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A cohomological study of local rings of embedding codepth 3

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ABSTRACT

The generating series of the Bass numbers $\mu_R^i = \text{rank}_k \text{Ext}_R^i(k, R)$ of local rings R with residue field k are computed in closed rational form, in case the embedding dimension e of R and its depth d satisfy $e - d \leq 3$. For each such R it is proved that there is a real number $\gamma > 1$, such that $\mu_R^{d+i} \geq \gamma \mu_R^{d+i-1}$ holds for all $i \geq 0$, except for $i = 2$ in two explicitly described cases, where $\mu_R^{d+2} = \mu_R^{d+1} = 2$. New restrictions are obtained on the multiplicative structures of minimal free resolutions of length 3 over regular local rings.

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0. Introduction

The paper concerns cohomological invariants of commutative noetherian local rings. Let R be such a ring, \mathfrak{m} its maximal ideal, and let d denote the depth of R and e the minimal number of generators of \mathfrak{m} . The number $e - d$ is called the *embedding codepth* of R . It is equal to the length of a minimal free resolution F of \widehat{R} over P , where \widehat{R} is the \mathfrak{m} -adic completion of R and P a regular local ring of dimension e , for which there is an isomorphism $\widehat{R} \cong P/I$; such an isomorphism always exists, due to Cohen's Structure Theorem. For $c \leq 2$ the structure of F , and hence that of \widehat{R} , is determined by the Hilbert–Burch Theorem.

This paper is mostly concerned with rings of codepth 3, so we assume $c = 3$ for the rest of the introduction. There exist then integers $l \geq 2$ and $n \geq 1$, such that

$$F = 0 \longrightarrow P^n \xrightarrow{\partial_3} P^{n+l} \xrightarrow{\partial_2} P^{l+1} \xrightarrow{\partial_1} P \longrightarrow 0. \quad (1)$$

The maps ∂_i are known in a few cases only. Buchsbaum and Eisenbud described them in [14] for $l = 2$, and in [15] when R is Cohen–Macaulay with $l = 3$ or $n = 1$. Brown determined ∂_i for certain Cohen–Macaulay rings with $n = 2$; see [13].

The proofs of those theorems use the fact that F can be turned into a graded–commutative DG (that is, differential graded) algebra; see [15]. Such a structure is not unique in general, but the isomorphism class of the graded k -algebra

$$A = F \otimes_P k, \quad \text{where } k = R/\mathfrak{m}, \quad (2)$$

is an invariant of R . The possible isomorphism classes were determined by Weyman [33] in characteristic zero and by Avramov et al. [11] in general. The remarkable fact is that for fixed l and n there exist only finitely many possibilities for A , described explicitly by simple multiplication tables. These are reviewed in Section 1, along with other background material.

We are interested in classifying non-Gorenstein rings. A natural tool for the task is provided by the *Bass numbers* $\mu_R^i = \text{rank}_k \text{Ext}_R^i(k, R)$, which are positive for all $i > d$ when R is not Gorenstein, but vanish when it is. The *Bass series* $I_R^R(t) = \sum_{i \geq 0} \mu_R^i t^i$ offers a useful format for recording the Bass numbers of R .

As our first result, [Theorem 2.1](#), we obtain in closed form expressions

$$I_R^R(t) = \frac{f(t)}{g(t)} \quad \text{and} \quad P_k^R(t) = \frac{(1+t)^{e-1}}{g(t)} \quad \text{with } f(t), g(t) \in \mathbb{Z}[t], \quad (3)$$

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where $P_k^R(t) = \sum_{i \geq 0} \text{rank}_k \text{Tor}_i^R(k, k) t^i$ is the Poincaré series of k . That such expressions exist follows from [11], via [18], and $g(t)$ was computed in [4].

For the goals of this paper we need the precise form of $f(t)$ as well. In Section 2 the series $I_R^R(t)$ and $P_k^R(t)$ are computed in parallel. Work in [6,2] reduces the problem to finding $I_A^A(t)$ and $P_k^A(t)$ for the algebra A in (2). To compute these series we use a battery of change-of-rings results, which are analogs of known theorems over local rings. Translation to the context of graded-commutative k -algebras requires changes in statements and proofs; these are discussed in Appendix.

It has long been known that for Gorenstein rings l is even, see [32], and that R is Gorenstein if and only if A has Poincaré duality, see [8], so $n = 1$. Furthermore, R is complete intersection if and only if A is an exterior algebra, see [1], and then $l = c - 1$. In Theorem 3.1 we prove that membership in each one of the remaining classes imposes new restrictions on the numbers l and n . The arguments introduce ideas that have not been applied earlier in this context, such as utilizing the DG module structure of $\text{Hom}_P(F, P)$ over the DG algebra F from (1), and analyzing the growth of the Betti numbers of \widehat{R} over complete intersection quotient rings of P .

In the first three sections the focus is on the structure of rings of codepth 3. The last section is motivated by open problems on the behavior of Bass sequences of local rings in general. In the introduction of [17], Christensen, Striuli, and Veliche collect precise questions and give a comprehensive survey of earlier results.

Theorem 4.1 gives complete answers in codepth 3: When R is not Gorenstein

$$\mu_R^{d+i} \geq \gamma \mu_R^{d+i-1} \tag{4}$$

holds for some real number $\gamma > 1$ and every integer $i \geq 1$, with a single exception:

$$\mu_R^{d+2} = \mu_R^{d+1} = 2 \quad \text{when } \widehat{R} \cong P/(wx, wy, z) \tag{5}$$

and w is P -regular, x, y is a P -regular sequence, and z is $P/(wx, wy)$ -regular. In particular, we recover the asymptotic information known from earlier work: the Bass sequence of R eventually is either constant or grows exponentially, see [4]; it is unbounded when R is Cohen–Macaulay, but not Gorenstein, see Jorgensen and Leuschke [23]; if it is unbounded, then (4) holds for $i \gg 0$, see Sun [30].

Neither the inequalities in (4), nor the description of the exceptions in (5), are formal consequences of the rational expressions in (3). In fact, extracting information on the Taylor coefficients of a rational function from expressions for its numerator and denominator is classically known to be a very hard problem.

Our approach is to prove first that $\mu_R^{d+i} > \mu_R^{d+i-1}$ holds, with the exceptions in (5), by drawing on three distinct sources—the expressions of the coefficients of $f(t)$ and $g(t)$ from Theorem 2.1, the relations between those coefficients implied by Theorem 3.1, and certain growth properties of the Betti numbers of k that are satisfied whenever R is not complete intersection. Once the growth of the Bass sequence is established, Theorem 4.1 easily follows from results in [4,30].

Since no additional effort is involved, all the results in the paper are stated and proved for local rings of embedding codepth at most 3.

1. Background

In this paper we say that (R, \mathfrak{m}, k) is a local ring if R is a commutative noetherian ring, \mathfrak{m} its unique maximal ideal, and $k = R/\mathfrak{m}$. Recall the invariants

$$\text{edim}R = \text{rank}_k(\mathfrak{m}/\mathfrak{m}^2) \quad \text{and} \quad \text{depth}R = \inf\{i \in \mathbb{Z} \mid \mu_R^i \neq 0\}.$$

1.1. The following notation is fixed for the rest of the paper:

$$e = \text{edim}R, \quad d = \text{depth}R, \quad c = e - d, \quad \text{and} \quad h = \dim R - d.$$

We write K for the Koszul complex on a minimal set of generators of \mathfrak{m} . It is a DG algebra over R , so its homology is a graded algebra with $H_0(K) = k$. We set

$$A = H(K)$$

and fix notation for the ranks of some k -vector spaces associated with A :

$$\begin{aligned} l &= \text{rank}_k A_1 - 1 & p &= \text{rank}_k(A_1^2) \\ m &= \text{rank}_k A_2 & q &= \text{rank}_k(A_1 \cdot A_2) \\ n &= \text{rank}_k A_3 & r &= \text{rank}_k(\delta_2) \end{aligned}$$

where $\delta_2 : A_2 \rightarrow \text{Hom}_k(A_1, A_3)$ is defined by $\delta_2(x)(y) = xy$ for $x \in A_2$ and $y \in A_1$.

1.2. Let \widehat{R} denote the \mathfrak{m} -adic completion of R . Cohen’s Structure Theorem yields $\widehat{R} \cong P/I$ for some regular local ring (P, \mathfrak{p}, k) with $\dim P = e$; that is, $I \subseteq \mathfrak{p}^2$.

When I can be generated by a regular sequence R is said to be complete intersection; this property is independent of the choice of presentation, see [16, 2.3.4(a)].

Let F be a minimal free resolution of \widehat{R} over P and L be the Koszul complex on a minimal generating set of the maximal ideal of P . There are natural maps

$$K = R \otimes_R K \xrightarrow{\cong} \widehat{R} \otimes_R K \xrightarrow{\cong} \widehat{R} \otimes_P L \xrightarrow{\cong} F \otimes_P L \xrightarrow{\cong} F \otimes_P k; \tag{1.2.1}$$

the symbol \cong denotes a quasi-isomorphism. In particular, an equality

$$\text{rank}_k A_i = \text{rank}_P F_i \tag{1.2.2}$$

holds for every integer i . The Auslander–Buchsbaum Equality now yields

$$\max\{i \mid A_i \neq 0\} = c = \text{pd}_P \widehat{R}. \tag{1.2.3}$$

Krull’s Principal Ideal Theorem gives the inequalities below; the first equality is (1.2.2) for $i = 1$; the third one comes from the catenarity of the regular ring P :

$$l + 1 = \text{rank}_k(I/pI) \geq \text{height}_P(I) = e - \dim R = c - h \geq 0. \tag{1.2.4}$$

By definition, the second inequality becomes an equality if and only if R is regular. Since P is Cohen–Macaulay, the first inequality becomes an equality precisely when I is generated by a regular sequence; that is, when R is complete intersection.

When $c \leq 3$, the equality $\sum_{i \geq 0} (-1)^i \text{rank}_P F_i = 0$, (1.2.2) and (1.2.3) give

$$m = l + n. \tag{1.2.5}$$

The following classification is the starting point for our work and is used throughout the paper. As always, \bigwedge_k denotes the exterior algebra functor. The functors Σ and $\text{Hom}_k(-, \Sigma^3 k)$ and the construction \ltimes are defined below, in 1.5.

