Cohomology of Isolated Invariant Sets under Perturbation

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INTRODUCTION

Ura and Kimura [20, 21] suggested the concept of an isolated invariant set for a flow, that is, an invariant set which is maximal in some neighborhood of itself. Any such closed neighborhood is called an isolating neighborhood.

Conley and Easton [8, 9, 13], in a smooth setting, show that each isolated invariant set admits a special neighborhood, called an isolating block. These neighborhoods have the advantage that their structure reveals properties of the invariant set inside ([9, 13] and Section 5 of this paper); in particular, the topological methods of Ważewski [22] can be applied.

These techniques have been applied to several problems: [4] and [14–16] deal with the two body and restricted three body problems; [3] and [17] deal with the behavior of a Hamiltonian system near a degenerate critical point; [10, 11, 18, 19] contain applications to the study of shock waves; and [7] generalizes the Morse–Smale inequalities. Further developments appear in [1], [2], [5], [6], and [23].

Churchill [1] has developed these ideas in the absence of smoothness, and with Conley has constructed an index, which, in the case of a nondegenerate rest point, carries the same information as the Morse index. This index, which is described in terms of an isolating block, takes the form of a homotopy class of pointed spaces or the associated cohomology algebra.

In this paper we consider the space of flows on a compact space. Usually we use the compact-open topology; however, occasionally we restrict our attention to smooth flows on a smooth manifold. A framework for studying changes in the structure of an isolated invariant set under perturbations of the flow is developed.

To do this, it is first required to define the continuation of the isolated invariant set as the flow changes. (We give an elementary description of this in 1.1) We topologize the invariant set space; i.e., the space whose points are pairs consisting of a flow with one of its isolated invariant sets. Two points in this space are in the same open set if their flows are close and they share a common isolating neighborhood. This space is not Hausdorff; however,
it does have some natural structure on it: the projection map on this space, defined by taking a pair onto its associated flow, is locally a homeomorphism.

In the smooth case, one isolated invariant set is the continuation of another if they can be connected by a path in the invariant set space; in the metric case, if they are in the same quasicomponent of the space. The difference in the latter case stems from the fact that we do not know that the space of flows is locally path connected. A fundamental result is that the index of an isolated invariant set is invariant under continuation. This makes use of the perturbation theorem for blocks [1].

One way to study changes in the structure of an invariant set under continuation is to examine changes in its cohomology algebra. In fact we are able to examine individual elements of the cohomology under continuation. We do this by introducing another space whose elements are triples consisting of a flow, one of its invariant sets, and some element of the cohomology of this invariant set. Two elements \((f_1, S_1, \alpha_1)\) and \((f_2, S_2, \alpha_2)\) are in the same open set if there is a common isolating neighborhood \(N\) for \((f_1, S_1)\) and \((f_2, S_2)\) and an element \(\alpha \in H^*(N)\) which is mapped to \(\alpha_1\) and \(\alpha_2\) under the map induced by the inclusions \(S_1 \subset N, S_2 \subset N\), respectively. Again, this space is not Hausdorff but the projection map to the invariant set space is locally a homeomorphism.

Our primary goal is to find methods of determining further details of this quite complicated space. The crucial step is the construction in Section 8 of the index space whose points are triples consisting of a flow, one of its invariant sets, and an element of the cohomology of the index of the invariant set. The projection map \(\pi\) to the invariant set space is again a local homeomorphism; but in addition, this space is locally a product in the sense that each point \((f, S)\) of the invariant set space has a neighborhood \(U\) so that \(\pi^{-1}(U)\) is homeomorphic to \(U \times \pi^{-1}(f, S)\). This implies that the structure of the index space is much simpler and more accessible.

In order to exploit this simplicity we relate the index space with the space containing invariant set cohomology by means of a third space, also of interest in itself, whose elements are triples \((f, S, \alpha)\) where \(\alpha\) is an element of the cohomology of a set related to the orbits asymptotic to or from \(S\).

The construction of this space and the index space in the smooth case parallels that of the space of invariant set cohomology with isolating blocks substituting for isolating neighborhoods. In the metric case, the construction is complicated by the fact that a block for one flow is not a block for a neighborhood of flows.

These three cohomology spaces are related by maps which if restricted to the inverse images of the projection maps on a point in the invariant set space, yield a long exact sequence of the cohomologies of the invariant set, the index of the invariant set, and the asymptotic set.
The important continuation properties of an invariant set are of the following nature: A nonempty invariant set is unstable if arbitrarily nearby continuations of it are empty; a nonzero element of the cohomology of an invariant set is unstable if arbitrarily nearby continuations are zero.

A point \((f, S)\) is a bifurcation point if, roughly speaking, there is some cohomology element of some invariant set for some flow which continues over points arbitrarily close to \((f, S)\) but not to \((f, S')\).

Unstable points and bifurcation points locate the beginning of gross changes in invariant set structure. Sections 5 and 10 deal with methods of identifying stable points and bifurcation points of these various spaces. Sections 1 and 10.8 are simple examples treating these ideas.

A flow might be called cohomologically stable if each invariant set has all its cohomology stable, and no invariant set is a bifurcation point. This is weaker than structural stability; in fact, some of the propositions of Section 10 lead us to conjecture that the set of cohomologically stable flows is residual.

1. **Examples**

**Example 1.1.** It is not hard to construct a path of flows \(f_\mu\) on \(E^3\), \(\mu \in [-1, 1]\), which is described below:

Suppose for \(\mu = -1\), that the only proper invariant set for \(f_\mu\) is a hyperbolic periodic orbit \(\pi\), whose stable and unstable manifolds have dimension one.

Poincaré's continuation methods imply that if \(\mu\) is near enough to \(-1\), there is a unique periodic orbit \(\pi_\mu\) for \(f_\mu\) near \(\pi\) whose period is near that of \(\pi\). \(\pi_\mu\) is traditionally called a continuation of \(\pi\). Now we suppose that \(\pi\) continues to \(\pi_\mu\) for all \(\mu \in [-1, 0)\) such that it is the only proper invariant set for \(f_\mu\). For \(\mu = 0\), we assume that the only proper invariant set is a degenerate rest point \(p_0\). (The diameter of \(\pi_\mu\) shrinks to zero as \(\mu\) tends to 0.) It then seems natural to call \(p_0\) a continuation of \(\pi\).

The first task of the paper, after the preliminary isolating block theory, is to generalize this idea of a continuation so that any isolated invariant set can be continued.

Suppose that for \(\mu = 1\), \((1) f_\mu\) has two hyperbolic rest points \(p_1, p_2\) of index 1 and 2, respectively, and that the only invariant sets of \(f_\mu\) are \(\{p_1\}\), \(\{p_2\}\) and \(\{p_1, p_2\}\). It follows that for \(\mu\) close enough to 1, \(f_\mu\) has unique hyperbolic critical points near \(p_1, p_2\) (again called \(p_1, p_2\)). We would naturally call these points continuations of \(p_1\) and \(p_2\).

Assume in fact that for \(\mu \in (0, 1]\), \((1)\) holds and that both \(p_1\) and \(p_2\) approach \(p_0\) as \(\mu\) tends to 0. One would call \(p_0\) a continuation of \(\{p_1 \cup p_2\}\) but not of \(\{p_1\}\) or \(\{p_2\}\).

The invariant set \(p_0\) for \(f_0\) is special in the sense that continuations arbitrarily...
nearby are more complicated. This complication occurs here in two ways. On the one hand, flows nearby have more invariant sets; on the other hand, arbitrarily close continuations of \( p_0 \) have more cohomology. For \( \mu < 0 \), this extra cohomology occurs in \( H^1(\mathbb{R}) \); for \( \mu > 0 \), it occurs in \( H^0(\mathbb{R}_+ \cup \mathbb{R}_-) \).

The second task of the paper is to establish means of studying this phenomenon more closely and to determine which points are or are not bifurcation points.

**Example 1.2.** A complementary notion to bifurcation is that of instability. The easiest example of this is the flow on \( \mathbb{R}^1 \) given by

\[
\dot{x} = |x|.
\]

This gives rise to a flow whose only proper invariant set is a degenerate rest point at \( x = 0 \). There are two other orbits, one to the left and one to the right of \( x = 0 \), and each has \( x \) increasing with time.

Arbitrarily small perturbations of this flow given by

\[
\dot{x} = |x| + \epsilon,
\]

where \( \epsilon > 0 \) have one orbit which has \( x \) increasing with time. In this case we shall see that the continuation of the invariant set \( \{0\} \) is the empty set; thus this is an example of an unstable isolated invariant set.

Another task of this paper is to distinguish different kinds of instability and to give conditions for stability.

### 2. Preliminaries

This paper examines the set \( F \) of continuous flows on a compact metric space \( X \). That is, \( \forall f \in F \) implies

(a) \( f: X \times \mathbb{R} \to X \) is continuous,

(b) \( f(p, 0) = p \) for all \( p \in X \), and

(c) \( f(p, t_1 + t_2) = f(f(p, t_1), t_2) \) for all \( p \in X, t_1, t_2 \in \mathbb{R} \).

\( \mathbb{R}, \mathbb{R}^+, \mathbb{R}^- \) will always denote the reals, nonnegative reals, and nonpositive reals, respectively.

When \( f \in F \) is fixed and \( A \subset X, T \subset \mathbb{R} \), then \( A \cdot T \) will denote \( F(A \times T) \); in particular, properties (b) and (c) above can be rewritten \( p \cdot 0 = p \) and \( p \cdot (t_1 + t_2) = (p \cdot t_1) \cdot t_2 \). This emphasizes the fact that \( f \) is a group action of the reals on \( X \).

An invariant set of \( f \) is a closed set \( I \subset X \) having the property \( I \cdot \mathbb{R} = I \). We shall consider only those invariant sets \( S \) which are isolated in the sense that there is an open set \( U \subset X \) containing \( S \) such that \( I \subset U \) is invariant.
implies $I \subseteq S$. Since $S$ must be closed, $U$ separates $S$ from invariant sets not contained in $S$. $\text{Iso}(f)$ denotes the set of isolated invariant sets of $f$.

A closed set $N$ is an isolating neighborhood for $f$ if $p \in \partial N$ implies $p \cdot \mathbb{R}$ is not contained in $N$. $\sigma(f, N)$ denotes the largest (under inclusion) invariant set of $f$ in $N$. One easily sees that $\sigma(f, N)$ is an isolated invariant set and that every isolated invariant set has an isolating neighborhood. The empty set is both an isolated invariant set and an isolating neighborhood.

**Proposition 2.1.** $\text{Iso}(f)$ is a semilattice.

**Proof.** We must show that if $S_1, S_2 \in \text{Iso}(f)$, then $S_1 \cap S_2 \in \text{Iso}(f)$. Choose $N_1, N_2$ isolating neighborhoods for $S_1, S_2$, respectively. Then $N_1 \cap N_2$ is an isolating neighborhood for $f$ and $\sigma(f, N_1 \cap N_2) = S_1 \cap S_2$. Thus $\text{Iso}(f)$ is a semilattice.

The topology used for $F$ is the compact-open topology, hence a subbasis for the topology on $F$ is given by sets which have the form

$$\{f \in F \mid f(K \times T) \subseteq U\}$$

for some $K, T$ compact subsets of $X, \mathbb{R}$, respectively, and $U$ open in $X$.

**Definition 2.2.** $\Phi(N)$ denotes the set of flows for which $N$ is an isolating neighborhood.

**Proposition 2.3.** $\Phi(N)$ is open in $F$.

The proof is an easy exercise in the compact-open topology which uses the compactness of $\partial N$.

**Theorem 2.4.** $F$ is a complete metric space with a countable basis.

**Proof.** The set of all continuous functions from $X \times [0, 1] \to X$ has a countable basis and is a complete metric space [12]. Since $f \in F$ is completely determined by its restriction to $X \times [0, 1]$, $F$ can be considered as a subspace. As such, it is closed and, therefore, inherits the two properties.

We consider occasionally one other setting, referred to as the smooth case. Here, $X$ is a compact manifold with some specified degree of smoothness, and $F$ is the set of flows with vector fields having the same degree of smoothness. We use the $C^0$ topology on the vector fields to induce a topology on $F$. Then $f_1$ is within $\varepsilon$ of $f_2$ if their corresponding vector fields $V_1, V_2$ have $\sup_{x \in X} \|V_1(x) - V_2(x)\| < \varepsilon$. One could also use a $C^r$ topology. The $C^0$ topology is finer than the topology induced by the compact-open topology of the nonsmooth case, and the theorems for the latter also apply to the
smooth case. In fact, Theorem 3.7(d) usually makes the proofs much easier in the smooth case.

We use the notations $\partial$, $\text{cl}$, $\text{int}$, $\text{Im}$, and $\text{ker}$ to denote respectively the boundary of a set, the closure of a set, the interior of a set, the image of a function, and the kernel of a homomorphism.

3. Isolating Blocks

**Definition 3.1.** $\Sigma \subset X$ is a *local surface of section*, or a *local section* for $f$ if for some $\delta > 0$, $f^{-1}(\delta \times \delta)$ is a homeomorphism with open range. In this case, $\delta$ is a *collar size* of $\Sigma$. We allow $\Sigma$ to be empty.

Note that if $\Sigma_0 \subset \Sigma$ is open (rel $\Sigma$), and $\delta_0 \leq \delta$, then $\Sigma_0$ is a local section with collar size $\delta_0$. Also note that in the smooth case, if $\Sigma$ is a submanifold of codimension 1 and the vector field is never tangent to $\Sigma$, then $\Sigma$ is a local section.

**Definition 3.2.** If $p \in D$, $\mathcal{O}^+(p, D, f)$ ($\mathcal{O}^-(p, D, f)$) denotes that component of $p \cdot \mathbb{R} \cap D$ ($p \cdot \mathbb{R} \cap D$) which contains $p$.

$$\mathcal{O}(p, D, f) = \mathcal{O}^+(p, D, f) \cup \mathcal{O}^-(p, D, f).$$

Also, $|\mathcal{O}^+(p, D, f)| \equiv \sup\{t \geq 0 \mid p \cdot [0, t] \subset \mathcal{O}^+(p, D, f)\}$ and

$$|\mathcal{O}^-(p, D, f)| \equiv \sup\{t \geq 0 \mid p \cdot [-t, 0] \subset \mathcal{O}^-(p, D, f)\}.$$

Then define $|\mathcal{O}(p, D, f)| \equiv |\mathcal{O}^+(p, D, f)| + |\mathcal{O}^-(p, D, f)|$.

Sometimes the reference to $f$ is suppressed if it does not lead to confusion.

**Definition 3.3.** $B \subset X$ is an *isolating block* for the flow $f$ if $B$ is closed and there are two local sections $\Sigma^+$ and $\Sigma^-$ with the following properties:

(a) $((\text{cl}\Sigma^+) - \Sigma^k) \cap B = \emptyset$;

(b) there is a $\delta > 0$ such that $p \in \Sigma^+ \cap B$ implies $p \cdot (-\delta, 0) \cap B = \emptyset$, and $p \in \Sigma^- \cap B$ implies $p \cdot (0, \delta) \cap B = \emptyset$; and

(c) $p \in \partial B - (\Sigma^+ \cup \Sigma^-)$ implies $\mathcal{O}(p, B) \subset \partial B$ and intersects $\Sigma^+$ and $\Sigma^-$.

Example 11.1 is an example of an isolating block for a hyperbolic rest point.

**Notes 3.4.** (a) Property 3.3(a) is not essential in the sense that if $\Sigma^+$, $\Sigma^-$ can be found satisfying 3.3(b) and (c), then $\Sigma^\pm \subset \Sigma^\pm$ can be found satisfying 3.3(a), (b), and (c).
(b) The definition in [1] implicitly includes the further assumption that \((\Sigma^+ \cap B) \cap (\Sigma^- \cap B) = \emptyset\). The definition in [9, 13] for the smooth case includes the assumption that \(\partial B = (B \cap \Sigma^+) \cup (B \cap \Sigma^-)\). The definitions in [23] for the smooth case are more general.

(c) \(B\) is an isolating neighborhood.

