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Growing Forests in Abelian p-Groups

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1. SIMPLY PRESENTED p-GROUPS

At the center of the theory of abelian p-groups are the classical theorems of Ulm, Zippin, and Kaplansky, going back to the thirties, that classify countable p-groups by their Ulm invariants: the uniqueness theorem is referred to as *Ulm's theorem*, the existence theorem as *Zippin's theorem*. For each ordinal α, the αth Ulm invariant of G can be defined as the dimension $f_G(\alpha)$ of the vector space (over the p-element field)

$$\frac{p^{\alpha}G[p]}{p^{\alpha+1}G[p]},$$

where $p^{\alpha}G$ is defined inductively by $p^{\beta}G = \bigcap_{\alpha<\beta} pp^{\alpha}G$, and $H[p] = \{x \in H : px = 0\}$. One also sets $p^{\infty}G = \bigcap_{\alpha} p^{\alpha}G$, and $f_G(\infty) = p^{\infty}G[p]$. As the structure of $p^{\infty}G$ is quite simple, and this subgroup is always a summand, attention is usually focused on *reduced* groups G, those for which $p^{\infty}G = 0$.

The *height* of an element x of G is defined by ht $x = \max\{\alpha : x \in p^{\alpha}G\}$, which always exists, and may be ∞. If G is finite, or any direct sum of cyclic groups, then $f_G(n)$ is the number of cyclic summands of G of order p^{n+1}. Zippin's theorem gives necessary and sufficient conditions on a function f (one says that f is *admissible*) for there to exist a countable p-group G with $f_G = f$.

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This theory was extended in the sixties to p-groups G that satisfy the following three equivalent conditions:

- G is simply presented,
- G is totally projective,
- G satisfies Hill's Axiom 3.

The first condition is attractive both for its pleasant graphical picture of trees, and because at the same time that it says what the groups are, it tells how to construct them. The other conditions are characterizing properties of this class of groups which, for all one could tell in advance, might conceivably not apply to any groups at all.

I will deal with the first and third conditions here. I feel that *simply presented* is the correct definition; indeed the purpose of this paper is to finish a development of the theory which exploits tree structures as much as possible. Hill's Axiom 3 allows a unified treatment of the fact that countable p-groups, and summands of simply presented p-groups, are simply presented, for it is immediate that Axiom 3 holds for both kinds of groups. We define a weak version of Axiom 3 just before Lemma 4. Roughly speaking, Axiom 3 says that the group has lots of *nice subgroups*—subgroups such that each coset contains an element of maximum height. This idea was isolated by Hill and derives from the fact that a countable p-group is the union of a chain of finite subgroups, and finite subgroups are nice—features that are used heavily in the classical proof of Ulm's theorem.

By a *torsion forest* we mean a set F together with a partial function $\pi : F \to F$ such that for each $x \in F$ there exists $n \in \mathbf{N}$ such that $\pi^n x$ is undefined. We refer to πx as the *parent* of x, and to x as a *child* of πx. Given a torsion forest F, we can construct a p-group $S(F)$ by taking F as a set of generators, setting $px = y$ if $\pi x = y$, and setting $px = 0$ if πx is undefined. A *simply presented* p-group is defined to be a group that is isomorphic to $S(F)$ for some torsion forest F.

The nonzero elements of a p-group form a torsion forest. A *partial p-basis* in a p-group is a subforest F of the nonzero elements such that the height of $\sum n_i x_i$ is the minimum of the heights of the x_i whenever the x_i are in F and $0 < n_i < p$. A *p-basis* is a partial p-basis that spans the group. It is not difficult to show that a p-group admits a p-basis if and only if it is simply presented. Thus to show that a p-group is simply presented, we grow a p-basis in the group.

Rogers [5] proved Ulm's theorem for simply presented p-groups by showing how to compute Ulm invariants directly from forests, and how to transform forests with the same Ulm invariants into each other (the W-theorem) in such a way that the corresponding groups were manifestly isomorphic. On this approach to Ulm's theorem, two important properties are left unaddressed: that summands of simply presented p-groups are simply presented, and that countable p-groups are simply presented. The former property is immediate for totally projective groups, and is certainly required for a complete theory; the latter is needed to derive the classical case from the general theory.

The notion of a simply presented p-group was introduced by Crawley and Hales, who carried out a complete, independent development in [1, 2]. This included Ulm's theorem, Zippin's theorem, and a proof that the class of such groups is closed under summands. Later, Hunter and Walker [4] gave a direct proof that countable p-groups are simply presented.

