which consists of the star $K_{1,\Delta}$ to which is appended a path of 0, 1, 2,..., $2\Delta - 13$, $2\Delta - 12$ or $2\Delta - 10$ edges.

Proof of Theorem 1. The proof is in two parts, according to the size of the blue color class.

Part I. If $\lfloor n/3 \rfloor \leqslant b \leqslant \lfloor n/2 \rfloor$, let b_3 be the number of blue vertices of degree 3 or more. Then the total number of edges is at least $3b_3$ so that $n - 1 \geqslant 3b_3$

$$b_3 \leqslant \lfloor (n-1)/3 \rfloor \leqslant \lfloor n/3 \rfloor \leqslant b$$

and we use

ALGORITHM I. Recolor with Crimson the $\lfloor n/3 \rfloor$ vertices of highest degree, then delete them and their incident edges. Since $b_3 \leqslant \lfloor n/3 \rfloor$ all blue vertices of degree 3 or more have been recolored and deleted. If all blue vertices of degree 2 have also been deleted, then the $b - \lfloor n/3 \rfloor$ remaining vertices are all of degree one, and just $b - \lfloor n/3 \rfloor$ edges remain. If some blue vertices of degree 2 remain, then at least $2\lfloor n/3 \rfloor$ edges have been deleted. In either case at most

$$\max\{\lfloor n/2 \rfloor - \lfloor n/3 \rfloor, n - 1 - 2\lfloor n/3 \rfloor\} \leqslant \lfloor (n+1)/3 \rfloor$$

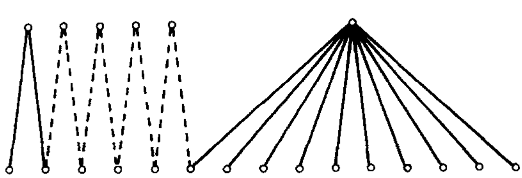

FIG. 1. Trees which are not equitably 3-colorable.

edges remain. By Lemma 2 with n replaced by $n - \lfloor n/3 \rfloor$ and

$$m \leqslant \lfloor (n+1)/3 \rfloor = n_2 \leqslant n_1 = \lfloor (n+2)/3 \rfloor$$

we can color n_1 of the vertices Amber and n_2 of the vertices Blue.

Note that in Algorithm I there are no restrictions on the relative sizes of n and Δ, other than $n \geqslant \Delta + 1$ and those implied by $b \geqslant \lfloor n/3 \rfloor$.

Part II. If $b \leqslant \lfloor n/3 \rfloor - 1$, we use

ALGORITHM II. This has four stages.

Stage 1. Recolor blue vertices with Crimson according to (a), (b), (c).

(a) Recolor a blue vertex of greatest degree.

(b) Continue to recolor blue vertices, so long as the following conditions (i) and (ii) can be satisfied:

(i) Define the *ambit* of a set of originally blue vertices as that set of amber vertices which are adjacent to one or more members of the set of blue vertices. Then we require that the ambit of each recolored vertex must contain a vertex (just one, since we are coloring a tree) which is in the ambit of the vertices already recolored so that, at each stage, the recolored vertices with their ambit and the edges connecting them form a single subtree.

(ii) At a given stage, suppose that i blue vertices have been recolored, and that their ambit contains j vertices. Then we require that the number i, together with the number $a - j$ of amber vertices *not* in the ambit of the i vertices, must satisfy the inequality

$$i + (a - j) \geqslant \lfloor n/3 \rfloor. \tag{1}$$

(c) Stop when $i = x$, $j = y$ satisfy inequality (1), but any further choice of a blue vertex (of degree d, say) would violate (1), so that

$$x + (a - y) \geqslant \lfloor n/3 \rfloor \tag{2}$$

and

$$(x+1) + a - (y + d - 1) \leqslant \lfloor n/3 \rfloor - 1$$

so that

$$x + a - y - \lfloor n/3 \rfloor + 3 \leqslant d \leqslant \Delta. \tag{3}$$

Note that if Stage 1 cannot be started, and if Δ_b is the greatest degree of a blue vertex, then we would have

$$1 + (a - \Delta_b) \leqslant \lfloor n/3 \rfloor - 1$$

$$\Delta \geqslant \Delta_b \geqslant a + 2 - \lfloor n/3 \rfloor = n - b + 2 - \lfloor n/3 \rfloor$$

$$\Delta \geqslant \Delta_b \geqslant n - (\lfloor n/3 \rfloor - 1) + 2 - \lfloor n/3 \rfloor = n - 2\lfloor n/3 \rfloor + 3 \tag{4}$$

and $n \leqslant 3\Delta - 11$ or $n = 3\Delta - 9$ contrary to the conditions of Theorem 1. So we may assume that $x \geqslant 1$.