(d) We standardize the following notation: \(b^\pm \equiv \Sigma^\pm \cap B = \Sigma^\pm \cap \partial B\),
\[
A^\pm \equiv \{ p \in B \mid ||\phi^+(p, B, f)|| = \infty \} \quad \text{(the set of orbits asymptotic in the \(\pm\) direction to \(\omega(f, B)\))}
\]
\[
a^\pm = A^\pm \cap \partial B = A^\pm \cap b^\pm,
A = A^+ \cup A^-,
S = \omega(f, B).
\]

(e) Observe that \(A^\pm, \partial B, b^\pm\) are all closed subsets of \(B\) and that, relative to \(\partial B\), \(a^\pm\) is closed in \(\text{int } b^\pm\).

(f) If \(p \in a^+(a^-)\), the \(\omega\)-limit (\(\alpha\)-limit) set of \(p\) is part of a nonempty invariant set interior to \(B\). Thus \(a^+ \cup a^- \neq \emptyset\) implies \(S \neq \emptyset\). Also, \(S \neq \emptyset\) implies \(a^+ \cup a^- \neq \emptyset\) if \(B\) is connected and \(b^+ \cup b^- \neq \emptyset\).

(g) \(S = A^+ \cap A^-\).

Often we shall be referring to more than one block at a time. In this case, we use subscripting, priming, etc. to distinguish among the blocks. We use the same distinguishing mark for the parts of a block as we do for the block itself.

Every isolated invariant set has an isolating block:

**Definition 3.5.** If \(S = \omega(f, B)\), and \(B\) is a block, we say \(B\) is a block for \((f, S)\). We denote the set of blocks for \((f, S)\) by \(\mathcal{B}(f, S)\).

**Theorem 3.6.** If \(N\) is an isolating neighborhood for \(f\) and \(S = \omega(f, N)\), then there is a block \(B \subset N\) for \((f, S)\) such that \(b^+ \cap b^- = \emptyset\).

For the proof see [1].

**Theorem 3.7.** In the smooth case, \(B\) can be chosen so that instead of \(b^+ \cap b^- = \emptyset\), we have

(a) \(B\) is a manifold with boundary \(b^+ \cup b^-\);

(b) \(B\) is smooth except at \(b^+ \cap b^-\);

(c) \(b^+\) and \(b^-\) are smooth manifolds of codimension 1 with boundary \(b^+ \cap b^-\); and

(d) \(B\) is an isolating block for an open set of flows in \(F\).

The proof is in [9] and [23].
The failure of 3.7(d) in the compact-open topology is the source of several technical complications.

If $N$ is an isolating neighborhood, $|\mathcal{O}(\cdot, N, f)|$ does not depend continuously on its first argument. For example, Fig. 1 shows $N$ having an orbit with an internal tangency on $\partial N$. Any point on this orbit segment in $N$ is a point of discontinuity of $|\mathcal{O}(\cdot, N, f)|$.

An isolating block is designed so that if an orbit is tangent to $\partial B$, it is externally tangent.

**Fig. 1.** Orbits near an internal tangency.

**Theorem 3.8.** If $B$ is a block for $f$, then $|\mathcal{O}(\cdot, N, f)|$ and $|\mathcal{O}^\pm(\cdot, N, f)|$ are continuous functions from $B$ into the extended reals $[0, \infty]$.

The proof can be found in [1], and for the smooth case in [9].

The following are immediate from 3.8.

**Corollary 3.9.** The functions of 3.8 can be used to define a strong deformation retraction of $B - A^\pm$ to $b^\pm$.

**Corollary 3.10.** If $b^+$ or $b^-$ is not a strong deformation retraction of $B$, then $S \neq \emptyset$.

From 3.7 and 3.8, we know that there are blocks in arbitrarily small neighborhoods of an isolated invariant set; we actually require that all these smaller blocks be modifications of one given block.

There are several ways of constructing new blocks from old:

**Proposition 3.11.** Let $B \in \mathcal{B}(f, S)$, let $U$ be a neighborhood of $a^\pm$ in $b^+$ and let $Y = b^+ - U$. Then $B_0 = \text{cl}(B - \mathcal{O}(Y, B))$ is also a block for $(f, S)$.

*Proof.* The sections $\Sigma^+, \Sigma^-$ used to define $B$ also work for $B_0$. 
**Definition 3.12.** If $B_0$ is obtained from $B$ as in 3.11, then $B_0$ is said to be a *shave* of $B$, and is said to be *obtained from $B$ by shaving $Y$*. (In Fig. 2a the shaded area $B_0$ is obtained from $B$ by shaving $Y$.)

The following technical lemma will allow us to use the flow to deform a block.

**Fig. 2a** (left). $B_0$, the shaded area, is obtained from $B$ by shaving $Y$.

**Fig. 2b** (right). $B_1$, the shaded area, is obtained from $B$ by squeezing $B^+$ and $B^-$.

**Lemma 3.13.** Let $\Sigma$ be a local section with collar size $2\delta$ and let $r: \Sigma \to [0, \infty)$ be continuous such that $p \in \Sigma$ implies

$$p \cdot (0, r(p)) \cap \Sigma \cdot [-2\delta, 0] = \emptyset.$$  

Then $\Sigma \cdot r = \{p \cdot r(p) \mid p \in \Sigma\}$ is also a local section.

**Proof.** Suppose $p_1, p_2 \in \Sigma$, $t_1, t_2 \in (-\delta, \delta)$, and

$$(p_1 \cdot r(p_1)) \cdot t_1 = (p_2 \cdot r(p_2)) \cdot t_2 \quad \text{with} \quad r(p_1) + t_1 \leq r(p_2) + t_2. \quad (2)$$

Then $p_1 = p_2 \cdot (r(p_2) + t_2 - r(p_1) - t_1) = p_2 \cdot T$ where $0 \leq T \leq r(p_2) + t_2$. But (1) implies $T \notin (0, r(p_2) + 2\delta]$, hence $T = 0$, and $p_1 = p_2$. It follows from (2) and the fact that $\Sigma$ is a local section, that $t_1 = t_2$.

Let $\theta: \Sigma \setminus (-\delta, \delta) \to (\Sigma \cdot r) \setminus (-\delta, \delta)$ map $p \cdot t$ into $(p \cdot r(p)) \cdot t$. Then $\theta$ maps open sets of $\Sigma \setminus (-\delta, \delta)$ into open sets of $X$: Fix $p \cdot t \in U$, an open (rel $X$) subset of $\Sigma \setminus (-\delta, \delta)$. Suppose there is a sequence $\{q_i\} \subset X - \theta(U)$ such that $q_i \to (p \cdot r(p)) \cdot t$. Then $q_i \cdot (-r(p)) \to p \cdot t \in U$, which implies that for almost all $i$, $q_i \cdot (-r(p)) \in U$. For these $i$, $q_i \cdot (-r(p))$ has the form $(p_t \cdot t_i)$ where $p_t \in \Sigma, t_t \subset (-\delta, \delta)$. It follows from injectivity of $f \mid \Sigma \times (-2\delta, 2\delta)$,
and openness of $\Sigma \cdot (-2\delta, 2\delta)$ that $p_i \rightarrow p$ and $t_i \rightarrow t$; so $r(p_i) \rightarrow r(p)$. Then, for almost all $i$, $(p_i \cdot t_i)(r(p) - r(p_i)) \in U$, which implies $\theta(p_i \cdot t_i) = q_i \in \theta(U)$. This contradicts $\{q_i\} \subset X \setminus \theta(U)$, so there can be no sequence $\{q_i\}$; hence $\theta(U)$ is open in $X$.

$\theta$ is continuous because it is the composition of continuous maps, one of which is the inverse of $f \mid \Sigma \times (-\delta, \delta)$.

Let $\psi$ be the restriction of $f$ to $(\Sigma \cdot r) \times (-\delta, \delta)$, a continuous map. The first paragraph implies that $\psi$ is injective. Continuity of $\theta$ implies that if $(U \cdot r) \times J$ is open in $\Sigma \cdot r \times (-\delta, \delta)$, then $U \times J$ is open in $\Sigma \times (-\delta, \delta)$. Since $\theta$ is open and $\theta(U \cdot J) = \psi((U \cdot r) \times J)$, it follows that $\psi$ is an open map. The second paragraph implies that $\psi(\Sigma \cdot r \times (-\delta, \delta)) = \theta(\Sigma \times (-\delta, \delta))$ is open in $X$. Therefore, $\psi$ is a homeomorphism with open range and the proof is complete.

**Definition 3.14.** Let $r: b^\pm \rightarrow [0, \infty)$ be continuous. $B^\pm$ is the $r$-collar or a collar of $b^\pm$ if $B^\pm = \{p \cdot t \mid t$ is between $r(p)$ and $0$, inclusively$, \}$, and $B^\pm \cap b^\pm = b^+ \cap b^-$. Also, $r$ is said to be the collar size of $B^\pm$.

**Lemma 3.15.** Let $B \in \mathcal{A}(f, S)$, and let $B^+$ be the $r$-collar of $b^+$. Then $B_1 \equiv cl(B - B^+)$ is also in $\mathcal{A}(f, S)$.

**Proof.** We use the notation of 3.3. Extend $r$ to $\Sigma^+$. Since

$$B^+ \cap b^- = b^+ \cap b^-$$

(where $r = 0$), there is an open (relz) subset $\Sigma_0^+$ which contains $b^+$ and satisfies (1) of 3.13 for some $\delta$. Then $\Sigma_0^+ \cdot r$ is a local section, and the conditions that $B_1$ be a block are easy to check.

Note that 3.15 remains true if $+$ is replaced by $-$.

**Definition 3.16.** $B_1$ of 3.15 is a squeeze of $B$, and is said to be obtained from $B$ by squeezing $B^\pm$. We also say $B_1$ is a squeeze of $B$ if $B_1$ is obtained from $B$ by first squeezing $B^+$, then squeezing $B^-$. (In Fig. 2b the shaded area $B_1$ is obtained from $B$ by squeezing $B^+$ and $B^-$.)

**Lemma 3.17.** Let $\Sigma_1$ and $\Sigma_2$ be local sections for $f$ with collar sizes greater than $3\delta > 0$. For $(i, j) = (1, 2), (2, 1)$, let $\Sigma_i' = \Sigma_i - K_i$ where

$$K_i = \{p \in \Sigma_i \mid p \cdot t \in cl \Sigma_i \text{ for some } t \in (0, 2\delta]\}.$$

Then $\Sigma = \Sigma_1' \cup \Sigma_2'$ is also a local section. (See Figs. 2c, 2d.)

**Proof.** Let $\psi$ be the restriction of $f$ to $\Sigma \times (-\delta, \delta)$, a continuous map. Suppose $\psi(p_1, t_1) = \psi(p_2, t_2)$ for $t_2 \geq t_1$, $p_1 \in \Sigma_1'$. Then $p_1 = p_2 \cdot (t_2 - t_1)$.
which implies \( p_2 \notin \Sigma_2 \); otherwise, \( p_2 \in K_2 \) which does not intersect \( \Sigma \). (\( \Sigma_1 \) has collar size \( \delta \).) Therefore, \( p_2 \in \Sigma_1' \). Since \( \Sigma_1 \) is a local section, \( (p_1, t_1) = (p_2, t_2) \). Hence \( \psi \) is injective.

Choose \( (p, t) \in U \times J \), an open subset of \( \Sigma \times (-\delta, \delta) \). Suppose \( p \in \Sigma_1 \) and \( t \in J_1 \), open in \( J \) with \( \text{cl} J_1 \subset J \). Then there is a neighborhood \( U_1 \) of \( p \) in \( \Sigma_1 \) such that \( U \cdot J \supset U_1 \cdot J_1 \), which is open (rel \( X \)); otherwise there is a sequence \( \{p_i\} \subset \Sigma_1', \{t_i\} \subset J_1 \), such that \( p_i \rightarrow p \), \( t_i \rightarrow t \in J \), and \( p_i \cdot t_i \notin U \cdot J \). It follows that \( p_i \notin \Sigma \) for almost all \( i \), hence \( p_i \in K_1 \) for these \( i \).

But this implies that the limit point \( p \) of \( \{p_i\} \) must be an element of either \( K_1 \) or \( \Sigma_2 \). The former case can be eliminated because \( K_1 \cap \Sigma = \emptyset \). In the latter case, there is a sequence \( \{s_i\} \subset \mathbb{R}^+ \) such that \( s_i \rightarrow 0 \), and \( p_i \cdot s_i \in \Sigma_2 \). It follows that \( p_i \cdot s_i \in \Sigma \). Hence for large \( i \), \( t_1 - s_1 \in J \) and

\[
(\psi(u) - s_i)(t_i - s_i) \in U \cdot J.
\]

This contradicts \( p_i \cdot t_i \notin U \cdot J \), so there is such a \( U_1 \subset \Sigma_1 \).

We have shown that \( \psi(U \times J) \) contains a neighborhood (rel \( X \)) of each of its points and hence is an open map with open (rel \( X \)) range. This completes the proof.

**Corollary 3.18.** If \( B_1 \) and \( B_2 \) are blocks for \( f \), then so is \( B = B_1 \cap B_2 \).

**Proof.** Let \( \Sigma_1^\pm, \Sigma_2^\pm \) be local sections used to define \( B_1, B_2 \), respectively. Let \( b^\pm = (b_1^\pm \cap B_2) \cup (b_2^\pm \cap B_1) \). Let \( \Sigma^\pm \) be the local section formed from \( \Sigma_1^\pm \cup \Sigma_2^\pm \) as in 3.17; let \( \Sigma^- \) be the local section formed from \( \Sigma_1^- \cup \Sigma_2^- \) as in 3.17 with \( f \) replaced by its reverse \( \hat{f} \) defined by \( \hat{f}(p, t) = f(p, -t) \). Then
\(\Sigma^+, \Sigma^-\) are local sections for \(f\) containing \(b^+\) and \(b^-\), respectively. Properties (a), (b), and (c) of 3.3 are easy to verify.

Note 3.18 (a). If \(b_i^+ \cap b_i^- = \emptyset\) \((i = 1, 2)\), then it is not necessarily true that \(b^+ \cap b^- = \emptyset\). However, \(B_0 = \{ p \in B \mid |\mathcal{C}(p, B)| \geq \delta > 0 \}\) is a block with \(b_0^+ \cap b_0^- = \emptyset\).

**Definition 3.19.** If \(B, B_0 \in \mathcal{B}(f, S)\), we write \(B_0 > B\) if \(B_0\) is a squeeze of a shave of \(B\). Note that if \(B_0 > B\), then \(B_0 \subset B\).

If \(B_0 \subset B\), the function \(r : b_0^+ \to [0, \infty)\) defined by \(r(p) = |\mathcal{C}(p, B - B_0)|\) is not necessarily continuous since an orbit segment in \(B\) might enter, leave, then reenter \(B_0\) before leaving \(B\). This cannot happen if \(B_0 > B\); and \(r\) is continuous and finite then. Consequently, if \(B_0 > B\), there is a collar of \(b^+\) containing \(b_0^+\); this is the reason the relation is introduced.

We will now show that the relation \(>\) in \(\mathcal{B}(f, S)\) is a partial ordering which makes \(\mathcal{B}(f, S)\) a directed set. This enables us to form direct limits over \(\mathcal{B}(f, S)\).

**Lemma 3.20.** If \(B, B_0 \in \mathcal{B}(f, S)\), then \(B_0 > B\) iff \(B_0 = \text{cl}(\text{int } B_0)\) (rel \(B\)) and (3) \(\mathcal{O}(p, B) \cap B_0 = \mathcal{O}(p, B_0)\) for all \(p \in B_0\).

**Proof.** Assume \(B_0 > B\), and that \(B_0\) is obtained from \(B\) by first shaving \(Y \subset b^+\) to obtain \(B_1\), and then squeezing collars \(B_1^\pm \equiv B^\pm \cap B_1\) of \(b_1^\pm\).

