Two Relations Between Oblique and Λ-Orthogonal Projectors

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ABSTRACT

Let $\mathfrak{L}$ and $\mathfrak{M}$ be any complementary subspaces. In this article, two relations established by T. N. E. Greville between the projector $\mathbf{P}_{\mathfrak{L}|\mathfrak{M}}$ on $\mathfrak{L}$ along $\mathfrak{M}$ and the orthogonal projectors on $\mathfrak{L}$ and $\mathfrak{M}$ are generalized by admitting any Λ-orthogonal projectors, with Λ being a positive definite matrix. Also, two representations of Λ are found for which, given $\mathfrak{L}$ and $\mathfrak{M}$, Λ-orthogonal projectors on $\mathfrak{L}$ become identical with $\mathbf{P}_{\mathfrak{L}|\mathfrak{M}}$.

1. INTRODUCTION

Let $\mathfrak{L}$ and $\mathfrak{M}$ be any two complementary subspaces of the n-dimensional complex vector space $\mathcal{C}^n$, i.e., such that $\mathfrak{L}\cap\mathfrak{M}=\{\mathbf{0}\}$ and $\mathfrak{L}\oplus\mathfrak{M}=\mathcal{C}^n$, and let Λ be any $n\times n$ positive definite matrix. Further, let $\mathbf{L}$ and $\mathbf{M}$ denote any matrices whose columns span the subspaces $\mathfrak{L}$ and $\mathfrak{M}$, respectively. We denote by $\mathbf{P}_{\mathfrak{L}|\mathfrak{M}}$ the projector on $\mathfrak{L}$ along $\mathfrak{M}$, and by $\mathbf{P}_{\mathfrak{L};\Lambda}$ the Λ-orthogonal projector on $\mathfrak{L}$. The former is uniquely determined by the equations $\mathbf{P}_{\mathfrak{L}|\mathfrak{M}}\mathbf{L}=\mathbf{L}$ and $\mathbf{P}_{\mathfrak{L}|\mathfrak{M}}\mathbf{M}=\mathbf{0}$, while the latter admits the representation

$$\mathbf{P}_{\mathfrak{L};\Lambda}=\mathbf{L}(\mathbf{L}^*\Lambda\mathbf{L})^-\mathbf{L}^*\Lambda, \tag{1}$$

the superscripts "$*$" and "$-$" denoting the conjugate transpose and a g-inverse of the matrix, respectively. If $\Lambda=\mathbf{I}$, the identity matrix, then the term "orthogonal", instead of "I-orthogonal", and the symbol $\mathbf{P}_{\mathfrak{L}}$, instead of $\mathbf{P}_{\mathfrak{L};\mathbf{I}}$, are used throughout this paper.

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Greville [2] proved that $\mathbf{P}_{\mathcal{L}|\mathcal{M}}$ can be expressed in terms of $\mathbf{P}_{\mathcal{L}}$ and $\mathbf{P}_{\mathcal{M}}$ as

$$\mathbf{P}_{\mathcal{L}|\mathcal{M}} = (\mathbf{I}-\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{L}})^{-1}(\mathbf{I}-\mathbf{P}_{\mathcal{M}})$$

$$= \mathbf{P}_{\mathcal{L}}(\mathbf{P}_{\mathcal{L}}+\mathbf{P}_{\mathcal{M}}-\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{L}})^{-1}. \tag{2}$$

In Sec. 2 of the present paper it is shown that this result can be strengthened by replacing $\mathbf{P}_{\mathcal{L}}$ with $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ and $\mathbf{P}_{\mathcal{M}}$ with $\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}$, with any $\mathbf{\Lambda}$. Moreover, a counterexample is given to establish the impossibility of a further generalization of the result consisting in the simultaneous use of $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ and $\mathbf{P}_{\mathcal{M};\mathbf{K}}$ when $\mathbf{K}\neq\mathbf{\Lambda}$. In the next section of the paper, a relation between oblique and $\mathbf{\Lambda}$-orthogonal projectors is considered from another point of view. Namely, two formulae for positive definite $\mathbf{\Lambda}$'s are derived for which, given $\mathcal{L}$ and $\mathcal{M}$, the projector $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ is identical with $\mathbf{P}_{\mathcal{L}|\mathcal{M}}$.