In [2], the summand question was resolved by modifying the proof of Ulm's theorem to show that if $A \oplus B$ and $A' \oplus B'$ are simply presented, A and A' have the same Ulm invariants, and B and B' have the same Ulm invariants, then A is isomorphic to A'. To finish off the summand question it was also necessary to show that the Ulm invariants of a summand are admissible, so that a simple presented group with those Ulm invariants could be constructed using Zippin's theorem.

In this paper we present a proof that summands of simply presented p-groups are simply presented that is independent of Ulm's theorem. The same techniques are used to show that countable p-groups are simply presented. In fact, we show that a summand of an Axiom-3 p-group is simply presented, thus settling both problems, and showing that Axiom-3 p-groups are simply presented, at one go.

2. CONSTRUCTING p-BASES

There are two problems involved in constructing a p-basis for a summand of a simply presented p-group. One is to keep the subgroup that is generated by the partial p-basis nice; the other is to ensure that the partial p-basis constructed at each stage can be extended. Of course the second is all we actually need, but the first is a minimum requirement because every subforest of a p-basis generates a nice subgroup. The first problem can be solved by coordinating our construction with a given p-basis of the original group—more generally, by coordinating our construction with a collection of nice subgroups of the original group. The second problem is solved by restricting the kind of forests constructed, so that there will always be enough relative Ulm invariants. Otherwise, it is entirely possible to run out of relative Ulm invariants, as the example in [4] of a nice, unextendable partial p-basis in a countable p-group shows.

If X is a valuated forest, then the *derived Ulm invariant* $g_X(\alpha)$ of X may be defined as the number of nodes x of X of value μ such that for some

$\beta < \alpha$, every child of x has value less than β. Call a partial p-basis F in a reduced p-group *suitable* if it is nice and every node of value $\alpha + 1$ has a child of value α. Recall that if H is a subgroup of G, then the *relative Ulm invariant* $f_{G,H}(\alpha)$ is defined as

$$\frac{\{x \in G : \text{ht } x \geq \alpha \text{ and ht } px > \alpha + 1\}}{\{x \in G : \text{ht } x > \alpha\} + \{x \in H : \text{ht } x \geq \alpha \text{ and ht } px > \alpha + 1\}}$$

where all heights are computed in G. Actually $f_{G,H}(\alpha)$ normally denotes the dimension of this vector space over $\mathbf{Z}/p\mathbf{Z}$, but it is often more convenient to think of it as the space itself.

The following may be derived from [3, Theorem 11].

LEMMA 1. *Let G be a reduced p-group, X a suitable partial p-basis of G, and H the subgroup of G generated by X. If $g_X(\alpha) \neq 0$, for some limit ordinal α, then for each $\beta < \alpha$ there exists γ with $\beta < \gamma < \alpha$ such that $f_{G,H}(\gamma) \neq 0$.*

Proof. As $g_X(\alpha) \neq 0$, there exists $x \in X$ such that ht $x = \alpha$, and, possibly increasing β, that every child of x in X has height less than β. Choose z in G such that $pz = x$ and ht $z > \beta$. Let w be an element of maximal height γ in $z + H$. If ht $pw = \gamma + 1$, then we can modify w by nodes of height γ so that ht $pw > \gamma + 1$ because each node of height $\gamma + 1$ has a child of height γ. Then w represents a nonzero element of $f_{G,H}(\gamma)$. ∎

So if H is generated by a suitable partial p-basis, then there are a lot of relative Ulm invariants around. It is relative Ulm invariants that we need to extend valuated forests. Recall that an element x is *H-proper* if ht $x \geq$ ht$(x + h)$ for each $h \in H$. We shall say that x is *F-proper*, for an arbitrary subset F, if x is $\langle F \rangle$-proper.

LEMMA 2. *Let F be a suitable partial p-basis in a reduced p-group G. Let $x \in F$ and $\alpha <$ ht x. Then there exists $y \in G$, such that $py = x$ and* ht $y \geq \alpha$, *and $F \cup \{y\}$ is a partial p-basis.*

Proof. If F contains y such that $py = x$ and ht $y \geq \alpha$, then we are done. Otherwise ht x is a limit ordinal and $g_F(\text{ht } x) \neq 0$, so, applying Lemma 1 there is an F-proper $x \in G[p]$ with $\alpha <$ ht $z <$ ht x (relative Ulm invariant). Choose y' so that $py' = x$ and ht $y' >$ ht z. Then $y = y' + z$ is the desired element. ∎

We can always give a node of value $\alpha + 1$ a child of value α if it does not have one. So if F' is a finite extension of a suitable forest, then F' has a finite suitable extension.