Recall that \(Y\) is closed in \(b^+ - a^+\); therefore \(p \in \partial(B - \mathcal{O}(Y, B))\) (rel \(B\)) iff \(p \in \mathcal{O}(\partial Y, B)(\text{rel } b^+)\). It follows that (rel \(X\)) \(\text{cl}(\text{int } B_1) = \text{cl}(B - \mathcal{O}(Y, B)) = B_1\).

Since \(B_1^+, B_1^-\) are closed, and \(B_0 = \text{cl}(B_1 - (B_1^+ \cup B_1^-))\), it can easily be shown that \(B_0 = \text{cl}(\text{int } B_0)\) (rel \(X\)).

Observe that if \(p \in B_0\), \(\mathcal{O}(p, B) = \mathcal{O}(p, B_1^+) \cup \mathcal{O}(p, B_1^-) \cup \mathcal{O}(p, B_0)\) where \(p^\pm \in \mathcal{O}(p, B) \cap B_1^\pm\). This implies that \(\mathcal{O}(p, B) \cap B_0 = \mathcal{O}(p, B_0)\) if \(p \in B_0\).

On the other hand, suppose that \(\text{cl}(\text{int } B_0) = B_0\) (rel \(X\)) and (3) both hold.

**Theorems 5.12.** If \(B, B_0 \in \mathcal{B}(f, S)\), then \(B_0 > B\) iff \(B_0 = \text{cl}(\text{int } B_0)\) (rel \(B\)) and (3) \(\mathcal{O}(p, B) \cap B_0 = \mathcal{O}(p, B_0)\) for all \(p \in B_0\).

**Proposition 3.21.** The relation \(>\) is transitive in \(\mathcal{B}(f, S)\), and is therefore a partial ordering.

**Theorem 3.22.** \(\mathcal{B}(f, S)\) is directed by \(>\); that is, if \(B_1, B_2 \in \mathcal{B}(f, S)\), there exists \(B \in \mathcal{B}(f, S)\) such that \(B > B_1, B_2\). In fact, \(B\) can be chosen to be a shave of \(B_1 \cap B_2\).
Proof. Let \( B_3 = B_1 \cap B_2 \). There is a neighborhood \( U \) of \( a_3^+ \) in \( b_3^+ \) such that if \( p \in U \), \( \partial(p, B_1) \cap \text{cl} U = \{p\} \). If not, there are sequences of points \( \{p_i\} \), \( \{p_i'\} \) in \( b_3^+ \), and a sequence of orbit segments \( \{\partial_i\} \) in \( B_1 \) such that

1. \( p_i' \) is the initial and \( p_i' \) is end point of \( \partial(p, B_1) \),
2. \( p_i' \to p' \in a_3^+ \), \( p_i'' \to p'' \in a_3^+ \),
3. \( \partial_i \) is not a subset of \( B_3 \) (i = 1, 2,...).

Choose \( p_i \in \partial_i - B_3 \) for \( i = 1, 2,... \), and let \( p \) be a limit point of \( \{p_i\} \).

Note that \( p \in B - \text{int} B_3 \). From (4) and (5) it follows that \( |\partial^+(p_i, B_1)| \to \infty \) and \( |\partial^-(p_i, B)| \to \infty \); which implies \( |\partial^+(p, B_3)| = |\partial^-(p, B_3)| = \infty \).

Therefore \( p \) is an element of an invariant set interior to \( B_1 \), but not to \( B_3 \). This contradicts the fact that \( B_1 \) and \( B_2 \) isolated the same invariant set.

Let \( B_3' = \text{cl}(B_3 - \partial(Y, B_3)) \) where \( Y = \partial b^+ \). From (4) and (5) it follows that \( |\partial^+(p, B_1)| \to \infty \) and \( |\partial^-(p, B)| \to \infty \); which implies \( |\partial^+(p, B_3)| = |\partial^-(p, B_3)| = \infty \).

Therefore \( p \) is an element of an invariant set interior to \( B_1 \), but not to \( B_3 \). This contradicts the fact that \( B_1 \) and \( B_2 \) isolated the same invariant set.

3.21 then implies \( B > B_1 \), also, and this proves the theorem.

Remarks 3.23. (a) If \( B \in \mathcal{B}(f, S) \) and \( W \subset B \) is a neighborhood of \( A \) in \( B \), it is not hard to show that there is a shave \( B_0 \) of \( B \) inside \( W \). Thus \( A = \cap B_0 \) and \( A^\pm = \cap b_0^\pm \) where \( B_0 \) ranges over all shaves of \( B \).

(b) If in 3.22, \( b_1^+ \cup b_1^- = \partial B_1 \) (i = 1, 2), \( B \) need not have the same property. However, if \( r: b^+ \to [0, \infty) \) be any collar of \( b^+ \) with \( r(p) = |\partial^+(p, B)| \) for \( p \in \partial b^+ \) (rel \( \partial B \)), and if \( B_0 \) is obtained from \( B \) by squeezing the \( r \)-collar of \( b^+ \), then \( B_0 > B_1 \), \( B_2 \), \( B_0 \) is a shave and a squeeze of \( B_1 \cap B_2 \), and \( b_0^+ \cup b_0^- = \partial B_0 \).

4. THE HOMOTOPY INDEX

In this section we develop the homotopy index \( i(f, S) \) of an element \( S \in \text{Iso}(f) \). This index takes the form of a homotopy type of a pointed space. It is a generalization, due mostly to C. C. Conley, of the index in [1].

Definitions 4.1. A pair \((Y, A)\) consists of a topological space \( Y \) and a closed subset \( A \subset Y \). If \((Y, A)\) and \((Z, B)\) are pairs, and \( g: Y \to Z \) is continuous such that \( g(A) \subset B \), we write \( g: (Y, A) \to (Z, B) \).

If \( Y \) is a topological space, and \( y \in Y \), \((Y, y)\) is a pointed space. If \((Y, y)\) and \((Z, z)\) are pointed spaces and \( g: Y \to Z \) is continuous such that \( g(y) = z \), then we write \( g: (Y, y) \to (Z, z) \).

If \((Y, A)\) is a pair, \( Y/A \) denotes the pointed space \((\overline{Y}, p)\) where \( \overline{Y} \) is obtained from \( Y \) by identifying \( A \) to a point \( p \). By convention, \( Y/A \) denotes \([Y \cup \{p\}]/\{p\}\), where \( p \) is disjoint from \( Y \).
Remarks 4.2. (a) $\mathcal{I}$ is a covariant functor from the category of pairs and mappings of pairs to the category of pointed spaces and mappings of pointed spaces. Thus, if $g: (Y, A) \rightarrow (Z, B)$, there is a corresponding map $\bar{g}: Y/A \rightarrow Z/B$ such that (1) if $g$ is the identity map, so is $\bar{g}$, and (2) $(g \circ h)^* = \bar{g} \circ \bar{h}$ if $h: (W, C) \rightarrow (Y, A)$.

(b) If $(Y, A)$ is a pair, there is a natural map $\varphi(Y, A) = \varphi: (Y, A) \rightarrow Y/A$, defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \notin A \\ \varphi & \text{if } x \in A \end{cases},$$

such that if $g: (Y, A) \rightarrow (Z, B)$, the following diagram commutes:

$$
\begin{array}{ccc}
(Y, A) & \xrightarrow{g} & (Z, B) \\
\downarrow \varphi(Y, A) & & \downarrow \varphi(Z, B) \\
Y/A & \xrightarrow{\bar{g}} & Z/B.
\end{array}
$$

Definitions 4.3. The pointed space $(Y, y)$ has the same homotopy type as the pointed space $(Z, z)$ means there are maps $g: (Z, z) \rightarrow (Y, y)$ and $h: (Y, y) \rightarrow (Z, z)$ such that $h \circ g$ and $g \circ h$ are homotopic to their respective identities. $g$ is called a homotopy equivalence.

We use $g^{-1}$ to denote $h$ even though the homotopy inverse is not unique.

Observe that if $(Y, y)$ and $(Z, z)$ have the same homotopy type, the homotopy axiom for cohomology implies that $H^*(Y, y) \approx H^*(Z, z)$ by the induced map $g^*$.

The following lemma is the main result of this section; it enables us to define the homotopy index. This lemma, and all the rest in the chapter, remain true if $\mathcal{I}$ is replaced by $\mathcal{J}$.

Lemma 4.4. If $B_0 \supset B$ in $\mathcal{B}(f, S)$, and $B^+$ is any collar of $b^+$ such that $b_0^+ \subset B^+$, then the inclusion induced maps $i: B/b^+ \rightarrow B/B^+$ and

$$i_0: B_0/b_0^+ \rightarrow R/R^+$$

are homotopy equivalences.

Definition 4.5. The homotopy index $i(f, S)$ is the pair $(i^+(f, S), i^-(f, S))$ where $i^\pm(f, S)$ is the homotopy type of $B/b^\pm$ for some $B \in \mathcal{B}(f, S)$. Lemma 4.4 and the fact that $\mathcal{B}(f, S)$ is directed imply that $i(f, S)$ is well defined.

The idea of the proof is as follows: Since squeezing a block does not change its homotopy type, we can assume that $B_0$ is a shave of $B$. Since all collars of $b^+$ have the same homotopy type as $b^+$, we use $B^+ - b^+$. The problem then
reduces to showing \( B_0/b_0^+ \to B/b^+ \) is a homotopy equivalence. But \( B_0/b_0^+ \to B_0 \cup b^+/b^+ \) is a homeomorphism, and since \( B_0 \) is a shave of \( B \), \((B, b^+)\) essentially (in the sense of 4.6) retracts to \((B_0 \cup b^+, b^+)\).

The proof of 4.4 requires several lemmas:

**Lemma 4.6.** Suppose \( D_0 \subseteq D \subseteq X \) are closed, and \( F: (D, D_0) \times [0, 1] \to (D, D_0) \) is continuous such that

\[
\text{(a)} \quad F(d, 0) = d \quad \text{for all } d \in D, \text{ and} \\
\text{(b)} \quad \text{if } (D, D_0) \equiv (F(D \times 1), F(D_0 \times 1)), \text{ then } F(D \times 1, 0) \subseteq D \text{ and } F(D_0 \times 1, 1) \subseteq D_0.
\]

Then \((D, D_0) \subseteq (D, D_0)\) induces a homotopy equivalence \( D/D_0 \to D/D_0 \).

**Proof.** Let \( i^{-1} \) be the map induced by \( F | D \times 1: D \to D \). Then \( i^{-1} \circ i \) is induced by \( F | D \times 1 \), which is homotopic by \( \{F | D \times 1\}_{t=0}^1 \to \) to the identity on \( D \). It follows from (b) that \( \{F | D \times t\} \) induces a homotopy of \( i^{-1} \circ i \) to the identity on \( D \).

\( i \circ i^{-1} \) is induced by \( F | D \times 1: D \to D \), which is also homotopic to the identity. Furthermore, (a) implies this homotopy induces a homotopy of \( i \circ i^{-1} \) to the identity of \( D \). Thus \( i^{-1} \) is a homotopy inverse of \( i \).

**Triangle Lemma.** Suppose \( X, Y, Z \) are topological spaces and \( a: X \to Y \), \( b: X \to Z \), \( c: Y \to Z \) are maps such that \( b = ca \). Then if any two of \( a, b, c \) are homotopy equivalences, then so is the third, and any triangular diagram formed with the maps \( a, b, c, a^{-1}, b^{-1}, c^{-1} \) is commutative, at least up to homotopy.

The proof is easy and is omitted.

**Lemma 4.7.** If \( B, B' \in \mathcal{A}(f, S) \) and \( B \subseteq B' \subseteq B^+ \) are collars for \( b^+ \), then \( B/B^+ \to B/B^+ \), induced by inclusion, is a homotopy equivalence.

**Proof.** Let \( \psi: B \to [0, 1] \) such that \( \psi^{-1}(0) \) is a closed neighborhood of \( A^- \cup b^- \), and \( \psi^{-1}(1) = B^+ \). Then define \( F: (B, B^+) \times [0, 1] \to (B, B^+) \) by

\[
F(p, t) = \begin{cases} 
\psi \cdot (-tv(p) | \delta^-(p, B)) & \text{if } p \in B - (A^- \cup b^-), \\
p & \text{otherwise}.
\end{cases}
\]

That \( F \) is continuous follows from 3.8, and 4.6 implies \( B/b^+ \to B/B^+ \) is a homotopy equivalence. Similarly, \( B/b^+ \to B/B^+ \) is a homotopy equivalence. 4.7 now follows from the Triangle Lemma.

**Lemma 4.8.** If \( B_0, B, B_0 \in \mathcal{A}(f, S) \), \( B_0 \) is a squeeze of \( B \), and \( B^+, B_0^+ \) are collars of \( b^+, b_0^+ \), respectively such that \( B^+ \subseteq B_0^+ \), then \( B_0/B_0^+ \to B/B^+ \) is a homotopy equivalence.
Proof. We prove 4.8 in the case $B_0^+ = B_0 \cap B^+$. Lemma 4.7 and the Triangle Lemma then imply 4.8 in general.

Let

$$F(p, s) = \begin{cases} p \cdot (s \mid \sigma(p, B^+ - B_0^+)) & \text{if } p \in B^+ - B_0^+, \\ p & \text{otherwise.} \end{cases}$$

This special case is now implied by 4.6.

**Lemma 4.9.** Suppose $B_0, B \in \mathcal{A}(f, S)$, $B_0$ is a shave of $B$, and $B_0^+ \subset B^+$ are collars of $b_0^+$ and $b^+$, respectively. Then $B_0/B^+ \to B/B^+$ is a homotopy equivalence.

**Proof.** In the commutative, inclusion induced diagram,

\[
\begin{array}{ccc}
B_0/B_0^+ & \leftarrow & B_0/b_0^+ \\
\downarrow & & \downarrow \\
B_0 \cup b^+/B_0^+ \cup b^+ & \leftarrow & B_0 \cup b^+/b^+,
\end{array}
\]

observe that the top horizontal map is a homotopy equivalence (4.7), and the right vertical map is a homeomorphism, hence the Triangle Lemma implies the diagonal is also a homotopy equivalence. Since the left vertical is also a homeomorphism, the Triangle Lemma implies the bottom horizontal map is also a homotopy equivalence. Thus all the maps are homotopy equivalences.

The second step of the proof is to show that $B_0 \cup b^+/b^+ \to B/B^+$ is a homotopy equivalence: let $\psi: B \to [0, 1]$ such that $\psi^{-1}(0)$ is a closed neighborhood of $A^-\subset B_0$ and $\psi^{-1}(1) = B^+ \cup \overline{B - B_0}$. Then let

$$F: (B, B^+) \times [0, 1] \to (B, B^+)$$

be defined by

$$F(p, s) = \begin{cases} p \cdot (-s\psi(p) \mid \sigma(p, B)) & \text{if } p \notin A^-, \\ p & \text{if } p \in A^-.
\end{cases}$$

Since $(F(B \times 1), F(B^+ \times 1)) = (F(B_0 \cup b^+ \times 1), F(b^+ \times 1))$, the Triangle Lemma and 4.6 imply that the following inclusion induced diagram consists of homotopy equivalences:

\[
\begin{array}{ccc}
B/B^+ & \leftarrow & F(B \times 1)/F(B^+ \times 1) \\
\downarrow & & \downarrow \\
B_0 \cup b^+/b^+.
\end{array}
\]
The final step of the proof involves the diagram

\[
\begin{array}{c}
D_0/B_0^+ \rightarrow D_0 \cup b^+ / B_0^+ \cup b^+ \leftarrow B_0 \cup b^+ / b^+ \\
\downarrow \\
B/B^+.
\end{array}
\]

The right horizontal and diagonal maps are homotopy equivalences (steps 1 and 2), so the middle vertical map is also a homotopy equivalence (Triangle Lemma). Since the left horizontal map is a homeomorphism, it follows that the left diagonal map, \( B_0 / B_0^+ \rightarrow B / B^+ \), is a homotopy equivalence. This completes the proof of 4.9.