1.3. If $c \leq 3$, then up to isomorphism A is described by the following table, where B, C , and D are graded k -algebras, and W a graded B -module with $(B_+)W = 0$:

| Class | [range] | c | A | B | C | D |
|---------------------|-----------------|-----|---------------|---|--|----------------------|
| C (c) | $[c \geq 0]$ | c | B | $\bigwedge_k \Sigma k^c$ | | |
| S | | 2 | $B \ltimes W$ | k | | |
| T | | 3 | $B \ltimes W$ | $C \ltimes \Sigma(C/C_{\geq 2})$ | $\bigwedge_k \Sigma k^2$ | |
| B | | 3 | $B \ltimes W$ | $C \ltimes \Sigma C_+$ | $\bigwedge_k \Sigma k^2$ | |
| G (r) | $[r \geq 2]$ | 3 | $B \ltimes W$ | $C \ltimes \text{Hom}_k(C, \Sigma^3 k)$ | $k \ltimes \Sigma k^r$ | |
| H (p, q) | $[p, q \geq 0]$ | 3 | $B \ltimes W$ | $C \otimes_k D$ | $k \ltimes (\Sigma k^p \oplus \Sigma^2 k^q)$ | $k \ltimes \Sigma k$ |

No two algebras A in the table are isomorphic, and neither are any two algebras B .

The table is compiled as follows. If $c \leq 1$, then $A_i = 0$ for $i > c$ and $A_1 \cong k^c$, by (1.2.2) and (1.2.3), whence $A \cong \bigwedge_k \Sigma k^c$. If $c = 2$, then F is given by the Hilbert–Burch Theorem; an explicit multiplication on F , see [5, 2.1.2], yields $A \cong \bigwedge_k \Sigma k^2$ or $A \cong k \ltimes W$. When $c = 3$ the possible isomorphism classes of A are determined in [33, Proof of 4.1] when k has characteristic 0, and in [11, 2.1] in general.

In some cases, the class of a ring and its structure determine each other:

1.4. Let R be a local ring with $\text{edim}R - \text{depth}R = c \leq 3$.

1.4.1. The ring R is complete intersection of codimension c if and only if it is in **C**(c), as proved by Assmus [1, 2.7], see also [16, 2.3.11]; for such rings $l = c - 1$.

1.4.2. The ring R is Gorenstein, but not complete intersection, if and only if it is in **G**(r) with $l = r - 1$ and $n = 1$; for such rings l is even and $l \geq 4$.

Indeed, R is Gorenstein if and only if A has Poincaré duality, by Avramov and Golod [8], and then l is even, by Watanabe [32, Theorem]; alternatively, see [15, 2.1] or [16, 3.4.1].

1.4.3. The ring R is Golod if and only if it is in **S** or in **H**(0, 0).

By definition, R is Golod if and only if all Massey products of elements of A_+ are trivial. The binary ones are just ordinary products. Massey products of three or more elements have degree at least 4, and $A_i = 0$ for $i \geq 4$. Thus, R is Golod if and only if $A_+^2 = 0$. By 1.3, this occurs precisely for the rings in **S** or **H**(0, 0).

We recall a modicum of notation and facts concerning DG modules.

1.5. Let E be a DG algebra over a commutative ring S . We assume $E_i = 0$ for $i < 0$ and that E is *graded-commutative*, meaning that $xy = (-1)^{ij}yx$ holds for all $x \in E_i$ and $y \in E_j$ and $x^2 = 0$ when i is odd. The DG algebra E acts on its module M from the left. All differentials have degree -1 . A *morphism* of DG modules is a degree zero E -linear map that commutes with the differentials; if it induces isomorphisms in homology in all degrees, it is called a *quasi-isomorphism*.

For every $s \in \mathbb{Z}$, set $(\Sigma^s M)_j = M_{j-s}$ for $j \in \mathbb{Z}$. The identity maps on M_j define a bijective map $\zeta^s : M \rightarrow \Sigma^s M$ of degree s . Setting $\partial^{\Sigma^s M}(\zeta^s(m)) = (-1)^s \zeta^s(\partial^M(m))$ and $x\zeta^s(m) = (-1)^{is} \zeta^s(xm)$ for every $x \in E_i$ turns $\Sigma^s M$ into a DG E -module.

Recall that $\text{Hom}_S(M, \Sigma^s S)$ denotes the DG E -module with $\text{Hom}_S(M, \Sigma^s S)_j = \text{Hom}_S(M_{s-j}, S)$, differential $\partial(\mu)(m) = (-1)^{j+1} \mu \partial(m)$ for $\mu \in \text{Hom}_S(M, \Sigma^s S)_j$, and E acting by $(x\mu)(m) = (-1)^{ij} \mu(xm)$ for $x \in E_i$. Set $M^* = \text{Hom}_S(M, S)$.

The *trivial extension* $E \ltimes M$ is the DG algebra with underlying complex $E \oplus M$ and product $(x, m)(x', m') = (xx', xm' + (-1)^{ij} x'm)$ for $x' \in E_{i'}$ and $m \in M_j$.

1.6. Let E be a DG algebra over S , and let M and N be DG E -modules

Modules $\text{Tor}_i^E(M, N)$ and $\text{Ext}_E^i(M, N)$ over the ring S are defined for every integer i , see [9, Section 1]. If E is a ring, considered as a DG algebra concentrated in degree 0, and M and N are E -modules, treated as DG modules in a similar way, then these derived functors coincide with the classical ones. When k is a field $E \rightarrow k$ is a homomorphism of DG algebras, and the k -vector spaces $\text{Tor}_i^E(M, k)$ and $\text{Ext}_E^i(k, N)$ have finite rank for each i and vanish for $i \ll 0$, we set

$$P_N^E(t) = \sum_{i \in \mathbb{Z}} \text{rank}_k \text{Tor}_i^E(M, k) \ t^i \in \mathbb{Z}[[t]][[t^{-1}]]. \tag{1.6.1}$$

$$I_E^N = \sum_{i \in \mathbb{Z}} \text{rank}_k \text{Ext}_E^i(k, N) \ t^i \in \mathbb{Z}[[t]][[t^{-1}]]. \tag{1.6.2}$$

Every morphism of DG algebras $\varepsilon : E' \rightarrow E$ induces natural homomorphisms of S -modules $\text{Tor}_i^{E'}(M, N) \rightarrow \text{Tor}_i^E(M, N)$ and $\text{Ext}_E^i(M, N) \rightarrow \text{Ext}_{E'}^i(M, N)$ for each $i \in \mathbb{Z}$. These maps are bijective when ε is a quasi-isomorphism.

A *graded algebra* over S is a DG algebra with zero differential; a *graded module* over a graded algebra is a DG module with zero differential.

1.7. Let K be the Koszul complex K described in 1.1. The natural map $K \rightarrow k$ turns k into a DG K -module. From [2, 3.2] and [6, 4.1], respectively, we get

$$P_k^R(t) = (1 + t)^e \cdot P_k^K(t) \quad \text{and} \quad I_R^K(t) = t^e \cdot I_K^K(t).$$

If the resolution F in 1.2 has a structure of DG algebra over P , then the natural surjection $F_0 = P \rightarrow k$ turns k into a DG F -module. The maps in (1.2.1) then are morphisms of DG algebras, so we get isomorphisms of DG algebras

$$F \otimes_P k = H(F \otimes_P k) \cong A. \tag{1.7.1}$$

The invariance under quasi-isomorphisms of the DG derived functors in 1.6 gives

$$P_k^K(t) = P_k^A(t) \quad \text{and} \quad I_k^K(t) = I_A^A(t).$$

When $c \leq 3$ we have $F_i = 0$ for $i > 3$, see (1.2.3), so F supports a structure of DG algebra over P by Buchsbaum and Eisenbud [15, 1.3]; see also [5, 2.1.4]. Thus, in this case we have

$$P_k^R(t) = (1 + t)^e \cdot P_k^A(t). \tag{1.7.2}$$

$$I_R^K(t) = t^e \cdot I_A^A(t). \tag{1.7.3}$$

Techniques for computing Poincaré series and Bass series over graded algebras are presented in Appendix, along with a number of examples.

2. Bass series

Our goal in this section is to prove the following result.

Theorem 2.1. *Let (R, \mathfrak{m}, k) be a local ring, set $d = \text{depth}R$ and $e = \text{edim}R$, and let l, n, p, q , and r be the numbers defined in 1.1. When $e - d = c \leq 3$ there are equalities*

$$P_k^R(t) = \frac{(1 + t)^{e-1}}{g(t)} \quad \text{and} \quad I_R^K(t) = t^d \cdot \frac{f(t)}{g(t)}$$

where $f(t)$ and $g(t)$ are polynomials in $\mathbb{Z}[t]$, listed in the following table:

| Class | $g(t)$ | $f(t)$ |
|----------------|------------------------------------|---|
| C(c) | $(1 - t)^c(1 + t)^{c-1}$ | $(1 - t)^c(1 + t)^{c-1}$ |
| S | $1 - t - lt^2$ | $l + t - t^2$ |
| T | $1 - t - lt^2 - (n - 3)t^3 - t^5$ | $n + lt - 2t^2 - t^3 + t^4$ |
| B | $1 - t - lt^2 - (n - 1)t^3 + t^4$ | $n + (l - 2)t - t^2 + t^4$ |
| G(r) | $1 - t - lt^2 - nt^3 + t^4$ | $n + (l - r)t - (r - 1)t^2 - t^3 + t^4$ |
| H(0, 0) | $1 - t - lt^2 - nt^3$ | $n + lt + t^2 - t^3$ |
| H(p, q) | $1 - t - lt^2 - (n - p)t^3 + qt^4$ | $n + (l - q)t - pt^2 - t^3 + t^4$ |
| | $p + q \geq 1$ | |

All of the Poincaré series and a smattering of the Bass series above are known:

Remark 2.2. Since rings in **C(c)** are complete intersection, see 1.4.1, the formula for $P_k^R(t)$ is due to Tate [31, Theorem 6]; we have $I_R^R(t) = t^d$ because R is Gorenstein.

When R is in **G(r)** with $r = l + 1$ and $n = 1$, it is Gorenstein by 1.4.2. The formula for $P_k^R(t)$ then is due to Wiebe [35, Satz 9]; the other formula gives $I_R^R(t) = t^d$.