Note also that the recoloring stops *before* all blue vertices are recolored, since $i = b, j = a$ violates (1):

$$i + (a - j) = b \leqslant \lfloor n/3 \rfloor - 1.$$

We illustrate Stage 1 with the tree in Fig. 2, for which $\Delta = 7$, $a = 20$, $b = 7$, $n = 27$, $\lfloor n/3 \rfloor = 9$. If we recolor the vertices in the order C_1, C_2, C_3, C_4, C_5, C_6, successive values of $i, j, a-j, i+a-j$ are shown in the left part of the display:

i	1	2	3	4	5	$6 = x$	1	2	3	4	$5 = x$
j	7	7	9	11	12	14	7	7	8	14	16
$a - j$	13	13	11	9	8	6	13	13	12	6	4
$i + a - j$	14	15	14	13	13	12	14	15	15	10	9

We cannot replace C_6 by B since this would lead to $i = 6, j = 18$, $a - j = 2$, $i + a - j = 8 < \lfloor n/3 \rfloor$, violating (1). But we could vary the order of recoloring to C_1, C_2, C_5, B followed by just one of C_3, C_4 or C_6, giving the values in the right part of the above display.

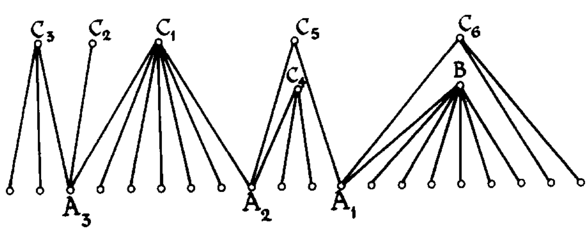

FIG. 2. Illustration of Stage 1 of Algorithm II.

Before we continue with Stage 2 of Algorithm II we observe that there are a few trees which can be equitably 3-colored, even though (4) holds and Algorithm II cannot be started. Two such trees (with $\Delta = 12$, $n = 24$ and 27) are depicted in Fig. 3. A coloring can be effected in these cases by an algorithm similar to Algorithm I. If a vertex C_1 of maximum degree Δ is

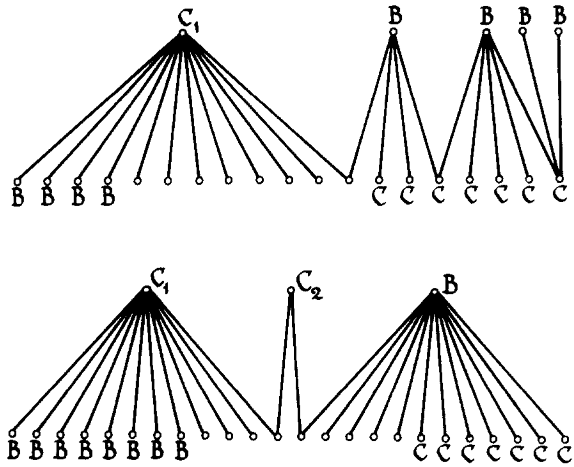

FIG. 3. Two equitably 3-colorable trees not covered by Theorem 1.

contained in an independent set $\{C_1, C_2, ..., C_t\}$, where $t = \lfloor n/3 \rfloor$, then deletion of this set and its incident edges leaves a bipartite graph with $\lceil 2n/3 \rceil$ vertices and $m = n - 1 - \Delta - (t-1) = n - \Delta - \lfloor n/3 \rfloor$ edges. By (4)

$$m \leqslant n - (n - 2\lfloor n/3 \rfloor + 3) - \lfloor n/3 \rfloor = \lfloor n/3 \rfloor - 3$$

so Lemma 2 tells us that there is an equitable 2-coloring of this bipartite graph and hence an equitable 3-coloring of the original tree.