**Proof of Theorem 4.4.** Let \( B_1 \) be a shave of \( B \) such that \( B_0 \) is a squeeze of \( B_1 \). Let \( B_1^+ = B^+ \cap B_1 \); then \( B_1^+ \supset B_0^+ \). The preceding lemmas imply that all maps in the inclusion induced diagram,

\[
\begin{array}{c}
B / b^+ \rightarrow B / B^+ \leftarrow B_1 / B_1^+ \\
\uparrow \\
B_0 / b_0^+.
\end{array}
\]

are homotopy equivalences. This proves 4.4.

**Example 4.10.** A hyperbolic rest point is an example of an isolated invariant set with a block \( B \) homeomorphic to a ball. If the rest point has a stable manifold of dimension \( n \) and an unstable manifold of dimension \( m \), then \( b^+ \) is homeomorphic to \( S^{n-1} \times [0, 1]^m \) and \( b^- \) to \( S^{m-1} \times [0, 1]^n \). (The case \( m = n = 1 \) is drawn in 9.1.) Thus the homotopy type of \( B / b^+ \) is the same as that of \( B / S^{n-1} \) (where \( S^{n-1} \), the sphere of dimension \( n - 1 \), is assumed to be embedded in \( \partial B \)). This homotopy type is the same as that of an \( S^n \). Similarly, \( B / b^- \) has the same homotopy type as an \( S^m \). The Morse index of a hyperbolic rest point is essentially the pair \( (n, m) \); hence \( \iota(f, S) \) is a generalization of the Morse index in the sense that the both carry the same information in the case of a hyperbolic rest point.

5. **Some Properties Common to All Invariant Sets Isolated by a Given Block**

Since we often take direct limits, we use a cohomology functor \( H^* \), such as the Čech cohomology, with the property that if \( K \) is a compact subset
of $X$, $\lim H^*(U) = H^*(K)$ where $U$ ranges over the set of closures of the neighborhoods of $K$.

The statements in this chapter usually concern only $b^+$ and not $b^-$. This is only for notational simplicity; the corresponding statement for $b^-$ will also be true.

The flow $f$ and $S \in \text{Iso}(f)$ are fixed throughout this section. We use the notation in the following diagram throughout the rest of the paper: Suppose $B \supset B_0$ are blocks (in later sections they will not have to be blocks for the same flow), and $B^+$ is a collar of $b^+$ such that $B^+ \supset b_0^+$. Then we have the commutative Diagram 5.1.

**Diagram 5.1.**

\[
\begin{array}{cccccc}
H^*(b^+) & \xrightarrow{\delta[b^+]} & H^*(B, b^+) & \longrightarrow & H^*(B) & \longrightarrow & H^*(b^+) \\
& \downarrow{m_1} & \downarrow{m_2} & & \downarrow{m_3} & & \\
H^*(b_0^+) & \xrightarrow{\delta[b_0^+]} & H^*(B_0, b_0^+) & \longrightarrow & H^*(B_0) & \longrightarrow & H^*(b_0^+) \\
\end{array}
\]

The horizontal rows are long exact sequences for pairs; $\delta[b^+]$ is the coboundary operator of degree 1. The vertical maps are all induced by inclusion, and the top vertical maps are all isomorphisms since $b^+$ is a strong deformation retraction of $B^+$.

Note that if $B_0 > B$ in $\mathcal{B}(f, S)$, we have shown (4.4) that $n_2$ is also an isomorphism, since $H^*(B, b^+) \approx H^*(B, b_0^+) \approx H^*(B, B^+)$. 

**Remark 5.1a.** We shall use $\rho[B, B_0]$ to denote one of the maps $n_i \circ m_i^{-1}$, $i = 1, 2, 3$. As the notation indicates, $\rho[B, B_0]$ does not depend on the particular $B^+$ used to define it. For example, if $\rho[B, B_0] = n_1 \circ m_1^{-1}$, and $\bar{B}^+$ is a second collar of $b^+$ containing $b_0^+$, then $B^+ \cup \bar{B}^+$ is a third, and the following diagram commutes:

\[
\begin{array}{cccc}
H^*(B^+) & \xrightarrow{m_1} & H^*(B^+) & \xleftarrow{\rho[B, B_0]} & H^*(B^+ \cup \bar{B}^+) & \xrightarrow{n_1} & H^*(b_0^+) \\
\end{array}
\]

It follows that the bottom map from $H^*(b^+)$ to $H^*(b_0^+)$ is the same as the top map, which is $\rho[B, B_0]$. 
Lemma 5.2. If $B_1 > B_0 > B$ in $\mathcal{A}(f, S)$, then $\rho[B_0, B_1] \circ \rho[B, B_0] = \rho[B, B_1]$.

Proof. We prove this in the case $\rho[B, B_0] = n_2 \circ m_1^{-1}$; the other cases are similar.

Choose collars $B^+, B_0^+$ of $b^+, b_0^+$, respectively, such that $B^+ \supset B_0^+ \supset b_1^+$. The lemma then follows from the inclusion induced commutative diagram

$$
\begin{array}{ccc}
H^*(b_1^+) & \longrightarrow & H^*(B_0^+) \\
\downarrow & & \downarrow \\
H^*(b^+) & \longleftrightarrow & H^*(B^+) \\
\downarrow & & \downarrow \\
H^*(b_0^+) & \longrightarrow & H^*(B_0^+) \\
\end{array}
$$

5.2. implies that $\{H^*(b_1^+), H^*(B_1, b_1^+)\}$, and $\{H^*(B_1)\}$, with the corresponding maps $\{\rho[B, B_0]\}$, where $B_0 > B$ ranges over $\mathcal{A}(f, S)$, are all direct systems under the partial ordering $\succ$.

Definition 5.3.

$$
H^*(\alpha(B, f, S)) \equiv \lim H^*(b^\pm), \quad H^*(\iota(B, f, S)) \equiv \lim H^*(B, b^\pm).
$$

Remarks 5.4. (a) One could define $H^*(\alpha(f, S)) = \lim H^*(B)$, but then $H^*(\alpha(f, S)) = H^*(S)$. We use the latter notation.

(b) If $B_0 > B$, it follows from 3.22 that there is a shave $B_1$ of $B$ such that $B_0$ is a squeeze of $B$. Then $b_1^+$ and $B_1^+$ have the same homotopy type as $b_0^+$. It follows that $H^*(\alpha^+(f, S)) \approx \lim H^*(b_1^+) = H^*(\alpha^+)$, if $B_1$ is allowed to range over only shaves of $B$. Thus without ambiguity one can think of $\alpha^-(f, S)$ as the homotopy type of the set $\alpha^\pm$ of any block $B \in \mathcal{A}(f, S)$. For similar reasons, $H^*(\iota^+(f, S)) \approx H^*(\iota^+(f, S))$, and $H^*(\alpha^+) \approx H^*(S)$.

(c) The commutativity of diagram 5.1 implies that the direct limit induces a long exact sequence

$$
H^*(\alpha^+(f, S)) \xrightarrow{\delta[f, S]} H^*(\iota^+(f, S)) \longrightarrow H^*(S) \longrightarrow H^*(\alpha^+(f, S)),
$$

where $\delta[f, S] = \lim \delta[b^\pm]$.

This sequence is exact because the direct limit of exact sequences is always exact.
exact. (1) is the same as the exact sequence for the pair \((A, a^+)\) in the sense that the following diagram commutes:

\[
\begin{array}{cccccc}
H^*(a^+) & \longrightarrow & H^*(A, a^+) & \longrightarrow & H^*(A) & \longrightarrow & H^*(a^+)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^*(a^+(f, S)) & \longrightarrow & H^*(\iota^+(f, S)) & \longrightarrow & H^*(S) & \longrightarrow & H^*(a^+(f, S))
\end{array}
\]

(d) From (1) one sees that an element of \(H^*(\iota^+(f, S))\) is either the image by \(\delta[f, S]\) of a nonzero element of \(H^*(a^+(f, S))\), or is mapped to a nonzero element of \(H^*(S)\). Since for each \(B \in \mathcal{B}(f, S)\), \(H^*(B, b^+) \cong H^*(\iota^+(f, S))\), each nonzero element of \(H^*(B, b^+)\) corresponds to an element either in \(H^*(a^+)\) or in \(H^*(S)\).

**DEFINITION 5.5.** \(H^*(\iota^+(f, S))\) are called the index algebras or simply the indices of \((f, S)\).

Some of the ambiguity of 5.4(d) can be removed:

**THEOREM 5.6.** If \(B \in \mathcal{B}(f, S)\), then Im(\(\delta[b^+]\)) injects into Im(\(\delta[f, S]\)) via the direct limit. Consequently, if \(0 \neq \beta \in H^*(b^+)\) and \(\delta[b^+](\beta) \neq 0\), then \(\beta\) is mapped nontrivially via direct limit to \(H^*(\iota^+(f, S))\).

**Proof.** This follows immediately from the commutativity of

\[
\begin{array}{ccc}
H^*(b^+) & \overset{\delta[b^+]}{\longrightarrow} & H^*(B, b^+) \\
\downarrow & & \downarrow \\
H^*(a^+(f, S)) & \overset{\delta[f, S]}{\longrightarrow} & H^*(\iota^+(f, S))
\end{array}
\]

and the fact that the right vertical map is injective.

Thus \(\beta \in H^*(b^+)\) represents nontrivial cohomology of \(H^*(a^+)\) if \(\delta[b^+](\beta) \neq 0\). However, we shall construct a counter-example to the complementary statement; that is, it is possible that an element of \(H^*(B, b^+)\) which is mapped nontrivially to \(H^*(\iota^+(f, S))\) which is mapped to zero in \(H^*(S)\).

**EXAMPLE 5.7.** It is not hard to write down a differential equation in the plane whose associated flow has orbits which look like those pictured in Fig. 3 in an annulus. \(S\) is shaded.

The annulus itself is an isolating block \(B\) for this flow, and \(H^1(B)\) is generated by one element \(\nu\). The upper semicircle of the outer boundary and the lower semicircle of the inner boundary form \(b^+\); thus \(H^1(b^+) = 0\). It follows from the exact sequence for the pair \((B, b^-)\) that \(\nu\) is the image
of some \( \beta \in H^*(B, b^+) \). However, \( H^1(S) = 0 \), so \( \beta \) is mapped by direct limit to some element \( \alpha \in H^1(\partial S) \) which is in turn mapped to zero in \( H^1(S) \).

5.6 gives cohomological information about the asymptotic set \( a^+ \) using only the exact sequence for \((B, b^+)\). The remainder of this chapter develops and adds to theorems in [9] of the same spirit as 5.6. We study some substructures of the index algebra which are related to \( H^*(a^+) \) and \( H^*(S) \). To do this we must establish the following notation: (2) Let \( B \in \mathcal{B}(f, S) \) let \( \tilde{c} \) be open and closed in \( b^+ \), and let \( d = b^+ - \tilde{c} \). If \( B > B \) in \( \mathcal{B}(f, S) \), let \( c, d \) be \( b^+ \cap \partial(\tilde{c}, B) \) and \( b^+ \cap \partial(d, B) \), respectively. \( a^+(\tilde{c}) = a^+ \cap \tilde{c} \).

Now observe (3) \( c \cap d = \emptyset \) and \( c \cup d = b^+ \). (4) \( H^*(b^+) = H_0 \oplus H_1 \) where \( H_1 = \ker\{H^*(b^+) \to H^*(c)\} \approx H^*(d) \) and \( H_0 = \ker\{H^*(b^+) \to H^*(d)\} \approx H^*(c) \). Henceforth, we relax the notation and use \( H^*(c) \) and \( H^*(d) \) for \( H_0 \) and \( H_1 \), respectively. (5) Suppose \( B_0 > B \in \mathcal{B}(f, S) \). Then, in the notation of 5.1, \( n_1 \circ m_1^{-1}(H^*(c)) \subset H^*(c_0) \); therefore \( \lim_{B \to B} H^*(c) \) is well defined. In fact, \( \lim_{B \to B} H^*(c(\tilde{c})) \approx H^*(a^+\tilde{c})) \).

**Definition 5.8.** \( H^*(a^+(\tilde{c}, f, S)) = \lim H^*(c) \) where the direct limit is taken over all \( B > B \) in \( \mathcal{B}(f, S) \). \( a^+(\tilde{c}, f, S) \) can be thought of as the homotopy type of \( a^+\tilde{c} \).

Note that \( H^*(a^+(\tilde{c}, f, S)) \subset H^*(a^+(f, S)) \) and is isomorphic to
\[
\ker\{H^*(a^+) \to H^*(a^+(d))\}.
\]
In fact, \( H^*(a^+(\tilde{c}, f, S)) \oplus H^*(a^+(d, f, S)) = H^*(a^+(f, S)) \).

**Definition 5.9.** \( \delta[\tilde{c}] = \delta[b^+] \mid H^*(c) \), and \( \delta[\tilde{c}, f, S] = \lim \delta[\tilde{c}] = \delta[f, S] \mid H^*(a^+(\tilde{c}, f, S)) \).
Theorem 5.10. If $B \succ B$ in $\mathcal{B}(f, S)$, then $\text{Im}(\delta[c])$ injects via the direct limit into $\text{Im}(\delta[c, f, S])$. Consequently, if $0 \neq \beta \in H^*(c)$ and $\delta[c](\beta) \neq 0$, then $\beta$ is mapped nontrivially via direct limit to $H^*(a^+[c, f, S])$.

The proof is similar to that of 5.6.

5.10 shows us elements of $H^*(c)$ which are mapped nontrivially into $H^*(a^+(c))$. An application of 5.10 occurs in 11.2. The following structure gives us elements in $H^*(B)$ which are mapped nontrivially to $H^*(S)$.

Definition 5.11. Let $D[c]$ be the homomorphism defined by the composition

$$
H^*(B) \xrightarrow{k} H^*(c) \xrightarrow{e^{-1}} H^*(b^+, d) \xrightarrow{\delta} H^*(B, b^+)
$$

where $k$ is induced by inclusion, $e$ is an excision isomorphism, and $\delta$ is the coboundary operator in the exact sequence for the triple $(B, b^+, d)$.

If $B \succ B_0$, the following diagram commutes (notation as in 5.1):

$$
\begin{array}{cccccc}
H^*(B) & \xrightarrow{k} & H^*(c) & \xrightarrow{e^{-1}} & H^*(b^+, d) & \xrightarrow{\delta} H^*(B, b^+), \\
\downarrow{m_0^{-1}} & & \downarrow{m_1^{-1}} & & \downarrow{\approx} & \\
H^*(B) & \xrightarrow{m_2^{-1}} & H^*(B, b^+) & & H^*(B, B^+) & \\
\downarrow{n_3} & & \downarrow{m_3^{-1}} & & \downarrow{n_3} & \\
H^*(B_0) & \xrightarrow{n_3} & H^*(b_0^+) & & H^*(B_0, b_0^+). & \\
\end{array}
$$

Here $C$ and $D$ are collars of $c$ and $d$, the maps marked $e$ are excisions, and those marked $\delta$ are coboundaries. The top row is $D[c]$, and the bottom, $D[c_0]$; therefore we can make the following definition:

Definition 5.12. $D[\bar{c}, f, S] \equiv \text{lim} D[c]$ where $B \succ B$ ranges over $\mathcal{B}(f, S)$.

Theorem 5.13. $\text{Im}(D[c])$ injects via the direct limit into $\text{Im}(D[\bar{c}, f, S])$. Consequently, if $\beta \in H^*(B)$ such that $D[c](\beta) \neq 0$, then $\beta$ includes nontrivially to $H^*(S)$.

The proof is similar to that of 5.6.

We now relate $\delta[f, S]$, $\delta[c, f, S]$, and $D[\bar{c}, f, S]$. As before, let $\bar{B} \in \mathcal{B}(f, S)$, and let $\bar{c}_1, \ldots, \bar{c}_n$ be mutually disjoint open and closed subsets of $b$ whose union is $b^+$. (For example, the components of $b^+$ if there are finitely many.) If $B \succ \bar{B}$, $\bar{c}_i = \partial(\bar{c}_i, \bar{B}) \cap b^+$ for $i = 1, \ldots, n$. If $\bar{f} \subset \{1, \ldots, n\}$, let $c = \bigcup_{i \in \bar{f}} \bar{c}_i$ and $d = \bigcup_{i \notin \bar{f}} \bar{c}_i$. 