It can be noted that the results of Sec. 2 are applicable in calculating $\mathbf{P}_{\mathcal{L}|\mathcal{M}}$ by the method of Greville when not $\mathbf{P}_{\mathcal{L}}$ and $\mathbf{P}_{\mathcal{M}}$ but $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ and $\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}$ are known for some $\mathbf{\Lambda}$. The results of Sec. 3, however, provide an alternative method for computing $\mathbf{P}_{\mathcal{L}|\mathcal{M}}$, viz. as a $\mathbf{\Lambda}$-orthogonal projector on $\mathcal{L}$ with a previously determined appropriate $\mathbf{\Lambda}$.

2. GENERALIZATION OF GREVILLE'S RESULT

The theorem of Greville [2], here quoted in (2), follows from the fact that the matrix $\mathbf{I}-\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{L}}$ is nonsingular whenever the subspaces $\mathcal{L}$ and $\mathcal{M}$ are disjoint. This is in fact a simple corollary from the result of Lent (cf. Ben-Israel and Greville [1, p. 200]) stating that the null space of $\mathbf{I}-\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{L}}$ is $\mathcal{L}\cap\mathcal{M}$. The lemma below shows that the statement is also true in a more general case where instead of orthogonal, $\mathbf{\Lambda}$-orthogonal projectors are used. It seems noteworthy that the lemma is proved without using explicitly the notion of the vector norm, which plays a critical role in the proof given by Lent.

Lemma. *Let $\mathcal{L}$ and $\mathcal{M}$ be any subspaces of $\mathcal{C}^n$, and $\mathbf{\Lambda}$ be any $n\times n$ positive definite matrix. Then the null space of $\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ is $\mathcal{L}\cap\mathcal{M}$.*

Proof. Let $\mathcal{N}$ stand for the null space of $\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$. If $\mathbf{x}\in\mathcal{L}\cap\mathcal{M}$, then $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}\mathbf{x}=\mathbf{x}$ and $\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{x}=\mathbf{x}$ regardless of $\mathbf{\Lambda}$. Hence $(\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}})\mathbf{x}=\mathbf{0}$ or $\mathbf{x}\in\mathcal{N}$. Thus, $\mathcal{L}\cap\mathcal{M}\subset\mathcal{N}$.

Conversely, if $\mathbf{x}\in\mathcal{N}$, then

$$\mathbf{x} = \mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}\mathbf{x}, \tag{3}$$

and therefore $\mathbf{x} \in \mathfrak{M}$. To prove that simultaneously $\mathbf{x} \in \mathfrak{L}$, premultiply (3) by $\mathbf{x}^*\boldsymbol{\Lambda}$ and utilize the fact that $\boldsymbol{\Lambda}\mathbf{P}_{\mathfrak{M};\Lambda}$ is Hermitian, which follows immediately from (1). Hence

$$(\boldsymbol{\Lambda}\mathbf{P}_{\mathfrak{M};\Lambda}\mathbf{x})^*\mathbf{P}_{\mathfrak{L};\Lambda}\mathbf{x} = \mathbf{x}^*\boldsymbol{\Lambda}\mathbf{x}.$$

But it has already been noted that $\mathbf{P}_{\mathfrak{M};\Lambda}\mathbf{x} = \mathbf{x}$. Therefore

$$\mathbf{x}^*\boldsymbol{\Lambda}(\mathbf{I}-\mathbf{P}_{\mathfrak{L};\Lambda})\mathbf{x} = \mathbf{0},$$

and then, since $\mathbf{I}-\mathbf{P}_{\mathfrak{L};\Lambda}$ is idempotent and $\boldsymbol{\Lambda}(\mathbf{I}-\mathbf{P}_{\mathfrak{L};\Lambda})$ is Hermitian,

$$\mathbf{x}^*(\mathbf{I}-\mathbf{P}_{\mathfrak{L};\Lambda})^*\boldsymbol{\Lambda}(\mathbf{I}-\mathbf{P}_{\mathfrak{L};\Lambda})\mathbf{x} = \mathbf{0}.$$