When R is Golod, $P_k^R(t)$ is given by Golod [19] and $I_R^R(t)$ by Avramov and Lescot [12]. In view of 1.4.3, this covers the rings R in **S** and **H(0, 0)**; for R in **S**, Scheja [28, Satz 9] computed $P_k^R(t)$ and Wiebe [35, Satz 8] calculated $I_R^R(t)$.

The formulas for $P_k^R(t)$ in the remaining cases were obtained in [4, 3.5].

In the proof that follows the series $P_k^R(t)$ and $I_R^R(t)$ are computed simultaneously and in a uniform manner. A separate calculation is needed for each class.

Proof of Theorem 2.1. By (1.7.2) and (1.7.3), it suffices to establish the equalities

$$\frac{1}{P_k^A(t)} = (1 + t) \cdot g(t) \quad \text{and} \quad \frac{I_A^A(t)}{P_k^A(t)} = t^{-c} \cdot (1 + t) \cdot f(t).$$

(Class **C(c)**). The formulas come from (A.3.1) and (A.4.1), respectively.

(Class **S**). The formulas come from (A.2.1) and (A.2.2), respectively.

(Class **T**). The exact sequence $0 \rightarrow \Sigma^2 k \rightarrow C \rightarrow C/C_{\geq 2} \rightarrow 0$ and formulas (A.1.1) and (A.1.3) give $P_{\Sigma(C/C_{\geq 2})}^C(t) = t(1 + t^3 P_k^C(t))$. Now (A.3.1) and (A.5.1) yield

$$\frac{1}{P_k^B(t)} = (1 - t^2)^2 \left(1 - t^2 \left(1 + t^3 \frac{1}{(1 - t^2)^2} \right) \right) = 1 - 3t^2 + 3t^4 - t^5 - t^6.$$

The isomorphism of k -algebras $B \cong E/E_{\geq 3}$ with $E = \bigwedge_k \Sigma k^3$ and (A.9.1) give

$$\frac{I_B^B(t)}{P_k^B(t)} = t^{-4} (1 - (1 - 3t^2 + 3t^4 - t^5 - t^6)) - t = 3t^{-2} - 3 + t^2.$$

Using formulas (A.5.1) and (A.8.1) we now obtain:

$$\begin{aligned} \frac{1}{P_k^A(t)} &= (1 - 3t^2 + 3t^4 - t^5 - t^6) - t((l - 2)t + (l + n - 3)t^2 + nt^3) \\ &= (1 + t)(1 - t - lt^2 - (n - 3)t^3 - t^5). \end{aligned}$$

$$\begin{aligned} \frac{I_A^A(t)}{P_k^A(t)} &= (3t^{-2} - 3 + t^2) + ((l - 2)t^{-1} + (l + n - 3)t^{-2} + nt^{-3}) \\ &= t^{-3}(1 + t)(n + lt - 2t^2 - t^3 + t^4). \end{aligned}$$

(Class **B**). Using (A.1.1), (A.1.3) and (A.3.1) we obtain

$$P_{\Sigma C_+}^C(t) = t \cdot P_{C_+}^C(t) = t \cdot t^{-1} \cdot (P_k^C(t) - 1) = \frac{1}{(1 - t^2)^2} - 1 = \frac{2t^2 - t^4}{(1 - t^2)^2}.$$

From formulas (A.3.1) and (A.5.1) we now get

$$\frac{1}{P_k^B(t)} = (1 - t^2)^2 - t(2t^2 - t^4) = 1 - 2t^2 - 2t^3 + t^4 + t^5.$$

Assume, for the moment, that there is an exact sequence of graded B -modules

$$0 \longrightarrow \Sigma^{-2}B_+ \oplus \Sigma^{-1}k \longrightarrow \Sigma^{-2}B \oplus \Sigma^{-3}B \longrightarrow B^* \longrightarrow 0 \tag{2.2.1}$$

where $B^* = \text{Hom}_k(B, k)$. We then have a string of equalities, where the first one comes from (A.1.2), and the second one from (A.1.1) and (A.1.3) applied to (2.2.1):

$$\begin{aligned} \frac{I_B^B(t)}{P_k^B(t)} &= \frac{P_{B^*}^B(t)}{P_k^B(t)} \\ &= \frac{t(t^{-2} \cdot t^{-1} \cdot (P_k^B(t) - 1) + t^{-1} \cdot P_k^B(t)) + t^{-3} + t^{-2}}{P_k^B(t)} \\ &= 1 + t^{-2} + t^{-3} \cdot \frac{1}{P_k^B(t)} \\ &= t^{-3} + t^{-2} - 2t^{-1} - 1 + t + t^2. \end{aligned}$$

Formulas (A.5.1) and (A.8.1) now yield:

$$\begin{aligned} \frac{1}{P_k^A(t)} &= (1 - 2t^2 - 2t^3 + t^4 + t^5) - t((l - 1)t + (l + n - 3)t^2 + (n - 1)t^3) \\ &= (1 + t)(1 - t - lt^2 - (n - 1)t^3 + t^4). \\ \frac{I_A^A(t)}{P_k^A(t)} &= (t^{-3} + t^{-2} - 2t^{-1} - 1 + t + t^2) + ((l - 1)t^{-1} + (l + n - 3)t^{-2} + (n - 1)t^{-3}) \\ &= t^{-3}(1 + t)(n + (l - 2)t - t^2 + t^4). \end{aligned}$$

It remains to construct the sequence (2.2.1). To do this we use the module structures on suspensions and dual modules, described in 1.5. Recall from 1.3 that $B = C \times \Sigma C_+$, with $C = \bigwedge_k \Sigma k^2$. Choose a basis $\{a_1, a_2\}$ for C_2 . With $b_i = (-1)^i \zeta(a_i)$ for $i = 1, 2$, $a_3 = a_1 a_2$ and $b_3 = a_1 b_2$ the set $\mathbf{a} = \{1, a_i, b_i\}_{1 \leq i \leq 3}$ is a k -basis for B . The non-zero products of elements of \mathbf{a} are listed below:

$$a_1 a_2 = -a_2 a_1 = a_3 \quad \text{and} \quad a_1 b_2 = a_2 b_1 = b_1 a_2 = b_2 a_1 = b_3. \tag{2.2.2}$$

Let $\alpha \in (B^*)_{-2}$, respectively, $\beta \in (B^*)_{-3}$ be the k -linear map that sends a_3 , respectively, b_3 to 1, and the remaining elements of \mathbf{a} to 0. The map defined by

$$\pi(\zeta^{-2}(x), \zeta^{-3}(y)) = x\alpha - (-1)^i y\beta$$

for $x \in B_i$ and $y \in B_{i+1}$ is a morphism $\pi : \Sigma^{-2}B \oplus \Sigma^{-3}B \rightarrow B^*$ of graded B -modules. Its image contains the basis of B^* dual to \mathbf{a} , so π is surjective.

Set $U = \text{Ker}(\pi)$. The surjectivity of π implies $\text{rank}_k U = 7$. Set

$$u_j = (\zeta^{-2}(a_j), (-1)^j \zeta^{-3}(b_j)), \quad v_j = (\zeta^{-2}(b_j), 0), \quad w = (0, \zeta^{-3}(a_3)),$$

and $\mathbf{u} = \{u_j, v_j, w\}_{j=1,2,3}$. It is easy to see that \mathbf{u} is in U and is linearly independent over k . Thus, \mathbf{u} is a k -basis of U , so there is an isomorphism of vector spaces

$$v : \Sigma^{-2}B_+ \oplus \Sigma^{-1}k \xrightarrow{\cong} U$$

satisfying $v(a_j) = u_j$ and $v(b_j) = v_j$ for $j = 1, 2, 3$, and $v(1) = w$. Simple calculations, using (2.2.2), yield $v(bu) = bv(u)$ for all $b \in \mathbf{a}$ and $u \in \mathbf{u}$. This means that v is B -linear, and so validates the exact sequence (2.2.1).

(Class $\mathbf{G}(r)$). Formulas (A.6.1) and (A.6.2) give

$$\begin{aligned} \frac{1}{P_k^B(t)} &= 1 - rt^2 - rt^3 + t^5. \\ \frac{I_B^B(t)}{P_k^B(t)} &= t^{-3} - rt^{-1} - r + t^2. \end{aligned}$$

From formulas (A.5.1) and (A.8.1) we now obtain:

$$\begin{aligned} \frac{1}{P_k^A(t)} &= (1 - rt^2 - rt^3 + t^5) - t((l + 1 - r)t + (l + n - r)t^2 + (n - 1)t^3) \\ &= (1 + t)(1 - t - lt^2 - nt^3 + t^4). \\ \frac{I_A^A(t)}{P_k^A(t)} &= (t^{-3} - rt^{-1} - r + t^2) + ((l + 1 - r)t^{-1} + (l + n - r)t^{-2} + (n - 1)t^{-3}) \\ &= t^{-3}(1 + t)(n + (l - r)t - (r - 1)t^2 - t^3 + t^4). \end{aligned}$$

(Class $\mathbf{H}(p, q)$). When $p = 0 = q$ we have $A = k \ltimes W$, so (A.5.1) and (A.7.1) give

$$\frac{1}{P_k^A(t)} = 1 - t((l+1)t + (l+n)t^2 + nt^3) = (1+t)(1-t-lt^2-nt^3).$$

$$\frac{I_A^A(t)}{P_k^A(t)} = (l+1)t^{-1} + (l+n)t^{-2} + nt^{-3} - t = t^{-3}(1+t)(n+lt+t^2-t^3).$$

When $(p, q) \neq (0, 0)$, using (A.1.4), (A.2.1) and (A.2.2) we get

$$\frac{1}{P_k^B(t)} = (1-pt^2-qt^3) \cdot (1-t^2).$$

$$I_B^B(t) = \frac{qt^{-2} + pt^{-1} - t}{1-pt^2-qt^3} \cdot t^{-1} = \frac{qt^{-3} + pt^{-2} - 1}{1-pt^2-qt^3}.$$

From (A.5.1) and (A.8.1) we now obtain

$$\frac{1}{P_k^A(t)} = (1-pt^2-qt^3)(1-t^2) - t((l-p)t + (l+n-p-q)t^2 + (n-q)t^3)$$

$$= (1+t)(1-t-lt^2-(n-p)t^3+qt^4).$$

$$\frac{I_A^A(t)}{P_k^A(t)} = (qt^{-3} + pt^{-2} - 1)(1-t^2) + ((l-p)t^{-1} + (l+n-p-q)t^{-2} + (n-q)t^{-3})$$

$$= t^{-3}(1+t)(n+(l-q)t-pt^2-t^3+t^4).$$

These formulas gives the desired expressions for $P_k^A(t)$ and $I_A^A(t)$. \square

3. Classification

We significantly tighten the classification of rings R of embedding codepth 3, recalled in 1.3, by proving that membership in each one of the classes described there imposes non-trivial restrictions on the numerical invariants of R . Comparison with existing examples raises intriguing questions, discussed at the end of the section.

