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Fixed Points of Compact Multifunctions

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The object of this note is to obtain two generalizations of the well-known fixed point theorem of Fan [1]. A slight modification of Fan's proof yields one; the second is then an easy corollary, which, though interesting, seems never to be mentioned in the literature. We conclude with a generalization of the minimax theorem.

Recall that a multifunction $F : X \to Y$ is a subset of $X \times Y$ with domain equal to X; equivalently, F is a point to set function assigning to each $x \in X$ a nonempty subset $F(x)$ of Y. F is upper semicontinuous (u.s.c.) if and only if the set $\{x \in X \mid F(x) \cap B \neq \varnothing\}$ is closed for each closed subset B of Y. Moreover, as is easily seen, if Y is a compact Hausdorff space, and if each value of F is closed, then F is u.s.c. if and only if F has closed graph, i.e., F is a closed subset of $X \times Y$.

We define a subset A of a locally convex space L to be *almost convex* if for any neighborhood V of 0, and for any finite set $\{w_1, ..., w_n\}$ of points of A there exist $z_1, ..., z_n \in A$ such that $z_i - w_i \in V$ for all i, and

$$\mathrm{co}\{z_1, ..., z_n\} \subset A.$$

THEOREM 1. *Let K be a nonvoid compact subset of a separated locally convex space L, and $G : K \to K$ be an u.s.c. multifunction such that $G(x)$ is closed for all z in K and convex for all x in some dense almost convex subset A of K. Then G has a fixed point.*

Proof. Let $\mathscr{V}$ be a local base of neighborhoods of 0 consisting of closed convex symmetric sets. For each $V \in \mathscr{V}$ let

$$F_V = \{x \in K \mid x \in G(x) + V\}.$$

To find a fixed point of G it is clearly sufficient (and necessary) to show $\cap\{F_V \mid V \in \mathscr{V}\} \neq \phi$. Since $F_U \cap F_V \supset F_{U \cap V}$ for all $U, V \in \mathscr{V}$, it is sufficient, by the compactness of K, to show that each F_V is closed and nonempty.

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So let $V \in \mathscr{V}$. Define multifunctions $G_V : K \to K$, $R_V : K \to K$ by

$$G_V(x) = (G(x) + V) \cap K,$$
$$R_V(x) = (x + V) \cap K, \qquad \text{if} \qquad x \in K.$$

Then $G_V = R_V \circ G$. Moreover, R_V is a closed subset of $K \times K$ since $R_V = \{(x, y) \in K \times K \mid y - x \in V\}$ and V is closed. Since K is compact, it follows that both R_V and G are u.s.c. Hence, G_V is u.s.c. and, in particular, is a closed subset of $K \times K$. Let Δ be the diagonal in $K \times K$. Then F_V is obtained by projecting the compact set $\Delta \cap G_V$ onto the domain of G_V. It follows that F_V is closed.

Now choose $z_1, \ldots, z_m \in A$ such that $K \subset \cup \{z_i + V \mid 1 \leqslant i \leqslant m\}$, and $C = \text{co}\{z_1, \ldots, z_m\} \subset A$. Define $H_V \subset C \times C$ by $H_V = G_V \cap (C \times C)$. For each $x \in C$, $H_V(x)$ is closed, convex (since $C \subset A$), and nonempty (since $G(x) + V$ contains some z_i). Moreover, H_V is a closed subset of $C \times C$ since G_V is closed. Thus H_V has a fixed point by Kakutani's fixed point theorem [2]. It belongs to F_V, which is thus not empty.

THEOREM 2. *Let T be a nonvoid convex subset of a separated locally convex space L. Let $F : T \to T$ be an u.s.c. multifunction such that $F(x)$ is closed and convex for all $x \in T$, and $F(T)$ is contained in some compact subset C of T. Then F has a fixed point.*

Proof. Without loss of generality, suppose L is complete (for the conditions on T and F remain unchanged in the completion of L). Let $A = \text{co}\, C$ and $K = \bar{A}$. Then K is compact, $A \subset T$, and $F(A) \subset C \subset A$. Let $H = F \cap (A \times A)$. Then H is a relatively closed subset of $A \times A$ and has the same values on A as F. Consider the relation $\bar{H} \subset K \times K$, with closure relative to $K \times K$. $\bar{H}$ is a multifunction from K to K, i.e., $\bar{H}^{-1}(K) = K$, since $\bar{H}^{-1}(K)$ is closed and contains A. Moreover, $\bar{H}(K) \subset C \subset A$ and $H = \bar{H} \cap (A \times A)$; so $\bar{H}(x) = H(x) = F(x)$ for all $x \in A$. Thus, by Theorem 1, $\bar{H}$ has a fixed point, say x, in K. But $x \in \bar{H}(x) \subset C \subset A$. So $x \in F(x)$.

We also conclude from Theorem 1 the following generalizations of Theorems 2 and 3 in [1].

THEOREM 3. *Let $\{L_\nu \mid \nu \in I\}$ be a family of separated locally convex spaces. For each $\nu \in I$, let A_ν be a dense almost convex subset of a compact subset K_ν of L_ν, let $A_\nu' = \Pi\{A_\lambda \mid \lambda \in I, \lambda \neq \nu\}$, and let $K_\nu' = \Pi\{K_\lambda \mid \lambda \in I, \lambda \neq \nu\}$. If $\{E_\nu \mid \nu \in I\}$ is a family of closed subsets of $K = \Pi\{K_\nu \mid \nu \in I\}$ such that the section $E_\nu(x_\nu') = \{x_\nu \in K_\nu \mid (x_\nu', x_\nu) \in E_\nu\}$ is convex for all $x_\nu' \in A_\nu'$ and nonempty for all $x_\nu' \in K_\nu'$, $\nu \in I$, then $\cap \{E_\nu \mid \nu \in I\} \neq \varnothing$.*

Proof. Each E_ν is a multifunction from K_ν' to K_ν. Define $F_\nu : K \to K_\nu$

by $F_\nu(x) = E_\nu(x_\nu')$, where x_ν' is the projection of x on K_ν'. Then F_ν is u.s.c. (being the composition of a continuous function and an u.s.c. multifunction), and consequently has closed graph.

Define $F : K \to K$ by $F(x) = \Pi\{F_\nu(x) \mid \nu \in I\}$. It is easy to check that F has closed graph, that $A = \Pi\{A_\nu \mid \nu \in I\}$ is almost convex, that $F(x)$ is convex for all $x \in A$, and that $F(x) \neq \varnothing$ for all $x \in K$. Thus, by Theorem 1, F has a fixed point. It belongs to each E_ν.

THEOREM 4. *Let K_1, K_2 be compact subsets of the separated locally convex spaces L_1, L_2, respectively, let A_1, A_2 be dense almost convex subsets of K_1, K_2, respectively, and let f be a continuous real-valued function on $K_1 \times K_2$. If for any $x_0 \in A_1$, $y_0 \in A_2$ the sets*

$$\{x \in K_1 \mid f(x, y_0) = \max_{\xi \in K_1} f(\xi, y_0)\}$$

and

$$\{y \in K_2 \mid f(x_0, y) = \min_{\eta \in K_2} f(x_0, \eta)\}$$

are convex, then

$$\max_{x \in K_1} \min_{y \in K_2} f(x, y) = \min_{y \in K_2} \max_{x \in K_1} f(x, y).$$

Proof. Using Theorem 3 with $I = \{1, 2\}$, the proof is the same as the proof of Theorem 3 in [1].

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