In view of the positive definiteness of $\boldsymbol{\Lambda}$, this is equivalent to $(\mathbf{I}-\mathbf{P}_{\mathfrak{L};\Lambda})\mathbf{x} = \mathbf{0}$ or to $\mathbf{x} \in \mathfrak{L}$. Thus, $\mathfrak{M} \subset \mathfrak{L} \cap \mathfrak{M}$, and the proof is completed. ■

Repeating the arguments of Greville [2, p. 831], but now with reference to the lemma above, we get the following

COROLLARY. *If $\mathfrak{L}$ and $\mathfrak{M}$ are disjoint subspaces of $\mathcal{C}^n$, then the matrix $\mathbf{I}-\mathbf{P}_{\mathfrak{M};\Lambda}\mathbf{P}_{\mathfrak{L};\Lambda}$ is nonsingular for any $n \times n$ positive definite $\boldsymbol{\Lambda}$.*

It is natural to ask if the result of the corollary remains true when the projectors involved are related to two different inner products, i.e., when $\mathbf{P}_{\mathfrak{M};\mathbf{K}}$, with $\mathbf{K} \neq \boldsymbol{\Lambda}$, is substituted for $\mathbf{P}_{\mathfrak{M};\Lambda}$. The following example disproves this conjecture.

Let $\mathbf{L}^* = (1 \;\; 1)$ and $\mathbf{M}^* = (1 \;\; 0)$. It is obvious that the subspaces $\mathfrak{L}$ and $\mathfrak{M}$ are disjoint. However, choosing

$$\boldsymbol{\Lambda} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

it follows from (1) that

$$\mathbf{P}_{\mathfrak{L};\Lambda} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_{\mathfrak{M};\mathbf{K}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and consequently,

$$\mathbf{I} - \mathbf{P}_{\mathfrak{M};\mathbf{K}}\mathbf{P}_{\mathfrak{L};\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

a singular matrix.

Now we are in a position to establish an extension of the theorem of Greville [2] quoted in (2).

THEOREM 1. *If $\mathcal{L}$ and $\mathcal{M}$ are complementary subspaces of $\mathcal{C}^n$, then*

$$\mathbf{P}_{\mathcal{L}|\mathcal{M}} = (\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}})^{-1}(\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}) \tag{4}$$

$$= \mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}(\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}+\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}})^{-1} \tag{5}$$

for any $n\times n$ positive definite matrix $\mathbf{\Lambda}$.

Proof. Let $\mathcal{L}^{\perp}$ denote the orthogonal complement of $\mathcal{L}$. It is obvious that

$$\mathbf{P}_{\mathcal{M}^{\perp}|\mathcal{L}^{\perp}}\mathbf{P}_{\mathcal{M}^{\perp};\mathbf{\Lambda}^{-1}} = \mathbf{P}_{\mathcal{M}^{\perp};\mathbf{\Lambda}^{-1}}. \tag{6}$$

Also, it can be verified that

$$\mathbf{P}_{\mathcal{M}^{\perp}|\mathcal{L}^{\perp}} = \mathbf{P}^{*}_{\mathcal{L}|\mathcal{M}} \quad \text{and} \quad \mathbf{P}_{\mathcal{M}^{\perp};\mathbf{\Lambda}^{-1}} = (\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}})^{*}.$$

Thus, taking conjugate transposes on both sides of (6) and utilizing the obvious equality $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L}|\mathcal{M}}=\mathbf{P}_{\mathcal{L}|\mathcal{M}}$, we get

$$(\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}})\mathbf{P}_{\mathcal{L}|\mathcal{M}} = \mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}.$$

From this (4) follows immediately, since the matrix $\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}}\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ is invertible due to the corollary given above.