Theorem 3.1. *Let (R, \mathfrak{m}, k) be a local ring with $\text{edim}R - \text{depth}R = c \leq 3$. When R is not Gorenstein the invariants from 1.1 satisfy the following relations.*

| Class | c | $h \leq$ | $l \geq$ | $n \geq$ | p | q | r |
|---------------------|-----|----------|------------------------------|------------------------------|-----|-----|-----|
| S | 2 | 1 | $2 - h$ | $0 = n$ | 0 | 0 | 0 |
| T | 3 | 1 | $3 - h$ | 2 | 3 | 0 | 0 |
| B | 3 | 1 | $4 - h$ | $2 - h$ | 1 | 1 | 2 |
| G (r) | 3 | 1 | $\max\{4 - h, r + 1\}$ | $2 - h$ | 0 | 1 | r |
| H (p, q) | 3 | 2 | $\max\{3 - h, p, q + 1, 2\}$ | $\max\{2 - h, p - 1, q, 1\}$ | p | q | q |

The notation used in the theorem remains in force for the rest of the section.

Remark 3.2. The entries in the columns for $c, p, q,$ and r are read off directly from the description of the graded algebra A in 1.3.

Some numerical equalities determine the structure of the ring R .

Corollary 3.3. *Assume $c = 3$ and R is not complete intersection. The following conditions then are equivalent.*

- (i) $l = q + 1$.
- (ii) $l = p$ and $n = q$.
- (iii) R is in $\mathbf{H}(p, q)$ with $n = p - 1$.
- (iv) $\widehat{R} \cong P/J + zR$, where (P, \mathfrak{p}, k) is a regular local ring, J an ideal of P with $J \subseteq \mathfrak{p}^2$ and $\text{rank}_k(J/\mathfrak{p}J) = l \geq 2$, and z a P/J -regular element in \mathfrak{p}^2 .

When l or n is small the theorem is complemented by more precise results.

3.4. Let R be a local ring with $c = 3$.

3.4.1. If $l = 2$, then by Avramov [2, Proof of 7.2] one of the following cases occurs:

- (a) $h = 0$ and R is in $\mathbf{C}(3)$.
- (b) $h = 1$ and R is in $\mathbf{H}(2, 1)$ with $n = 1$, or in $\mathbf{H}(0, 0)$ or $\mathbf{H}(1, 0)$ with $n \geq 1$, or in $\mathbf{H}(2, 0)$ or \mathbf{T} with $n \geq 2$.
- (c) $h = 2$ and R is in $\mathbf{H}(0, 0)$ with $n \geq 1$.

3.4.2. If $l = 3$ and $h = 0$, then by Avramov [3, Proof of Thm. 2] R is in one of the classes:

- (a) $\mathbf{H}(3, 2)$ with $n = 2$.
- (b) \mathbf{T} with odd $n \geq 3$.
- (c) $\mathbf{H}(3, 0)$ with even $n \geq 4$.

3.4.3. If $l \geq 4, h = 0, n = 2$, and $p > 0$, then by Brown [13, 4.5] R is in one of the classes:

- (a) \mathbf{B} with even l .
- (b) $\mathbf{H}(1, 2)$ with odd l .

We start the proof of the theorem with some general considerations.

Lemma 3.5. Write \widehat{R} in the form P/I , with P regular and $\dim P = e$; see 1.2. When R is not complete intersection the following assertions hold.

- (1) $l \geq c - h = e - \dim R \geq 1$ hold.
- (2) $l = 1$ implies $c = 2$ and $h = 1$; furthermore, $I = (wx, wy)$ where w is an \widehat{R} -regular element and x, y is an \widehat{R} -regular sequence.
- (3) $h \leq 2$ holds; furthermore, $h = 2$ implies that R is in $\mathbf{H}(0, 0)$.
- (4) If R is not Gorenstein and $c = 3$, then $n \geq 2 - h$.

Proof. (1) Formula (1.2.4) gives $l + 1 > c - h = e - \dim R \geq 1$.

(2) When $l = 1$ the ideal I is minimally generated by two elements, see (1.2.4); say, $I = (u, v)$. As the regular local ring P is factorial, we have $u = wx$ and $v = wy$ with relatively prime x, y . The sequence x, y is regular, so $\text{pd}_P \widehat{R} = 2$. As \widehat{R} is not complete intersection, the element w is non-zero and not invertible, so $h = 1$.

(3) From (1) we see that $h \leq c - 1 \leq 2$ holds, and equality implies $e - \dim R = 1$. The ring R then is Golod by Avramov [5, 5.2.5], and so it is in $\mathbf{H}(0, 0)$ by 1.4.3.

(4) For any maximal R -regular sequence \mathfrak{x} standard results, see [16, 1.6.16], give

$$A_3 \cong ((\mathfrak{x} : \mathfrak{m}) / (\mathfrak{x})) \cong \text{Hom}_{R/(\mathfrak{x})}(k, R/(\mathfrak{x})) \cong \text{Ext}_R^d(k, R) \neq 0,$$

so $n = \mu_R^d \geq 1$. Now recall that R is Gorenstein if and only if $h = 0$ and $\mu_R^d = 1$. \square

It is proved in [11] that for several classes of local rings R , including those of embedding codepth at most 3, there is a complete intersection ring Q and a Golod homomorphism $Q \rightarrow R$. This was used to show that for every finite module M over such a ring, $P_M^R(t)$ represents a rational function with fixed denominator.

In the proof of the next lemma we turn the tables: By applying the formulas for $P_k^Q(t)$ and $P_k^R(t)$ from Theorem 2.1 we express the Betti numbers $\beta_i^Q(R)$ in terms of the numerical invariants of R , defined in 1.1, then use information on the asymptotic behavior of Betti numbers over complete intersections, obtained in [7].

Lemma 3.6. Set $\tau_R = 1$ for R in \mathbf{T} and $\tau_R = 0$ otherwise.

If $c = 3$ and R is not complete intersection, then the following dichotomy holds:

- (a) $l \geq q + 2$ and $n \geq p - \tau_R$, or
- (b) $l = q + 1$ and $n = p - 1 - \tau_R$.

Proof. Parts (1) and (2) of Lemma 3.5 imply $l \geq 2$ and $h \leq 2$. If $h = 2$, then Lemma 3.5(3) shows that R is in $\mathbf{H}(0, 0)$. When R is in $\mathbf{H}(0, 0)$ the first pair holds and the second fails. Until the end of the proof we assume $h \leq 1$ and $p + q \geq 1$.

We choose an isomorphism $\widehat{R} \cong P/I$, as in 1.2. By taking a close look at some arguments in [11], we set out to show next that I contains a regular sequence x, y , such that for $Q = P/(x, y)$ the induced map $Q \rightarrow \widehat{R}$ is a Golod homomorphism.

For R in \mathbf{T} such a sequence is found in the proof of [11, 6.1]. It is also shown there that if R is in $\mathbf{G}(r)$, \mathbf{B} , or $\mathbf{H}(p, q)$ with $p + q \geq 1$, then for some $x \in I$ and $\overline{P} = P/xP$ the map $\overline{P} \rightarrow R$ is Golod. As the ideal $\overline{I} = I/xP$ of \overline{P} has positive height, we can choose a minimal generator y of \overline{I} so that its image in \overline{R} is regular. The natural map from $Q = \overline{P}/y\overline{P}$ to R is Golod by Avramov et al. [11, 5.13]. By the definition of Golod homomorphisms, see Levin [25], the following equality then holds:

$$P_R^Q(t) = \frac{1}{t} \cdot \left(1 + t - P_k^Q(t) \cdot \frac{1}{P_k^R(t)} \right). \tag{3.6.1}$$

Inspecting the tabulated values of p and q , see Remark 3.2, we note that the various forms of $P_k^R(t)$ listed in Theorem 2.1 admit a uniform expression, namely,

$$P_k^R(t) = \frac{(1+t)^{e-1}}{1-t-lt^2-(n-p)t^3+qt^4-\tau_R t^5}. \tag{3.6.2}$$

Now Q is in $\mathbf{C}(2)$, so $P_k^Q(t)$ is given by Theorem 2.1, hence (3.6.1) and (3.6.2) yield

$$\begin{aligned} (1-t) \cdot P_R^Q(t) &= \frac{1-t}{t} \left(1+t - \frac{(1+t)^{e-2}}{(1-t)^2} \cdot \frac{1-t-lt^2-(n-p)t^3+qt^4-\tau_R t^5}{(1+t)^{e-1}} \right) \\ &= \frac{1-t}{t(1-t)^2(1+t)} \left((1-t^2)^2 - (1-t-lt^2-(n-p)t^3+qt^4-\tau_R t^5) \right) \\ &= \frac{1}{t(1-t^2)} \left(t + (l-2)t^2 + (n-p)t^3 - (q-1)t^4 + \tau_R t^5 \right) \\ &= \frac{1+(n-p)t^2+\tau_R t^4}{1-t^2} + \frac{(l-2)t-(q-1)t^3}{1-t^2} \\ &= 1+(l-2)t+(n+1-p)t^2 + \sum_{i=1}^{\infty} (l-1-q)t^{2i+1} + \sum_{i=1}^{\infty} (n+1-p+\tau_R)t^{2i+2}. \end{aligned}$$

The composite equality of formal power produces numerical equalities

$$\begin{aligned} l-1-q &= \beta_{2i+1}^Q(\widehat{R}) - \beta_{2i}^Q(\widehat{R}) \quad \text{for all } i \geq 1, \\ n+1-p+\tau_R &= \beta_{2i+2}^Q(\widehat{R}) - \beta_{2i+1}^Q(\widehat{R}) \quad \text{for all } i \geq 1. \end{aligned} \tag{3.6.3}$$

The ring Q being complete intersection, the sequence of Betti numbers of each finite Q -module is eventually either strictly increasing or constant, see [7, 8.1] or [5, 9.2.1(5)]. Thus, the left-hand sides of the equalities in (3.6.3) are either both positive or both equal to zero. This is just a rewording of the desired conclusion. \square

The proof of the next result, with its use of a DG module structure on a minimal P -free resolution of a dualizing complex for \widehat{R} , presents independent interest.