To find such a 3-coloring, pick C_1 of degree Δ and color it Crimson. Delete C_1 together with its Δ adjacent vertices and their incident edges. Let $a_1, a_2, ... a_\Delta$ be the numbers of vertices in the larger of the two natural color classes in each of the resulting Δ components (some of which may be empty). Then C_1 is contained in an independent set of size $t = \lfloor n/3 \rfloor$ just if

$$\sum_{l=1}^{\Delta} a_l \geqslant t - 1. \tag{5}$$

If (5) holds, then the proof of Lemma 2 shows us how to find an equitable 2-coloring of the $n - t$ vertices which are not in the (Crimson) independent set.

We resume the proof of Theorem 1.

Stage 2 of Algorithm II is to recolor with Crimson $\lfloor n/3 \rfloor - x$ of the amber vertices which are *not* in the ambit of those already recolored Crimson. This is possible, since (2) implies that $a - y \geqslant \lfloor n/3 \rfloor - x$. The total number of Crimson vertices is now $\lfloor n/3 \rfloor$.

Stage 3 consists in recoloring with Blue $\lfloor (n+1)/3 \rfloor - (b-x)$ of the amber vertices which *are* in the ambit of the x vertices, but are *not* in the ambit of the remaining $b-x$ blue vertices. We prove that this is possible, i.e., that

$$\lfloor (n+1)/3 \rfloor - (b-x) \leqslant y - z \tag{6}$$

where z is the number of amber vertices adjacent to *both* Crimson and blue ones, i.e., z is the cardinality of the intersection of the ambits of both the x recolored vertices and the $b-x$ blue ones. In Fig. 2, z is either 1 (A_1) or 2 (two of A_1, A_2, A_3). If (6) is false, we have

$$¿ \qquad z \geqslant y - \lfloor (n+1)/3 \rfloor + (b-x) + 1 \qquad ? \tag{7}$$

On the other hand, since the z amber vertices belong to a subtree, they are adjacent to at least z unchosen blue vertices whose degrees are bounded below by (3). These unchosen blue vertices are each adjacent to $d-1$ amber vertices *not* in the ambit of the Crimson ones, so

$$z(d-1) \leqslant a - y$$

and, by (3),

$$z(x + a - y - \lfloor n/3 \rfloor + 2) \leqslant a - y$$

which, when combined with (7), gives

$$(y - \lfloor (n+1)/3 \rfloor + b - x + 1)(x + a - y - \lfloor n/3 \rfloor + 2) \leqslant a - y. \tag{8}$$

The sum of the two factors on the left of (8) is $a + b - \lfloor (n+1)/3 \rfloor - \lfloor n/3 \rfloor + 3 = \lfloor (n+11)/3 \rfloor$, which is independent of x and y. So (8) must hold for at least one of the two extreme values of $y - x$ given by (2) and (3). The first of these may be written

$$y - x \leqslant a - \lfloor n/3 \rfloor \tag{9}$$

and (3) may be written

$$y - x \geqslant a - \lfloor n/3 \rfloor + 3 - d. \tag{10}$$

If (6) is false, i.e., if (7) holds, we have (since the definition of z and the absence of cricuits in a tree imply that $z \leqslant b - x$)

$$b - x \geqslant z \geqslant y - \lfloor (n+1)/3 \rfloor + (b-x) + 1$$

so that $y \leqslant \lfloor (n+1)/3 \rfloor - 1$ and it follows that

$$d \leqslant \Delta_b \leqslant y \leqslant \lfloor (n-2)/3 \rfloor \tag{11}$$

and (10) gives us

$$y - x \geqslant a - \lfloor n/3 \rfloor + 3 - \lfloor (n-2)/3 \rfloor. \tag{12}$$