Now, replace $\mathcal{L}$, $\mathcal{M}$ and $\mathbf{\Lambda}$ in (4) by $\mathcal{M}^{\perp}$, $\mathcal{L}^{\perp}$ and $\mathbf{\Lambda}^{-1}$, respectively, and take conjugate transposes on both sides. Then

$$\mathbf{P}_{\mathcal{L}|\mathcal{M}} = \mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}\{\mathbf{I}-(\mathbf{I}-\mathbf{P}_{\mathcal{M};\mathbf{\Lambda}})(\mathbf{I}-\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}})\}^{-1},$$

from which (5) results by elementary calculations. ■

3. DERIVATION OF Λ'S FOR WHICH $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}=\mathbf{P}_{\mathcal{L}|\mathcal{M}}$

In this section, two representations of a positive definite matrix $\mathbf{\Lambda}$ are found such that, given $\mathcal{L}$ and $\mathcal{M}$, the $\mathbf{\Lambda}$-orthogonal projector on $\mathcal{L}$ coincides with the projector on $\mathcal{L}$ along $\mathcal{M}$.

THEOREM 2. *Let* $\mathcal{L}$ *and* $\mathcal{M}$ *be any complementary subspaces of* $\mathcal{C}^n$. *Then, for any positive numbers* λ *and* μ,

$$\mathbf{\Lambda}_1 = \lambda \mathbf{P}^*_{\mathcal{L}|\mathcal{M}} \mathbf{P}_{\mathcal{L}|\mathcal{M}} + \mu \mathbf{P}^*_{\mathcal{M}|\mathcal{L}} \mathbf{P}_{\mathcal{M}|\mathcal{L}} \tag{7}$$

and

$$\mathbf{\Lambda}_2 = \lambda \mathbf{P}_{\mathcal{L}^\perp} + \mu \mathbf{P}_{\mathcal{M}^\perp} \tag{8}$$

are positive definite matrices such that $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}_1} = \mathbf{P}_{\mathcal{L}|\mathcal{M}} = \mathbf{P}_{\mathcal{L};\mathbf{\Lambda}_2}$.

Proof. It is easily seen from (1) that $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}}$ is the projector on $\mathcal{L}$ along $\mathcal{N}^\perp$, where $\mathbf{N} = \mathbf{\Lambda L}$. Hence it follows that $\mathbf{P}_{\mathcal{L};\mathbf{\Lambda}} = \mathbf{P}_{\mathcal{L}|\mathcal{M}}$ if and only if $\mathcal{N}^\perp = \mathcal{M}$, i.e., if $\mathbf{\Lambda}$ is a positive definite solution of the equation

$$\mathbf{M}^* \mathbf{\Lambda L} = \mathbf{0}. \tag{9}$$

It is a direct consequence of the properties of the projectors involved that $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, defined in (7) and (8), satisfy (9). Therefore, since they are obviously nonnegative definite matrices, it only remains to establish their nonsingularity. But, since $r(\mathbf{\Lambda}_1) = r([\mathbf{P}_{\mathcal{L}|\mathcal{M}} \vdots \mathbf{P}_{\mathcal{M}|\mathcal{L}}])$ and $r(\mathbf{\Lambda}_2) = r([\mathbf{P}_{\mathcal{L}^\perp} \vdots \mathbf{P}_{\mathcal{M}^\perp}])$, the nonsingularity follows from the fact that $\mathcal{L}$ and $\mathcal{M}$ are complementary subspaces of $\mathcal{C}^n$. ∎

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REFERENCES

1 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, New York, 1974.

2 T. N. E. Greville, Solutions of the matrix equation $XAX = X$ and relations between oblique and orthogonal projectors, *SIAM J. Appl. Math.* 26:828–832 (1974).

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