Lemma 3.7. *If $c = 3$ and R is not Gorenstein, then $l \geq r + 1$ holds.*

Proof. There is nothing to prove for R in \mathbf{T} , as then $r = 0$; see Remark 3.2.

By the same remark, rings in \mathbf{B} have $p = q = 1$ and $r = 2$. Case (b) in Lemma 3.6 then cannot hold, as it implies $n = 0$, and case (a) gives $l \geq 3 = r + 1$.

Rings R in $\mathbf{H}(p, q)$ have $r = q$, see Remark 3.2, and Lemma 3.6 gives $l \geq q + 1$.

For the rest of the proof we assume that R is in $\mathbf{G}(r)$. Thus, its Koszul homology algebra A has the form $A = B \rtimes W$, where B is a Poincaré duality k -algebra with

$$\text{rank}_k B_1 = r = \text{rank}_k B_2, \quad \text{rank}_k B_3 = 1, \quad B_1 \cdot B_1 = 0$$

and W is a graded B -module with $B_+ W = 0$. For every graded B -module N , set $N' = \text{Hom}_k(N, \Sigma^3 k)$ and endow this graded vector space with the natural B -module structure described in 1.5.

Choose $\beta \in (B')_0$ with $\text{Ker}(\beta) = B_{\leq 2}$. As B has Poincaré duality, the homomorphism of left graded B -modules $\alpha: B \rightarrow B'$ with $\alpha(1) = \beta$ is bijective; thus,

$$A' = B\beta \oplus W' \quad \text{and} \quad \alpha: B \cong B\beta \tag{3.7.1}$$

as graded B -modules, where B act on A -modules through the inclusion $B \subseteq A$.

As we may assume that R is complete, we fix a Cohen presentation $R \cong P/I$, a minimal resolution F of R as a P -module, a DG P -algebra structures on F ; see 1.2. Set $F' = \text{Hom}_P(F, \Sigma^3 P)$ and turn F' into a DG F -module, as in 1.5.

Using (1.7.1) to identify the graded algebras $F \otimes_P k$ and A , we get isomorphisms

$$F' \otimes_P k \cong \text{Hom}_P(F, k \otimes_P \Sigma^3 P) \cong \text{Hom}_k(F \otimes_P k, \Sigma^3 k) = A'$$

of graded A -modules. Choose $\xi \in F'_0$, so that these maps send $\xi \otimes 1$ to $(\beta, 0) \in A'$; see (3.7.1). The morphism $\phi: F \rightarrow F'$ of left DG F -modules with $\phi(1) = \xi$ satisfies

$$(\phi \otimes_P k)|_B = \alpha \quad \text{and} \quad (\phi \otimes_P k)|_W = 0. \tag{3.7.2}$$

Let Y denote the mapping cone of ϕ . We have $H_i(F) = 0$ for $i \geq 1$ by choice, and $H_i(F') = \text{Ext}_P^{3-i}(R, P) = 0$ for $i \geq 2$ because $h \leq 1$ holds by Lemma 3.5(3). The exact sequences $H_{i-1}(F) \rightarrow H_i(Y) \rightarrow H_i(F')$ now yield $H_i(Y) = 0$ for $i \geq 2$.

(Class **T**). Lemma 3.5(1) gives $l \geq 3 - h$. Rings in **T** have $q = 0$, so case (b) in Lemma 3.6 implies $l = 1$; since $c = 3$, this is ruled out by Lemma 3.5(2). Thus, the inequalities (a) of Lemma 3.6 hold, and they give $n \geq 2$.

(Class **B**). Lemma 3.5(4) gives $n \geq 2 - h \geq 1$, while Lemma 3.7 yields $l \geq r + 1 = 3$. By 3.4.2, the class **B** contains no ring with $h = 0$ and $l = 3$, so $l \geq 4 - h$ holds.

(Class **G**(r)). Here $p = 0$, so case (b) in Lemma 3.6 gives $n = -1$, which is absurd. Thus, case (a) holds, whence $l \geq 3$. By 3.4.2, in **G**(r) there are no rings with $h = 0$ and $l = 3$, hence $l \geq 4 - h$ holds. So does $l \geq r + 1$, by Lemma 3.7.

(Class **H**(p, q)). Parts (1) and (3) of Lemma 3.5 give $l \geq \max\{3 - h, 2\}$.

By definition, $A = (C \otimes_k D) \rtimes W$ with $C = k \rtimes (\Sigma k^p \oplus \Sigma^2 k^q)$, $D = k \rtimes \Sigma k$, and $C_+ W = 0 = D_+ W$. The relations $A_1 \supseteq C_1 \cong C_1 \otimes_k D_1 = A_1 \cdot A_1$ imply $l \geq p$, while $A_3 \supseteq A_1 \cdot A_2 = C_2 \otimes_k D_1 \cong C_2$ yield $n \geq q$. On the other hand, from Lemma 3.6 we obtain the inequalities $l \geq q + 1$ and $n \geq p - 1$. \square

Proof of Corollary 3.3. When $l = q + 1$ the values of q and bounds for l in Theorem 3.1 show that R is in **H**(p, q). Lemma 3.6 now gives $n = p - 1$, so (i) implies (iii).

If (iii) holds, then we have a string $l \geq p = n + 1 \geq q + 1 = l$, where the inequalities come from Theorem 3.1, the first equality is given by Lemma 3.6, and the second one holds by hypothesis. We get $l = p$ and $n = q$, which is (ii).

Assuming that (ii) holds, we see from Theorem 3.1 that R is in **H**(p, q). The description of A in 1.3 then yields $A \cong C \otimes_k D$ with $D = k \rtimes \Sigma k$. In particular, if a is a non-zero element in $1 \otimes_k D_1$, then A is free as a graded module over its subalgebra generated by a . Now [4, 3.4] shows that (iv) holds.

When (iv) holds $\text{Tor}_i^P(P/J, P/zP) = 0$ for $i \geq 1$, so we have isomorphisms

$$\begin{aligned} A &\cong \text{Tor}^P(P/(J + zP), k) \\ &\cong \text{Tor}^P(P/J, k) \otimes_k \text{Tor}^P(P/zP, k) \\ &\cong \text{Tor}^P(P/J, k) \otimes_k (k \rtimes \Sigma k) \end{aligned}$$

of graded k -algebras. They imply $\text{Tor}_2^P(P/J, k) \otimes_k \Sigma k \cong A_3$ and $\text{pd}_P(P/J) = 2$ the latter because $A_i = 0$ holds for $i > 3$ by (1.2.3). We get a string of equalities

$$q = \text{rank}_k A_3 = \text{rank}_k \text{Tor}_2^P(P/J, k) = \text{rank}_k(J/\mathfrak{p}J) - 1 = l - 1,$$

where the third one comes from (1.2.4) and (1.2.5). Thus, (iv) implies (i). \square

To complete the classification of rings R with $\dim R - \text{depth} R = 3$ along the lines of 1.3 and the results in this section, one needs to determine for those rings all the restrictions satisfied by the invariants in 1.1. This leads to:

Question 3.8. Which sextuples (h, l, n, p, q, r) , allowed by Theorem 3.1, Corollary 3.3, or the results cited in 3.4, are realized by some local ring R with $c = 3$?

The list of available answers is not long and runs as follows.

3.9. Let (P, \mathfrak{p}, k) be a regular local with $\dim P = e \geq 3$ and x_1, \dots, x_e a minimal set of generators of \mathfrak{p} . We describe rings $R = P/I$ with $c = 3$ by specifying I .

3.9.1. The rings admitted by 1.4.2, 3.4.1, 3.4.2, 3.4.3 are realized by ideals I constructed in [15, 6.2], [2, 7.7], [3, Rem. (1), p. 171], and [13, 3.4, 3.6], respectively.

3.9.2. The following sextuples (h, l, n, p, q, r) with $l = q + 1$ are realized:

- (a) $(0, 2, 1, 3, 1, 3)$ by $I = (x_1^2, x_2^2, x_3^2)$.
- (b) $(0, l, l - 1, l, l - 1, l - 1)$ by $I = (x_1, x_2)^{l-1} + (x_3^2)$ for each $l \geq 3$.
- (c) $(1, l, l - 1, l, l - 1, l - 1)$ by $x_1(x_1, x_2)^{l-1} + (x_3^2)$ for each $l \geq 2$.

There are no other sextuples with $l = q + 1$, by Corollary 3.3 and Lemma 3.5(3).

3.9.3. Every sextuple $(2, l, n, 0, 0, 0)$ with $l \geq 2$ and $n \geq 1$ is realized when $k = \mathbb{C}$.

Indeed, for each such pair (l, n) Weyman [34] shows that $P = \mathbb{C}[[x_1, \dots, x_e]]$ contains an ideal J with $\text{rank}_k \text{Tor}_1^P(P/J, k) = l + 1$ and $\text{rank}_k \text{Tor}_3^P(P/J, k) = n$. On the other hand, if w is a P -regular element, then P/J with $J = wI$ realizes the sextuple $(2, l, n, 0, 0, 0)$, since for each $i \geq 1$ there are isomorphisms of vector spaces

$$\text{Tor}_i^P(P/I, k) \cong \text{Tor}_{i-1}^P(I, k) \cong \text{Tor}_{i-1}^P(wI, k) \cong \text{Tor}_i^P(P/J, k),$$

and Shamash [29, Thm. (3), p. 467] shows that P/wI is Golod; see also [5, 5.2.5].

There are no other sextuples with $h = 2$, by Lemma 3.5(3).

The only known examples in **G**(r) are the Gorenstein rings. We propose:

Conjecture 3.10. If R is in **G**(r) for some $r \geq 2$, then R is Gorenstein.

Lemma 3.7 is a first step towards a verification of this statement. If proved in full, it will eliminate an entire family from the classification in Theorem 3.1.

Another elusive class is **B**, for which the only examples are those in [13]. Rings in **T** appear in several situations, and the families **H**(*p*, *q*) seem to be ubiquitous.