If we write $y - x = a - \lfloor n/3 \rfloor - u$, then (9) and (12) show that

$$0 \leqslant u \leqslant \lfloor (n-11)/3 \rfloor$$

and (8), which we wish to examine at these extreme values, becomes

$$(\lfloor (n+5)/3 \rfloor - u)(u+2) \leqslant a - y. \tag{13}$$

If $u > 0$, the product on the left side of (13) is at least $3\lfloor (n+2)/3 \rfloor$, a contradiction. If $u = 0$, (13) is $2\lfloor (n+5)/3 \rfloor \leqslant a - y$ and (3) gives

$$2\lfloor (n+5)/3 \rfloor \leqslant \lfloor n/3 \rfloor - x - 3 + d$$

and since $x \geqslant 1$ we have $y \geqslant d \geqslant \lfloor (n+20)/3 \rfloor$ and

$$a = a - y + y \geqslant 2\lfloor (n+5)/3 \rfloor + \lfloor (n+20)/3 \rfloor$$

again a contradiction. Finally, if $u = \lfloor (n-11)/3 \rfloor$, (13) with (3) gives

$$5\lfloor (n-5)/3 \rfloor \leqslant a - y \leqslant \lfloor n/3 \rfloor - x - 3 + d$$

and (11), with $x \geqslant 1$, gives

$$5\lfloor (n-5)/3 \rfloor \leqslant \lfloor n/3 \rfloor - 4 + \lfloor (n-2)/3 \rfloor$$

which is once more a contradiction if $n \geqslant 5$, and we know that trees with less that five vertices can be equitably 3-colored.

Stage 4 confirms the color Blue for the $b - x$ blue vertices and the color Amber for the remaining $n - \lfloor n/3 \rfloor - \lfloor (n+1)/3 \rfloor = \lfloor (n+2)/3 \rfloor$ amber vertices. This completes the proof of Theorem 1. Notice that in the course of our proof we have also established the following assertion:

THEOREM 3. *Algorithms* I *and* II *will serve to equitably* 3-*color all trees which can be so colored.*

The assertion of Theorem 1 is equivalent to the existence of $\lceil n/3 \rceil$ or $\lfloor n/3 \rfloor$ independent vertices whose omission leaves at most $\lfloor (n-1)/3 \rfloor$ edges. It is curious that we cannot prove the result in this way.

PROBLEM. Given n, Δ and $m \leqslant n/2$, find the maximum number M such that any tree on n vertices with maximum degree Δ contains m independent vertices incident with at least M edges.

We say that a graph with n vertices is *k-colorable with color classes of sizes* $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ if $n_1 + n_2 + \cdots + n_k = n$ and there is a partition of the vertices into k parts of sizes n_l, $1 \leqslant l \leqslant k$, so that no two vertices in the same part are adjacent.

THEOREM 4. *A tree with n vertices, maximum degree Δ and natural color classes of sizes a and b, $a \geqslant b$, $a+b=n$, is 3-colorable with color classes of sizes r, s, t, $r \geqslant s \geqslant t$, $r+s+t=n$, provided $\Delta \leqslant r+2$ and $b \leqslant t + st/(r+t-1)$. Also if $s \leqslant b \leqslant s + st/(r+s-1)$.*

Proof. This is similar to that of Theorem 1. The two parts are for the cases $t \leqslant b \leqslant t + st/(r+t-1)$ and $b \leqslant t-1$.

Part I. $t \leqslant b \leqslant t + st/(r+t-1)$. Choose t blue vertices of highest degree, recolor them Crimson, and delete them and their incident edges. At least $(t/b)(n-1)$ edges have been deleted, so there remains a bipartite graph with $n-t=r+s$ vertices, and at most $(n-1)(1-t/b)$ edges. By Lemma 2 this can be colored with r Amber vertices and s Blue ones, provided $r \geqslant s \geqslant (n-1)(1-t/b)$, i.e., $bs \geqslant (n-1)(b-t)$, $b(n-s-1) \leqslant (n-1)t$ or $b \leqslant t(n-1)/(r+t-1)$, as assumed.
If $b \geqslant s$, then this argument goes through with s and t interchanged.

Part II. If $b \leqslant t-1$ we use an algorithm similar to Algorithm II, but the roles of Blue and Crimson are interchanged: we first select the vertices of the middle sized color class.

Stage 1. Confirm Blue as the color of successive blue vertices, so long as the following conditions (i) and (ii) can be satisfied:

(i) The ambit of each confirmed vertex must contain a vertex in the ambit of those already confirmed.