THEOREM 5.14. (a) $D[c] = -D[d]$, so $\text{Im } D[c] = \text{Im } D[d]$.

(b) $\text{Im } D[c] = \text{Im } \delta[c] \cap \text{Im } \delta[d]$.

(c) $\text{Im } \delta[c] = \sum_{i=1}^{n} \text{Im } [c_i]$; in particular, $\text{Im } [b^+] = \sum_{i=1}^{n} \text{Im } [c_i]$.

(d) $\text{Im } \delta[b^+] / \text{Im } [c] \approx \text{Im } \delta[d] / \text{Im } D[c]$.

(e) $\text{Im } D[c] = \left( \sum_{i=1} \text{Im } \delta[c_i] \right) \cap \left( \sum_{i \neq j} \text{Im } \delta[c_i] \right)$.

Remark. The direct limit of each of the statements above is also true.

Proof. We have the following commutative diagram:

$$
\begin{array}{ccc}
H^*(B) & \xrightarrow{e} & H^*(d) \\
\downarrow e & & \downarrow e \\
H^*(b^+, d) & \xrightarrow{\delta[b^+]} & H^*(b^+, c) \\
\downarrow \delta & & \downarrow \delta \\
H^*(B, b^+) & & 
\end{array}
$$

The maps marked $e$ are excision isomorphisms, those marked $\delta$ are coboundary maps. All others are induced by inclusion. The map from $H^*(B)$ to $H^*(B, b^+)$ around the left side is $D[c]$; around the right, $D[d]$. All sequences are exact at $H^*(b^+)$. (a) is just the hexagonal lemma of algebraic topology, and (b) follows easily from the diagram and (a). (c) follows from the fact that $H^*(c) = \bigoplus_{i \in J} H^*(c_i)$. Since $\text{Im } \delta[b^+] = \sum_{i=1}^{n} \text{Im } \delta[c_i] = \text{Im } \delta[c] + \text{Im } \delta[d]$, (d) follows from the classical isomorphism theorem for modules: If $H_1$ and $H_2$ are submodules, then $(H_1 + H_2)/H_1 \approx H_2/(H_1 \cap H_2)$.

To prove (e), let $\alpha \in \text{Im } D[c]$. By (b) and (c), $\alpha = \sum_{i \in J} \alpha_i = \sum_{i \neq j} \alpha_i$ where $\alpha_i \in \text{Im } \delta[c_i]$. Subtracting the two equations, we find that

$$0 = \sum_{i \neq j} \alpha_i - \sum_{i \neq j} \alpha_i,$$

which implies each $\alpha_i \in \sum_{i \neq j} \text{Im } \delta[c_i] = \text{Im } \delta[c_i']$, where $c_i' \equiv \bigcup_{j \neq i} c_j$. Thus each $\alpha_i \in \text{Im } \delta[c_i] \cap \text{Im } \delta[c_i'] = \text{Im } D[c_i]$. 

It follows that
\[ \text{Im } D[c] \subset \left( \sum_{i \in I} \text{Im } D[c_i] \right) \cap \left( \sum_{i \in J} \text{Im } D[c_i] \right). \]

The opposite inclusion follows from the fact that \( \text{Im } D[c_i] \subset \text{Im } \delta[c_i] \) and hence (c) implies \( \sum_{i \in I} \text{Im } D[c_i] \cap \sum_{i \in J} \text{Im } D[c_i] \subset \text{Im } \delta[c] \cap \text{Im } \delta[d] = \text{Im } D[c] \). This completes the proof.

Example 11.2 contains an application of 5.13 and 5.14.

We now restrict ourselves to the smooth case only and assume that our blocks have the form of Theorem 3.7. In particular, (6) \( H^*(B, b^+) \) is finitely generated; (7) The duality theorems hold for \( B \). Using coefficients in a field so that \( H^p \) is the vector space dual of \( H_0 \), we have \( H^p(B, B - C) \approx H^{n-p}(C) \) if \( C \subset B; \ H^p(B, b^+) \approx H^{n-p}(B, b^-) \), and \( H^p(B) \approx H^{n-p}(B, \partial B) \).

**Definition 5.15.** Let \( c \) denote the cup product of cohomology elements, and let \( d \) denote the last duality map in (7). Define \( Q[B] \) to be the composition
\[ H^*(B, b^+) \otimes H^*(B, b^-) \xrightarrow{c} H^*(B, \partial B) \xrightarrow{d} H^*(B). \]

**Theorem 5.16.** If \( 0 \neq v \in \text{Im } Q[B] \), then \( v \) injects nontrivially by inclusion into \( H^*(S) \).

**Proof.** Let \( B_0 > B \) in \( \mathcal{B}(f, S) \), and choose collars \( B^+, B^- \) of \( b^+, b^- \), respectively, such that \( B^+ \cup B^- = B - \text{int } B_0 \). The following diagram commutes:

\[
\begin{array}{ccc}
H^*(B, b^+) \otimes H^*(B, b^-) & \xrightarrow{c} & H^*(B, \partial B) \xrightarrow{d} H^*(B) \\
\uparrow & & \uparrow \\
H^*(B, B^+) \otimes H^*(B, B^-) & \xrightarrow{c} & H^*(B, B - \text{int } B_0) \xrightarrow{d} H^*(B) \\
\downarrow & & \downarrow \\
H^*(B_0, b_0^+) \otimes H^*(B_0, b_0^-) & \xrightarrow{c} & H^*(B_0, \partial B_0) \xrightarrow{d} H^*(B_0). \\
\end{array}
\]

The vertical maps are all induced by inclusion and the ones on the left are isomorphisms. The map marked \( e \) is an excision isomorphism. One sees from the diagram that (8) \( \text{Im } Q[B_0] \supset \text{Im } k \circ Q[B] \), and (9) if \( 0 \neq v \in \text{Im } Q[B] \), then \( k(v) \neq 0 \) in \( \text{Im } Q[B_0] \). The theorem now follows from the fact that \( H^*(S) \) is the direct limit of \( H^*(B) \).

\( Q[B] \) is computed in 11.2 and 11.4 for specific cases.

**Definition 5.17.** \( Q[f, S] : H^*(\iota^+ \otimes \iota^-(f, S)) \rightarrow H^*(S) \) is the direct limit of the maps \( Q[B] \) for \( B \in \mathcal{B}(f, S) \). Then \( \text{Im } Q[f, S] = \lim \text{Im } Q[B] \).

**Remarks 5.18.** (a) \( \text{Im } Q[f, S] \) is finitely generated even if \( H^*(S) \) is infinitely generated. This follows from (6).
(b) $\text{Im} Q[f, S]$ is the same as the image of the composition $c \circ d$ in the commutative diagram

$$
\begin{array}{c}
H^*(B, B - A^-) \otimes H^*(B, B - A^+) \xrightarrow{c} H^*(B, B - S) \xrightarrow{d} H^*(S) \\
\downarrow \\
H^*(B, b^+) \otimes H^*(B, b^-) \rightarrow H^*(B, \partial B) \rightarrow H^*(B).
\end{array}
$$

The vertical maps are induced by inclusion and the left one is an isomorphism (3.9).

We proceed to relate (in the smooth case) $\delta[c], D[c]$, and $Q[B]$. We use the notation $|V|$ to denote the dimension of a vector space $V$. Let $c$ be open and closed in $b^+$ (possibly $c = b^+$), and let $j[c]: H^*(B, c) \rightarrow H^*(B)$ be induced by inclusion:

**Theorem 5.19.**

(a) $|\text{Im} Q[B] \cap \text{Im} j[c]| + |\text{Im} \delta[c]| \leq |H^*(B, c)|$.

(b) $|\text{Im} Q[B] \cap \text{Im} j[c]| + |\text{Im} D[c]| \leq |H^*(B)|$.

(c) $|\text{Im} Q[B] \cap \text{Im} j[b^+]| \leq |\text{Im} Q[B] \cap \text{Im} j[c]|$.

(d) The direct limit of (a), (b), and (c) also hold.

Note that the case $c = b^+$ in (a) is interesting since $|H^*(B, b^+)| = |H^*(i^+(f, S))|$.

**Proof.** (a) follows from the exactness of the sequence for the pair $(B, c)$; (b) follows from the fact that $\text{Im} j[c] \subseteq \ker D[c]$; and (c) follows from the fact that $j[b^+] = j^* \circ j[c]$ where $j: (B, c) \rightarrow (B, b^+)$.

6. THE SPACE OF INVARIANT SETS AND THE INVARIANCE OF THE INDEX

**Definition 6.1.** Let $\mathcal{S} = \{(f, S) \mid f \in F, S \in \text{Iso}(f)\}$.

We put a topology on $\mathcal{S}$ which allows us to discuss perturbations of a point $(f, S) \in \mathcal{S}$. Recall (2.2, 2.3) that $\Phi(N) = \{f \in F \mid N \text{ is an isolating neighborhood for } f\}$ is an open subset of $F$.

**Definition 6.2.** If $N \subseteq X$ is closed, define $\gamma[N]: \Phi(N) \rightarrow \mathcal{S}$ by $\gamma[N](f) = (f, \phi(f, N)) \in \mathcal{S}$.

The topology is designed so that $\gamma[N]$ is a homeomorphism.

**Lemma 6.3.** The set $\{\text{Im}(\gamma[N] \mid \Phi) \mid N \subseteq X \text{ is closed, } \Phi \subseteq F \text{ is open}\}$ is a basis for a topology on $\mathcal{S}$; that is, if $N_1, N_2 \subseteq X$ are closed, $\Phi_1, \Phi_2$ are open in $F$, and $(f, S) \in \text{Im}(\gamma[N_1] \mid \Phi_1) \cap \text{Im}(\gamma[N_2] \mid \Phi_2)$, there is a closed $N \subseteq X$ and a neighborhood $\Phi \subseteq F$ such that $(f, S) \in \text{Im}(\gamma[N] \mid \Phi) \subseteq \text{Im}(\gamma[N_1] \mid \Phi_1) \cap \text{Im}(\gamma[N_2] \mid \Phi_2)$.
Proof. By hypothesis, $N_1$ and $N_2$ isolate $S$ for the flow $f$, so $N = N_1 \cap N_2$ also isolates $S$. It follows that (1) for each $x \in N_i - N$, there is a $T_x \in \mathbb{R}$ such that $f(x, T_x) \in X - N_i (i = 1, 2)$.

Using compactness, one can find a neighborhood $\Phi' \subset F$ of $f$, each element of which has property (1). Therefore $f' \in \Phi'$ implies $\sigma(f'_0, N_0) = \sigma(f', N_2) = \sigma(f', N)$. Let $\Phi = \Phi' \cap \Phi_1 \cap \Phi_2$, and observe that $\gamma[N_1] | \Phi = \gamma[N_2] | \Phi = \gamma[N] | \Phi$.

Remark 6.4. In this topology $\mathcal{P}$ is $T_1$ but not Hausdorff. For example, if $X = \mathbb{R}$, and $f$ is the flow of 1.2, then $S = \{0\}$ is a rest point and is isolated, but any neighborhood of $(f, S)$ must intersect each neighborhood of $(f, \phi)$.

Definition 6.5. $(f, S)$ is a continuation of $(f', S')$ if $(f, S)$ and $(f', S')$ are in the same quasicomponent of $\mathcal{P}$; that is, there is no separation of $\mathcal{P}$ into nonempty, closed and open subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $(f, S) \in \mathcal{S}_1$ and $(f', S') \in \mathcal{S}_2$.

To show that the index is invariant under continuation, we need the following restatement of Churchill's perturbation theorem for blocks [1].

Theorem 6.6. Suppose $B \in \mathcal{B}(f, S)$ such that $b^+ \cap b^- = \emptyset$, and that $B^\pm$ is the $\delta$-collar of $b^\pm$ where $\delta > 0$ is a constant such that $B^+ \cap B^- = \emptyset$. Then there is a block $B_0 > B$ in $\mathcal{B}(f, S)$ and a neighborhood $\Phi$ of $f$ in $F$ such that

(a) $f \in \Phi$ implies there are blocks $B_0$ and $B_0$ for $f$, collars $B_0 \pm$ of $b_0 \pm$ (rel $f$) and a collar $B_0^+$ of $b_0^+$ so that the following inclusions hold:

$$ (B_0^-, B_0^+) \subset (B_0, B_0^-) \subset (B_1^-, B_1^+) \subset (B, B^\pm), $$

(b) $\sigma(f, B_0) = \sigma(f, B_0)$.

Note. We further restrict $\Phi$ so that $B$ is an isolating neighborhood for all flows in $\Phi$ and so that if $p \in \text{cl}(B - B_0)$, $f \in \Phi$, then $f(p, t) \notin B$ for some $t \in \mathbb{R}$. Then $\sigma(f, B_0) = \sigma(f, B)$.

Lemma 6.7. The inclusions in 6.6(a) induce homotopy equivalences $B_0 | B_0^+ \rightarrow B_0 | B_0^+ \rightarrow B_1 | B_1^+ \rightarrow B | B^+.$

Proof. We have the following diagram induced from 6.6(a):

$$
\begin{array}{c}
\begin{array}{c}
B | B^+ \leftarrow B_1 | B_1^+ \\
\nearrow \hspace{2cm} \searrow \\
\end{array}
\begin{array}{c}
\begin{array}{c}
B_0 | B_0^+ \leftarrow B_2 | B_2^+ \\
\uparrow \hspace{2cm} \downarrow \\
\end{array}
\end{array}
\end{array}
$$
From 4.4, we have that \( i \) and \( \tilde{i} \) are homotopy equivalences. The Triangle Lemma implies that 6.7 is proved if we show that \( k \) is a homotopy equivalence. We show \( k^{-1} = \tilde{i}^{-1}k\tilde{i}^{-1} \) is a homotopy inverse of \( k \):

\[
k^{-1}k = k^{-1}(\tilde{i}j) \sim \tilde{i}^{-1}\tilde{k}j = \tilde{i}^{-1} \sim \text{identity};
\]
and \( kk^{-1} = j_1ik^{-1} \sim j_1k\tilde{i}^{-1} = ii^{-1} \sim \text{identity}.\]

**Corollary 6.8.** Suppose \( B, B^\pm \) satisfy the hypotheses of 6.6. Then there is a neighborhood \( \mathcal{U} = \mathcal{U}(B, B^+) \subseteq \mathcal{S} \) of \((f, S)\) with the property that \((\tilde{f}, \tilde{S}) \in \mathcal{U}\) implies there is a block \( \tilde{B} \in \mathcal{B}(\tilde{f}, \tilde{S}) \) such that \((B, b^\pm) \subseteq (B, B^\pm)\), and these inclusions induce homotopy equivalences \( \tilde{B}/\tilde{b}^\pm \to B/B^\pm \).

**Proof.** \( \mathcal{U} = \text{Im}(\gamma[B] \mid \Phi) \) where \( \Phi \) is defined in the Note of 6.6. The Corollary now follows from 6.7.

**Corollary 6.9.** If \((f, S) \) and \((f', S') \in \mathcal{U}(B, B^+)\), then \( i(f, S) = i(f', S') \).

**Theorem 6.10.** If \((f', S') \) is a continuation of \((f, S)\), then \( i(f, S) = i(f', S') \).

**Proof.** 6.9 implies that the set of all points of \( \mathcal{S} \) with a given index is open and contains all its boundary points. 6.10 follows from this.

### 7. Continuation of the Cohomology of an Invariant Set

In order to describe the behavior of an invariant set under perturbation, we use its cohomology algebra. There is a natural way to follow an element of this cohomology as the invariant set continues. The construction in this section models the construction of \( \mathcal{S} \).

The natural language to use is that of sheaves.

**Definition 7.1.** A continuous surjective map \( \pi: \mathcal{A} \to Z \) is a sheaf if for each \( \alpha \in \mathcal{A} \), there is a neighborhood \( \mathcal{U} \) of \( \alpha \) such that \( \pi \mid \mathcal{U} \) is a homeomorphism onto its range. Sometimes \( \pi \) is suppressed, and \( \mathcal{A} \) is said to be a sheaf over \( Z \).