4. Bass numbers

The following theorem is the third main result of this paper.

Theorem 4.1. *Let (R, \mathfrak{m}, k) be a local ring, and set $e = \text{edim}R$ and $d = \text{depth}R$. When $e - d \leq 3$ and R is not Gorenstein there is real number $\gamma_R > 1$, such that*

$$\mu_R^{d+i} \geq \gamma_R \mu_R^{d+i-1} \quad \text{holds for every } i \geq 1, \tag{4.1.1}$$

with two exceptions for $i = 2$: If there exists an isomorphism

$$\widehat{R} \cong P/(wx, wy) \quad \text{or} \tag{4.1.2}$$

$$\widehat{R} \cong P/(wx, wy, z), \tag{4.1.3}$$

where (P, \mathfrak{p}, k) is an e -dimensional regular local ring, w a P -regular element, x, y a P -regular sequence, and z a $P/(wx, wy)$ -regular element in \mathfrak{p}^2 , then

$$\mu_R^{d+2} = \mu_R^{d+1} = 2.$$

In particular, when R is Cohen–Macaulay the inequalities (4.1.1) hold for all i .

The theorem should be viewed in the context of a number of problems raised in recent publications, sometimes under the hypothesis that R is Cohen–Macaulay. We say that a sequence (a_i) of real numbers is said to have *strongly exponential growth* if $\beta^i \geq a_i \geq \alpha^i$ hold for all $i \gg 0$ for some real numbers $\beta \geq \alpha > 1$.

Questions 4.2. Assume that (R, \mathfrak{m}, k) is a non-Gorenstein local ring.

- (1) Determine the number $\inf\{j \in \mathbb{Z} \mid \mu_R^{d+i} > \mu_R^{d+i-1} \text{ for all } i \geq j\}$. (See [17, 1.3].)
- (2) Does $\mu_R^{d+1} > \mu_R^d$ always hold? (See [23, 2.6].)
- (3) Does $\mu_R^i \geq 2$ hold for all $i > \dim R$? (See [17, 1.7].)
- (4) Does the sequence (μ_R^i) have strongly exponential growth? (See [23, p. 647].)

All of these questions are open in general. Here is a list of the known answers:

Remark 4.3. Assume that (R, \mathfrak{m}, k) is not Gorenstein.

- (1) An inequality $\mu_R^{d+i} > \mu_R^{d+i-1}$ holds for $i \geq 1$ in the following cases:
 - (a) $\mathfrak{m}^3 = 0$; see [17, 5.1].
 - (b) $R \cong Q/(0 : \mathfrak{q})$ for some Gorenstein local ring (Q, \mathfrak{q}, k) ; see [17, 6.2].
 - (c) $R \cong S \times_k T$ with $S \neq k \neq T$, except when S is a discrete valuation ring, and either $\text{edim}T = 1 > \dim T$ or $\text{edim}T = 2 = \dim T$; see [17, 3.3].
 - (d) R is Golod, except when $e - d = 2$ and $\mu_R^d = 1$; see [17, 2.4].
 In addition, $\mu_R^{d+i} > \mu_R^{d+i-1}$ is known to hold in the following cases:
 - (e) for $i \geq 3$ if R is among the exceptions in (c) and (d); see [17, 2.5, 3.2].
 - (f) for $i \gg 0$ if R is Cohen–Macaulay with $e - d \leq 3$; see [23, 1.1].
- (2) holds when R is Cohen–Macaulay and is generically Gorenstein; see [23, 2.3].
- (3) holds when R is a domain, see [26, p. 67], or is Cohen–Macaulay; see [17, 1.6].
- (4) holds in cases (a) through (f) of (1); see the references given above.

For rings with $e - d \leq 3$, Theorem 4.1 provides sharp answers to all the questions in 4.2. The next remark shows that the theorem also implies 4.3(1)(e).

Remark 4.4. Let S be a discrete valuation ring and (T, \mathfrak{t}, k) a local ring.

If $\text{edim}T = 1$ and $\mathfrak{t}^s = 0 \neq \mathfrak{t}^{s-1}$, then $R = S \times_k T$ satisfies $\widehat{R} \cong P/(wx, w^s)$, where (P, \mathfrak{p}, k) is a regular local ring and $\{w, x\}$ is a minimal generating set for \mathfrak{p} .

If T is regular of dimension 2, then $R = S \times_k T$ satisfies $\widehat{R} \cong P/(wx, wy)$, where (P, \mathfrak{p}, k) is a regular local ring and $\{w, x, y\}$ is a minimal generating set for \mathfrak{p} .

In preparation for the proof of Theorem 4.1, we establish a technical result where the hypotheses are made on the Bass series of R and the Poincaré series of k , not on the ring R itself. The argument relies on general properties of $P_k^R(t)$.

Lemma 4.5. Let (R, m, k) be a local ring and let $d, e, l, m,$ and p be as in 1.1. Assume there exist polynomials $f(t)$ and $g(t)$ in $\mathbb{Z}[t]$, such that

$$P_k^R(t) = \frac{(1+t)^{e-1}}{g(t)} \quad \text{and} \quad I_R^R(t) = t^d \cdot \frac{f(t)}{g(t)}. \tag{4.5.1}$$

(1) If $\sum_{i=0}^\infty a_i t^i$ is the Taylor expansion of $(f(t) - g(t))/(1 - t^2)$, then

$$\begin{aligned} \mu_R^d &= a_0 + 1, \\ \mu_R^{d+1} - \mu_R^d &= a_1 - 1, \\ \mu_R^{d+2} - \mu_R^{d+1} &= a_2 + (l-1)a_0. \end{aligned}$$

In case $l \geq 1$ and a_i is non-negative for $i \geq 1$ the following inequalities hold:

$$\mu_R^{d+i} - \mu_R^{d+i-1} \geq a_i + (l-1)a_{i-2} \geq a_i \quad \text{for } i \geq 2.$$

(2) If $\sum_{i=0}^\infty b_i t^i$ is the Taylor expansion of $f(t)(1+t^3)^s/(1-t^2)^2$, where s is an integer satisfying $0 \leq s \leq m-p$, then

$$\begin{aligned} \mu_R^d &= b_0, \\ \mu_R^{d+1} - \mu_R^d &= b_1, \\ \mu_R^{d+2} - \mu_R^{d+1} &= b_2 + (l-2)b_0. \end{aligned}$$

In case $l \geq 2$ and b_i is non-negative for $i \geq 1$ the following inequalities hold:

$$\mu_R^{d+i} - \mu_R^{d+i-1} \geq b_i + (l-2)b_{i-2} \geq b_i \quad \text{for } i \geq 2.$$

Remark. One has $m-p = \text{rank}_k A_2 - \text{rank}_k(A_1)^2 = \text{rank}_k(A_2/(A_1)^2) \geq 0$.

Proof. Recall that the Poincaré series of k can be written as a product

$$P_k^R(t) = \frac{(1+t)^e(1+t^3)^{m-p}}{(1-t^2)^{l+1}} \cdot \frac{\prod_{i=2}^\infty (1+t^{2i+1})^{\varepsilon_{2i+1}}}{\prod_{i=1}^\infty (1-t^{2i+2})^{\varepsilon_{2i+2}}} \tag{4.5.2}$$

with non-negative integers $\varepsilon_j \geq 0$; see [21, 3.1.2(ii), 3.1.3] or [5, 7.1.4, 7.1.5].

To compare consecutive Bass numbers, we will use the identity

$$\sum_{i \in \mathbb{Z}} (\mu_R^{d+i} - \mu_R^{d+i-1}) t^i = (1-t) \frac{I_R^R(t)}{t^d}. \tag{4.5.3}$$

(1) In view of (4.5.2), for $j \geq 0$ there exist non-negative integers c_j , such that

$$P_k^R(t) = \frac{(1+t)^e}{(1-t^2)^{l+1}} \left(1 + \sum_{j=3}^\infty c_j t^j \right).$$

Formulas (4.5.1) and (4.5.3) give equalities

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (\mu_R^{d+i} - \mu_R^{d+i-1}) t^i &= \left(1 + \frac{f(t) - g(t)}{g(t)} \right) (1-t) \\ &= 1-t + \frac{f(t) - g(t)}{1-t^2} \frac{1}{(1-t^2)^{l-1}} \left(1 + \sum_{j=3}^\infty c_j t^j \right) \\ &= 1-t + \left(\sum_{i=0}^\infty a_i t^i \right) \left(1 + (l-1)t^2 + \sum_{j=3}^\infty d_j t^j \right) \end{aligned}$$

with $d_j \geq 0$ for $j \geq 3$. They yield $\mu_R^d = a_0 + 1$ and the expressions for $\mu_R^{d+i} - \mu_R^{d+i-1}$ when $i = 1, 2$. In case $l \geq 1$, and $a_i \geq 0$ holds for $i \geq 1$, we get the following relations, where \succcurlyeq denotes a coefficient-wise inequality of formal power series

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (\mu_R^{d+i} - \mu_R^{d+i-1}) t^i &\succcurlyeq 1-t + \left(\sum_{i=0}^\infty a_i t^i \right) (1 + (l-1)t^2) \\ &= a_0 + (a_1 - 1)t + \sum_{i=2}^\infty (a_i + (l-1)a_{i-2}) t^i. \end{aligned}$$

They imply the desired lower bounds for $\mu_R^{d+i} - \mu_R^{d+i-1}$ when $i \geq 2$.

(2) In view of (4.5.2), we can write $P_k^R(t)$ in the form

$$P_k^R(t) = \frac{(1+t)^e(1+t^3)^s}{(1-t^2)^{l+1}} \left(1 + \sum_{j=3}^{\infty} c_j t^j \right)$$

with non-negative integers c_j . Formulas (4.5.1) and (4.5.3) give equalities

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (\mu_R^{d+i} - \mu_R^{d+i-1}) t^i &= \frac{f(t)(1+t^3)^s}{(1-t^2)^2} \frac{1}{(1-t^2)^{l-2}} \left(1 + \sum_{j=3}^{\infty} c_j t^j \right) \\ &= \left(\sum_{j=0}^{\infty} b_j t^j \right) \left(1 + (l-2)t^2 + \sum_{j=3}^{\infty} d_j t^j \right) \end{aligned}$$

with $d_j \geq 0$ for $j \geq 3$. They yield $\mu^d = b_0$, and the expressions for $\mu_R^{d+i} - \mu_R^{d+i-1}$ when $i = 1, 2$. When $l \geq 2$, and $b_i \geq 0$ holds for $i \geq 1$, we also have

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (\mu_R^{d+i} - \mu_R^{d+i-1}) t^i &\succcurlyeq \left(\sum_{i=0}^{\infty} b_i t^i \right) (1 + (l-2)t^2) \\ &= b_0 + b_1 t + \sum_{i=2}^{\infty} (b_i + (l-2)b_{i-2}) t^i. \end{aligned}$$

The desired lower bounds for $\mu_R^{d+i} - \mu_R^{d+i-1}$ when $i \geq 2$ follow from here. \square

The next lemma is the major step towards the proof of Theorem 4.1.