(ii) When i vertices have been confirmed and their ambit contains j vertices, we require that

$$i + (a-j) \geqslant s. \tag{14}$$

Stop when $i=x$, $j=y$ satisfy (14), but confirmation of any other blue vertex (of degree d, say) would violate (14), so that

$$x + (a-y) \geqslant s \tag{15}$$

and

$$x + 1 + a - (y+d-1) \leqslant s-1$$

so that

$$x + a - y - s + 3 \leqslant d \leqslant \Delta \tag{16}$$

We need not start Stage 1 with a blue vertex of highest degree, but note that Stage 1 cannot be started if (14) is violated for *every* blue vertex, i.e., $1 + (a - d) \leqslant s - 1$ for every blue vertex, i.e., $d \geqslant a - s + 2$ for every blue vertex. This implies that $b(a - s + 2) \leqslant n - 1$. But $b(a - s + 2) \geqslant n$ unless $b = 1$, and such trees are proportionally 3-colorable just if $t = 1$.

Note also that confirmation stops before all blue vertices are confirmed, since $i = b, j = a$ violates (14): $i + (a - j) = b \leqslant t - 1 \leqslant s - 1$.

Stage 2. Recolor with Blue $s - x$ of the amber vertices which are *not* in the ambit of those vertices confirmed as Blue. This is possible, since (15) implies that $a - y \geqslant s - x$ and $x < b \leqslant t - 1 \leqslant s - 1 < s$.

Stage 3. Recolor with Crimson the $b - x$ blue vertices whose color has not been confirmed and then recolor with Crimson $t - (b - x)$ of the amber vertices which *are* in the ambit of the x confirmed Blue vertices but are *not* in that of $b - x$ recolored ones. We need to show that

$$0 \leqslant t - (b - x) \leqslant y - z, \tag{17}$$

where z is, as before, the number of vertices in both of the ambits of the x and of the $b - x$ vertices. The first inequality holds since $b \leqslant t - 1 < t + 1 \leqslant t + x$. Suppose that the second inequality in (17) is false, so that

$$z \geqslant y - t + (b - x) + 1 \qquad ? \tag{18}$$

From this, much as in the proof of Theorem 1, we deduce that

$$(y - t + b - x + 1)(x + a - y - s + 2) \leqslant a - y, \tag{19}$$

where the sum of the factors on the left is $a + b - s - t + 3 = r + 3$, again independent of x and y. If we write $y - x = a - s - u$, then, as before

$$0 \leqslant u \leqslant d - 3$$

and (19) has to hold for at least one of the extreme values of $y - x$. The left side of (19) is $(a - s - u - t + b + 1)(u + 2) = (r + 1 - u)(u + 2)$, so $u = 0$ gives $2r + 2 \leqslant a - y$ and, by (16), $a - y \leqslant s + d - 3 - x \leqslant s + d - 4$ since $x \geqslant 1$ and this contradicts $s \leqslant r$ and $d \leqslant \Delta \leqslant r + 2$. On the other hand $u = d - 3$ yields $(r + 4 - d)(d - 1) \leqslant a - y \leqslant s + d - 4$ which again contradicts $d \leqslant r + 2$.

Stage 4. Confirm the color Amber for the remaining $a-(s-x)-(t-b+x)=a+b-s-t=n-s-t=r$ amber vertices.

This completes the proof of theorem 4.

Bennet Manvel has observed that, since the critical cases in proportional coloring are stars and paths, which are the trees with least and greatest diameters, it may be possible to obtain an alternative characterization of proportionally colorable trees in terms of their diameters.

Since almost all trees are equitably 3-colorable, we should expect, a fortiori, that they are equitably k-colorable for all $k \geqslant 3$. Details of proof are elusive, but we believe that if the maximum degree of a tree is written in the form $\Delta=(q+2)(k-2)+r$, with $3 \leqslant r \leqslant k$ and $q \geqslant -3$, then the tree is equitably k-colorable if its number of vertices is $n \geqslant \max(\Delta+1, \Delta+2q+2)$ or if $r=3$, $q>0$ and $n=\Delta+2q$.

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