LEMMA 3. *Let F be a suitable partial p-basis in a reduced p-group G. Let g be an element of G. Then we can extend F finitely to a suitable partial p-basis F' such that g is in the span of F'.*

Proof. We may assume that pg is in the span of F, say $pg = \sum u_i x_i$ where $0 < u_i < p$. As F is nice, we may assume that g is F-proper. By Lemma 2 we may extend F finitely to F' so that there exist $y_i \in F'$ such that $x_i = py_i$ and $\mathrm{ht}\, y_i \geq \mathrm{ht}\, g$, and $y_i \in F$ unless x_i is a limit ordinal, in which case $\mathrm{ht}\, y_i > \mathrm{ht}\, g$. Then g is F'-proper, so we may assume that $pg = 0$. Adjoin g to F'. ∎

A *nice system* in a p-group G is a set $\mathcal{N}$ of nice subgroups such that

- $\mathcal{N}$ is closed under unions of chains,
- if $N \in \mathcal{N}$, and S is countable, then there exists $N' \in \mathcal{N}$ containing N and S so that N'/N is countable.

Note that 0, as the union of the empty chain, is in $\mathcal{N}$. If G is a countable p-group, then the set $\{0, G\}$ is a nice system. The set of subgroups generated by subtrees of a p-basis from a nice system in a simply presented p-group. A group satisfies *Axiom* 3 if it has a nice system.

LEMMA 4. *Let $\mathcal{N}$ be a nice system in $G = A_1 \oplus A_2$. If $T_i \subset A_i$ are suitable partial p-bases such that $\langle T_1 \rangle \oplus \langle T_2 \rangle$ is in $\mathcal{N}$, and g is an element of G, then for each i there is a suitable partial p-basis T_i' extending T_i such that $g \in \langle T_1' \rangle \oplus \langle T_2' \rangle \in \mathcal{N}$.*

Proof. We construct sequences of suitable partial p-bases $T_i^0 \subset T_i^1 \subset T_i^2 \subset \cdots$ and subgroups N_n in $\mathcal{N}$, such that

- $T_i^0 = T_i$ for each i, and $g \in N_0$.
- $\langle T_1^n \rangle \oplus \langle T_2^n \rangle$ is a subgroup of countable index in $\langle T_1^{n+1} \rangle \oplus \langle T_2^{n+1} \rangle$ for each n.
- $\langle T_1^n \rangle \oplus \langle T_2^n \rangle \subset N_n \subset \langle T_1^{n+1} \rangle \oplus \langle T_2^{n+1} \rangle$ for each n.

Setting $T_i' = \bigcup T_i^n$, and noting that the union of a chain of suitable forests is suitable, completes the proof.

As $\langle T_1 \rangle \oplus \langle T_2 \rangle$ is in $\mathcal{N}$, there exists N_0 in $\mathcal{N}$, containing g, such that $\langle T_1 \rangle \oplus \langle T_2 \rangle$ is of countable index in N_0. Given that $\langle T_1^n \rangle \oplus \langle T_2^n \rangle$ is a subgroup of countable index in N_n, for each i let S_i be a countable subset of A_i such that $N_n \subset (\langle T_1^n \rangle + \langle S_1 \rangle) \oplus (\langle T_2^n \rangle + \langle S_2 \rangle)$, and construct T_1^{n+1} and T_2^{n+1} so that $\langle T_i^{n+1} \rangle$ contains S_i by repeated application of Lemma 4. Finally, given that N_n in $\mathcal{N}$ is of countable index in $\langle T_1^{n+1} \rangle \oplus \langle T_2^{n+1} \rangle$, there exists N_{n+1} in $\mathcal{N}$ such that $\langle T_1^{n+1} \rangle \oplus \langle T_2^{n+1} \rangle$ is of countable index in N_{n+1} by the definition of a nice system. ∎

THEOREM 5. *Let G be an abelian p-group that satisfies Axiom* 3. *Then any summand of G is simply presented.*

Proof. Let $\mathscr{N}$ be a nice system in $G = A_1 \oplus A_2$ and consider the set of pairs (T_1, T_2) where T_i is a suitable partial p-basis in A_i and $\langle T_1 \rangle \oplus \langle T_2 \rangle$ is in $\mathscr{N}$. Partial order these pairs by extension. They are clearly closed under unions of chains. Zorn's lemma produces a maximal element (T_1, T_2), and Lemma 4 says that $\langle T_1 \rangle \oplus \langle T_2 \rangle = G$. ∎

As countable p-groups and simply presented p-groups satisfy Axiom 3, this shows that countable p-groups and summands of simply presented p-groups are simply presented.

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