If \( z \in Z \), \( \pi^{-1}(z) \) is the stalk over \( z \), denoted \( \mathcal{A}_z \). If \( Y \subseteq Z \), \( \mathcal{A} \mid Y = \bigcup_{y \in Y} \mathcal{A}_y \).

If \( \mathcal{A} \) and \( \mathcal{A}' \) are sheaves over \( Z \), then \( \mathcal{A} \times \mathcal{A}' \) is the sheaf over \( Z \) such that \( (\mathcal{A} \times \mathcal{A}')_z = \mathcal{A}_z \times \mathcal{A}'_z \) for all \( z \in Z \).

**Theorem 7.2.** \( \mathcal{S} \) is a sheaf over \( F \), and intersection is a continuous operation from \( \mathcal{S} \times \mathcal{S} \) to \( \mathcal{S} \).

The proof is straightforward and follows from 6.3.
Definition 7.3.  $\mathcal{H}[x] = \{(f, S, \sigma) \mid (f, S) \in \mathcal{S}, \sigma \in H^*(S)\}$.

Definition 7.4.  For each closed $N \subset X$, and each $\nu \in H^*(N)$, define $\gamma(N, \nu) : \text{Im}(\gamma[N]) \to \mathcal{H}[x]$ by $\gamma(N, \nu)(f, S) = (f, S, j^*\nu)$ where $j : S \subset N$.  $\gamma(N, \nu)$ is said to be a section over $(f, S)$ or through $(f, S, \sigma)$.

The set of isolating neighborhoods for $(f, S)$ is directed by inclusion, so we can form the direct system $\{H^*(N), \rho[N, N']\}$ where $\rho$ is induced by inclusion $N' \subset N$. The direct limit of this system is $\{H^*(S), \rho[N, S]\}$ where $\rho[N, S]$ is induced by $S \subset N$. In particular, (1) if $\sigma \in H^*(S)$, then there is an isolating neighborhood $N$ and $\nu \in H^*(N)$ such that $\rho[N, S]\nu = \sigma$, and (2) if $\nu \in H^*(N)$ and $\rho[N, S]\nu = 0$, there is an isolating neighborhood $N' \subset N$ such that $\rho[N, N'](\nu) = 0$.

Lemma 7.5.  Suppose $(f, S, \sigma) \in \gamma(N_1, \nu_1)(\mathcal{U}_1) \cap \gamma(N_2, \nu_2)(\mathcal{U}_2)$ where $\mathcal{U}_1$ and $\mathcal{U}_2$ are neighborhoods of $(f, S)$ in $\mathcal{S}$. Then there is a neighborhood $\mathcal{U}$ of $(f, S)$ in $\mathcal{S}$ such that $\gamma(N_1, \nu_1) | \mathcal{U} = \gamma(N_2, \nu_2) | \mathcal{U}$.

Proof.  It follows from (2) that we can choose $N \subset N_1 \cap N_2$ such that $\rho[N_1, N]\nu_1 = \rho[N_2, N]\nu_2 = \nu$. Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \text{Im}(\gamma[N])$. Then, on $\mathcal{U}$ each of the sections agrees with $\gamma[N, \nu] | \mathcal{U}$.

Theorem 7.6.  $\mathcal{H}[x]$ is a sheaf over $\mathcal{S}$ with a basis for the topology being $\{\text{Im}(\gamma[N, \nu]) | \mathcal{U} \mid N \text{ closed}, \nu \in H^*(N), \text{ and } \mathcal{U} \text{ open in } \mathcal{S}\}$. The binary operations of addition and cup product are continuous.

The proof is straightforward from 7.5.

Remark 7.7.  $\mathcal{H}[x]$ is also a sheaf over $F$ and further continuous binary operations can be defined on the stalks over $F$, such as the following:

$$(f, S_1, \sigma_1) \ast (f, S_2, \sigma_2) = (f, S_1 \cap S_2, j_1^*\sigma_1 + j_2^*\sigma_2)$$

where $j_i$ is induced by the inclusions $S_1 \cap S_2 \subset S_i (i = 1, 2)$.

8. The Exact Sequence of Sheaves

Here we construct the index sheaf and the sheaf involving $H^*(a^+)$. In the smooth case, we use 3.7(d), and define topologies for them in the same way we did for $\mathcal{H}[x]$ except that isolating blocks must replace isolating neighborhoods. However, 3.7(d) is false in the compact-open topology, so we use 6.8 instead. Consequently, construction is slightly more complicated than the one given above but is essentially the same. In fact we can topologize
all the sheaves simultaneously; afterwards, it is a simple matter to use the
exact sequence 5.4(c) to define an exact sequence of sheaves.

Whenever we invoke 6.6 or 6.8, we assume that the hypotheses hold for
\((B, B^+)\), since 3.6 and 3.18(a) allow us to choose them in this way.

**Definition 8.1.** \(\mathcal{H}[\alpha] = \{(f, S, \alpha) \mid (f, S) \in \mathcal{S}, \alpha \in H^*(\iota^+(f, S))\},\)
\(\mathcal{H}[\alpha^+] = \{(f, S, \alpha) \mid (f, S) \in \mathcal{S}, \alpha \in H^*(\iota^+(f, S))\}.\)

**Definition 8.2.** Fix a point \((f, S) \in \mathcal{S}\), and choose a block \(B \in \mathcal{B}(f, S)\)
with \(b^+ \cap b^- = \emptyset\), and a collar \(B^+\) of \(b^+.\) Let \(\mathcal{U} = \mathcal{U}(B, B^+)\) be a neighborhood
of \((f, S)\) in \(\mathcal{S}\) as described in 6.8. Then for each \((f', S') \in \mathcal{U}\), we can choose
a block \(B' \in \mathcal{B}(f, S)\) such that the map \(B'/b^+ \to B/B^+\) induced by inclusion
is a homotopy equivalence. Recall that \(B/b^+ \to B/B^+\) is also a homotopy
equivalence. Thus, instead of using one block for an open set of flows, we
use a collection \(\{B'\}\), one block for each flow in \(\mathcal{U}\). These blocks, in the sense
of 6.8, are all the same.

We also make definition 8.2 when \(+\) is replaced by \(-\). In fact, all statements
in the remainder of this section remain true when \(+\) is replaced by \(-\).

Let \((\mathcal{B}, \mathcal{C}(B))\) denote one of \((\alpha, B), (\alpha^+, b^+), (\iota^+, (B, b^+)).\)

For each \(\beta \in H^*(\mathcal{C}(B))\), we define a function, called a section,
\(\gamma = \gamma[B, B^+, \beta, \{B'\}] : \mathcal{U} \to \mathcal{H}[\alpha]\)
by \(\gamma(f', S') = (f', S', \lim \beta')\) where the direct limit is taken over \(\mathcal{B}(f', S')\),
and where \(\beta' = \rho[B, B'][\beta]. \rho[B, B'] : H^*(\mathcal{C}(B)) \to H^*(\mathcal{C}(B'))\) is defined by
one of the compositions \(m^{-1} \circ n^{-1}\) in 5.1.

**Remark 8.3.** We have chosen \(\mathcal{U}\) so that \(n_2 \circ m^{-1}\) is always an isomorphism.
Consequently, the map from \(H^*(\iota^+(f, S))\) to \(H^*(\iota^+(f', S'))\) defined by
mapping \(\lim \beta\) to \(\lim \beta'\) is an isomorphism.

Of course, 8.3 was proven when we showed that the index is invariant under
continuation. However, it is important to show that if \(\alpha'\) is the image of an
element \(\alpha\) of \(H^*(\iota^+(f, S))\) under this isomorphism, then \((f', S', \alpha')\) continues
\((f, S, \alpha).\) This we do after showing that continuations are well defined.

Eventually we shrink the domain of \(\gamma\) to a smaller set \(\mathcal{V}\) which depends
only on \(B\) and \(B^+.\)

What we want now is the analog of 7.5, which we obtain in several steps.

**Lemma 8.4.** Let \((f, S), (f', S') \in \mathcal{S}\) and let \(B, \bar{B}\) be elements of \(\mathcal{B}(f, S)\)
and \(\mathcal{B}(f', S')\), respectively, with \(B \subset \bar{B}.\) Suppose \(\gamma = \gamma[B, B^+, \beta, \{B'\}]\) and
\(\gamma' = \gamma[B, B^+, \beta, \{B'\}]\), where \(B^+ \subset \bar{B}^+.\) Then, if \(\gamma(f, S) = \gamma(f', S)\), \(\gamma\) agrees
with \(\gamma'\) on the intersection of their domains.
Proof. We have the following commutative triangle:

\[
\begin{array}{c}
\text{H}^*(\mathcal{C}(B')) \\
\text{H}^*(\mathcal{C}(B)) \\
\text{H}^*(\mathcal{C}(B)) \leftarrow \text{H}^*(\mathcal{C}(B)).
\end{array}
\]

Since \( \tilde{\gamma}(f, S) = \gamma(f, S), j(B) = \beta \). Therefore, the image of \( \beta \) in \( \text{H}^*(\mathcal{C}(B')) \) is the same as the image of \( \tilde{\beta} \) in \( \text{H}^*(\mathcal{C}(B')) \). Since \( B' \) is the block in \( \{B'\} \) chosen for \( f' \), and \( f' \) is arbitrary, 7.11 follows.

The following shows the extent to which \( \gamma \) is independent of \( \{B'\} \).

**Lemma 8.5.** There is a neighborhood \( \mathcal{V} = \mathcal{V}(B, B^+) \) of \((f, S)\) in \( \mathcal{U}(B, B^+) \) such that \( \gamma_1 = \gamma(B, B^+, \beta, \{B_1\}) = \gamma_2 = \gamma(B, B^+, \beta, \{B_2\}) \) for all choices of collections \( \{B_1\} \) and \( \{B_2\} \).

Proof. To simplify notation, let \((\mathcal{C}, \mathcal{C}(B)) = (a^+, b^+)\). The proof of the other cases is similar to this one.

There is an open neighborhood \( \Phi \) of \( f \) in \( F \) such that \( f' \in \Phi \) implies \( \mathcal{C}^{-}(B^+, B, f') \subset B - (A^- \cup b^-) \). Let \( \mathcal{V} = \{(f', S') \in \mathcal{U} | f' \in \Phi \} \). Then \( \mathcal{V} \) depends only on \((B, B^+)\).

If \((f', S') \in \mathcal{V} \) and \( B_1', B_2' \in \mathcal{U}(f', S') \) such that \((B_i', B_i'^+) \subset (B, B^+) \) \((i = 1, 2)\), then there is a block \( B_3' > B_1', B_2' \) for \((f', S') \) such that \( b_3'^+ \subset b_1'^+ \cup b_2'^+ \) (3.22). Let \( B_i'^+ \) be a collar of \( b_i'^+ \) containing \( b_3'^+ \) and contained in \( B^+ \). \( B_i'^+ \) is a compact subset of \( B - (A^- \cup b^-) \), hence there is a collar \( \bar{B}^+ \subset B_1'^+ \cup B_2'^+ \) of \( b^+(\text{rel} f) \). The situation is summarized by the following inclusion induced diagram:

\[
\begin{array}{c}
\text{H}^*(B_1'^+) \xrightarrow{i} \text{H}^*(b_1'^+)
\\
\text{H}^*(a^+(f', S')) \xleftarrow{\text{H}^*(b_3'^+)} \text{H}^*(B^+) \xrightarrow{i} \text{H}^*(b^+).\n\end{array}
\]

The maps marked \( i \) are isomorphisms, hence invertible. It is easy to check that the top path and the bottom are each the same as the middle path from \( \text{H}^*(b^+) \) to \( \text{H}^*(a^+(f', S')) \). \( \gamma_1(f', S') \) takes \( \beta \) along the top path, \( \gamma_2(f', S') \) along the bottom: thus \( \gamma_2(f', S') = \gamma_1(f', S') \) for all \((f', S') \in \mathcal{V} \). This completes the proof.
Lemma 8.6. Let \( \tilde{\gamma} = \gamma[\bar{B}, B^+, \beta, \{B\}] \), where \( B \in \mathcal{B}(f, S) \). Suppose \((f, S) \in \mathcal{V}(\bar{B}, B^+) \), and let \( B \in \{B\} \) be the block chosen for \((f, S)\). Choose \( B^+ \subset B^+ \) a collar of \( b^+ \), and form the section \( \gamma = \gamma[\bar{B}, B^+, \beta, \{B\}] \) where \( \beta = \rho[\bar{B}, B]\). Then \( \gamma \) agrees with \( \tilde{\gamma} \) on
\[
\mathcal{V}(\bar{B}, B^+) \cap \mathcal{V}(B, B^+).
\]

Proof. 8.5 allows us to use the set \( \{B\} \) in place of \( \{B'\} \) to define \( \gamma \) over \( \mathcal{V}(B, B^+) \cap \mathcal{V}(B', B^+) \). The conclusion now follows from 8.4.

Here is the analog of 7.5:

Lemma 8.7. Any two sections \( \gamma_1, \gamma_2 \) agree over an open set of \( \mathcal{S} \). In other words, the set \( \{\text{Im}(\gamma | \mathcal{U}) \mid \gamma \text{ is a section, } \mathcal{U} \text{ open in } \mathcal{S} \} \) forms a basis for a topology on \( \mathcal{X} \).

Proof. Let \((f, S) \in \mathcal{S} \) such that \( \gamma_1(f, S) = \gamma_2(f, S) \). \( \gamma_1 \) and \( \gamma_2 \) could be defined using blocks for distinct points \((f_1, S_1), (f_2, S_2) \in \mathcal{S} \). However, 8.6 implies there are sections of the form \( \gamma_i = \gamma[B_i, B_i^+, \beta_i, \{B_i\}] \) \( i = 1, 2 \), where \( B_1 \) and \( B_2 \) are each blocks for \((f, S)\), such that \( \gamma_i = \gamma_i \) on a neighborhood of \((f, S)\) \( i = 1, 2 \). Choose \( B_3 > B_1, B_2 \); then, enlarging \( B_1^+, B_2^+ \) to contain \( B_3^+ \) if necessary (5.1a), choose \( B_3 \) a collar of \( B_3^+ \) in \( B_1^+ \cap B_2^+ \). Now form a section \( \gamma_3 = \gamma[B_3, B_3^+, \beta_3, \{B_3\}] \) where \( \beta_3 = \rho[B_1, B_3] \beta_1 = \rho[B_2, B_3] \beta_2 \). (We can choose \( B_3 \) small enough so that \( \beta_3 = \rho[B_1, B_3] \beta_i \), \( i = 1, 2 \), because \( \lim \beta_i = \lim \beta_2 \).) Finally, 8.6 implies that \( \gamma_1 \) and \( \gamma_2 \) agree with \( \gamma_3 \) on an open set containing \((f, S)\); from this it follows that \( \gamma_1 \) agrees with \( \gamma_2 \) on an open set, and the lemma is proved.

Theorem 8.8. \( \mathcal{X}(\mathcal{C}, \mathcal{C}), \mathcal{X}(\mathcal{C}), \) and \( \mathcal{X}(\mathcal{C}^+) \) are cohomology sheaves over \( \mathcal{S} \) with the topology given in 7.14, and the algebraic operations in the stalks are continuous.

Proof. \( \pi: \mathcal{X}(\mathcal{C}) \rightarrow \mathcal{S} \) defined by \( \pi(f, S, \alpha) = (f, S) \) has a local inverse \( \gamma[B, B^+, \beta, \{B\}] \) defined on \( \mathcal{V}(B, B^+) \) where \( B \in \mathcal{B}(f, S) \) and \( \lim \beta = \alpha \). That both maps are continuous and the operations are continuous follow directly from 8.7.

Corollary 8.9. The sheaf \( \mathcal{X}(\mathcal{C}^+) \) is locally a product; that is, if \((f, S) \in \mathcal{S}, \mathcal{X}(\mathcal{C}^+) \mid \mathcal{V}(B, B^+) \approx \mathcal{V}(B, B^+) \times \mathcal{X}(\mathcal{C}^+)_{(f, S)} \).