Lemma 4.6. *If (R, \mathfrak{m}, k) is a non-Gorenstein local ring with $e - d \leq 3$, then*

$$\mu_R^{d+i} \geq \mu_R^{d+i-1} + 1 \quad \text{holds for } i \geq 1,$$

unless $i = 2$ and \widehat{R} is described by (4.1.2) or (4.1.3), and then

$$\mu_R^{d+2} = \mu_R^{d+1} = 2.$$

Proof. Once again, there are several different cases to consider.

(Class **S**). Theorem 2.1 gives $(1-t-lt^2)I_R^R(t) = t^d(l+t-t^2)$, hence

$$\begin{aligned} \mu_R^d &= l, \\ \mu_R^{d+1} - \mu_R^d &= 1, \\ \mu_R^{d+2} - \mu_R^{d+1} &= l^2 - 1, \\ \mu_R^{d+i} - \mu_R^{d+i-1} &= l\mu_R^{d+i-2} \geq 2 \quad \text{for } i \geq 3. \end{aligned}$$

We get $\mu_R^{d+i} \geq \mu_R^{d+i-1} + 1$ for all $i \geq 1$, except when $i = 1$ and $l = 1$, and then $\mu_R^{d+2} = \mu_R^{d+1} = 2$. Furthermore, $l = 1$ implies (4.1.2) by Lemma 3.5(2).

For the rest of the proof we assume $c = 3$ and let $f(t)$ and $g(t)$ be the polynomials from Theorem 2.1, satisfying $P_k^R(t) = (1+t)^{e-1}/f(t)$ and $I_S^S(t) = t^d f(t)/g(t)$.

(Class **T**). The value of $f(t)$ from Theorem 2.1 provides the first equality below:

$$\begin{aligned} \frac{f(t)}{(1-t^2)^2} &= \frac{n+lt-2t^2-t^3+t^4}{(1-t^2)^2} \\ &= (n-2t^2+t^4) \sum_{j=0}^{\infty} (j+1)t^{2j} + (lt-t^3) \sum_{j=0}^{\infty} (j+1)t^{2j} \\ &= n + \sum_{j=0}^{\infty} ((l-1)j+l)t^{2j+1} + \sum_{j=1}^{\infty} (n-1)(j+1)t^{2j} \end{aligned}$$

Theorem 3.1 gives $l, n \geq 2$, so Lemma 4.5(2) applies with $s = 0$ and yields

$$\begin{aligned} \mu_R^{d+1} - \mu_R^d &= l - 1 \geq 1, \\ \mu_R^{d+2j} - \mu_R^{d+2j-1} &\geq (n-1)(j+1) \geq 2 \quad \text{for } j \geq 1, \\ \mu_R^{d+2j+1} - \mu_R^{d+2j} &\geq (l-1)j + l \geq 2 \quad \text{for } j \geq 1. \end{aligned}$$

(Classes **B**, **G**(r), and **H**($0, 0$)). **Theorem 3.1** provides uniform expressions,

$$f(t) = n + (l - r)t - (r - 1)t^2 + (p - 1)t^3 + qt^4 \quad \text{and}$$

$$g(t) = 1 - t - lt^2 - (n - p)t^3 + qt^4,$$

for the polynomials that appear in **Theorem 2.1**. Using them, we obtain

$$\frac{f(t) - g(t)}{1 - t^2} = \frac{(n - 1)(1 + t^3) + (l + 1 - r)(t + t^2)}{1 - t^2}$$

$$= (n - 1) + (l + 1 - r)t + (l + n - r) \left(\sum_{j=2}^{\infty} t^j \right).$$

Lemma 3.7 gives $l + n - r \geq l + 1 - r \geq 2$, so the series above has non-negative coefficients. Thus, **Lemma 4.5**(1) applies and yields

$$\mu_R^{d+1} - \mu_R^d = l - r \geq 1,$$

$$\mu_R^{d+i} - \mu_R^{d+i-1} \geq l + n - r \geq 2 \quad \text{for } i \geq 2.$$

(Class **H**(p, q) with $p + q \geq 1$). **Theorem 2.1** gives

$$\frac{f(t)(1 + t^3)}{(1 - t^2)^2} = \frac{n + (l - q)t - pt^2 - t^3 + t^4}{(1 - t^2)^2} (1 + t^3)$$

$$= n + (l - q)t + (2n - p)t^2 + \sum_{i=3}^{\infty} b_i t^i,$$

where for $i \geq 3$ the numbers b_i are defined by the formulas

$$b_{2j+1} = (l - q + n - p)j + l + p - q - 2 \quad \text{for } j \geq 1,$$

$$b_{2j} = (l + n - p - q)j - l + n + q + 1 \quad \text{for } j \geq 2.$$

By **Theorem 3.1**, we have $l - q \geq 1$, $n - p \geq -1$, and $n \geq 1$, hence

$$2n - p = n + (n - p) \geq n - 1 \geq 0,$$

$$b_{2j+1} \geq b_3 = n - 2 + 2(l - q) \geq n \geq 1 \quad \text{for } j \geq 1,$$

$$b_{2j} \geq b_4 = n + 2(n - p) + (l - q) \geq n \geq 1 \quad \text{for } j \geq 2.$$

Thus, **Lemma 4.5**(2) applies with $s = 1$. With the preceding inequalities, it gives

$$\mu_R^{d+1} - \mu_R^d = l - q \geq 1,$$

$$\mu_R^{d+2} - \mu_R^{d+1} = 2n - p + (l - 2)n = ln - p \geq l(n - 1) \geq 0,$$

$$\mu_R^{d+i} - \mu_R^{d+i-1} \geq b_i \geq 1 \quad \text{for } i \geq 3.$$

We conclude that $\mu_R^{d+i} \geq \mu_R^{d+i-1} + 1$ holds for all $i \geq 1$, except when $i = 2$ and $ln - p = l(n - 1) = 0$. To finish the proof, we unravel this special case.

The last two equalities force $n = 1$ and $l = p$. Now **Theorem 3.1** gives the inequalities in the string $2 = n + 1 \geq p = l \geq 2$, whence $l = p = n + 1 = 2$. Thus, we have shown that condition (iii) in **Corollary 3.3** holds with $n = p - 1 = 1$. From condition (ii) in that corollary we get $q = n = 1$, so the formulas above yield

$$\mu_R^{d+2} = \mu_R^{d+1} = \mu_R^d + l - q = n + l - q = 2.$$

On the other hand condition (iv) gives an isomorphism $\widehat{R} \cong P/(J + zR)$, where (P, \mathfrak{p}, k) is a regular local ring, J is an ideal of P contained in \mathfrak{p}^2 and minimally generated by 2 elements, and z is an element of \mathfrak{p}^2 that is regular on P/J . Since R is not complete intersection, neither is P/J , which means that $J = (wx, wy)$ for some non-zero element w in \mathfrak{p} and P -regular sequence x, y . Thus, (4.1.3) holds. \square

Proof of Theorem 4.1. We may assume that R is complete. A construction of Foxby, see [18, 3.10], then yields a finite R -module N , such that

$$\mu_R^{d+i} = \beta_i^R(N) \quad \text{for all } i \geq \dim R - d. \tag{4.7.1}$$

By [4, 1.4 and 1.6], when $c \leq 3$ the Betti sequence of every finite R -module either has strongly exponential growth or is eventually constant. Since R is not Gorenstein, **Lemma 4.6** rules out the second case for the module N in (4.7.1). Thus, $\beta_i^R(N) \geq \alpha^i$ holds for some real number $\alpha > 1$ and all $i \gg 0$.

The series $P_N^R(t)$ converges in a circle of radius $\rho > 0$, see [5, 4.1.5]. As ρ is equal to $\limsup_i \{1/\sqrt[i]{\beta_k^i(N)}\}$ we get $0 < \rho \leq 1/\alpha < 1$. Fix a real number β satisfying

$$1/\rho > \beta > 1. \tag{4.7.2}$$

Sun [30, 1.2(c)] proved that there is an integer f , such that

$$\beta_i^R(N) \geq \beta \beta_{i-1}^R(N) \text{ holds for all } i \geq f + 1. \tag{4.7.3}$$

Set $j = \max\{3, \dim R - d, f\}$ and define real numbers γ' and γ'' by the formulas

$$\begin{aligned} \gamma' &= \min \{ \beta, \mu_R^{d+1}/\mu_R^d, \min\{\mu_R^{d+i}/\mu_R^{d+i-1}\}_{3 \leq i \leq j} \}, \\ \gamma'' &= \min \{ \gamma', \mu_R^{d+2}/\mu_R^{d+1} \}. \end{aligned}$$

In view of (4.7.1) and (4.7.3), the following inequalities then hold:

$$\mu_R^{d+i} \geq \begin{cases} \gamma' \mu^{d+i-1} & \text{for } i = 1 \text{ and } i \geq 3, \\ \gamma'' \mu^{d+i-1} & \text{for } i \geq 1. \end{cases}$$

From (4.7.2) and Lemma 4.6 we see that $\gamma'' > 1$ holds unless \widehat{R} satisfies (4.1.2) or (4.1.3), else $\gamma' > 1$ holds and $\mu_R^{d+2} = \mu_R^{d+1} = 2$. This is the desired result. \square

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Appendix. Graded algebras

Here k denotes a field and B a graded k -algebra that is graded-commutative, has $B_0 = k$ and $B_i = 0$ for $i < 0$, and $\text{rank}_k B$ is finite; set $B_+ = B_{\geq 1}$.

In addition, M and N denote finitely generated graded B -modules; we set

$$H_M(t) = \sum_{i \in \mathbb{Z}} \text{rank}_k M_i t^i.$$

We treat B as a DG algebra and M, N as DG B -modules, all with zero differentials. As a consequence, $\text{Tor}_i^B(M, N)$ and $\text{Ext}_B^i(M, N)$ are formed as in 1.6. The k -spaces $\text{Tor}_i^B(M, N)$ and $\text{Ext}_B^i(M, N)$ are finite for each $i \in \mathbb{Z}$ and zero for $i \ll 0$, so Poincaré series $P_M^B(t)$ and Bass series I_B^N are defined; see (1.6.1) and (1.6.2).

Here we assemble a collection of such series, used in the body of the paper. Their computations rely on analogs of results concerning finite modules over local rings.

A.1. For the graded B -modules $\Sigma^s N$ and $N^* = \text{Hom}_k(N, k)$, see 1.5, one has

$$P_{\Sigma^s N}^B(t) = t^s \cdot P_N^B(t) \quad \text{and} \quad I_B^{\Sigma^s N}(t) = t^{-s} \cdot I_B^N. \tag{A.1.1}$$

$$P_{N^*}^B(t) = I_B^N(t) \quad \text{and} \quad I_B^{N^*}(t) = P_N^B(t). \tag{A.1.2}$$

An exact sequence $0 \rightarrow N \rightarrow G \rightarrow M \rightarrow 0$ with G free and $N \subseteq B_+ G$ yields

$$P_N^B(t) = t^{-1} \cdot (P_M^B(t) - H_{M/B_+ M}(t)). \tag{A.1.3}$$

If $B = C \otimes_k D$ and $M = T \otimes_k U$, where T is a graded C -module and U a graded D -module, then the Künneth Formula gives

$$P_M^B(t) = P_T^C(t) \cdot P_U^D(t) \quad \text{and} \quad I_B^M = I_C^T \cdot I_D^U. \tag{A.1.4}$$

The formulas above suffice to compute $P_k^B(t)$ and/or $I_B^B(t)$ in some simple cases.

Example A.2. If $B = k \ltimes W$ for some graded k -vector space $W \neq 0$, then:

$$P_k^B(t) = \frac{1}{1 - t \cdot H_W(t)}. \tag{A.2.1}$$

$$\frac{I_B^B(t)}{P_k^B(t)} = H_W(t^{-1}) - t. \tag{A.2.2}$$

Indeed, $B_+^2 = 0$ implies $P_{B_+}^B(t) = H_W(t) \cdot P_k^B(t)$, so (A.2.1) follows from (A.1.3). As $B_+(B^*) = (B^*)_{\geq 0}$, any lifting to B^* of some basis of the k -space $(B_+)^*$ minimally generates B^* over B . Thus, there is an exact sequence of graded B -modules

$$0 \rightarrow U \rightarrow B \otimes_k (B_+)^* \rightarrow B^* \rightarrow 0$$

with $U \subseteq B_+ \otimes_k (B_+)^*$, hence $B_+U = 0$. From this and (A.1.2) we obtain

$$I_B^B(t) = P_{B^*}^B(t) = H_U(t) \cdot t \cdot P_k^B(t) + H_W(t^{-1}),$$

because $H_{(B_+)^*}(t) = H_W(t^{-1})$. Since $H_B(t) = H_W(t) + 1$, the sequence also gives

$$H_U(t) = (H_W(t) + 1) \cdot H_W(t^{-1}) - (H_W(t^{-1}) + 1) = H_W(t) \cdot H_W(t^{-1}) - 1,$$

because $H_{B^*}(t) = H_B(t^{-1})$. From the last two formulas and (A.2.1), we get

$$\frac{I_B^B(t)}{P_k^B(t)} = t \cdot (H_W(t) \cdot H_W(t^{-1}) - 1) + H_W(t^{-1}) \cdot (1 - t \cdot H_W(t)) = H_W(t^{-1}) - t.$$

Example A.3. If $B = \bigwedge_k V$, where $V_i = 0$ for all even i , then there is an equality

$$P_k^B(t) = \prod_{i \in \mathbb{Z}} \frac{1}{(1 - t^{i+1})^{\text{rank}_k V_i}}. \tag{A.3.1}$$

Indeed, set $c = \text{rank}_k V$. When $c = 1$ the isomorphism $\bigwedge_k \Sigma^i k \cong k \times \Sigma^i k$ and (A.2.1) give the desired expression. For $c \geq 2$ it is obtained by induction, using the isomorphism $\bigwedge_k (V' \oplus V'') \cong \bigwedge_k V' \otimes_k \bigwedge_k V''$ and (A.1.4).

Example A.4. When B has Poincaré duality in degree s there is an equality

$$I_B^B(t) = t^{-s}. \tag{A.4.1}$$

Indeed, the condition on B means that the $B_i \rightarrow \text{Hom}_k(B_{s-i}, B_s)$, induced by the products $B_i \times B_{s-i} \rightarrow B_s$, are bijective for all $i \in \mathbb{Z}$. This implies an isomorphism $B^* \cong \Sigma^{-s} B$ of graded B -modules, so (A.1.1) and (A.1.2) give

$$I_B^B(t) = P_{B^*}^B(t) = P_{\Sigma^{-s} B}^B(t) = t^{-s} \cdot P_B^B(t) = t^{-s}.$$

The next result is an analog of a theorem of Gulliksen; see [20, Thm. 2]. The original proof, or the one for [22, Cor. 2], carries over essentially without changes.

A.5. If $B = C \rtimes W$ for some graded k -algebra C and graded C -module W , then

$$\frac{1}{P_k^B(t)} = \frac{1}{P_k^C(t)} - t \cdot \frac{P_W^C(t)}{P_k^C(t)}. \tag{A.5.1}$$

Example A.6. If $B = C \rtimes \Sigma^s(C^*)$ with $C = k \rtimes W$, then the following hold:

$$P_k^B(t) = \frac{1}{1 - t \cdot H_W(t) - t^{s+1} \cdot H_W(t^{-1}) + t^{s+2}}. \tag{A.6.1}$$

$$\frac{I_B^B(t)}{P_k^B(t)} = \frac{1 - t \cdot H_W(t) - t^{s+1} \cdot H_W(t^{-1}) + t^{s+2}}{t^s}. \tag{A.6.2}$$

Indeed, the isomorphism of graded B -modules $\Sigma^s(C^*) \cong (\Sigma^{-s} C)^*$ and (A.1.2) give $P_{\Sigma^s(C^*)}^C(t) = I_C^{\Sigma^{-s} C}(t) = t^s \cdot I_C^C(t)$. Now (A.5.1), (A.2.1) and (A.2.2) yield

$$\frac{1}{P_k^B(t)} = \frac{1}{P_k^C(t)} - t \cdot t^s \cdot \frac{I_C^C(t)}{P_k^C(t)} = \frac{1 - t \cdot H_W(t) - t^{s+1} \cdot H_W(t^{-1}) + t^{s+2}}{t^s}.$$

Since B has Poincaré duality in degree s , (A.4.1) and (A.6.1) imply (A.6.2).

The following analog of a result of Lescot, see [24, 1.8(2)], can be proved along the lines of the original argument, but subtle changes are needed. Instead of going into those details, we refer to [10] for a direct proof covering both cases.

A.7. If $B_+ \neq 0$, then the following equality holds:

$$\frac{I_B^B(t)}{P_k^B(t)} = \frac{I_{B_+}^{B_+}(t)}{P_k^{B_+}(t)} - t. \tag{A.7.1}$$

In the last two examples we adapt the arguments for [24, 3.2(1) and 1.9].

Example A.8. If $B = C \times W$ for some graded k -algebra C with $C_+ \neq 0$ and graded C -module W with $C_+W = 0$, then the following equality holds:

$$\frac{I_B^B(t)}{P_k^B(t)} = \frac{I_C^C(t)}{P_k^C(t)} + H_W(t^{-1}). \quad (\text{A.8.1})$$

Indeed, C is an algebra retract of B . The proof of [22, Thm. 1] transfers *verbatim* and gives $P_N^B(t)/P_k^B(t) = P_N^C(t)/P_k^C(t)$ for each graded C -module N , viewed as a B -module via the natural homomorphism $B \rightarrow C$. By (A.1.2), this implies that $I_B^N(t)/P_k^B(t) = I_C^N(t)/P_k^C(t)$ holds as well. Since $B_+ = C_+ \oplus W$ as graded B -modules, using the preceding equality and (A.7.1) (twice) we obtain

$$\frac{I_B^B(t)}{P_k^B(t)} = \frac{I_{B_+}^{C_+}(t)}{P_k^B(t)} + \frac{I_B^W(t)}{P_k^B(t)} - t = \frac{I_C^{C_+}(t)}{P_k^C(t)} + H_W(t^{-1}) - t = \frac{I_C^C(t)}{P_k^C(t)} + H_W(t^{-1}).$$

Example A.9. If $B = E/E_{>s}$, where E is a graded k -algebra that has Poincaré duality in degree s , then the following equality holds:

$$\frac{I_B^B(t)}{P_k^B(t)} = t^{-s-1} \cdot \left(1 - \frac{1}{P_k^B(t)}\right) - t. \quad (\text{A.9.1})$$

Indeed, set $(-)' = \text{Hom}_k(-, \Sigma^s k)$. Applying the functor $(-)'$ to the exact sequence $0 \rightarrow B_+ \rightarrow B \rightarrow k \rightarrow 0$ we get $(B_+)' \cong B'/B'_{>s}$ as graded B -modules. Since $E \cong E'$ as graded E -modules, $(-)'$ applied to $0 \rightarrow \Sigma^s k \rightarrow E \rightarrow B \rightarrow 0$ gives $B' \cong E_+$, hence $B'/B'_{>s} \cong E_+/E_{>s} \cong B_+$, and thus $B_+ \cong (B_+)' \cong (\Sigma^{-s} B_+)^*$.

Now from formulas (A.1.2), (A.1.3) and (A.7.1) we obtain

$$\frac{I_B^B(t)}{P_k^B(t)} + t = \frac{I_B^{(\Sigma^{-s} B_+)^*}(t)}{P_k^B(t)} = \frac{P_{\Sigma^{-s} B_+}^B(t)}{P_k^B(t)} = t^{-s} \cdot t^{-1} \cdot \frac{P_k^B(t) - 1}{P_k^B(t)}.$$

All the labeled formulas in this appendix are used in computations in Section 2.

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