Proof. The sheaf isomorphism is defined on each stalk by the composition of isomorphisms: \( H^*(\mathcal{C}^+(f', S')) \approx H^*(B', b^+) \approx H^*(B, b^+) \approx H^*(\mathcal{C}^+(f, S)) \). Thus \((f', S', \lim \beta') \rightarrow (f, S, \lim \beta) \) where \( \beta' = \rho[B, B'] \beta \). It is easy to check that this map is continuous.
8.10. A sheaf \( \pi : \mathfrak{U} \rightarrow Z \) is Hausdorff on stalks if for each \( z \in Z, \alpha_1, \alpha_2 \in \mathfrak{U}_z \), there are open sets \( \mathcal{U}_1, \mathcal{U}_2 \) such that \( \alpha_i \in \mathcal{U}_i \) (\( i = 1, 2 \)), and \( \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset \).

\( \mathcal{U} \) is not Hausdorff, so \( \mathcal{H}\left[\mathfrak{U}\right] \) is not Hausdorff; however, since \( \mathcal{H}\left[\mathfrak{U}\right] \) is locally a product, it is Hausdorff on stalks.

**Definition 8.11.** The long exact sequence of sheaves,

\[
\mathcal{H}[a^\pm] \xrightarrow{\delta} \mathcal{H}[\epsilon^\pm] \rightarrow \mathcal{H} \rightarrow \mathcal{H}[a^\pm],
\]

is defined so that over each point \((f, S) \in \mathcal{U}\), the sequence is

\[
H^*(\mathfrak{U}(f, S)) \rightarrow H^*(\epsilon^\pm(f, S)) \rightarrow H^*(S) \rightarrow H^*(a^\pm(f, S)).
\]

**Theorem 8.12.** The sheaf maps in 8.11 are continuous; in fact, they are local homeomorphisms.

**Proof.** The proof is similar for each of the maps; we show only that \( \delta \) is continuous:

Let \( \text{Im} \gamma \) be a basis open set in \( \mathcal{H}[\epsilon^\pm] \), and let \((f, S, \alpha^\pm) \in \mathcal{H}[a^\pm] \) such that \( \delta(f, S, \alpha^\pm) = (f, S, \alpha) \in \text{Im} \gamma \). There is a block \( B \), and \( \beta \in H^*(B, \mathfrak{U}) \) such that \( \gamma = \gamma[B, \mathfrak{U}, \beta] \) agrees with \( \gamma \) on a neighborhood of \((f, S) \) in \( \mathcal{U} \).

There is a block \( B_0 > B \), and \( \beta_0^\pm \in H^*(\mathfrak{U}) \), such that if \( \beta_0 = \delta(B_0)(\beta_0^\pm) \), then \( \lim \beta_0^\pm = \alpha^\pm \), \( \lim \beta_0 = \alpha \), and \( \gamma_0 = \gamma[B_0, \beta_0^\pm, \beta] \) agrees with \( \gamma \) and \( \gamma_0 \) on a neighborhood \( \mathcal{U} \) of \((f, S) \) in \( \mathcal{U} \). Then \( \delta^{-1}(\text{Im} \gamma) \supset \text{Im}(\gamma[B_0, \beta_0^\pm, \beta] \mid \mathcal{U}) \), a neighborhood of \((f, S, \alpha^\pm) \) in \( \mathcal{H}[a^\pm] \). We have shown that \( \delta^{-1}(\text{Im} \gamma) \) contains a neighborhood of each of its points and is therefore open. It follows that \( \delta \) is continuous.

It is easy to check that any continuous sheaf map is a local homeomorphism, and the theorem follows.

9. Subsheaves and Sheaf Maps

In Section 5, we developed methods of determining from a given block cohomology elements related to all invariant sets which fit in this block. In this section we will fit these elements into the sheaf structures already established. In the next, we show how they determine some local properties of \( \mathcal{H}[x] \) and \( \mathcal{H}[a^\pm] \).

Fix a block \( B \) and a component \( c \) of \( \mathfrak{U}^+ \). Recall that \( a^\pm(f, S, c) \) is the homotopy type of \( a^\pm \cap c \). In the natural way, consider \( H^*(a^\pm \cap c) \) a direct summand of \( H^*(a^\pm) \). We want to form a subsheaf \( \mathcal{H}[a^\pm(c)] \) of \( \mathcal{H}[a^\pm] \) in a way similar to that used to form \( \mathcal{H}[a^\pm] \). However, one must first choose
Let \( B \in \mathcal{B}(f, S) \), and \( \epsilon \) open and closed in \( b^+ \), then define \( H^\kappa(\alpha^+(\epsilon, f, S)) \equiv \lim H^\kappa(c_0) \) where the direct limit is taken over all \( B_0 > B \) in \( \mathcal{B}(f, S) \) (see Definition 5.8). To define \( \mathcal{H}[\alpha^+(\epsilon)] \) over a neighborhood \( \mathcal{V} \subset \mathcal{P} \), the choice of \( B \) and \( \epsilon \) must be made for all \((f, S) \in \mathcal{V}\). Then \( \mathcal{H}[\alpha^+(\epsilon)] \equiv \{(f, S, \alpha) \mid (f, S) \in \mathcal{V}, \alpha \in H^\kappa(\alpha^+(\epsilon, f, S))\} \) depends on \( \{B\} \) and \( \{\epsilon\} \). To insure that \( \mathcal{H}[\alpha^+(\epsilon)] \subset \mathcal{H}[\alpha^+] \) is open, and therefore a subsheaf, we make the choices of \( \{B\} \) and \( \{\epsilon\} \) in the following way:

Let \((f, S) \in \mathcal{P}, B \in \mathcal{B}(f, S)\), and \( B^+ \) a collar of \( b^+ \). Fix \( \epsilon \) open and closed in \( b^+ \); define \( C \subset B^+ \) to be \( \partial^+(\epsilon, B^+, f) \), a collar of \( \epsilon \). For each

\[(f, S) \in \mathcal{V}(\bar{B}, \bar{B}^+) = \mathcal{V},\]

choose \( B \) as in 6.8, and let \( \epsilon \equiv b^+ \cap C \). Then \( \epsilon \) is open and closed in \( b^+ \).

**Definition 9.1.**

\[\mathcal{H}[\alpha^+(\epsilon)] \equiv \{(f, S, \alpha) \mid (f, S) \in \mathcal{V}, \alpha \in H^\kappa(\alpha^+(\epsilon, f, S))\}.\]

Here \( \mathcal{V} \subset \mathcal{V}(\bar{B}, \bar{B}^+) \) is a neighborhood of \((f, S) \) defined in the proof of 9.2.

**Theorem 9.2.** \( \mathcal{H}[\alpha^+(\epsilon)] \) is an open subset of \( \mathcal{H}[\alpha^+] \) and therefore a subsheaf.

The problem is that we must be able to continue a given open and closed subset of \( a^+ \) as a flow changes in such a way that the continuation of cohomology of a component is the cohomology of the continuation of a component. This cannot be done globally; for example: a closed path of flows may be defined by rotating the system

\[\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x,\]

a saddle point in the plane, gradually through 180° back to itself. (This is a homotopically nontrivial loop in \( F \).) If the unit disc is used for an isolating block for the initial (and final) flow, one sees that the two components of \( a^+ \) are interchanged by the rotation. Hence if the subsheaf \( \mathcal{H}[a_0] \) (where \( a_0 \) is one of the components of \( a^+ \)) is to be defined globally, it must contain \( H^\kappa(a^+ - a_0) \) also, which defeats the purpose of the construction.

**Proof.** We define \( \mathcal{V} \subset \mathcal{V}(B, B^+) \): let \( \Phi \subset F \) be a neighborhood of \( f \) such that \( f \in \Phi \) implies (3) \( \partial(B^+ - \bar{C}, \bar{B}, f) \cap \bar{C} = \emptyset \).

Then \( \mathcal{V} \equiv \{(f, S) \in \mathcal{V}(B, B^+) \mid f \in \Phi\}; \mathcal{V} \) depends only on \((B, B^+)\).

Let \((f, S, \alpha) \in \mathcal{H}[\alpha^+(\epsilon)]\). Choose \( B_0 > B \) in \( \mathcal{B}(f, S) \) such that there is \( \beta_0 \in H^\kappa(c_0) \) with \( \lim \beta_0 = \alpha \). (Recall that \( c_0 \equiv b_0^+ \cap \partial^+(\epsilon, B) \).) Choose a collar \( R_0^+ \) of \( b_0^+ \) and let \( C_0 = R_0^+ \cap \partial^+(c_0, B_0) \).
From (3) and the definition of $c_0$, it follows that

$$\partial^{-1}(\text{cl}(B_0^+ - C_0), \bar{B}, f) \cap C = \emptyset.$$  \hspace{1cm} (4)

Therefore, there is a neighborhood $\Phi_0 \subset \Phi$ of $f$ such that $f' \in \Phi_0$ implies (4) is true if $f'$ replaces $f$. Let $\mathcal{U} = \{(f', S') \in \mathcal{V} \cap \mathcal{V}(B_0, B_0^+) \mid f' \in \Phi_0\}$, a neighborhood of $(f, S)$ in $\mathcal{V}$. It follows that if $B_0' \in \mathcal{B}(f', S')$, $(f', S') \in \mathcal{U}$ such that $(B_0', b_0'^+) \subset (B_0, B_0^+)$, then $b_0'^+ \cap C_0 = \partial^+(b_0'^+ \cap C, B', f') \cap B_0^+$ where $(B', b_0^+) \subset (B, B^+)$. Therefore, $\text{Im}(\gamma[B_0, B_0^+, \beta_0, U]) \subset \mathcal{H}[\omega^+(\bar{c})]$.

Since $\mathcal{H}[\omega^+(\bar{c})]$ contains a neighborhood in $\mathcal{H}[\omega^+]$ of each of its points, it is open; and the proof is complete.

**Corollary 9.3.** The following are also open, continuous sheaf maps.

(a) $\delta[\bar{c}]: \mathcal{H}[\omega^+(\bar{c})] \to \mathcal{H}[\omega^+] | \mathcal{V}$ defined by $\delta[\bar{c}](f, S, \alpha) = \delta[c, f, S](\alpha)$.

(b) $D[\bar{c}]: \mathcal{H}[\alpha] | \mathcal{V} \to \mathcal{H}[\omega^+] | \mathcal{V}$ defined by $D[\bar{c}](f, S, \alpha) \equiv D[\bar{c}, f, S](\alpha)$.

The proof is similar to 8.12.

Observe that these sheaves over $\mathcal{S}$ are also sheaves over $F$.

**10. Stability and Bifurcation**

In Section 5 we saw that for a fixed $(f, S) \in \mathcal{S}$, $B \in \mathcal{B}(f, S)$, we could use certain homomorphisms to determine nontrivial elements of the cohomology of $\omega^+, \omega^+(c)$, or $S$. The elements so determined have the additional property that they persist under perturbations. This happens because these elements are determined from properties of a block and a block is stable (in the sense of 6.8 and 3.7(d)) under perturbations. We make this more precise.

**Definition 10.1.** If $\mathcal{H}$ is one of the cohomology sheaves defined in Sections 7 or 8, and $(f, S, \alpha) \in \mathcal{H}$, we say $(f, S, \alpha)$ is stable if either $\alpha = 0$ or there is a section $\gamma$ through $(f, S, \alpha)$ and a neighborhood $\mathcal{U}$ of $(f, S)$ in $\mathcal{S}$ such that $\gamma(f', S') = (f', S', \alpha')$ and $\alpha' \neq 0$ for all $(f', S') \in \mathcal{U}$. Otherwise, $(f, S, \alpha)$ is unstable. Equivalently, $(f, S, \alpha)$ is unstable if it cannot be separated from $(f, S, 0)$ by open sets.

**Proposition 10.2.** (a) The stable elements of $\mathcal{H}$ form an open dense subset of $\mathcal{H}$.

(b) Every element of $\mathcal{H}[\omega^+]$ is stable.

**Proof.** The set of unstable elements is exactly the point set boundary of $\{(f, S, 0) \in \mathcal{H} \mid (f, S) \in \mathcal{S}\}$, the zero section. The zero section is an open
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This proves (a). (b) follows from the fact that $\mathcal{H}[t^\pm]$ is locally a product.

**Proposition 10.3.** Suppose $\psi: \mathcal{H} \rightarrow \mathcal{H}'$ is a continuous sheaf homomorphism. If $\psi(f, S, \alpha_1), \psi(f, S, \alpha_2)$ can be separated by disjoint open sets, then so can $(f, S, \alpha_1), (f, S, \alpha_2)$. In particular, if $\psi(f, S, \alpha) \neq 0$ and is stable in $\mathcal{H}'$, then $(f, S, \alpha)$ is stable in $\mathcal{H}$.

*Proof.* Suppose $\mathcal{U}_1, \mathcal{U}_2$ separate $\psi(f, S, \alpha_1)$ and $\psi(f, S, \alpha_2)$. Then $\psi^{-1}(\mathcal{U}_1), \psi^{-1}(\mathcal{U}_2)$ separate $(f, S, \alpha_1), (f, S, \alpha_2)$. 10.3 follows.

**Corollary 10.4.** If $\psi: \mathcal{H} \rightarrow \mathcal{H}[t^\pm]$, and if $\psi(f, S, \alpha) \neq 0$, then $(f, S, \alpha)$ and all its continuations are stable.

*Proof.* 10.3 and 10.2(b) imply $(f, S, \alpha)$ is stable. 10.3 also implies that the global zero section $\gamma_0$ in $\mathcal{H}[t^\pm]$ is both open and closed, so $\psi^{-1}(\gamma_0)$ is also open and closed. Since $(f, S, \alpha) \notin \psi^{-1}(\gamma_0)$, the same holds for all continuations of $(f, S, \alpha)$. It follows that if $(f', S', \alpha')$ is a continuation of $(f, S, \alpha)$, then $\psi(f', S', \alpha') \neq 0$ and hence, by the first sentence, is stable. This proves 10.4.

**Corollary 10.5.** Let $B \in \mathcal{B}(f, S)$. Then

(a) Every nonzero element of $\text{Im } \delta[B^\pm]$ represents a nonzero stable element of $\mathcal{H}[\alpha^\pm]$.

(b) Let $c$ be open and closed in $B^\pm$. Then every nonzero element of $\text{Im } \delta[c]$ represents a nonzero stable element of $\mathcal{H}[\alpha^\pm(c)]$, and every nonzero element of $\text{Im } D[c]$ represents a nonzero stable element of $\mathcal{H}[\alpha^\pm(c)]$.

(c) Any continuation of an element represented as in (a) or (b) above is also nonzero and stable.

This follows from 10.4, 8.12 and 9.3.

In order to show that $Q$ (see 5.15) determines stable elements of $\mathcal{H}[\alpha]$, we gain examine the smooth case.

**Definition 10.6.** $Q: \mathcal{H}[t^\pm] \otimes \mathcal{H}[t^-] \rightarrow \mathcal{H}[\alpha]$ is defined by

$$Q(f, S, \alpha^+ \otimes \alpha^-) = Q(f, S)(\alpha^+ \otimes \alpha^-).$$

**Theorem 10.7.** $Q$ is a continuous open map of sheaves, so $\text{Im } Q$ is a subsheaf of $\mathcal{H}[\alpha]$. Furthermore, each element of $\text{Im } Q$ is stable in $\mathcal{H}[\alpha]$.

*Proof.* If $Q(f, S, \alpha^+ \otimes \alpha^-) = (f, S, \sigma)$, then there is a block $B \in \mathcal{B}(f, S)$ satisfying 3.7(a), (b), (c), and (d), such that there exist $\beta^\pm \in H^*(B, b^\pm)$ with $\beta = Q(B)(\beta^+ \otimes \beta^-)$ being mapped to $\sigma$ under inclusion $B \supset S$, and
\[ \lim \beta^\pm = \sigma^\pm. \]

Then \( Q^{-1}(\text{Im } y[B, \beta]) \supset \text{Im}(y[B, \beta^+] \mid \mathcal{U}) \otimes \text{Im}(y[B, \beta^-] \mid \mathcal{U}) \)

where \( \mathcal{U} \) is a neighborhood of \((f, S)\) in \( \mathcal{H} \) such that \( B \) is a block for all points in \( \mathcal{U} \). It follows that \( Q \) is continuous. Every continuous map of sheaves is a local homeomorphism. Thus each nonzero point of \( \text{Im } Q \) has a neighborhood \( \mathcal{U} \) on which an inverse \( Q^{-1}: \mathcal{H}' \rightarrow \mathcal{H} \) is defined. Each element of \( \mathcal{H}' \) is stable, so 10.3 implies each point of \( \mathcal{U} \) is also stable. This completes the proof.

To motivate our definition of a bifurcation point, we re-present Example 1.1 in a more precise way:

**Example 10.8.** Let \( P: \mathbb{R}^3 \rightarrow \mathbb{R} \) be defined by \( P(x, y, z) = z^3 - x(x^2 + y^2) \). Then \( \nabla P(x, y, z) = (-2zx, -2zy, 3x^2 - (x^2 + y^2)) \). The flow generated by the vectorfield \( \nabla P \) in the unit ball can be pictured by rotating Fig. 4a around the indicated axis. One can check that the unit ball is an isolating block \( B \), and \( b^+ \) is the union of a disc and an annulus on the unit sphere \( \partial B \). (See Fig. 4b.) One sees that \( S \) is an isolated degenerate rest point.

It is not hard to write down a perturbation of \( \nabla P \) whose orbits in the unit ball can be pictured by rotating Fig. 5a around the indicated axis. In this flow, the isolated rest point has bifurcated into a periodic orbit (Fig. 5b). \( B \) is still an isolating block for this flow.

Another perturbation of \( \nabla P \) yields orbits in \( B \) like those pictured in Fig. 6. Here the rest point has bifurcated into two hyperbolic rest points. Again \( B \) is a block.
Let \( f_\mu, \mu \in [-1, 1] \) be a path of flows in \( F \), where \( f_0 \) is generated by the vector field \( VP \), where for \( \mu < 0, f_\mu \) looks like Fig. 5b, and for \( \mu > 0, f_\mu \) looks like Fig. 6. Thus as \( \mu \) increases from \(-1\) to \(1\), the periodic orbit shrinks to a rest point which then splits into two hyperbolic rest points.

\( f_\mu \) can be lifted to \( \mathcal{S} \) to the path \((f_\mu, S(\mu))\) where \( S(\mu) = \delta(f_\mu, B) \). The bifurcation of the degenerate rest point \( S(0) \) into two hyperbolic rest points \( \{p_1(\mu), p_2(\mu)\} = S(\mu) (\mu > 0) \) is already reflected in the structure of \( \mathcal{S} \):

One computes from \( B \) that \( H^*(\iota^*(f_0, S(0))) = (0, \mathbb{R}, \mathbb{R}, 0) \). The homology
index for a hyperbolic critical point having stable manifold of dimension $r$ is easily seen (from 4.10) to be $(0, \ldots, 0, \mathbb{R}, 0, \ldots, 0)$ where $\mathbb{R}$ occurs in the $r$-th place. Thus $(f_\mu, p_\mu(\mu))$ is not a continuation of $(f(0), S(0))$ for any $\mu$ since the index is invariant under continuation. Thus the paths $(f_\mu, p_\mu(\mu))_{\mu > 0}$ in $\mathcal{S}$ cannot be defined for $\mu \to 0$.

The bifurcation of $S(0)$ into the periodic orbit is reflected in the structure of $\mathcal{H}[\mathcal{A}]$. Let $0 \neq \alpha(p)$ generate $H^1(S(p)) (\mu < 0)$ such that $(f_\mu, S(\mu), \alpha(p))$ is a path in $\mathcal{H}[\mathcal{A}]$. This path cannot be defined continuously for $\mu = 0$, for if it could, there would be an $\epsilon > 0$ and a block $\overline{B}$ for $(f_\mu, S(\mu)), -\epsilon \leq \mu \leq 0$, and $\overline{\beta} \in H^*(\overline{B})$ such that $\overline{\beta}$ is mapped to $\alpha(p)$ under each inclusion $S(\mu) \subset \overline{B}$. This cannot happen since then $\overline{B}$ must contain the continuum $\{S(\mu)\}_{\mu = -\epsilon, 0}$ which forces $\overline{\beta} = 0$.

The bifurcation occurring for $\mu \geq 0$ also appears in $\mathcal{H}[\mathcal{A}]$ in a similar way.

**Definition 10.9.** Let $\mathcal{H}$ be one of the previously defined cohomology sheaves over $\mathcal{I}$. Then $(f, S) \in \mathcal{I}$ is an $\mathcal{H}$-bifurcation point if there is a closed connected $\mathcal{C}^* \subset \mathcal{H}$ with projection $\mathcal{C} \subset \mathcal{I}$ such that $(f, S) \in \text{cl}(\mathcal{C}) \subset \mathcal{I}$.

In the example above, $(f_0, S(0))$ is an $\mathcal{H}[\mathcal{A}]$-bifurcation point because $(f_\mu, S(\mu), \alpha(p))_{\mu \in [-\epsilon, 0]}$ is a closed and connected subset of $\mathcal{H}[\mathcal{A}]$, but $(f_0, S(0))$ is a limit point of the projection.

The concepts of instability and bifurcation are complementary in that $(f, S, \alpha)$ unstable implies the stalks arbitrarily nearby are smaller. $(f, S)$ is a bifurcation point implies that stalks of points arbitrarily near $(f, S)$ are larger.

**Corollary 10.10.** $\mathcal{H} [\mathcal{I}]$ contains no bifurcation points.

This follows from the fact that $\mathcal{H}[\mathcal{I}]$ is locally a product.

**Proposition 10.11.** Suppose $\mathcal{I}_0 \subset \mathcal{I}$, $\pi: \mathcal{H} \to \mathcal{I}_0$, $\pi': \mathcal{H}' \to \mathcal{I}_0$ are sheaves and $\psi: \mathcal{H} \to \mathcal{H}'$ is a sheaf homomorphism. If $\mathcal{C}^*$ is closed and connected in $\mathcal{H}$, then $\pi'(\text{cl}(\psi(\mathcal{C}^*))) = \psi(\mathcal{C}^*)$ contains only $\mathcal{H}$-bifurcation points.

**Proof.** $\text{cl}(\pi'(\mathcal{C}^*)) = \pi'(\text{cl}(\psi(\mathcal{C}^*))) \supset \pi'(\text{cl}(\psi(\mathcal{C}^*))) = \psi(\mathcal{C}^*)$.

**Definition 10.12.** Such a bifurcation point is called a $\psi$-bifurcation point.

**Proposition 10.13.** In the notation of 10.11, assume $\mathcal{H}'$ is locally a product and has countable stalks. Then the $\psi$-bifurcation points are a subset of a nowhere dense countable union of closed subsets of $\mathcal{I}_0$.

Note that since $X$ is compact and metric the stalks of any of the cohomology sheaves are countable if coefficients are taken in the integers.
Proof. It is enough to prove 10.13 when $\mathcal{H}' = \mathcal{H}_0 \times \mathcal{H}_{(f,S)}'$ for some $(f,S) \in \mathcal{H}_0$. Let $\mathcal{H}_{(f,S)}'$ be $\{\alpha_i\}_{i=1}^\infty$, and define $\mathcal{U}_i = \{(f',S') \in \mathcal{H}_0 | \phi \mathcal{H}_{(f',S')}$ contains $\alpha_i\}$. Observe that each $\mathcal{U}_i$ is open and that the set of $\psi$ bifurcation points is contained in $\bigcup_i \partial \mathcal{U}_i$. Each $\partial \mathcal{U}_i$ is closed and nowhere dense and since $F$ is a Baire space, so is $\mathcal{F}$. It follows that $\bigcup_i \partial \mathcal{U}_i$ is a nowhere dense set.

**Example 10.14.** We show that 10.8 is an example of a $D[\varepsilon]$-bifurcation point. In Fig. 4b, one sees that $b^+$ is the union of two components. Denote one of them by $c$. Then the map $D[c]$ is defined over $\mathcal{H}[c] \cup \{(f_\mu, S(\mu))\}$ where $-1 \leq \mu \leq 1$, since $B$ is a block for all the pairs $(f_\mu, S(\mu))$.

Note that for each $0 < \mu < 1$, the orbit of the point $p = (0,0,-1) \in B$ leaves $B$ at the point $(0,0,+1) \in b^-$. Thus we can shave a small neighborhood $Y$ of $p$ to obtain a new block $B_\mu$, which is a torus. Let $c_\mu$ denote the remainder of $c$ after the shave. In Fig. 7 $c_\mu$ is the south polar cap and $S$ is a periodic orbit encircling the tube.

We shall compute, in Example 11.2, that in such a case

$$\text{Im}(D[c_\mu, f_\mu, S(\mu)]) \subset H^2(\varepsilon^+(f_\mu, S(\mu)))$$

has a nontrivial second cohomology (implying the existence of nontrivial first level cohomology of $H^*(S(\mu))$. Since $S(0)$ has no first order cohomology, it follows that $(f_\mu, S(\mu))_{\mu \geq 0}$ cannot be continuously defined for $\mu = 0$ or else the image of this path by $D[c]$ in $\mathcal{H}[\varepsilon^+]$ would be a path continuing nontrivial cohomology of $\text{Im} D$ to zero, contradicting 10.5(b). Thus $(f_0, S(0))$ is a $D[\varepsilon]$-bifurcation point.

In Example 11.2 we also find that $\text{Im}(\delta[c_\mu])$ contains nonzero second order cohomology. By arguments similar to the above, we see that $(f_0, S(0))$ is also a $\mathcal{H}(\alpha^+[c])$-bifurcation point.

![Fig. 7. A shave of $B$ containing the periodic orbit.](image-url)
11. Examples

Example 11.1. An isolating block $B$ for a hyperbolic rest point in two dimensions might look like the disc in Fig. 8. $b^+$ is the disjoint union of two intervals $c$, $d$. Note that $c$ is drawn so that $c \cap b^- = \emptyset$. One could draw it so that $c \cup d \cup b^- = \partial B$.

The rest point $S$, together with its stable manifold form $A^+$; $A^-$ is $S$ with its unstable manifold. Notice that $b^+$ is a strong deformation retraction (along orbits of the flow) of $B - A^-$, as 3.9 implies.

Example 11.2. Suppose a flow $f$ admits an isolating block $B$ which is a solid torus with the following structure:

We use polar coordinates on $B$; that is let

$$B' = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\},$$

and

$$B = \{(x, \psi) \mid x \in B', 0 \leq \psi \leq 2\pi\}$$

with the usual identifications making $R$ a solid torus.

Then $b^{+'} = \{(1, \theta) \in B' \mid \pi/4 \leq \theta \leq 3\pi/4$, or $5\pi/4 \leq \theta \leq 7\pi/4\}$, $b^- = \text{cl}\{(1, \theta) \in B' \mid (1, \theta) \notin b^{-}\}$, and $b^\pm = \{(x, \psi) \mid x \in b^{\pm}\}$. Thus $b^\pm$ is the disjoint union of two annuli each encircling the hole in the solid torus. $B$ is obtained by rotating $B'$ around the axis indicated in Fig. 9.

A block with such a structure can be found in 3 dimensions around a hyperbolic periodic orbit whose stable manifold has dimension 2.
One might ask the following questions concerning an arbitrary flow \( f \) having \( B \) for a block:

1. Must there be an invariant set \( S \) inside \( B \)?
2. Can \( S \) be a finite collection of rest points or must it encircle the hole as in the case of the hyperbolic periodic orbit?
3. What is the structure of \( a^+ \) and \( a^- \)?

In general, one might ask what properties one can deduce about \( f \) in \( B \) knowing information about \( f \) only on \( \partial B \).

The homotopy type of the space \( \overline{B} \) is that of \( S^2 \lor S^1 \) and thus

\[
H^*(B, b^+) = (H^0(B, b^+), H^1(B, b^+), \ldots) = (0, \mathbb{R}, \mathbb{R}, 0, \ldots).
\]

This implies that \( b^+ \) is not a strong deformation retraction of \( B \), and hence \( S \neq \varnothing \).

Let \( c \) denote one of the components of \( b^+ \); \( d \) the other. Since \( B \) retracts to \( d \), \( H^*(B, d) = (0, 0, \ldots) \). Therefore, \( \text{Im}(\delta[c]) \), which is the same as

\[
\ker(H^*(B, b^+) \to H^*(B, d)),
\]

is \( H^*(B, b^+) \); by symmetry \( \text{Im}(\delta[d]) = H^*(B, b^+) \). This implies that \( H^*(a^+ \cap c) \) and \( H^*(a^+ \cap d) \) must contain at least \((\mathbb{R}, \mathbb{R}, 0, 0, \ldots)\) (5.10); in particular, the component of \( a^+ \) inside \( c \) must encircle the hole in \( B \), and any path connecting the two boundary components of \( c \) must intersect \( a^+ \cap c \). By symmetry, similar statements hold for \( d \) and the components of \( b^- \).

By 5.14(b), \( \text{Im} D[c] = H^*(B, b^+) \). It follows from 5.13 that \( H^*(S) \) must
contain at least \((\mathbb{R}, \mathbb{R}, 0, 0,...)\). Therefore, \(S\) must encircle the hole in the torus, and if \(D\) is a disc in \(B\) whose boundary in \(B\) has the form \(\{(1, \theta, \psi) | 0 \leq \theta \leq 2\pi\}\) for some fixed \(\psi\), then \(D \cap S \neq \emptyset\). In particular, \(S\) cannot be a finite set of rest points.

One can also obtain this information using \(Q[B]\). Note that \((B, \partial B) \approx (B', \partial B') \times S^1\), and \((B, b^+) \approx (B', b^{+'}\) \times S^1\). Therefore, \(H^*(B, \partial B) \approx H^*(B', \partial B') \times H^*(S^1)\) and \(H^*(B, b^+) \approx H^*(B', b^{+'}) \times H^*(S^1)\). It follows from duality that there are elements \(\alpha^+\) and \(\alpha^-\) in \(H^1(B', b^{+'})\) and \(H^1(B', b^-)\) whose cup product is the generator \(\alpha\) of \(H^2(B', \partial B')\). Then the cup product of \(\alpha^+ \otimes 1\) in \(H^1(B, b^+)\) with \(\alpha^- \otimes 1\) in \(H^1(B, b^-)\) is the element \(\alpha \otimes 1\) in \(H^2(B, \partial B)\), where 1 denotes the generator of \(H^0(S^1)\). \(\alpha \otimes 1\) corresponds via duality to an element of \(H^1(B)\) which injects nontrivially into \(H^1(S)\) by inclusion (5.16).

The theorems of Section 10 imply these cohomology elements are all stable under perturbation.

**Example 11.3.** It is reasonable to call an isolated invariant set \(S\) for the flow \(f\) on an attractor (repellor) if it has an isolating block \(B\) with \(b^-(b^+)\) empty. In such a case, \(S\) has all the cohomology of \(B\) and this cohomology is stable. To see this, observe that the squeezes of \(B\) form set \(a\) of isolating blocks cofinal in \(\mathcal{B}(f, S)\), each one of which is a strong deformation retraction of \(B\). Therefore, \(H^*(S) = H^*(B)\). The perturbation theorem for blocks implies that all cohomology of \(H^*(B, b^-) = H^*(B)\) is stable.

In the smooth case, this is reflected in the map \(Q[B]\). \(H^*(B, b^+) \otimes H^*(B, b^-)\) reduces to \(H^*(B, \partial B) \otimes H^*(B)\), and \(H^*(B)\) has an identity element for cup product. Therefore, \(\text{Im}(Q[B]) = H^*(B)\), and 5.16 implies all the cohomology of \(B\) injects nontrivially into \(H^1(S)\). 10.7 implies these elements are stable.

Also note that in the case of an attractor, \(b^+ = a^+\).

In this example, \(Q\) tells us everything about \(S\) and \(D\) tells us nothing if \(b^+\) has only one component.

**References**